

# On the $K$ -theory of higher rank graph $C^*$ -algebras

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ABSTRACT. Given a row-finite  $k$ -graph  $\Lambda$  with no sources we investigate the  $K$ -theory of the higher rank graph  $C^*$ -algebra,  $C^*(\Lambda)$ . When  $k = 2$  we are able to give explicit formulae to calculate the  $K$ -groups of  $C^*(\Lambda)$ . The  $K$ -groups of  $C^*(\Lambda)$  for  $k > 2$  can be calculated under certain circumstances and we consider the case  $k = 3$ . We prove that for arbitrary  $k$ , the torsion-free rank of  $K_0(C^*(\Lambda))$  and  $K_1(C^*(\Lambda))$  are equal when  $C^*(\Lambda)$  is unital, and for  $k = 2$  we determine the position of the class of the unit of  $C^*(\Lambda)$  in  $K_0(C^*(\Lambda))$ .

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## 1. Introduction

In [31] Spielberg realised that a crossed product algebra  $C(\Omega) \rtimes \Gamma$ , where  $\Omega$  is the boundary of a certain tree and  $\Gamma$  is a free group, is isomorphic to a Cuntz–Krieger algebra [6, 5]. Noticing that such a tree may be regarded as an affine building of type  $\tilde{A}_1$ , Robertson and Steger studied the situation when a group  $\Gamma$  acts simply transitively on the vertices of an affine building of type  $\tilde{A}_2$  with boundary  $\Omega$  [23]. They found that the corresponding crossed

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product algebra  $C(\Omega) \rtimes \Gamma$  is generated by two Cuntz–Krieger algebras. This led them to define a  $C^*$ -algebra  $\mathcal{A}$  via a finite sequence of finite 0-1 matrices (i.e., matrices with entries in  $\{0, 1\}$ )  $M_1, \dots, M_r$  satisfying certain conditions (H0)–(H3), such that  $\mathcal{A}$  is generated by  $r$  Cuntz–Krieger algebras, one for each  $M_1, \dots, M_r$ . Accordingly they named their algebras higher rank Cuntz–Krieger algebras, the rank being  $r$ .

Kumjian and Pask [13] noticed that Robertson and Steger had constructed their algebras from a set,  $W$  of (*higher rank*) *words* in a finite *alphabet*  $A$  — the common index set of the 0-1 matrices — and realised that  $W$  could be thought of as a special case of a generalised directed graph — a higher rank graph or  $k$ -graph, where  $k$  is the rank. Subsequently, Kumjian and Pask associated a  $C^*$ -algebra,  $C^*(\Lambda)$  to a higher rank graph  $\Lambda$  and showed that  $\mathcal{A} \cong C^*(W)$  [13, Corollary 3.5 (ii)]. Moreover, they derived a number of results elucidating the structure of higher rank graph  $C^*$ -algebras. They show in [13, Theorem 5.5] that a simple, purely infinite  $k$ -graph  $C^*$ -algebra  $C^*(\Lambda)$  may be classified by its  $K$ -theory. This is a consequence of  $C^*(\Lambda)$  satisfying the hypotheses of the Kirchberg–Phillips classification theorem ([12, 18]). Moreover, criteria on the underlying  $k$ -graph  $\Lambda$  were found that decided when  $C^*(\Lambda)$  was simple and purely infinite (see [13, Proposition 4.8, Proposition 4.9] and [28, Proposition 8.8]). Thus a step towards the classification of  $k$ -graph  $C^*$ -algebras is the computation of their  $K$ -groups.

In [25, Proposition 4.1] Robertson and Steger proved that the  $K$ -groups of a rank 2 Cuntz–Krieger algebra is given in terms of the homology of a certain chain complex, whose differentials are defined in terms of  $M_1$  and  $M_2$ . Their proof relied on the fact that a rank 2 Cuntz–Krieger algebra is stably isomorphic to a crossed product of an AF-algebra by  $\mathbb{Z}^2$ . We will generalise their method to provide explicit formulae for the  $K$ -groups of 2-graph  $C^*$ -algebras and to gain information on the  $K$ -groups of  $k$ -graph  $C^*$ -algebras for  $k > 2$ .

The rest of this paper is organised as follows. We begin in §2 by recalling the fundamental definitions relating to higher rank graphs and their  $C^*$ -algebras we will need from [13].

In §3 we use the fact that the  $C^*$ -algebra of a row-finite  $k$ -graph  $\Lambda$  with no sources is stably isomorphic to a crossed product of an AF algebra,  $B$ , by  $\mathbb{Z}^k$  ([13, Theorem 5.5]) to apply a theorem of Kasparov [11, 6.10 Theorem] to deduce that there is a homological spectral sequence ([34, Chapter 5]) converging to  $K_*(C^*(\Lambda))$  with initial term given by  $E_{p,q}^2 \cong H_p(\mathbb{Z}^k, K_q(B))$  (see [3] for the definition of the homology of a group  $G$  with coefficients in a left  $G$ -module  $M$ , denoted by  $H_*(G, M)$ ). We will see that it suffices to compute  $H_*(\mathbb{Z}^k, K_0(B))$ . It transpires that  $H_*(\mathbb{Z}^k, K_0(B))$  is given by the so-called vertex matrices of  $\Lambda$ . These are matrices with nonnegative integer entries that encode the structure of the category  $\Lambda$ . Next we assemble the results of §3 and state them in our main theorem, Theorem 3.15. We then specialise to the cases  $k = 2$  and  $k = 3$ . For  $k = 2$  a complete description

of the  $K$ -groups in terms of the vertex matrices can be given. For  $k = 3$  we illustrate how Theorem 3.15 can be used to give a description of the  $K$ -groups of 3-graph  $C^*$ -algebras under stronger hypotheses.

In section §4 we consider the  $K$ -theory of unital  $k$ -graph  $C^*$ -algebras. We show that the torsion-free rank of  $K_0(C^*(\Lambda))$  is equal to that of  $K_1(C^*(\Lambda))$  when  $C^*(\Lambda)$  is unital and give formulae for the torsion-free rank and torsion parts of the  $K$ -groups of 2-graph  $C^*$ -algebras.

We conclude with §5, in which we consider some immediate applications to the classification of  $k$ -graph  $C^*$ -algebras by means of the Kirchberg–Phillips classification theorem. We also consider some simple examples of  $K$ -group calculations using the results derived in the previous sections.

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## 2. Preliminaries

By the usual slight abuse of notation we shall let the set of morphisms of a small category  $\Lambda$  be denoted by  $\Lambda$  and identify an object of  $\Lambda$  with its corresponding identity morphism. Also note that a monoid  $M$  (and hence a group) can be considered as a category with one object and morphism set equal to  $M$ , with composition given by multiplication in the monoid. For convenience of notation, we shall denote a monoid and its associated category by the same symbol.

The following notation will be used throughout this paper. We let  $\mathbb{N}$  denote the abelian monoid of nonnegative integers and we let  $\mathbb{Z}$  be the group of integers. For a positive integer  $k$ , we let  $\mathbb{N}^k$  be the product monoid viewed as a category. Similarly, we let  $\mathbb{Z}^k$  be the product group viewed, where appropriate, as a category. Let  $\{e_i\}_{i=1}^k$  be the canonical generators of  $\mathbb{N}^k$  as a monoid and  $\mathbb{Z}^k$  as a group. Moreover, we choose to endow  $\mathbb{N}^k$  and  $\mathbb{Z}^k$  with the coordinatewise order induced by the usual order on  $\mathbb{N}$  and  $\mathbb{Z}$ , i.e., for all  $m, n \in \mathbb{Z}^k$   $m \leq n \iff m - n \in \mathbb{N}^k$ .

We will denote by  $\mathbb{K}(\mathcal{H})$  the  $C^*$ -algebra of compact operators on a Hilbert space  $\mathcal{H}$ . Where the Hilbert space  $\mathcal{H}$  is separable and of infinite dimension we write  $\mathbb{K}$  for  $\mathbb{K}(\mathcal{H})$ .

The concept of a *higher rank graph* or *k-graph* ( $k = 1, 2, \dots$  being the rank) was introduced by A. Kumjian and D. Pask in [13]. We recall their definition of a *k-graph*.

**Definition 2.1** ([13, Definitions 1.1]). A *k-graph* (rank  $k$  graph or higher rank graph)  $(\Lambda, d)$  consists of a countable small category  $\Lambda$  (with range and source maps  $r$  and  $s$  respectively) together with a functor  $d : \Lambda \rightarrow \mathbb{N}^k$  satisfying the *factorisation property*: for every  $\lambda \in \Lambda$  and  $m, n \in \mathbb{N}^k$  with  $d(\lambda) = m + n$ , there are unique elements  $\mu, \nu \in \Lambda$  such that  $\lambda = \mu\nu$  and  $d(\mu) = m$ ,  $d(\nu) = n$ . For  $n \in \mathbb{N}^k$  and  $v \in \Lambda^0$  we write  $\Lambda^n := d^{-1}(n)$ ,  $\Lambda(v) := r^{-1}(v)$  and  $\Lambda^n(v) := \{\lambda \in \Lambda^n \mid r(\lambda) = v\}$ .

**Definition 2.2** ([13, Definitions 1.4]). A *k-graph*  $\Lambda$  is *row-finite* if for each  $n \in \mathbb{N}^k$  and  $v \in \Lambda^0$  the set  $\Lambda^n(v)$  is finite. We say that  $\Lambda$  has *no sources* if  $\Lambda^n(v) \neq \emptyset$  for all  $v \in \Lambda^0$  and  $n \in \mathbb{N}^k$ .

Unless stated otherwise, we will assume that each higher rank graph in this paper is row-finite with no sources. Furthermore, we shall denote such a generic higher rank graph by  $(\Lambda, d)$  (or more succinctly  $\Lambda$  with the understanding that the degree functor will be denoted by  $d$ ).

We refer to [15] as an appropriate reference on category theory. There is no need for a detailed knowledge of category theory as we will be interested in the combinatorial graph-like nature of higher rank graphs. As the name suggests a higher rank graph can be thought of as a higher rank analogue of a directed graph. Indeed, every 1-graph is isomorphic (in the natural sense) to the category of finite paths of a directed graph ([13, Example 1.3]). By [13, Remarks 1.2]  $\Lambda^0$  is the set of identity morphisms of  $\Lambda$ . Indeed it is fruitful to view  $\Lambda^0$  as a set of vertices and  $\Lambda$  as a set of (coloured) paths with composition in  $\Lambda$  being concatenation of paths. This viewpoint is discussed further in [7, 20].

In the sequel we will use the following higher rank graph constructions devised by Kumjian and Pask. For further examples of *k-graphs* see, for example, [13, 20, 21, 29].

**Examples 2.3.** (1) Let  $\Delta_k$  be the category with morphism set equal to  $\{(m, n) \in \mathbb{Z}^k \times \mathbb{Z}^k \mid m \leq n\}$ , object set equal to  $\{(m, m) \mid m \in \mathbb{Z}^k\}$ , structure maps defined by  $r(m, n) = m$ ,  $s(m, n) = n$  and composition defined by  $(m, l)(l, n) = (m, n)$  for all  $m, l, n \in \mathbb{Z}^k$  such that  $m \leq l \leq n$ . One may define a degree functor  $d : \Delta_k \rightarrow \mathbb{N}^k$  by  $d(m, n) = n - m$  so that  $(\Delta_k, d)$  is a *k-graph*. Furthermore, it is straightforward to check that  $(\Delta_k, d)$  is row-finite and has no sources.

(2) The product higher rank graph ([13, Proposition 1.8]): Let  $(\Lambda_1, d_1)$  and  $(\Lambda_2, d_2)$  be rank  $k_1$  and  $k_2$  graphs respectively. Then their product higher rank graph  $(\Lambda_1 \times \Lambda_2, d_1 \times d_2)$  is a  $(k_1 + k_2)$ -graph, where  $\Lambda_1 \times \Lambda_2$  is the product category and the degree functor  $d_1 \times d_2 : \Lambda_1 \times \Lambda_2 \rightarrow \mathbb{N}^{k_1 + k_2}$  is given by  $d_1 \times d_2(\lambda_1, \lambda_2) = (d_1(\lambda_1), d_2(\lambda_2)) \in \mathbb{N}^{k_1} \times \mathbb{N}^{k_2}$  for  $\lambda_1 \in \Lambda_1$  and  $\lambda_2 \in \Lambda_2$ .

- (3) The skew-product higher rank graph ([13, Definition 5.1]): Given a countable group  $G$ , a  $k$ -graph  $\Lambda$  and a functor  $c : \Lambda \rightarrow G$ , the *skew-product  $k$ -graph*  $G \times_c \Lambda$  consists of a category with object set identified with  $G \times \Lambda^0$  and morphism set identified with  $G \times \Lambda$ . The structure maps are given by:  $s(g, \lambda) = (gc(\lambda), s(\lambda))$  and  $r(g, \lambda) = (g, r(\lambda))$ . If  $s(\lambda) = r(\mu)$  then  $(g, \lambda)$  and  $(gc(\lambda), \mu)$  are composable in  $G \times_c \Lambda$  and  $(g, \lambda)(gc(\lambda), \mu) = (g, \lambda\mu)$ . The degree map is given by  $d(g, \lambda) = d(\lambda)$ . Furthermore,  $G$  acts freely on  $G \times_c \Lambda$  by  $g \cdot (h, \lambda) \mapsto (gh, \lambda)$  for all  $g, h \in G$  and  $\lambda \in \Lambda$  (see [13, Remark 5.6] and its preceding paragraph).

To each row-finite  $k$ -graph with no sources, Kumjian and Pask associated an unique  $C^*$ -algebra in the following way.

