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Extension of linear operators and Lipschitz maps into $\mathcal{C}(K)$-spaces

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Abstract. We study the extension of linear operators with range in a $\mathcal{C}(K)$-space, comparing and contrasting our results with the corresponding results for the nonlinear problem of extending Lipschitz maps with values in a $\mathcal{C}(K)$-space. We give necessary and sufficient conditions on a separable Banach space X which ensure that every operator $T : E \to \mathcal{C}(K)$ defined on a subspace may be extended to an operator $\widetilde{T} : X \to \mathcal{C}(K)$ with $\|\widetilde{T}\| \le (1+\epsilon)\|T\|$ (for any $\epsilon > 0$). Based on these we give new examples of such spaces (including all Orlicz sequence spaces with separable dual for a certain equivalent norm). We answer a question of Johnson and Zippin by showing that if E is a weak*-closed subspace of ℓ_1 then every operator $T : E \to \mathcal{C}(K)$ can be extended to an operator $\widetilde{T} : \ell_1 \to \mathcal{C}(K)$ with $\|\widetilde{T}\| \le (1+\epsilon)\|T\|$. We then show that ℓ_1 has a universal extension property: if X is a separable Banach space containing ℓ_1 then any operator $T : \ell_1 \to \mathcal{C}(K)$ can be extended to an operator $\widetilde{T} : X \to \mathcal{C}(K)$ with $\|\widetilde{T}\| \le (1+\epsilon)\|T\|$; this answers a question of Speegle.

Contents

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1. Introduction

In this paper we study extensions of linear operators from separable Banach spaces into $\mathcal{C}(K)$-spaces where K is compact metric. This subject has a long history but many quite simply stated problems remain open. The first result in this direction is Sobczyk's theorem [37], which states that c_0 is 2-separably injective, i.e., if X is a separable Banach space and $T_0 : E \to c_0$ is a bounded operator defined on a subspace then T_0 has an extension T with $\|T\| \leq 2\|T_0\|$. Later Zippin [40] showed the converse that c_0 is the only separably injective space.

If one replaces c_0 by $\mathcal{C}(K)$ with K an arbitrary compact metric space then a number of special results are known (if X is separable it will then follow that one has the same results for an arbitrary $\mathcal{C}(K)$-space). In 1971, Lindenstrauss and Pełczyński showed that every operator $T_0 : E \to \mathcal{C}(K)$ where E is a subspace of c_0, can be extended to $T; c_0 \to \mathcal{C}(K)$ with $\|T\| \leq (1+\epsilon)\|T_0\|$. Similar results with $\|T\| = \|T_0\|$ are known for ℓ_p where $1 < p < \infty$ (Zippin [41]). On the other hand, the fact that $\mathcal{C}(K)$-spaces are not separably injective in general implies that a similar result is false in ℓ_1. Johnson and Zippin [16] showed that $T_0 : E \to \mathcal{C}(K)$ can be extended if E is weak*-closed and give an estimate of $\|T\| \leq (3+\epsilon)\|T_0\|$. A partial converse was given by the author in [18]: if E is a subspace of ℓ_1 such that every bounded operator $T : E \to \mathcal{C}(K)$ can be extended and ℓ_1/E has a UFDD then ℓ_1/E is isomorphic to the dual of a subspace of c_0. For recent survey of this problem we refer to Zippin [42].

We here aim to compare and contrast this linear extension problem with the corresponding nonlinear problem for Lipschitz maps. In fact if E is a closed subset of any Banach space X and $F_0 : E \to \mathcal{C}(K)$ is a Lipschitz map then there is always a Lipschitz extension $F : X \to \mathcal{C}(K)$ by a result of Lindenstrauss [26]; furthermore in [19] we showed that one can take the Lipschitz constant of F, $\mathrm{Lip}(F) \leq 2\mathrm{Lip}(F_0)$. Nevertheless if one considers the existence of an almost isometric extension ($\mathrm{Lip}(F) \leq (1+\epsilon)\mathrm{Lip}(F_0)$) the results seem to parallel the linear theory.

We now discuss the contents of the paper. We will restrict ourselves to the context of *real* Banach spaces. In order to discuss our results let us introduce the following definitions (see §2 for full description of our terminology). If X, Y are Banach spaces and E is a closed subspace of X (respectively, closed subset of X) then (E, X) has the linear (λ, Y)-extension property (respectively the Lipschitz (λ, Y)-extension property) if every linear operator $T_0 : E \to Y$ (respectively every Lipschitz map $F_0 : E \to Y$) has a linear extension $T : X \to Y$ with $\|T\| \leq \lambda\|T_0\|$ (respectively a Lipschitz extension $F : X \to Y$ with $\mathrm{Lip}(F) \leq \lambda\mathrm{Lip}(F_0)$).

In §2 we introduce our terminology and discuss some elementary facts about extension problems. In §3 we recall the theory of types introduced by Krivine and Maurey [25], and discuss some special classes of Banach spaces which will play a role in the latter part of the paper; these include spaces with properties (M) and (M^*), introduced by the author in the study of M-ideals [17] and some new properties (L) and (L^*). In §4 we relate the theory of types to the existence of Lipschitz extensions by recasting the results of [19] in terms of the behavior of types.