**Definition 2.4.** ([13, Definitions 1.5]) Let  $\Lambda$  be a row-finite  $k$ -graph with no sources. Then  $C^*(\Lambda)$  is defined to be the universal  $C^*$ -algebra generated by a family  $\{s_\lambda \mid \lambda \in \Lambda\}$  of partial isometries satisfying:

- (i)  $\{s_v \mid v \in \Lambda^0\}$  is a family of mutually orthogonal projections.
- (ii)  $s_{\lambda\mu} = s_\lambda s_\mu$  for all  $\lambda, \mu \in \Lambda$  such that  $s(\lambda) = r(\mu)$ .
- (iii)  $s_\lambda^* s_\lambda = s_{s(\lambda)}$  for all  $\lambda \in \Lambda$ .
- (iv) for all  $v \in \Lambda^0$  and  $n \in \mathbb{N}^k$  we have  $s_v = \sum_{\lambda \in \Lambda^n(v)} s_\lambda s_\lambda^*$ .

For  $\lambda \in \Lambda$ , define  $p_\lambda := s_\lambda s_\lambda^*$ . A family of partial isometries satisfying (i)–(iv) above is called a *\*-representation* of  $\Lambda$ .

We consider the following  $C^*$ -algebras associated with the constructions noted in Examples 2.3, which will be useful in the sequel.

- Examples 2.5.** (1) Let  $\Delta_k$  be the row-finite  $k$ -graph with no sources defined in Examples 2.3.1. Let  $e_{m,n} := s_{(m,q)} s_{(n,q)}^*$ ,  $m, n \in \mathbb{Z}^k$ ,  $q := \sup\{m, n\}$ . Since  $\{e_{m,n} \mid m, n \in \mathbb{Z}^k\}$  is a complete system of matrix units, we have  $C^*(\Delta_k) \cong \mathbb{K}(\ell^2(\mathbb{Z}^k))$  (cf. [13, Examples 1.7 (ii)]).
- (2) Let  $(\Lambda_i, d_i)$  be a row-finite  $k_i$ -graph with no sources for  $i = 1, 2$ . Then

$$C^*(\Lambda_1 \times \Lambda_2) \cong C^*(\Lambda_1) \otimes C^*(\Lambda_2)$$

by [13, Corollary 3.5 (iv)].<sup>1</sup>

- (3) Let  $G$  be a countable group,  $\Lambda$  a row-finite  $k$ -graph with no sources and  $c : \Lambda \rightarrow G$  a functor. Then the action of  $G$  on  $G \times_c \Lambda$  described in Examples 2.3.3 induces an action  $\beta : G \rightarrow \text{Aut}(C^*(G \times_c \Lambda))$  such that  $\beta_g(s_{(h,\lambda)}) = s_{(gh,\lambda)}$ . Furthermore  $C^*(G \times_c \Lambda) \rtimes_\beta G \cong C^*(\Lambda) \otimes \mathbb{K}(\ell^2(G))$  [13, Theorem 5.7].

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<sup>1</sup>The  $C^*$ -algebra of a row-finite higher rank graph with no sources is nuclear [13, Theorem 5.5].

### 3. The $K$ -groups of $k$ -graph $C^*$ -algebras

For the remainder of this paper we shall denote by  $B_\Lambda$  (or simply  $B$  when there is no ambiguity) the  $C^*$ -algebra of the skew-product of a row-finite  $k$ -graph  $(\Lambda, d)$ , with no sources, by  $\mathbb{Z}^k$  via the degree functor regarded as a functor into  $\mathbb{Z}^k$ , i.e.,  $B := C^*(\mathbb{Z}^k \times_d \Lambda)$ , and by  $\beta$  the action of  $\mathbb{Z}^k$  on  $B$  as described in Examples 2.5.3. Note that by [13, Corollary 5.3 and Theorem 5.5] and Takesaki–Takai duality [32] (or [13, Theorem 5.7], cf. Examples 2.5.3),  $C^*(\Lambda)$  is stably isomorphic to the crossed product of an AF-algebra,  $B$ , by  $\mathbb{Z}^k$ , i.e.,  $B \rtimes_\beta \mathbb{Z}^k \cong C^*(\Lambda) \otimes \mathbb{K}$ . Therefore

$$K_0(C^*(\Lambda)) \cong K_0(B \rtimes_\beta \mathbb{Z}^k).$$

It will be useful for us in the sequel to have an explicit description of how an isomorphism  $K_0(C^*(\Lambda)) \rightarrow K_0(B \rtimes_\beta \mathbb{Z}^k)$  acts on the  $K_0$ -class of a canonical projection  $p_v$ ,  $v \in \Lambda^0$  in  $C^*(\Lambda)$ . To this end, we prefer to investigate how the isomorphism acts by using an alternative approach to that outlined above. This we do below by using a standard technique in  $k$ -graph  $C^*$ -algebra theory, namely by utilising the gauge-invariant uniqueness theorem for  $k$ -graph  $C^*$ -algebras [13, Theorem 3.4].

**Theorem 3.1.** *Let  $\Lambda$  be a row-finite  $k$ -graph with no sources. Then there exists a group isomorphism  $\Phi_0 : K_0(B \rtimes_\beta \mathbb{Z}^k) \rightarrow K_0(C^*(\Lambda))$  such that  $\Phi_0([i_B(p_{(0,v)})]) = [p_v]$  for all  $v \in \Lambda^0$  (where we adopt the notation used in [19] for crossed product  $C^*$ -algebras).*

**Proof.** Let  $(B \rtimes_\beta \mathbb{Z}^k, i_B, i_{\mathbb{Z}^k})$  be a crossed product for the dynamical system  $(B, \mathbb{Z}^k, \beta)$  in the sense of [19]. One checks that

$$\{t_{(\lambda, (m, n))} \mid (\lambda, (m, n)) \in \Lambda \times \Delta\}$$

is a  $*$ -representation of  $\Lambda \times \Delta$ , where  $\Delta := \Delta_k$  (see Examples 2.3.1), and for  $(\lambda, (m, n)) \in \Lambda \times \Delta$  we let

$$t_{(\lambda, (m, n))} := i_B(s_{(m, \lambda)})i_{\mathbb{Z}^k}(m + d(\lambda) - n).$$

Moreover,  $C^*(t_\xi \mid \xi \in \Lambda \times \Delta) = B \rtimes_\beta \mathbb{Z}^k$ . Thus by the universal property of  $C^*(\Lambda \times \Delta)$ , there exists a  $*$ -homomorphism  $\pi : C^*(\Lambda \times \Delta) \rightarrow B \rtimes_\beta \mathbb{Z}^k$  such that  $\pi(s_\xi) = t_\xi$  for all  $\xi \in \Lambda \times \Delta$ . Let  $\alpha : \mathbb{T}^k \rightarrow \text{Aut}(B)$  denote the canonical gauge action on  $B$  (see [13, §3]) and let  $\hat{\beta} : \mathbb{T}^k \rightarrow \text{Aut}(B \rtimes_\beta \mathbb{Z}^k)$  denote the dual action of  $\beta$ . There exists an action  $\tilde{\alpha}$  of  $\mathbb{T}^k$  on  $B \rtimes_\beta \mathbb{Z}^k$  such that  $i_B \alpha_z = \tilde{\alpha}_z i_B$  for all  $z \in \mathbb{T}^k$ . It is clear that setting  $\gamma_{(z_1, z_2)} := \tilde{\alpha}_{z_1 z_2} \hat{\beta}_{z_2}^{-1}$  for all  $(z_1, z_2) \in \mathbb{T}^k \times \mathbb{T}^k$  defines an action  $\gamma$  of  $\mathbb{T}^{2k}$  on  $B \rtimes_\beta \mathbb{Z}^k$ . Moreover, it satisfies  $\pi \alpha_z^\times = \gamma_z \pi$  for all  $z \in \mathbb{T}^{2k}$  where  $\alpha^\times$  is the canonical gauge action on  $C^*(\Lambda \times \Delta)$ . Clearly  $\pi(p_v) = 0$  for all  $v \in \Lambda \times \Delta$ , hence by the gauge-invariant uniqueness theorem [13, Theorem 3.4] we see that  $\pi : C^*(\Lambda \times \Delta) \rightarrow B \rtimes_\beta \mathbb{Z}^k$  is a  $*$ -isomorphism.

For the zero element  $0 \in \mathbb{Z}^k$ , we see that  $p_{(0,0)}$  is a minimal projection in  $C^*(\Delta) \cong \mathbb{K}$  (cf. Examples 2.5.1). Therefore, the homomorphism given by  $x \mapsto x \otimes p_{(0,0)}$  induces an isomorphism  $K_0(C^*(\Lambda)) \cong K_0(C^*(\Lambda) \otimes C^*(\Delta))$ . The latter is isomorphic to  $K_0(C^*(\Lambda \times \Delta))$  (see 2.5.2) and  $K_0(B \rtimes_{\beta} \mathbb{Z}^k)$ . Let  $\Psi : K_0(C^*(\Lambda)) \rightarrow K_0(B \rtimes_{\beta} \mathbb{Z}^k)$  be the composition of the preceding group isomorphisms. Then it follows easily that  $\Psi([p_v]) = [i_B(p_{(0,v)})]$ . Setting  $\Phi_0 := \Psi^{-1}$  completes the proof.  $\square$

Therefore we may apply [11, 6.10 Theorem] to describe the  $K$ -groups of  $C^*(\Lambda)$  by means of a homological spectral sequence with initial term given by  $H_p(\mathbb{Z}^k, K_q(B))$ , i.e., the homology of the group  $\mathbb{Z}^k$  with coefficients in the left  $\mathbb{Z}^k$ -module  $K_q(B)$  [3], where the  $\mathbb{Z}^k$ -action is given by  $e_i \cdot m = K_0(\beta_{e_i})(m)$  for  $i = 1, \dots, k$ , (cf. the proof of [25, Proposition 4.1]). First, we recall the definition of a homology spectral sequence from [34, §5] and the notion of convergence (see also [14]).<sup>2</sup>

**Definition 3.2.** A *homology spectral sequence* (starting at  $E^a$ ) consists of the following data:

- (1) a family  $\{E_{p,q}^r\}$  of modules defined for all integers  $p, q$  and  $r \geq a$ ,
- (2) maps  $d_{pq}^r : E_{p,q}^r \rightarrow E_{p-r, q+r-1}^r$  that are differentials in the sense that  $d_{p-r, q+r-1}^r d_{pq}^r = 0$ ,
- (3) isomorphisms  $E_{pq}^{r+1} \rightarrow \ker(d_{pq}^r) / \text{im}(d_{p+r, q-r+1}^r)$ .

We will denote the above data by  $\{(E^r, d^r)\}$ . The *total degree* of the term  $E_{pq}^r$  is  $n = p + q$ . The homology spectral sequence is said to be *bounded* if for each  $n$  there are only finitely many nonzero terms of total degree  $n$  in  $\{E_{pq}^r\}$ , in which case, for each  $p$  and  $q$  there is an  $r_0$  such that  $E_{pq}^r \cong E_{pq}^{r+1}$  for all  $r \geq r_0$ . We write  $E_{pq}^{\infty}$  for this stable value of  $E_{pq}^r$ .

We say that a bounded spectral sequence *converges to*  $\mathcal{K}_*$  if we are given a family of modules  $\{\mathcal{K}_n\}$ , each having a finite filtration

$$0 = F_s(\mathcal{K}_n) \subseteq \dots \subseteq F_{p-1}(\mathcal{K}_n) \subseteq F_p(\mathcal{K}_n) \subseteq F_{p+1}(\mathcal{K}_n) \subseteq \dots \subseteq F_t(\mathcal{K}_n) = \mathcal{K}_n,$$

and we are given isomorphisms  $E_{pq}^{\infty} \rightarrow F_p(\mathcal{K}_{p+q}) / F_{p-1}(\mathcal{K}_{p+q})$ .

**Lemma 3.3** (cf. [25, Proposition 4.1]). *There exists a spectral sequence  $\{(E^r, d^r)\}$  converging to  $K_*(C^*(\Lambda)) := \{\mathcal{K}_n\}_{n \in \mathbb{Z}}$  where*

$$\mathcal{K}_n := \begin{cases} K_0(C^*(\Lambda)) & \text{if } n \text{ is even,} \\ K_1(C^*(\Lambda)) & \text{if } n \text{ is odd.} \end{cases}$$

Moreover, for  $p, q \in \mathbb{Z}$ ,

$$E_{p,q}^2 \cong \begin{cases} H_p(\mathbb{Z}^k, K_0(B)) & \text{if } p \in \{0, 1, \dots, k\} \text{ and } q \text{ is even,} \\ 0 & \text{otherwise,} \end{cases}$$

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<sup>2</sup>The reader will notice that the definition is presented in a less general form than in [34, §5], but is adequate for our purposes.

$E_{p,q}^\infty \cong E_{p,q}^{k+1}$  and  $E_{p,q}^{k+1} = 0$  if  $p \in \mathbb{Z} \setminus \{0, 1, \dots, k\}$  or  $q$  is odd.

**Proof.** The first assertion follows from an application of [11, 6.10 Theorem] to  $B \rtimes_\beta \mathbb{Z}^k$ , which is  $*$ -isomorphic to  $C^*(\Lambda) \otimes \mathbb{K}$  by the proof of Theorem 3.1, after noting that  $K_*(B \rtimes_\beta \mathbb{Z}^k)$  coincides with its “ $\gamma$ -part” since the Baum–Connes Conjecture with coefficients in an arbitrary  $C^*$ -algebra is true for the amenable group  $\mathbb{Z}^k$  for all  $k \geq 1$ .

By the proof of [11, 6.10 Theorem],  $K_*(B \rtimes_\beta \mathbb{Z}^k) \cong K_*(D)$  for some  $C^*$ -algebra  $D$  which has a finite filtration by ideals:  $0 \subset D_0 \subset D_1 \subset \dots \subset D_k = D$  since the dimension of the universal covering space of the classifying space of  $\mathbb{Z}^k$  is  $k$ .

Here, we are considering the spectral sequence  $\{(E^r, d^r)\}$  in homology  $K_*$  associated with the finite filtration  $0 \subset D_0 \subset D_1 \subset \dots \subset D_k = D$  of  $D$  ([27, §6]) which has  $E_{p,q}^1 = K_{(p+q \bmod 2)}(D_p/D_{p-1})$  where  $D_n = 0$  for  $n < 0$  and  $D_n = D$  for  $n \geq k$ . It follows easily that  $E_{p,q}^r = 0$  for  $p \in \mathbb{Z} \setminus \{0, 1, \dots, k\}$ , for all  $q \in \mathbb{Z}$  and for all  $r \geq 1$  and  $E_{p,q}^\infty \cong E_{p,q}^{k+1}$  (see also [27, Theorem 2.1]). This combined with Kasparov’s calculation in the proof of [11, 6.10 Theorem], giving  $E_{p,q}^2 \cong H_p(\mathbb{Z}^k, K_q(B))$ , along with the observation that  $K_q(B) = 0$  for odd  $q$ , since  $B$  is an AF-algebra, proves the second assertion.  $\square$

Now we will compute  $H_*(\mathbb{Z}^k, K_0(B))$  in terms of the combinatorial data encoded in  $\Lambda$ . First, let us examine the structure of  $B$ , and hence  $K_0(B)$ , in a little more detail.

**Lemma 3.4.** *Let  $\Lambda$  be a row-finite  $k$ -graph with no sources. Then*

$$B = \overline{\bigcup_{n \in \mathbb{Z}^k} B_n},$$

where

$$\begin{aligned} B_n &= \overline{\text{span}\{s_\lambda s_\mu^* \mid \lambda, \mu \in \mathbb{Z}^k \times_d \Lambda, s(\lambda) = s(\mu) = (n, v) \text{ for some } v \in \Lambda^0\}} \\ &\cong \bigoplus_{v \in \Lambda^0} B_n(v), \end{aligned}$$

where

$$\begin{aligned} B_n(v) &:= \overline{\text{span}\{s_\lambda s_\mu^* \mid \lambda, \mu \in \mathbb{Z}^k \times_d \Lambda, s(\lambda) = s(\mu) = (n, v)\}} \\ &\cong \mathbb{K}(\ell^2(s^{-1}(v))) \end{aligned}$$

for all  $v \in \Lambda^0$  and  $n \in \mathbb{Z}^k$ .

**Proof.** Follows immediately from the proofs of [13, Lemma 5.4, Theorem 5.5] and the observation that for all  $n \in \mathbb{Z}^k$  and  $v \in \Lambda^0$ ,  $s^{-1}((n, v)) \subset \mathbb{Z}^k \times_d \Lambda$  may be identified with  $s^{-1}(v) \subset \Lambda$  via  $(n - d(\lambda), \lambda) \mapsto \lambda$  for all  $\lambda \in s^{-1}(v)$ .  $\square$



**Definition 3.5.** Let  $\mathbb{Z}\Lambda^0$  be the group of all maps from  $\Lambda^0$  into  $\mathbb{Z}$  that have finite support under pointwise addition. For each  $u \in \Lambda^0$ , we denote by  $\delta_u$  the element of  $\mathbb{Z}\Lambda^0$  defined by  $\delta_u(v) = \delta_{u,v}$  (the Kronecker delta) for all  $v \in \Lambda^0$ . Note that  $\mathbb{Z}\Lambda^0$  is a free abelian group with free set of generators  $\{\delta_u \mid u \in \Lambda^0\}$ .

**Definition 3.6** (cf. [13, §6]). Define the vertex matrices of  $\Lambda$ ,  $M_i$ , by the following. For  $u, v \in \Lambda^0$  and  $i = 1, 2, \dots, k$ ,

$$M_i(u, v) := |\{\lambda \in \Lambda^{e_i} \mid r(\lambda) = u, s(\lambda) = v\}|.$$

**Remarks 3.7.** By the factorisation property, the vertex matrices of a  $k$ -graph pairwise commute [13, §6].

**Convention 3.8.** Given a matrix  $M$  with integer entries and index set  $\Lambda^0$ , by slight abuse of notation we shall, on occasion, regard  $M$  as the group endomorphism  $\mathbb{Z}\Lambda^0 \rightarrow \mathbb{Z}\Lambda^0$ , defined in the natural way as

$$(Mf)(u) = \sum_{v \in \Lambda^0} M(u, v)f(v) \text{ for all } u \in \Lambda^0, f \in \mathbb{Z}\Lambda^0.$$

**Lemma 3.9** (cf. [25] Lemma 4.5 and [17] Proposition 4.1.2). *For all  $n, m \in \mathbb{Z}^k$  such that  $m \leq n$ , let  $A_m := \mathbb{Z}\Lambda^0$ . Moreover, define homomorphisms  $j_{nm} : A_m \rightarrow A_n$  by  $j_{nm}(f) = f$ ,  $j_{m+e_i, m} := M_i^t$  for all  $f \in A_m$ ,  $u \in \Lambda^0$ ,  $i \in \{1, \dots, k\}$ , and  $j_{m+e_i+e_j, m} := j_{m+e_i+e_j, m+e_i}j_{m+e_i, m}$  for all  $j \in \{1, \dots, k\}$ . Then  $(A_m; j_{nm})$  is a direct system of groups and*

$$K_0(B) \cong A := \varinjlim (A_m; j_{nm}).$$

**Proof.** It follows from Remarks 3.7 that the connecting homomorphisms are well-defined and that  $(A_m; j_{nm})$  is a direct system. From Lemma 3.4 we deduce that

$$K_0(B) \cong \varinjlim (K_0(B_n); K_0(\iota_{n,m})),$$

where, for  $m, n \in \mathbb{Z}^k$  with  $m \leq n$ ,  $\iota_{n,m} : B_m \rightarrow B_n$  are the inclusion maps [33, Proposition 6.2.9]. We also deduce that for  $n \in \mathbb{Z}^k$  and  $v \in \Lambda^0$ ,  $K_0(B_n(v))$  is isomorphic to  $\mathbb{Z}$  and is generated by the equivalence class comprising all minimal projections in  $B_n(v)$ , of which  $p_\xi$  is a member for any  $\xi \in s^{-1}(n, v)$ . Therefore,  $K_0(B_n) \cong \bigoplus_{v \in \Lambda^0} K_0(B_n(v))$  is generated by  $\{[p_{(n,v)}]_n \mid v \in \Lambda^0\}$ , where  $[\cdot]_n$  denotes the equivalence classes of  $K_0(B_n)$  for all  $n \in \mathbb{Z}^k$ . Thus the map  $\psi_n : A_n \rightarrow K_0(B_n)$  given by  $f \mapsto \sum_{u \in \Lambda^0} f(u)[p_{(n,u)}]_n$  is a group isomorphism for all  $n \in \mathbb{Z}^k$ .

The embedding  $\iota_{n,m} : B_m \rightarrow B_n$  is given by

$$\iota_{n,m}(s_{(m-d(\lambda), \lambda)} s_{(m-d(\mu), \mu)}^*) = \sum_{\alpha \in \Lambda^{n-m}(s(\lambda))} s_{(m-d(\lambda), \lambda\alpha)} s_{(m-d(\mu), \mu\alpha)}^*$$

for all  $\lambda, \mu \in \Lambda$ . Therefore,

$$\begin{aligned} K_0(\iota_{n+e_i, n}) ([p_{(n, v)}]_n) &= [\iota_{n+e_i, n} (p_{(n, v)})]_{n+e_i} = \left[ \sum_{\alpha \in \Lambda^{e_i}(v)} p_{(n, \alpha)} \right]_{n+e_i} \\ &= \sum_{u \in \Lambda^0} M_i(v, u) [p_{(n+e_i, u)}]_{n+e_i} \end{aligned}$$

and

$$\begin{aligned} K_0(\iota_{n+e_i, n}) \left( \sum_{v \in \Lambda^0} f(v) [p_{(n, v)}]_n \right) &= \sum_{u \in \Lambda^0} \left( \sum_{v \in \Lambda^0} M_i(v, u) f(v) \right) [p_{(n+e_i, u)}]_{n+e_i} \\ &= \sum_{u \in \Lambda^0} J_{n+e_i, n}(f)(u) [p_{(n+e_i, u)}]_{n+e_i}. \end{aligned}$$

Thus the following square commutes for all  $i \in \{1, 2, \dots, k\}$ :

$$\begin{array}{ccc} A_n & \xrightarrow{J_{n+e_i, n}} & A_{n+e_i} \\ \psi_n \downarrow & & \downarrow \psi_{n+e_i} \\ K_0(B_n) & \xrightarrow{K_0(\iota_{n+e_i, n})} & K_0(B_{n+e_i}). \end{array}$$

The result follows.  $\square$

Henceforth, we shall follow the notation introduced in Lemma 3.9 and its proof.

Now we begin to examine the action of  $\mathbb{Z}^k$  on  $K_0(B)$  in terms of the description of  $K_0(B)$  provided by Lemma 3.9.

**Lemma 3.10** (cf. [25] Lemma 4.10). *Fix  $i \in \{1, \dots, k\}$  and define a homomorphism  $\phi_{i, n} : A_n \rightarrow A_n$  by  $\phi_{i, n} := M_i^t$  for all  $n \in \mathbb{Z}^k$ . Let  $\phi_i : A \rightarrow A$  be the homomorphism induced by the system of homomorphisms  $\{\phi_{i, n} \mid n \in \mathbb{Z}^k\}$ . Then  $\psi \phi_i = K_0(\beta_{e_i}) \psi$ , where  $\psi : A \rightarrow K_0(B)$  is the isomorphism constructed in Lemma 3.9.*

**Proof.** It follows from Remarks 3.7 that  $\phi_{i, n} J_{nm} = J_{nm} \phi_{i, m}$  for all  $m, n \in \mathbb{Z}^k$  so that  $\phi_i$  is well-defined for all  $i \in \{1, \dots, k\}$ .

Now, we let  $\tilde{\psi} : A \rightarrow \varinjlim (K_0(B_m); K_0(\iota_{n, m}))$  be the unique isomorphism such that  $K_0(\iota_n) \circ \psi_n = \tilde{\psi} \circ J_n$  for all  $n \in \mathbb{Z}^k$  where

$$\begin{aligned} \psi_n : A_n &\xrightarrow{\cong} K_0(B_n) \\ f &\mapsto \sum_{u \in \Lambda^0} f(u) [p_{(n, u)}]_n \end{aligned}$$

(proof of Lemma 3.9). Then  $\psi : A \rightarrow K_0(B)$  is the composition of  $\tilde{\psi}$  with the canonical isomorphism  $\varinjlim (K_0(B_n); K_0(\iota_{n, m})) \cong K_0(B)$ .

We will show that

$$\begin{array}{ccc} K_0(B_n) & \xrightarrow{K_0(\iota_n)} & K_0(B) \\ \tilde{\phi}_{i,n} \downarrow & & \downarrow K_0(\beta_{e_i}) \\ K_0(B_n) & \xrightarrow{K_0(\iota_n)} & K_0(B) \end{array}$$

commutes for all  $i = 1, 2, \dots, k$  and  $n \in \mathbb{Z}^k$  where  $\iota_n : B_n \rightarrow B$  is the inclusion map for all  $n \in \mathbb{Z}^k$  and  $\tilde{\phi}_{i,n} = \psi_n \circ \phi_{i,n} \circ \psi_n^{-1}$ . The lemma then follows from the universal properties of direct limits.

Fix  $i \in \{1, \dots, k\}$  and  $n \in \mathbb{Z}^k$ . Since  $K_0(B_n)$  is generated by

$$\{[p_{(n,v)}]_n \mid v \in \Lambda^0\},$$

it suffices to show that  $K_0(\beta_{e_i}) \circ K_0(\iota_n)([p_{(n,v)}]_n) = K_0(\iota_n) \circ \tilde{\phi}_{i,n}([p_{(n,v)}]_n)$  for all  $v \in \Lambda^0$ . To see that this holds let  $v \in \Lambda^0$ . Then

$$K_0(\beta_{e_i}) \circ K_0(\iota_n)([p_{(n,v)}]_n) = K_0(\beta_{e_i})([p_{(n,v)}]) = [p_{(n+e_i,v)}],$$

while

$$\begin{aligned} K_0(\iota_n) \circ \tilde{\phi}_{i,n}([p_{(n,v)}]_n) &= \sum_{u \in \Lambda^0} M_i(v, u) [p_{(n,u)}] \\ &= \sum_{\alpha \in \Lambda^{e_i}(v)} [p_{(n+e_i,\alpha)}] = [p_{(n+e_i,v)}]. \quad \square \end{aligned}$$

Having established a description of  $K_0(B)$  as a left  $\mathbb{Z}^k$ -module in terms of the structure of  $\Lambda$ , we are almost in a position to describe  $H_*(\mathbb{Z}^k, K_0(B))$ . First we recall some relevant notions from homological algebra.

It will be convenient to use multiplicative notation for the free abelian group  $\mathbb{Z}^k$ , generated by  $k$  generators. Thus we set

$$G := \langle s_i \mid s_i s_j = s_j s_i \text{ for all } i, j \in \{1, \dots, k\} \rangle$$

and  $R := \mathbb{Z}G$ , the group ring of  $G$  [3]. We shall compute  $H_*(\mathbb{Z}^k, K_0(B))$  by means of a Koszul resolution  $K(\mathbf{x})$  for an appropriate regular sequence  $\mathbf{x}$  on  $R$ , [34, Corollary 4.5.5].

By a regular sequence on  $R$  we mean a sequence  $\mathbf{x} = \{x_i\}_{i=1}^n$  of elements of  $R$  such that:

- (a)  $(x_1, \dots, x_n) \neq R$ .
- (b) For  $i = 1, \dots, n$ ,  $x_i \notin \mathcal{Z}(R/(x_1, \dots, x_{i-1}))$ .

Here, we regard  $R$  as an  $R$ -module, write  $(x_1, \dots, x_j)$  for the ideal of  $R$  generated by  $\{x_i\}_{i=1}^j$ , and write  $\mathcal{Z}(M)$  for the set of zero-divisors on a  $R$ -module  $M$ , i.e.,  $\mathcal{Z}(M) := \{r \in R \mid r \cdot m = 0 \text{ for some nonzero } m \in M\}$  (see [10, §3.1] for more details).

It is straightforward to check that for any finite set  $\{t_1, \dots, t_k\}$  of generators of  $G$ , the subset  $\mathbf{x} = \{x_i\}_{i=1}^k$ , where  $x_i := 1 - t_i$  for  $i = 1, \dots, k$ , is a regular sequence on  $R$ .

Following [34, §4.5] we will describe the Koszul complex,  $K(\mathbf{x})$ , in terms of the exterior algebra of a free  $R$ -module [2]. It will be convenient for us to describe the terms of the exterior algebra as follows.

**Definition 3.11.** For any nonnegative integer  $l$ , let  $\mathcal{E}^l(R^k)$  denote the  $l^{\text{th}}$  term of the exterior algebra of the free  $R$ -module  $R^k := \bigoplus_{i=1}^k R_i$ , over  $R$ , where  $R_i = R$  for  $i = 1, \dots, k$ . Moreover, for any negative integer  $l$ , let  $\mathcal{E}^l(R^k) = \{0\}$ . For  $l \in \mathbb{Z}$  let

$$N_l := \begin{cases} \{(\mu_1, \dots, \mu_l) \in \{1, \dots, k\}^l \mid \mu_1 < \dots < \mu_l\} & \text{if } l \in \{1, \dots, k\}, \\ \{\star\} & \text{if } l = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

For  $l \in \{1, \dots, k\}$  and  $\mu = (\mu_1, \dots, \mu_l) \in N_l$ ,  $i = 1, \dots, l$  we let

$$\mu^i := \begin{cases} (\mu_1, \mu_2, \dots, \mu_{i-1}, \mu_{i+1}, \mu_{i+2}, \dots, \mu_l) \in N_{l-1} & \text{if } l \neq 1, \\ \star & \text{if } l = 1. \end{cases}$$

For  $n = 1, 2, \dots$  and  $r \in \mathbb{Z}$ , let

$$\binom{n}{r} := \begin{cases} \frac{n!}{(n-r)!r!} & \text{if } 0 \leq r \leq n, \\ 0 & \text{if } r < 0 \text{ or } r > n. \end{cases}$$

Using the above notation we may describe the  $l^{\text{th}}$ -term ( $l \in \mathbb{Z}$ ) of the exterior algebra over  $R^k$ ,  $\mathcal{E}^l(R^k)$ , as being generated by the set  $N_l$  as a free  $R$ -module and having rank  $\binom{k}{l}$ .

Now let  $K(\mathbf{x})$  be the chain complex

$$0 \longleftarrow \mathcal{E}^0(R^k) \longleftarrow \mathcal{E}^1(R^k) \longleftarrow \dots \longleftarrow \mathcal{E}^k(R^k) \longleftarrow 0$$

where the differentials  $\mathcal{E}^l(R^k) \longrightarrow \mathcal{E}^{l-1}(R^k)$  are given by mapping

$$\mu \mapsto \sum_{j=1}^l (-1)^{j+1} x_{\mu_j} \mu^j \quad \text{for all } \mu = (\mu_1, \dots, \mu_l) \in N_l,$$

if  $l \in \{1, \dots, k\}$  and the zero map otherwise. By [34, Corollary 4.5.5]  $K(\mathbf{x})$  is a free resolution of  $R/I$  over  $R$  where  $I$  is the ideal of  $R$  generated by  $\mathbf{x}$ . It is well-known (see, e.g., [34, Chapter 6],[3, §I.2]) that  $I = \ker \epsilon$  where  $\epsilon : R \longrightarrow \mathbb{Z} : g \mapsto 1$  is the augmentation map of the group ring  $R = \mathbb{Z}G$ . Thus we have a free (and hence projective) resolution of  $\mathbb{Z}$  over  $\mathbb{Z}G$ , which we may use to compute  $H_*(G, K_0(B))$  (see [3, Chapter III]).

**Lemma 3.12.** *Following the above notation, we have  $H_*(G, K_0(B))$  isomorphic to the homology of the chain complex*

$$\mathcal{B} : 0 \longleftarrow K_0(B) \longleftarrow \dots \longleftarrow \bigoplus_{N_i} K_0(B) \longleftarrow \dots \longleftarrow K_0(B) \longleftarrow 0,$$

where the differentials  $\tilde{\partial}_l : \bigoplus_{N_l} K_0(B) \longrightarrow \bigoplus_{N_{l-1}} K_0(B)$  ( $l \in \{1, \dots, k\}$ ) are defined by

$$\bigoplus_{\mu \in N_l} m_\mu \mapsto \bigoplus_{\lambda \in N_{l-1}} \sum_{\mu \in N_l} \sum_{i=1}^l (-1)^{i+1} \delta_{\lambda, \mu^i} (m_\mu - K_0(\beta_{\mu_i})(m_\mu)).$$

(Recall that the  $G$ -action on  $K_0(B)$  is given by  $s_i \cdot m = K_0(\beta_{e_i})(m)$  for all  $m \in K_0(B)$ ,  $i = 1, \dots, k$ .)

**Proof.** By definition  $H_*(G, K_0(B)) \cong H_*(K(\mathbf{x}) \otimes_G K_0(B))$ , where the latter chain complex is obtained by applying the functor  $- \otimes_G K_0(B)$  termwise to the chain complex  $K(\mathbf{x})$ . The lemma follows via the canonical isomorphism between  $\mathcal{E}^l(R^k) \otimes_G K_0(B)$  and  $\bigoplus_{N_l} K_0(B)$  ( $l \in \{0, \dots, k\}$ ), setting  $t_i := s_i^{-1}$  ( $i = 1, \dots, k$ ) as our generators of  $G$  to obtain  $\mathbf{x}$  as described above.  $\square$

For  $m, n \in \mathbb{Z}^k$  with  $m \leq n$ , let  $\mathcal{A}^{(n)}$  be the chain complex

$$0 \longleftarrow A_n \longleftarrow \cdots \longleftarrow \bigoplus_{N_l} A_n \longleftarrow \cdots \longleftarrow A_n \longleftarrow 0,$$

where  $l = 0, \dots, k$ ,  $A_n = \mathbb{Z}\Lambda^0$  and for  $l > 0$  the differentials

$$\partial_l^{(n)} : \bigoplus_{N_l} A_n \longrightarrow \bigoplus_{N_{l-1}} A_n$$

are defined by

$$\bigoplus_{\mu \in N_l} m_\mu \mapsto \bigoplus_{\lambda \in N_{l-1}} \sum_{\mu \in N_l} \sum_{i=1}^l (-1)^{i+1} \delta_{\lambda, \mu^i} (m_\mu - \phi_{\mu_i, n}(m_\mu)),$$

where for  $i = 1, \dots, k$  and  $n \in \mathbb{Z}^k$ ,  $\phi_{i, n}$  is the homomorphism defined in Lemma 3.10. Furthermore, let  $(\tau_m^n)_p : \mathcal{A}_p^{(m)} \longrightarrow \mathcal{A}_p^{(n)}$  be the homomorphism defined by

$$(\tau_m^n)_p \left( \bigoplus_{\mu \in N_p} m_\mu \right) = \bigoplus_{\mu \in N_p} J_{nm}(m_\mu)$$

for all  $p \in \{0, 1, \dots, k\}$  and the trivial map for  $p \in \mathbb{Z} \setminus \{0, 1, \dots, k\}$  (see Lemma 3.9).

Following [30, Chapter 4, §1], by a chain map  $\tau : \mathcal{C} \longrightarrow \mathcal{C}'$  we mean a collection  $\{\tau_p : \mathcal{C}_p \longrightarrow \mathcal{C}'_p\}$  of homomorphisms that commute with the differentials in the sense that commutativity holds in each square:

$$\begin{array}{ccc} \mathcal{C}_p & \longrightarrow & \mathcal{C}_{p-1} \\ \tau_p \downarrow & & \downarrow \tau_{p-1} \\ \mathcal{C}'_p & \longrightarrow & \mathcal{C}'_{p-1} \end{array}$$

Recall that there is a category of chain complexes whose objects are chain complexes and whose morphisms are chain maps. Moreover, the category of chain complexes admits direct limits.

**Lemma 3.13.** *Following the above notation, for each  $m, n \in \mathbb{Z}^k$  the system of homomorphisms  $\{(\tau_m^n)_p \mid p \in \mathbb{Z}\}$  defines a chain map  $\tau_m^n : \mathcal{A}^{(m)} \rightarrow \mathcal{A}^{(n)}$  such that  $(\mathcal{A}^{(n)}; \tau_m^n)$  is a direct system in the category of chain complexes. Furthermore,  $(\mathcal{B}; \gamma_n)$  is a direct limit for  $(\mathcal{A}^{(n)}; \tau_m^n)$ , where  $\gamma_n : \mathcal{A}^{(n)} \rightarrow \mathcal{B}$  is given by  $(\gamma_n)_p(\bigoplus_{\mu \in N_p} m_\mu) = \bigoplus_{\mu \in N_p} \psi j_n(m_\mu)$  for all  $p \in \{0, 1, \dots, k\}$  and the trivial map otherwise.*

**Proof.** That  $\tau_m^n : \mathcal{A}^{(m)} \rightarrow \mathcal{A}^{(n)}$  is a chain map for all  $m, n \in \mathbb{Z}^k$ , follows immediately from the fact that  $\phi_{i,n} j_{nm} = j_{nm} \phi_{i,m}$  for all  $i \in \{1, \dots, k\}$  and  $m, n \in \mathbb{Z}^k$  (cf. proof of Lemma 3.10). That  $(\mathcal{A}^{(n)}; \tau_m^n)$  is a direct system of chain complexes follows immediately from the fact that  $(A_m; j_{nm})$  is a direct system of groups (by Lemma 3.9).

Note that since  $K_0(\beta_{e_i})\psi = \psi\phi_i$  for all  $i \in \{1, \dots, k\}$  by Lemma 3.10, a direct calculation shows that  $\tilde{\partial}_p(\gamma_n)_p = (\gamma_n)_{p-1}\partial_p^{(n)}$  for all  $p \in \mathbb{Z}^k$ , thus  $\gamma_n : \mathcal{A}^{(n)} \rightarrow \mathcal{B}$  is a chain map for all  $n \in \mathbb{Z}^k$ . The fact that  $\gamma_m = \gamma_n \tau_{nm}$  for all  $m, n \in \mathbb{Z}^k$  follows immediately by construction of the maps.

Now suppose that  $(\mathcal{A}; \tau_n)$  is a direct limit for  $(\mathcal{A}^{(n)}; \tau_m^n)$ . Then by the above and the universal property of direct limits, there exists a morphism  $\gamma : \mathcal{A} \rightarrow \mathcal{B}$  such that  $\gamma \tau_m^n = \gamma_n$  for all  $m, n \in \mathbb{Z}^k$ . In order to show that  $\gamma$  is an isomorphism it suffices to show that each  $(\gamma)_p : \mathcal{A}_p \rightarrow \mathcal{B}_p$  is an isomorphism for all  $p \in \mathbb{Z}$ . This follows immediately from the fact that  $\psi : A \rightarrow K_0(B)$  is an isomorphism (Lemma 3.9) and that direct limits commute with (finite) direct sums in the category of abelian groups in the obvious way. We have shown, therefore, that  $(\mathcal{B}; \gamma_n)$  is a direct limit for  $(\mathcal{A}^{(n)}; \tau_m^n)$  in the category of chain complexes.  $\square$

Note that each chain complex  $\mathcal{A}^{(n)}$  does not actually depend on  $n \in \mathbb{Z}^k$ , thus for ease of notation we let  $\mathcal{D}$  denote this common chain complex with differentials  $\partial_p := \partial_p^{(n)}$  for all  $p \in \mathbb{Z}$ .

**Theorem 3.14.** *Using the above notation, the homology of  $\mathbb{Z}^k$  with coefficients in the left  $\mathbb{Z}^k$ -module  $K_0(B)$  is given by the homology of the chain complex  $\mathcal{D}$ , i.e., we have  $H_*(G, K_0(B)) \cong H_*(\mathcal{D})$ .*

**Proof.** The homology functor commutes with direct limits ([30, Chapter 4, §1, Theorem 7]), therefore it follows that

$$H_*(G, K_0(B)) \cong \varinjlim (H_*(\mathcal{A}^{(n)}); H_*(\tau_m^n)).$$

Thus, it suffices to prove that  $H_p(\tau_m^{m+e_j})$  is the identity map for all  $p \in \mathbb{Z}$ ,  $m \in \mathbb{Z}^k$ ,  $j \in \{1, \dots, k\}$ . To prove that this is true, we fix  $j \in \{1, \dots, k\}$  and show that

$$(3.1) \quad y - (\tau_m^{m+e_j})_p y \in \text{im } \partial_{p+1} \quad \text{for } y \in \ker \partial_p \quad \text{and } p \in \{0, 1, \dots, k\}.$$

Here, we recall that if  $y = \bigoplus_{\mu \in N_p} y_\mu$ , then

$$(\tau_m^{m+e_j})_p y = \bigoplus_{\mu \in N_p} M_j^t y_\mu,$$

so for a given  $p \in \{0, 1, \dots, k\}$ , (3.1) says

$$(3.2) \quad \bigoplus_{\mu \in N_p} (1 - M_j^t) y_\mu \in \text{im } \partial_{p+1} \quad \text{for} \quad y = \bigoplus_{\mu \in N_p} y_\mu \in \ker \partial_p.$$

The case  $p = 0$  is obvious, so we concentrate on the case  $p \in \{1, \dots, k\}$ .

For  $\mu \in N_p$ , let  $\kappa(\mu) := \sum_{i=1}^p \delta_{j, \mu_i} i$ , i.e., if  $j$  is a component of  $\mu$ , then  $\kappa(\mu)$  denotes the unique  $i \in \{1, \dots, p\}$  such that  $\mu_i = j$ ; otherwise  $\kappa(\mu) = 0$ . Now fix a  $\mu' = (\mu'_1, \dots, \mu'_p) \in N_p$  and let  $y = \bigoplus_{\mu \in N_p} y_\mu$  be in  $\ker \partial_p$ .

First, suppose that  $\kappa(\mu') > 0$  and let  $\eta = (\mu')^{\kappa(\mu')}$ . Then

$$\begin{aligned} 0 &= \partial_p(y)_\eta = \sum_{\mu \in N_p} \sum_{i=1}^p (-1)^{i+1} \delta_{\eta, \mu^i} (1 - M_{\mu_i}^t) y_\mu \\ &= \sum_{i=1}^p (-1)^{i+1} \delta_{\eta, (\mu')^i} (1 - M_{\mu_i}^t) y_{\mu'} + \sum_{\substack{\mu \in N_p \\ \mu \neq \mu'}} \sum_{i=1}^p (-1)^{i+1} \delta_{\eta, \mu^i} (1 - M_{\mu_i}^t) y_\mu \\ &= (-1)^{\kappa(\mu')+1} (1 - M_j^t) y_{\mu'} + \sum_{\substack{\mu \in N_p \\ \mu \neq \mu'}} \sum_{i=1}^p (-1)^{i+1} \delta_{\eta, \mu^i} (1 - M_{\mu_i}^t) y_\mu, \end{aligned}$$

so that

$$(1 - M_j^t) y_{\mu'} = \sum_{\substack{\mu \in N_p \\ \mu \neq \mu'}} \sum_{i=1}^p (-1)^{i+\kappa(\mu')+1} \delta_{\eta, \mu^i} (1 - M_{\mu_i}^t) y_\mu.$$

Note that the case  $p = k$  (where  $\kappa(\mu)$  is necessarily greater than zero) follows immediately from the above. Therefore, we may assume that  $p \in \{1, \dots, k-1\}$ , in which case we claim that

$$\bigoplus_{\mu \in N_p} (1 - M_j^t) y_\mu = \partial_{p+1} \left( \bigoplus_{\lambda \in N_{p+1}} z_\lambda \right)$$

where  $z_\lambda = \sum_{i=1}^{p+1} (-1)^{i+1} \delta_{\lambda_i, j} y_{\lambda^i}$  for all  $\lambda \in N_{p+1}$ .

Now

$$\partial_{p+1} \left( \bigoplus_{\lambda \in N_{p+1}} z_\lambda \right)_{\mu'} = \sum_{\lambda \in N_{p+1}} \sum_{i,r=1}^{p+1} (-1)^{i+r+2} \delta_{\mu', \lambda^i} \delta_{j, \lambda_r} (1 - M_{\lambda_i}^t) y_{\lambda^r}$$

$$\begin{aligned}
&= \sum_{\substack{\lambda \in N_{p+1} \\ \kappa(\lambda) > 0}} \sum_{i=1}^{p+1} (-1)^{i+\kappa(\lambda)} \delta_{\mu', \lambda^i} (1 - M_{\lambda_i}^t) y_{\lambda^{\kappa(\lambda)}} \\
&= \sum_{\substack{\lambda \in N_{p+1} \\ \kappa(\lambda) > 0}} \left\{ \sum_{i=1}^{\kappa(\lambda)-1} (-1)^{i+\kappa(\mu')+1} \delta_{\mu', \lambda^i} (1 - M_{\lambda_i}^t) y_{\lambda^{\kappa(\lambda)}} \right. \\
&\quad \left. + \sum_{i=\kappa(\lambda)+1}^{p+1} (-1)^{i+\kappa(\mu')} \delta_{\mu', \lambda^i} (1 - M_{\lambda_i}^t) y_{\lambda^{\kappa(\lambda)}} \right\} \\
&= \sum_{\substack{\mu \in N_p \\ \kappa(\mu)=0}} \sum_{i=1}^p (-1)^{i+\kappa(\mu')+1} \delta_{\eta, \mu^i} (1 - M_{\mu_i}^t) y_{\mu}
\end{aligned}$$

since for every  $\lambda \in N_{p+1}$  such that  $\kappa(\lambda) > 0$  and for every  $i \in \{1, \dots, p+1\} \setminus \{\kappa(\lambda)\}$  we have:

- (1)  $\delta_{\mu', \lambda^{\kappa(\lambda)}} = 0$ ,
- (2)  $\kappa(\lambda) = \begin{cases} \kappa(\mu') + 1 & \text{if } \mu' = \lambda^i \text{ with } i < \kappa(\lambda), \\ \kappa(\mu') & \text{if } \mu' = \lambda^i \text{ with } \kappa(\lambda) < i, \end{cases}$
- (3)  $\mu' = \lambda^i \iff \eta = \begin{cases} (\lambda^{\kappa(\lambda)})^i & \text{if } i < \kappa(\lambda), \\ (\lambda^{\kappa(\lambda)})^{i-1} & \text{if } \kappa(\lambda) < i, \end{cases}$

and if  $\mu \in N_p$  then  $\kappa(\mu) = 0$ ,  $\eta = \mu^i$  for some  $i \in \{1, \dots, p\} \iff \mu \neq \mu'$ ,  $\eta = \mu^i$  for some  $i \in \{1, \dots, p\}$ .

Hence,

$$\begin{aligned}
\partial_{p+1} \left( \bigoplus_{\lambda \in N_{p+1}} z_{\lambda} \right)_{\mu'} &= \sum_{\substack{\mu \in N_p \\ \mu \neq \mu'}} \sum_{i=1}^p (-1)^{i+\kappa(\mu')+1} \delta_{\eta, \mu^i} (1 - M_{\mu_i}^t) y_{\mu} \\
&= (1 - M_j^t) y_{\mu'}.
\end{aligned}$$

Now, if  $\kappa(\mu') = 0$ ,  $p \in \{1, \dots, k-1\}$  and  $\bigoplus_{\lambda \in N_{p+1}} z_{\lambda}$  is as before, then

$$\begin{aligned}
\partial_{p+1} \left( \bigoplus_{\lambda \in N_{p+1}} z_{\lambda} \right)_{\mu'} &= \sum_{\lambda \in N_{p+1}} \sum_{i,r=1}^{p+1} (-1)^{i+r+2} \delta_{\mu', \lambda^i} \delta_{j, \lambda^r} (1 - M_{\lambda_i}^t) y_{\lambda^r} \\
&= (-1)^{\kappa(\xi)+\kappa(\xi)+2} (1 - M_{\xi^{\kappa(\xi)}}^t) y_{\xi^{\kappa(\xi)}} \\
&= (1 - M_j^t) y_{\mu'},
\end{aligned}$$

where  $\xi$  is the unique element of  $N_{p+1}$  satisfying  $\kappa(\xi) > 0$  and  $\xi^{\kappa(\xi)} = \mu'$ .  $\square$

Combining the results of this section we get the following theorem.



**Theorem 3.15.** *Let  $\Lambda$  be a row-finite  $k$ -graph with no sources. Then there exists a spectral sequence  $\{(E^r, d^r)\}$  converging to  $K_*(C^*(\Lambda))$  with  $E_{p,q}^\infty \cong E_{p,q}^{k+1}$  and*

$$E_{p,q}^2 \cong \begin{cases} H_p(\mathcal{D}) & \text{if } p \in \{0, 1, \dots, k\} \text{ and } q \text{ is even,} \\ 0 & \text{otherwise,} \end{cases}$$

where  $\mathcal{D}$  is the chain complex with

$$\mathcal{D}_p := \begin{cases} \bigoplus_{\mu \in N_p} \mathbb{Z}\Lambda^0 & \text{if } p \in \{0, 1, \dots, k\}, \\ 0 & \text{otherwise,} \end{cases}$$

and differentials

$$\partial_p : \mathcal{D}_p \longrightarrow \mathcal{D}_{p-1} : \bigoplus_{\mu \in N_p} m_\mu \mapsto \bigoplus_{\lambda \in N_{p-1}} \sum_{\mu \in N_p} \sum_{i=1}^p (-1)^{i+1} \delta_{\lambda, \mu^i} (1 - M_{\mu^i}^t) m_\mu$$

for  $p \in \{1, \dots, k\}$ .

Specialising Theorem 3.15 to the case when  $k = 2$  gives us explicit formulae to compute the  $K$ -groups of the  $C^*$ -algebras of row-finite 2-graphs with no sources.

**Proposition 3.16.** *Let  $\Lambda$  be a row-finite 2-graph with no sources and vertex matrices  $M_1$  and  $M_2$ . Then there is an isomorphism*

$$\Phi : \text{coker}(1 - M_1^t, 1 - M_2^t) \oplus \ker \begin{pmatrix} M_2^t - 1 \\ 1 - M_1^t \end{pmatrix} \longrightarrow K_0(C^*(\Lambda))$$

such that  $\Phi((\delta_u + \text{im } \partial_1) \oplus 0) = [p_u]$  for all  $u \in \Lambda^0$  (see Definition 3.5) and where we regard

$$\begin{aligned} (1 - M_1^t, 1 - M_2^t) : \mathbb{Z}\Lambda^0 \oplus \mathbb{Z}\Lambda^0 &\longrightarrow \mathbb{Z}\Lambda^0, \\ \begin{pmatrix} M_2^t - 1 \\ 1 - M_1^t \end{pmatrix} : \mathbb{Z}\Lambda^0 &\longrightarrow \mathbb{Z}\Lambda^0 \oplus \mathbb{Z}\Lambda^0, \end{aligned}$$

as group homomorphisms defined in the natural way.

Moreover, we have

$$K_1(C^*(\Lambda)) \cong \ker(1 - M_1^t, 1 - M_2^t) / \text{im} \begin{pmatrix} M_2^t - 1 \\ 1 - M_1^t \end{pmatrix}.$$

**Proof.** The Kasparov spectral sequence converging to  $K_*(C^*(\Lambda))$  of Theorem 3.15 has  $E_{p,q}^\infty \cong E_{p,q}^3$  for all  $p, q \in \mathbb{Z}$ . However, it follows from  $E_{p,q}^2 = 0$  for odd  $q$  that the differential  $d^2$  is the zero map and  $E_{p,q}^3 \cong E_{p,q}^2 \cong H_p(\mathcal{D})$  for all  $p \in \{0, 1, \dots, k\}$  and even  $q$ , where  $\mathcal{D}$  is the chain complex

$$0 \longleftarrow \mathbb{Z}\Lambda^0 \xleftarrow{\partial_1} \mathbb{Z}\Lambda^0 \oplus \mathbb{Z}\Lambda^0 \xleftarrow{\partial_2} \mathbb{Z}\Lambda^0 \longleftarrow 0$$

with  $\partial_1 = (1 - M_1^t, 1 - M_2^t)$  and  $\partial_2 = \begin{pmatrix} M_2^t - 1 \\ 1 - M_1^t \end{pmatrix}$  for a suitable choice of bases.

Convergence of the spectral sequence to  $K_*(C^*(\Lambda))$  and the above means that we have the following finite filtration of  $\mathcal{K}_1 = K_1(C^*(\Lambda))$ :

$$0 = F_0(\mathcal{K}_1) \subseteq F_1(\mathcal{K}_1) = F_2(\mathcal{K}_1) = \mathcal{K}_1,$$

with  $F_1(\mathcal{K}_1) \cong H_1(\mathcal{D})$ . Hence,  $K_1(C^*(\Lambda)) \cong H_1(\mathcal{D})$  as required.

Now, we could proceed to obtain an isomorphism of  $K_0(C^*(\Lambda))$  by use of the spectral sequence; however we choose to use the Pimsner–Voiculescu sequence in order to deduce relatively easily the action of the isomorphism on basis elements.

By applying the Pimsner–Voiculescu sequence to  $B \rtimes \mathbb{Z}$  and

$$B \rtimes \mathbb{Z}^2 \cong (B \rtimes \mathbb{Z}) \rtimes \mathbb{Z}$$

in succession, we may deduce that  $K_0(i_B) : K_0(B) \longrightarrow K_0(B \rtimes \mathbb{Z}^2)$  factors through an injection  $\Phi_1 : H_0(\mathcal{B}) \longrightarrow K_0(B \rtimes \mathbb{Z}^2)$ , where  $i_B$  is the canonical injection  $i_B : B \longrightarrow B \rtimes \mathbb{Z}^2$  and  $\mathcal{B}$  is the chain complex defined in Lemma 3.12. We may also deduce that there is an exact sequence that constitutes the first row of the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_0(\mathcal{B}) & \xrightarrow{\Phi_1} & K_0(B \rtimes \mathbb{Z}^2) & \longrightarrow & H_2(\mathcal{B}) & \longrightarrow & 0 \\ & & \Phi_2 \downarrow & & \downarrow \Phi_0 & & \downarrow & & \\ 0 & \longrightarrow & H_0(\mathcal{D}) & \longrightarrow & K_0(C^*(\Lambda)) & \longrightarrow & H_2(\mathcal{D}) & \longrightarrow & 0. \end{array}$$

where all downward arrows represent isomorphisms. In particular,

$$\Phi_0 : K_0(C^*(\Lambda)) \longrightarrow K_0(B \rtimes \mathbb{Z}^2)$$

is the isomorphism constructed in Theorem 3.1 and

$$\Phi_2 : H_0(\mathcal{B}) \longrightarrow H_0(\mathcal{D})$$

is one of the isomorphisms in Theorem 3.14.

Now  $H_2(\mathcal{D}) \subseteq \mathbb{Z}\Lambda^0$  is a free abelian group, thus the exact sequences split and we have an isomorphism  $\Phi : H_0(\mathcal{D}) \oplus H_2(\mathcal{D}) \longrightarrow C^*(\Lambda)$  such that  $\Phi(g \oplus 0) = \Phi_0 \Phi_1 \Phi_2^{-1}(g)$  for all  $g \in H_0(\mathcal{D})$ . It is straightforward to check that  $\Phi(\delta_u + \text{im } \partial_1 \oplus 0) = [p_u]$  for all  $u \in \Lambda^0$ . Thus the theorem is proved.  $\square$

Evidently complications arise when  $k > 2$ , however it is worth noting that under some extra assumptions on the vertex matrices it is possible to determine a fair amount about the  $K$ -groups of higher rank graph  $C^*$ -algebras. For example, the case  $k = 3$  is considered below.

**Proposition 3.17.** *Let  $\Lambda$  be a row-finite 3-graph with no sources. Consider the following group homomorphisms defined by block matrices:*

$$\begin{aligned} \partial_1 &= (1 - M_1^t \ 1 - M_2^t \ 1 - M_3^t) : \bigoplus_{i=1}^3 \mathbb{Z}\Lambda^0 \longrightarrow \mathbb{Z}\Lambda^0, \\ \partial_2 &= \begin{pmatrix} M_2^t - 1 & M_3^t - 1 & 0 \\ 1 - M_1^t & 0 & M_3^t - 1 \\ 0 & 1 - M_1^t & 1 - M_2^t \end{pmatrix} : \bigoplus_{i=1}^3 \mathbb{Z}\Lambda^0 \longrightarrow \bigoplus_{i=1}^3 \mathbb{Z}\Lambda^0, \\ \partial_3 &= \begin{pmatrix} 1 - M_3^t \\ M_2^t - 1 \\ 1 - M_1^t \end{pmatrix} : \mathbb{Z}\Lambda^0 \longrightarrow \bigoplus_{i=1}^3 \mathbb{Z}\Lambda^0. \end{aligned}$$

There exists a short exact sequence:

$$0 \longrightarrow \text{coker } \partial_1 / G_0 \longrightarrow K_0(C^*(\Lambda)) \longrightarrow \ker \partial_2 / \text{im } \partial_3 \longrightarrow 0,$$

and

$$K_1(C^*(\Lambda)) \cong \ker \partial_1 / \text{im } \partial_2 \oplus G_1,$$

where  $G_0$  is a subgroup of  $\text{coker } \partial_1$  and  $G_1$  is a subgroup of  $\ker \partial_3$ .

**Proof.** By Theorem 3.15, there exist short exact sequences

$$\begin{aligned} 0 \longrightarrow E_{0,0}^4 \longrightarrow K_0(C^*(\Lambda)) \longrightarrow E_{2,-2}^4 \longrightarrow 0, \\ 0 \longrightarrow E_{1,0}^4 \longrightarrow K_1(C^*(\Lambda)) \longrightarrow E_{3,-2}^4 \longrightarrow 0. \end{aligned}$$

However, since  $E_{p,q}^4 = 0$  if  $p \in \mathbb{Z} \setminus \{0, 1, 2, 3\}$ , the only nonzero components of the differential  $d^3$  are  $d_{3,q}^3 : E_{3,q}^3 \longrightarrow E_{0,q+2}^3$ , where  $q \in 2\mathbb{Z}$ . Moreover, as in the proof of Proposition 3.16, the differential  $d^2$  is the zero map. Thus we have

$$\begin{aligned} E_{1,0}^4 \cong E_{1,0}^3 \cong E_{1,0}^2 \cong H_1(\mathcal{D}), \quad E_{0,0}^3 \cong E_{0,0}^2 \cong H_0(\mathcal{D}), \\ E_{2,-2}^4 \cong E_{2,-2}^3 \cong E_{2,-2}^2 \cong H_2(\mathcal{D}), \quad E_{3,-2}^3 \cong E_{3,-2}^2 \cong H_3(\mathcal{D}). \end{aligned}$$

Also note that  $E_{3,-2}^4$  is isomorphic to  $\ker d_{3,-2}^3 \subseteq E_{3,-2}^3$ , which is a subgroup of the free abelian group  $H_3(\mathcal{D}) \cong \ker \partial_3$ . Thus  $E_{3,-2}^4$  is itself a free abelian group, from which we deduce that the exact sequence for  $K_1(C^*(\Lambda))$  splits. Hence the result follows by setting  $G_0$  to be the image of  $\text{im } d_{3,-2}^3 \subseteq E_{0,0}^3$  under the isomorphism  $E_{0,0}^3 \longrightarrow E_{0,0}^2 \longrightarrow H_0(\mathcal{D})$ , and  $G_1$  to be the image of  $\ker d_{3,-2}^3 \subseteq E_{3,-2}^3$  under the isomorphism  $E_{3,-2}^3 \longrightarrow E_{3,-2}^2 \longrightarrow H_3(\mathcal{D})$ .  $\square$

Now we consider two cases for which we can describe the  $K$ -groups of the  $C^*$ -algebra of a row finite 3-graph with no sources in terms of its vertex matrices by the immediate application of Proposition 3.17.

**Corollary 3.18.** *In addition to the hypothesis of Proposition 3.17:*

(1) *If  $\partial_1$  is surjective then*

$$K_0(C^*(\Lambda)) \cong \ker \partial_2 / \text{im } \partial_3,$$

$$K_1(C^*(\Lambda)) \cong \ker \partial_1 / \text{im } \partial_2 \oplus \ker \partial_3.$$

(2) *If  $\bigcap_{i=1}^3 \ker(1 - M_i^t) = 0$  then*

$$K_1(C^*(\Lambda)) \cong \ker \partial_1 / \text{im } \partial_2$$

*and there exists a short exact sequence*

$$0 \longrightarrow \text{coker } \partial_1 \longrightarrow K_0(C^*(\Lambda)) \longrightarrow \ker \partial_2 / \text{im } \partial_3 \longrightarrow 0.$$

**Proof.** To prove (1), note that we have  $0 = \text{coker } \partial_1$  thus the exact sequence for  $K_0(C^*(\Lambda))$  collapses to give the result for  $K_0(C^*(\Lambda))$ . Also note that  $0 = \text{coker } \partial_1 = H_0(\mathcal{D}) \cong E_{0,0}^3$ . Therefore  $d_{3,-2}^3 : E_{3,-2}^3 \longrightarrow E_{3,0}^3$  is the zero map and  $\ker d_{3,-2}^3 = E_{3,-2}^3 \cong E_{3,-2}^2 \cong H_3(\mathcal{D}) = \ker \partial_3$ . Therefore,  $G_1$  in Proposition 3.17 is  $\ker \partial_3$  and (1) is proved.

To prove (2), if  $\bigcap_{i=1}^3 \ker(1 - M_i^t) = 0$  then  $\ker \partial_3 = 0$ , which implies that  $G_1$  in Proposition 3.17 is the trivial group. It also follows that  $E_{3,-2}^3 = 0$  so that  $\text{im } d_{3,-2}^3 = 0$  and  $G_0$  in Proposition 3.17 is the trivial group. Whence (2) follows immediately from Proposition 3.17.  $\square$

**Remarks 3.19.** (i) One may recover [16, Theorem 3.1] from Theorem 3.15 by setting  $k$  equal to 1.

(ii) By [13, Corollary 3.5 (ii)] a rank  $k$  Cuntz–Krieger algebra ([24, 25]) is isomorphic to a  $k$ -graph  $C^*$ -algebra. Thus, Proposition 3.16 generalises [25, Proposition 4.1], the proof of which inspired the methods used throughout this paper.

(iii) By showing that the  $C^*$ -algebra of a row-finite 2-graph,  $\Lambda$ , with no sources and finite vertex set, satisfying some further conditions, is isomorphic to a rank 2 Cuntz–Krieger algebra, Allen, Pask and Sims used Robertson and Steger’s [25, Proposition 4.1] result to calculate the  $K$ -groups of  $C^*(\Lambda)$  [1, Theorem 4.1]. Moreover, in [1, Remark 4.7. (1)] they note that their formulae for the  $K$ -groups holds for more general 2-graph  $C^*$ -algebras, namely the  $C^*$ -algebras of row-finite 2-graphs,  $\Lambda$ , with no sinks (i.e.,  $s^{-1}(v) \cap \Lambda^n \neq \emptyset$  for all  $n \in \mathbb{N}^k$ ,  $v \in \Lambda^0$ ) nor sources and finite vertex set.

(iv) The notion of associating a  $C^*$ -algebra,  $C^*(\Lambda)$ , to a  $k$ -graph  $\Lambda$  was generalised by Raeburn, Sims and Yeend [20] to include the case where  $\Lambda$  is finitely-aligned; a property identified by them to enable an appropriate  $C^*$ -algebra to be constructed. The family of finitely-aligned  $k$ -graphs and their associated  $C^*$ -algebras admit  $k$ -graphs with sources and those that are not row-finite. In [8], Farthing devised a method of constructing, from an arbitrary finitely-aligned  $k$ -graph  $\Lambda$  with sources, a row-finite  $k$ -graph with no sources,  $\bar{\Lambda}$ , which contains  $\Lambda$

as a subgraph. If, in addition,  $\Lambda$  is row-finite then Farthing showed that  $C^*(\tilde{\Lambda})$  is strong Morita equivalent to  $C^*(\Lambda)$  and thus has isomorphic  $K$ -groups to those of  $C^*(\Lambda)$ . Therefore, in principle, the results in this paper could be extended to the case where  $\Lambda$  is row-finite but with sources.

#### 4. The $K$ -groups of unital $k$ -graph $C^*$ -algebras

Recall that if  $\Lambda$  is a row-finite higher rank graph with no sources then  $\Lambda^0$  finite is equivalent to  $C^*(\Lambda)$  being unital ([13, Remarks 1.6 (v)]). Thus in this section we specialise in the case where the vertex set of our higher rank graph, hence each vertex matrix, is finite. We will continue to denote the Kasparov spectral sequence converging to  $K_*(C^*(\Lambda))$  of the previous section by  $\{(E^r, d^r)\}$  and we shall denote the torsion-free rank of an abelian group  $G$  by  $r_0(G)$  (see, e.g., [9]).

**Proposition 4.1.** *If  $\Lambda$  is a row-finite higher rank graph with no sources and  $\Lambda^0$  finite then  $K_0(C^*(\Lambda))$  and  $K_1(C^*(\Lambda))$  have equal torsion-free rank.*

**Proof.** Let the rank of the given higher rank graph  $\Lambda$  be  $k$  and let  $|\Lambda^0| = n$ .

Since  $E_{p,q}^\infty \cong E_{p,q}^{k+1}$  for all  $p, q \in \mathbb{Z}$  and  $E_{p,q}^{k+1} = 0$  if  $p \in \mathbb{Z} \setminus \{0, 1, \dots, k\}$  or  $q$  odd by Lemma 3.3, it follows from the definition of convergence of  $\{(E^r, d^r)\}$  (Definition 3.2) that there exist finite filtrations,

$$0 = F_{-1}(\mathcal{K}_0) \subseteq E_{0,0}^{k+1} \cong F_0(\mathcal{K}_0) \subseteq F_1(\mathcal{K}_0) \subseteq \dots \subseteq F_{k-1}(\mathcal{K}_0) \subseteq F_k(\mathcal{K}_0) = \mathcal{K}_0,$$

and

$$0 = F_0(\mathcal{K}_1) \subseteq E_{1,0}^{k+1} \cong F_1(\mathcal{K}_1) \subseteq F_2(\mathcal{K}_1) \subseteq \dots \subseteq F_{k-1}(\mathcal{K}_1) \subseteq F_k(\mathcal{K}_1) = \mathcal{K}_1$$

of  $\mathcal{K}_0 = K_0(C^*(\Lambda))$  and  $\mathcal{K}_1 = K_1(C^*(\Lambda))$  respectively, such that

$$E_{p,q}^{k+1} \cong F_p(\mathcal{K}_{p+q})/F_{p-1}(\mathcal{K}_{p+q}).$$

Thus,

$$\begin{aligned} r_0(K_0(C^*(\Lambda))) &= r_0(F_k(\mathcal{K}_0)) = r_0(F_{k-1}(\mathcal{K}_0)) + r_0(E_{k,-k}^{k+1}) = \dots \\ &= r_0(F_0(\mathcal{K}_0)) + \sum_{s \geq 1} r_0(E_{s,-s}^{k+1}) = \sum_{s \in \mathbb{Z}} r_0(E_{s,-s}^{k+1}), \end{aligned}$$

and

$$\begin{aligned} r_0(K_1(C^*(\Lambda))) &= r_0(F_k(\mathcal{K}_1)) = r_0(F_{k-1}(\mathcal{K}_1)) + r_0(E_{k,-k+1}^{k+1}) = \dots \\ &= r_0(F_1(\mathcal{K}_1)) + \sum_{s \geq 2} r_0(E_{s,-s+1}^{k+1}) = \sum_{s \in \mathbb{Z}} r_0(E_{s,-s+1}^{k+1}). \end{aligned}$$

Now we claim that

$$\sum_{s \in \mathbb{Z}} r_0(E_{s,-s}^{k+1}) - r_0(E_{s,-s+1}^{k+1}) = \sum_{s \in \mathbb{Z}} r_0(E_{s,-s}^2) - r_0(E_{s,-s+1}^2).$$

To see that this holds it is sufficient to prove that for all  $r \geq 2$  we have

$$\sum_{s \in \mathbb{Z}} r_0(E_{s,-s}^{r+1}) - r_0(E_{s,-s+1}^{r+1}) = \sum_{s \in \mathbb{Z}} r_0(E_{s,-s}^r) - r_0(E_{s,-s+1}^r).$$

Recall that for all  $r \geq 1$ ,  $p, q \in \mathbb{Z}$ ,  $E_{p,q}^{r+1} \cong Z(E^r)_{p,q}/B(E^r)_{p,q}$  where  $Z(E^r)_{p,q} = \ker d_{p,q}^r$  and  $B(E^r)_{p,q} = \text{im } d_{p+r,q-r+1}^r$ . Thus

$$\begin{aligned} r_0(E_{p,q}^{r+1}) &= r_0(Z(E^r)_{p,q}) - r_0(B(E^r)_{p,q}) \\ &= r_0(Z(E^r)_{p,q}) - r_0(E_{p+r,q-r+1}^r) + r_0(Z(E^r)_{p+r,q-r+1}) \end{aligned}$$

for all  $r \geq 1$ ,  $p, q \in \mathbb{Z}$ . Moreover, it follows from the definition of the Kasparov spectral sequence that given any  $r \geq 1$  and  $p, q, q' \in \mathbb{Z}$  with  $q = q' \pmod{2}$  there exist isomorphisms

$$\begin{aligned} \rho : E_{p,q}^r &\longrightarrow E_{p,q'}^r, \\ \sigma : E_{p-r,q+r-1}^r &\longrightarrow E_{p-r,q'+r-1}^r, \end{aligned}$$

such that  $d_{p,q'}^r \circ \rho = \sigma \circ d_{p,q}^r$ . Therefore,

$$\begin{aligned} \sum_{s \in \mathbb{Z}} r_0(E_{s,-s}^{r+1}) - r_0(E_{s,-s+1}^{r+1}) &= \sum_{s \in \mathbb{Z}} \{r_0(Z(E^r)_{s,-s}) - r_0(Z(E^r)_{s+r,-s-r+2}) \\ &\quad - r_0(Z(E^r)_{s,-s+1}) + r_0(Z(E^r)_{s+r,-s-r+1}) \\ &\quad + r_0(E_{s+r,-s-r+2}^r) - r_0(E_{s+r,-s-r+1}^r)\} \\ &= \sum_{s \in \mathbb{Z}} r_0(E_{s,-s}^r) - r_0(E_{s,-s+1}^r) \end{aligned}$$

for all  $r \geq 1$ . Combining the above gives

$$r_0(K_0(C^*(\Lambda))) - r_0(K_1(C^*(\Lambda))) = \sum_{s \in \mathbb{Z}} r_0(E_{s,-s}^2) - r_0(E_{s,-s+1}^2).$$

Now, recall that for all  $p \in \mathbb{Z}$  and  $q \in 2\mathbb{Z}$ ,

$$E_{p,q}^2 \cong H_p(\mathbb{Z}^k, K_0(B)) \cong \ker \partial_p / \text{im } \partial_{p+1}$$

by Theorem 3.14. Therefore,

$$\begin{aligned}
 r_0(K_0(C^*(\Lambda))) - r_0(K_1(C^*(\Lambda))) &= \sum_{s \in \mathbb{Z}} r_0(E_{2s, -2s}^2) - r_0(E_{2s+1, -2s}^2) \\
 &= \sum_{s \in \mathbb{Z}} r_0(\ker \partial_{2s}) - r_0(\operatorname{im} \partial_{2s+1}) - r_0(\ker \partial_{2s+1}) + r_0(\operatorname{im} \partial_{2s+2}) \\
 &= \sum_{s \in \mathbb{Z}} r_0(\ker \partial_{2s}) - r_0 \left( \left( \bigoplus_{N_{2s+1}} \mathbb{Z}\Lambda^0 \right) / \ker \partial_{2s+1} \right) - r_0(\ker \partial_{2s+1}) \\
 &\quad + r_0 \left( \left( \bigoplus_{N_{2s+2}} \mathbb{Z}\Lambda^0 \right) / \ker \partial_{2s+2} \right) \\
 &= \sum_{s \in \mathbb{Z}} r_0(\ker \partial_{2s}) - \binom{k}{2s+1} n + r_0(\ker \partial_{2s+1}) - r_0(\ker \partial_{2s+1}) \\
 &\quad + \binom{k}{2s+2} n - r_0(\ker \partial_{2s+2}) \\
 &= \sum_{s \in \mathbb{Z}} \left\{ \binom{k}{2s} - \binom{k}{2s-1} \right\} n \\
 &= \sum_{s \in \mathbb{Z}} \left\{ \binom{k-1}{2s} + \binom{k-1}{2s-1} - \binom{k-1}{2s-1} - \binom{k-1}{2s-2} \right\} n = 0. \quad \square
 \end{aligned}$$

**Corollary 4.2.** *If  $\Lambda$  is a row-finite higher rank graph with no sources and  $\Lambda^0$  is finite then there exists a nonnegative integer  $r$  such that for  $i = 0, 1$ ,*

$$K_i(C^*(\Lambda)) \cong \mathbb{Z}^r \oplus T_i$$

for some finite group  $T_i$ , where  $\mathbb{Z}^0 := \{0\}$ .

**Proof.** It is well-known that if  $B$  is a finitely generated subgroup of an abelian group  $A$  such that  $A/B$  is also finitely generated then  $A$  must be finitely generated too [9]. Now, for all  $p, q \in \mathbb{Z}$ ,  $E_{p,q}^{k+1}$  is isomorphic to a subquotient of the finitely generated abelian group  $E_{p,q}^2 \cong H_p(\mathbb{Z}^k, K_q(B))$ , so  $E_{p,-p}^{k+1}$  is also finitely generated. Moreover,  $E_{0,i}^{k+1} \cong F_0(K_i(C^*(\Lambda)))$  and for  $p \in \{1, 2, \dots, k\}$ ,  $E_{p,-p+i}^{k+1} \cong F_p(K_i(C^*(\Lambda))) / F_{p-1}(K_i(C^*(\Lambda)))$ , which implies that  $K_i(C^*(\Lambda)) = F_k(K_i(C^*(\Lambda)))$  is finitely generated. The result follows from Proposition 4.1 by noting that every finitely generated abelian group  $A$  is isomorphic to the direct sum of a finite group with  $\mathbb{Z}^r$ , where  $r = r_0(A)$  (see, e.g., [9, Theorem 15.5]).  $\square$

**Remarks 4.3.** Note that it is well-known that when  $k = 1$  we always have  $T_1 = 0$  in the above, i.e.,  $K_1(C^*(\Lambda))$  is torsion-free. However, for  $k > 1$ ,  $K_1(C^*(\Lambda))$  may contain torsion elements.

Formulae for the torsion-free rank and torsion parts of the  $K$ -groups of unital  $C^*$ -algebras of row-finite 2-graphs with no sources can be given in

terms of the vertex matrices (cf. [25, Proposition 4.13]). This we do in Proposition 4.4 below.

**Proposition 4.4** (cf. [25, Proposition 4.13]). *Let  $\Lambda$  be a row-finite 2-graph with no sources and finite vertex set. Then*

$$\begin{aligned} r_0(K_0(C^*(\Lambda))) &= r_0(K_1(C^*(\Lambda))) \\ &= r_0(\text{coker}(1 - M_1^t, 1 - M_2^t)) + r_0(\text{coker}(1 - M_1, 1 - M_2)), \\ \text{tor}(K_0(C^*(\Lambda))) &\cong \text{tor}(\text{coker}(1 - M_1^t, 1 - M_2^t)), \\ \text{tor}(K_1(C^*(\Lambda))) &\cong \text{tor}(\text{coker}(1 - M_1, 1 - M_2)). \end{aligned}$$

**Proof.** We have already seen in Proposition 4.1 that the torsion-free rank of the  $K_0$ -group and  $K_1$ -group of a  $k$ -graph are equal so it is sufficient to calculate the torsion-free rank of  $K_0(C^*(\Lambda))$ . Let  $n := |\Lambda^0|$ . By Proposition 3.16 we have

$$\begin{aligned} r_0(K_0(C^*(\Lambda))) &= r_0(\text{coker}(1 - M_1^t, 1 - M_2^t)) + r_0\left(\ker\begin{pmatrix} 1 - M_1^t \\ 1 - M_2^t \end{pmatrix}\right) \\ &= r_0(\text{coker}(1 - M_1^t, 1 - M_2^t)) + n - r_0(\text{im}(1 - M_1, 1 - M_2)) \\ &= r_0(\text{coker}(1 - M_1^t, 1 - M_2^t)) + r_0(\text{coker}(1 - M_1, 1 - M_2)). \end{aligned}$$

Furthermore, the assertion about the torsion part of  $K_0(C^*(\Lambda))$  is obvious. The torsion part of  $K_1(C^*(\Lambda))$  is given by

$$\text{tor}(K_1(C^*(\Lambda))) \cong \text{tor}\left(\ker(1 - M_1^t, 1 - M_2^t) / \text{im}\begin{pmatrix} M_2^t - 1 \\ 1 - M_1^t \end{pmatrix}\right),$$

which is clearly isomorphic to  $\text{tor}(\text{coker}\begin{pmatrix} M_2^t - 1 \\ 1 - M_1^t \end{pmatrix})$ . However, by reduction to Smith normal forms,  $\text{coker}\begin{pmatrix} M_2^t - 1 \\ 1 - M_1^t \end{pmatrix}$  is isomorphic to  $\text{coker}(1 - M_1, 1 - M_2)$ .  $\square$

**Remarks 4.5.** We note that, in the case where  $\Lambda$  is a row-finite 3-graph with no sources and finite vertex set, and with  $\partial_1, \partial_2$  defined as in Proposition 3.17, it is straightforward to show that if  $\partial_1$  is surjective then

$$K_0(C^*(\Lambda)) \cong K_1(C^*(\Lambda)) \cong \mathbb{Z}^m,$$

where  $m := r_0(\ker \partial_2) - |\Lambda^0| = r_0(\text{coker } \partial_2) - |\Lambda^0|$  (with  $\mathbb{Z}^0 := 0$ ).

## 5. Applications and examples

We begin this section with two corollaries to the results in the preceding section, which facilitate the classification of the  $C^*$ -algebras of row-finite 2-graphs with no sources. We then end the paper with some simple illustrative examples.



**Corollary 5.1.** *Let  $\Lambda$  be a row-finite 2-graph with no sources, finite vertex set and vertex matrices  $M_1$  and  $M_2$ . Then there exists an isomorphism*

$$\Phi : \operatorname{coker}(1 - M_1^t, 1 - M_2^t) \oplus \ker \begin{pmatrix} M_2^t - 1 \\ 1 - M_1^t \end{pmatrix} \longrightarrow K_0(C^*(\Lambda))$$

such that  $\Phi(e + \operatorname{im} \partial_0) = [1]$ , where  $e(v) = 1$  for all  $v \in \Lambda^0$ .

**Proof.** Follows from Proposition 3.16 and that  $\sum_{u \in \Lambda^0} p_u = 1$ .  $\square$

**Remarks 5.2.** We note that the  $C^*$ -algebra of a row-finite  $k$ -graph  $\Lambda$ , with no sources, is separable, nuclear and satisfies the UCT [26]. If in addition the  $C^*$ -algebra is simple and purely infinite we say that it is a Kirchberg algebra, and note that by the Kirchberg–Phillips classification theorem ([12, 18]) it is classifiable by its  $K$ -theory (see [13, Theorem 5.5]). We also note that conditions on the underlying  $k$ -graph have been identified, which determine whether the  $C^*$ -algebra is simple ([13, Proposition 4.8], [22, Theorem 3.2]) and purely infinite ([28, Proposition 8.8]).

**Corollary 5.3.** *Let  $\Lambda$  and  $\Delta$  be two row-finite 2-graphs with no sources. Furthermore, suppose that  $C^*(\Lambda)$  and  $C^*(\Delta)$  are both simple and purely infinite, and that  $\Lambda$  and  $\Delta$  share the same vertex matrices. Then*

$$C^*(\Lambda) \cong C^*(\Delta).$$

**Proof.** Let  $\Lambda$  and  $\Delta$  be two 2-graphs satisfying the hypothesis. Then, by Proposition 3.16 their  $K$ -groups are isomorphic.

Suppose that the vertex set of  $\Lambda$  (and hence that of  $\Delta$ ) is infinite. Then  $C^*(\Lambda)$  and  $C^*(\Delta)$  are both nonunital, and thus stable, Kirchberg algebras with isomorphic  $K$ -groups. Thus by the Kirchberg–Phillips classification theorem  $C^*(\Lambda) \cong C^*(\Delta)$ .

In the case where the vertex set of  $\Lambda$  (and hence that of  $\Delta$ ) is finite,  $C^*(\Lambda)$  and  $C^*(\Delta)$  are both unital Kirchberg algebras with isomorphic  $K$ -groups. Furthermore, by Corollary 5.1 we see that the isomorphism of  $K$ -groups maps the  $K_0$ -class of the unit of one of the  $C^*$ -algebras onto that of the other. Therefore, by the Kirchberg–Phillips classification theorem we conclude that  $C^*(\Lambda) \cong C^*(\Delta)$  and the result is proved.  $\square$

**Examples 5.4.** (1) Let  $\Lambda$  be a row-finite 2-graph with no sources. Suppose that the vertex matrices of  $\Lambda$  are both equal to  $M$ , say. By Proposition 3.16 and [16, Theorem 3.1] we have:

$$\begin{aligned} K_i(C^*(\Lambda)) &\cong \mathbb{Z}\Lambda^0 / \operatorname{im}(1 - M^t) \oplus \ker(1 - M^t) \\ &\cong K_0(C^*(E)) \oplus K_1(C^*(E)), \end{aligned}$$

for  $i = 1, 2$ , where  $E$  is the 1-graph with vertex matrix  $M$ . The isomorphism for  $K_0(C^*(\Lambda))$  is immediately obvious. That for  $K_1(C^*(\Lambda))$  is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} + \operatorname{im} \begin{pmatrix} M^t - 1 \\ 1 - M^t \end{pmatrix} \mapsto (x + \operatorname{im}(1 - M^t)) \oplus (x + y)$$

for all  $\begin{pmatrix} x \\ y \end{pmatrix} \in \ker(1 - M^t, 1 - M^t)$ .

- (2) Fix nonzero  $n_1, n_2 \in \mathbb{N}^2$  and let  $\Lambda$  be a 2-graph with one vertex and vertex matrices  $M_1 = (n_1)$  and  $M_2 = (n_2)$  respectively. By Proposition 3.16, we have  $K_0(C^*(\Lambda)) \cong K_1(C^*(\Lambda)) \cong \mathbb{Z}/g\mathbb{Z}$ , where  $g$  is the greatest common divisor of  $n_1 - 1$  and  $n_2 - 1$ .

Note that we recover the  $K$ -groups of tensor products of Cuntz algebras [4] by letting  $\Lambda$  be the product 2-graph of two 1-graphs each with one vertex and a finite number of edges (i.e., morphisms of degree 1). We also note that tensor products of two Cuntz algebras are not the only examples of  $C^*$ -algebras of such 2-graphs  $\Lambda$  (cf. [13, §6]). However, by Corollary 5.3, they are, up to  $*$ -isomorphism, the only examples of Kirchberg algebras arising from row-finite 2-graphs with one vertex.

- (3) For each positive integer  $n$ , let  $\mathbf{O}_n$  be the 1-graph with 1 vertex,  $\star$ , and  $n$  edges,  $\alpha_1, \alpha_2, \dots, \alpha_n$ . Let  $c : \mathbf{O}_3 \times \mathbf{O}_n \rightarrow \mathbb{Z}$  be the unique functor that satisfies  $c(\alpha_i, \star) = \delta_{i,1}$  ( $i = 1, 2, 3$ ) and  $c(\star, \alpha_i) = 1$  ( $i = 1, \dots, n$ ). Define  $\Lambda$  to be the 2-graph  $\mathbb{Z} \times_c (\mathbf{O}_3 \times \mathbf{O}_n)$ . Let  $T_i := 1 - M_i^t$ , where  $M_1$  and  $M_2$  are the vertex matrices of  $\Lambda$ . Then

$$(5.1) \quad T_1 \delta_u = -\delta_u - \delta_{u+1} \text{ and}$$

$$(5.2) \quad T_2 \delta_u = \delta_u - n\delta_{u+1},$$

where we identify  $\Lambda^0$  with  $\mathbb{Z}$ .

Clearly,  $\ker \begin{pmatrix} -T_2 \\ T_1 \end{pmatrix} = 0$ . Now consider  $\text{coker}(T_1, T_2)$  and for each  $g \in \mathbb{Z}\Lambda^0$  let  $[g]$  be the image of  $g$  under the natural homomorphism  $\mathbb{Z}\Lambda^0 \rightarrow \text{coker}(T_1, T_2)$ . By (5.1) and (5.2) we have  $(n+1)[\delta_u] = 0$ . Therefore,  $\text{coker}(T_1, T_2)$  is a cyclic group, generated by  $[\delta_0]$  say, whose order divides  $n+1$ . We claim that  $\rho[\delta_0] \neq 0$  for each  $\rho = 1, \dots, n$ . Suppose the contrary, then we have

$$\rho\delta_0 = T_1 x + T_2 y$$

for some  $x, y \in \mathbb{Z}\Lambda^0$  and  $\rho \in \{1, \dots, n\}$ . Thus, for each  $u \in \Lambda^0$  we have

$$\rho\delta_0(u) = -x(u) - x(u-1) + y(u) - ny(u-1).$$

Since  $x, y \in \mathbb{Z}\Lambda^0$ , there exists  $N$  such that  $x(u) = y(u) = 0$  if  $|u| > N$ , which we assume, without loss of generality, to be greater than zero. It follows that

$$y(u) = \begin{cases} x(u), & u = -N, \\ x(u) + (n+1) \sum_{j=0}^{N-1+u} n^{N-1+u-j} x(-N+j), & -N+1 \leq u \leq -1, \\ x(u) + (n+1) \sum_{j=0}^{N-1+u} n^{N-1+u-j} x(-N+j) + \rho n^u, & u \geq 0. \end{cases}$$

Setting  $u = N + 1$ , we arrive at the contradiction  $(n + 1)|\rho n^{N+1}$ . Therefore, by Proposition 3.16  $K_0(C^*(\Lambda)) \cong \mathbb{Z}/(n + 1)\mathbb{Z}$ .

Now we turn our attention to  $\ker(T_1, T_2)/\operatorname{im} \begin{pmatrix} -T_2 \\ T_1 \end{pmatrix}$ . Suppose that  $x \oplus y \in \ker(T_1, T_2)$ , then there exists  $N$  such that  $x(u) = y(u) = 0$  if  $|u| > N$ ,

$$(5.3) \quad y(-N) = x(-N) \text{ and}$$

$$(5.4) \quad y(u) = x(u) + (n + 1) \sum_{j=-N}^{u-1} n^{u-1-i} x(j) \text{ for all } u \geq -N + 1.$$

Let  $P : \mathbb{Z}\Lambda^0 \oplus \mathbb{Z}\Lambda^0 \longrightarrow \mathbb{Z}\Lambda^0$  be the projection onto the second component, i.e.,  $P(x \oplus y) = y$ . From (5.3) and (5.4) we see that  $P$  is injective on  $\ker(T_1, T_2)$  and thus induces an isomorphism

$$\ker(T_1, T_2)/\operatorname{im} \begin{pmatrix} -T_2 \\ T_1 \end{pmatrix} \cong P(\ker(T_1, T_2))/\operatorname{im} T_1.$$

Moreover,  $P(\ker(T_1, T_2)) = \{y \in \mathbb{Z}\Lambda^0 \mid \sum_{j \in \mathbb{Z}} (-1)^j y(j) = 0\}$ . Now given  $y \in P(\ker(T_1, T_2))$ , define  $z : \Lambda^0 \longrightarrow \mathbb{Z}$  by

$$z(u) = 0 \text{ if } u < -N,$$

$$z(u) = \sum_{j=-N}^u (-1)^{u-j+1} y(j) \text{ if } u \geq -N.$$

Then it is straightforward to show that  $z \in \mathbb{Z}\Lambda^0$  and  $T_1 z = y$ , and thus  $\ker(T_1, T_2)/\operatorname{im} \begin{pmatrix} -T_2 \\ T_1 \end{pmatrix}$  is the trivial group. Therefore, by Proposition 3.16  $K_1(C^*(\Lambda)) = 0$ .

Note that  $\Lambda$  satisfies the hypotheses of [13, Proposition 4.8] and [28, Proposition 8.8] and thus  $C^*(\Lambda)$  is a (stable) Kirchberg algebra. It now follows from the Kirchberg–Phillips classification theorem that  $C^*(\Lambda)$  is  $*$ -isomorphic to the stabilized Cuntz algebra  $\mathcal{O}_{n+2} \otimes \mathbb{K}$ .

(4) Let  $c : \mathbf{O}_3 \times \mathbf{O}_3 \times \mathbf{O}_3 \longrightarrow \mathbb{Z}_2$  be the unique functor that satisfies<sup>3</sup>

$$\begin{array}{lll} c(\alpha_1, \star, \star) = 0, & c(\star, \alpha_1, \star) = 0, & c(\star, \star, \alpha_1) = 1, \\ c(\alpha_2, \star, \star) = 0, & c(\star, \alpha_2, \star) = 0, & c(\star, \star, \alpha_2) = 1, \\ c(\alpha_3, \star, \star) = 1, & c(\star, \alpha_3, \star) = 1, & c(\star, \star, \alpha_3) = 1. \end{array}$$

Then the vertex matrices of  $\Lambda := \mathbb{Z}_2 \times_c (\mathbf{O}_3 \times \mathbf{O}_3 \times \mathbf{O}_3)$  are

$$M_1 = M_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}.$$

<sup>3</sup>We extend the definition of the product of two higher rank graphs (Examples 2.3.2) to the product higher rank graph of three higher rank graphs in the natural way. Note that if  $\Lambda_i$  is a  $k_i$ -graph for  $i = 1, 2, 3$  then both  $(\Lambda_1 \times \Lambda_2) \times \Lambda_3$  and  $\Lambda_1 \times (\Lambda_2 \times \Lambda_3)$  are clearly pairwise isomorphic as  $(k_1 + k_2 + k_3)$ -graphs to  $\Lambda_1 \times \Lambda_2 \times \Lambda_3$ .

Following the notation in Proposition 3.17, for  $i = 1, 2, 3$ , we have

$$\begin{aligned} \partial_1 &= \begin{pmatrix} -1 & -1 & -1 & -1 & 1 & -3 \\ -1 & -1 & -1 & -1 & -3 & 1 \end{pmatrix}, & \partial_3 &= \begin{pmatrix} 1 & -3 \\ -3 & 1 \\ 1 & 1 \\ 1 & 1 \\ -1 & -1 \\ -1 & -1 \end{pmatrix}. \\ \partial_2 &= \begin{pmatrix} 1 & 1 & -1 & 3 & 0 & 0 \\ 1 & 1 & 3 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & 3 \\ -1 & -1 & 0 & 0 & 3 & -1 \\ 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 & -1 \end{pmatrix}, \end{aligned}$$

To compute the  $K$ -groups of  $C^*(\Lambda)$  we reduce the relevant matrices to their Smith normal forms (for a (not necessarily square) matrix  $M$  we shall denote its Smith normal form by  $S(M)$ ). In particular,

$$U_1 \partial_1 V_1 = S(\partial_1) = S(\partial_3)^t = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \end{pmatrix}$$

for some invertible matrices  $U_1, V_1$ . Thus,  $\text{coker } \partial_1 \cong \mathbb{Z}/4\mathbb{Z}$ ,  $\ker \partial_3 = 0$  and Corollary 3.18 can be applied to deduce that there exists a short exact sequence

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow K_0(C^*(\Lambda)) \longrightarrow \ker \partial_2 / \text{im } \partial_3 \longrightarrow 0$$

and  $K_1(C^*(\Lambda)) \cong \ker \partial_1 / \text{im } \partial_2$ . Now

$$U_2 \partial_2 V_2 = S(\partial_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

for some invertible matrices  $U_2, V_2$ , and we see that there exists an isomorphism (induced by  $V_2^{-1}$  or by  $U_1$ ) of  $\ker \partial_2 / \text{im } \partial_3$  onto  $\mathbb{Z}/4\mathbb{Z}$ . Furthermore, we can now see that  $\ker \partial_1 / \text{im } \partial_2$  is isomorphic to  $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ . Hence,  $K_0(C^*(\Lambda))$  is a group order 16 and  $K_1(C^*(\Lambda)) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ .<sup>4</sup>

We note that it is, perhaps, slightly surprising that  $C^*(\Lambda)$  shares, at least, one of its  $K$ -groups with that of  $\mathcal{O}_5 \otimes \mathcal{O}_5 \otimes \mathcal{O}_5$ , given that this relationship is not obvious at the level of  $C^*$ -algebras.

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<sup>4</sup>It is well-known that there are, up to isomorphism, 5 abelian groups of order 16.

(5) Let  $c : \mathcal{O}_2 \times \mathcal{O}_3 \times \mathcal{O}_3 \longrightarrow \mathbb{Z}_2$  be the unique functor that satisfies

$$\begin{aligned} c(\alpha_1, \star, \star) &= 0, & c(\star, \alpha_1, \star) &= 0, & c(\star, \star, \alpha_1) &= 1, \\ c(\alpha_2, \star, \star) &= 1, & c(\star, \alpha_2, \star) &= 1, & c(\star, \star, \alpha_2) &= 1, \\ & & c(\star, \alpha_3, \star) &= 1, & c(\star, \star, \alpha_3) &= 1. \end{aligned}$$

Then the vertex matrices of  $\Lambda := \mathbb{Z}_2 \times_c (\mathcal{O}_3 \times \mathcal{O}_3 \times \mathcal{O}_3)$  are

$$M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}.$$

Now, for  $i = 1, 2, 3$ , we have

$$\begin{aligned} \partial_1 &= \begin{pmatrix} 0 & -1 & 0 & -2 & 1 & -3 \\ -1 & 0 & -2 & 0 & -3 & 1 \end{pmatrix}, & \partial_3 &= \begin{pmatrix} 1 & -3 \\ -3 & 1 \\ 0 & 2 \\ 2 & 0 \\ 0 & -1 \\ -1 & 0 \end{pmatrix}. \\ \partial_2 &= \begin{pmatrix} 0 & 2 & -1 & 3 & 0 & 0 \\ 2 & 0 & 3 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 3 \\ -1 & 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & -1 & 0 & -2 \\ 0 & 0 & -1 & 0 & -2 & 0 \end{pmatrix}, \end{aligned}$$

As in the previous example we compute the Smith normal form of  $\partial_1$  (and hence that of  $\partial_3$ ) first, to determine whether Corollary 3.18 is applicable. We find that

$$S(\partial_1) = S(\partial_3)^t = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

thus  $\text{coker } \partial_1 \cong \ker \partial_3 \cong 0$  and we may apply Corollary 3.18 to deduce that

$$K_0(C^*(\Lambda)) = \ker \partial_2 / \text{im } \partial_3 \quad \text{and} \quad K_1(C^*(\Lambda)) = \ker \partial_1 / \text{im } \partial_2.$$

Now,

$$S(\partial_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It follows that both  $\ker \partial_2 / \text{im } \partial_3$  and  $\ker \partial_1 / \text{im } \partial_2$  are trivial. Thus, the  $K$ -groups of  $C^*(\Lambda)$  are isomorphic to those of the Cuntz algebra  $\mathcal{O}_2$ . Furthermore, It is clear that  $\Lambda$  satisfies the hypotheses of [13, Proposition 4.8] and [28, Proposition 8.8], and therefore  $C^*(\Lambda)$  is an

unital Kirchberg algebra (cf. Remarks 5.2). Applying the Kirchberg–Phillips classification theorem, we conclude that  $C^*(\Lambda) \cong \mathcal{O}_2$ .

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