In §5, we turn to the problem of extending linear operators with range in c_0. We show that for $1 < \lambda \leq 2$, if E is a linear subspace of a separable Banach space X then (E, X) has the linear (λ, c_0)-extension property if and only it has the Lipschitz (λ, c_0)-extension property. One remarkable fact that emerges is that (E, X) has the linear (λ, c_0)-extension property if and only if (E, X) has the corresponding Lipschitz (λ, c_0)-property; for either, it suffices to check all pairs (E, F) where $E \subset F \subset X$ and $\dim F/E = 1$. We then characterize spaces X with the property that (E, X) has the linear (λ, c_0)-extension property for every closed subspace E.

In §6 we consider the same problem for c in place of c_0. Here the results are necessarily somewhat more complicated and the problem of extending Lipschitz maps is not equivalent to extending linear maps. Indeed c is a 2-Lipschitz absolute retract but the extension constant for linear maps from separable Banach spaces into c is 3 (McWilliams [30]).

In §7 we find that by restricting attention to the almost isometric case we can obtain quite satisfactory results. We obtain a description of separable spaces X so that for every subspace E the pair (E, X) has the linear $(1 + \epsilon, \mathcal{C}(K))$-extension property for every $\mathcal{C}(K)$-space and every $\epsilon > 0$. This property is equivalent to the corresponding property for Lipschitz extensions and can be described in terms of types on X^*. As a particular example we see that spaces with properties (M^*) or (L^*) have this extension property. In particular since Orlicz sequence spaces with separable dual can be renormed to have property (M) we obtain an extension theorem for such spaces. In §8 we consider isometric extensions. Johnson and Zippin [16] had shown that that if X is uniformly smooth then the existence of almost isometric linear extensions implies the existence of isometric extensions; they asked if this is true when X is merely smooth and reflexive. We present a generalization of their result and an extension to the Lipschitz case, but give a counterexample to their question.

In §9 we consider the problem of extending operators from weak*-closed subspaces of a separable dual space. Here we improve a result of Johnson and Zippin [16] who showed that if E is a weak*-closed subspace of ℓ_1 then (E, ℓ_1) has the $(3+\epsilon, \mathcal{C}(K))$-extension property (for every $\epsilon > 0$ and every K). We show that $3+\epsilon$ can be reduced to $1 + \epsilon$.

In §10 we turn our attention to universal extensions. We consider separable Banach spaces X such that if $Y \supset X$ and Y is separable then (X, Y) always has the linear $(1 + \epsilon, \mathcal{C}(K))$-extension property; such spaces are said to have the separable universal linear $\mathcal{C}$-AIEP. This class was first considered by Speegle [38] who showed that such a space cannot be uniformly smooth. However no infinite-dimensional examples of such spaces were known. We show that ℓ_1 and moreover any dual of a subspace of c_0 has this property. We also give examples to show the class contains some spaces not of this form (at least isometrically).

In §11 we attempt to describe those separable Banach spaces X which have the property that whenever M is a metric space containing X then the pair (X, M) has the Lipschitz $(1 + \epsilon, \mathcal{C}(K))$-extension property for every $\epsilon > 0$ and every K; in this case we say that X has the universal Lipschitz $\mathcal{C}$-AIEP. We show that this is related to a property we call the 1-positive Schur property; this draws heavily on work of Odell and Schlumprecht [32]. We also show that under certain other mild conditions the universal Lipschitz $\mathcal{C}$-AIEP is equivalent to the separable universal linear c-AIEP (i.e., the corresponding linear property but just for the space c in place of all $\mathcal{C}(K)$-spaces).

2. Extension properties

We remind the reader that all Banach spaces are assumed real. Suppose X, Y are Banach spaces and E is a closed subspace of X. We say that (E, X) has the (λ, Y)-*linear extension property* (linear (λ, Y)-EP) if for every bounded linear operator $T_0 : E \to Y$ there is a bounded linear extension $T : X \to Y$ with $\|T\| \leq \lambda \|T_0\|$. If (E, X) has the linear (λ, Y)-EP for every closed linear subspace E we say that X has the linear (λ, Y)-EP. We say that X has the [*almost*] *isometric linear* Y-*extension property* (linear Y-[A]IEP) if it has the linear $(1, Y)$-EP [respectively, the linear (λ, Y)-EP for every $\lambda > 1$].

Y is said to be λ-*injective* if every pair (E, X) has the (λ, Y)-linear extension property. There are no separable 1-injective spaces but Sobczyk's theorem [37] implies that c_0 is 2-*separably injective* i.e., every pair (E, X) with X separable has the $(2, c_0)$-linear extension property. A result of Zippin [40] shows that the converse is true, i.e., every Banach space which is separably injective is isomorphic to c_0.

The spaces $\mathcal{C}(K)$ are, in general, not separably injective, but there has been some considerable work on establishing conditions on a pair (E, X) so that (E, X) has the $(\lambda, \mathcal{C}(K))$-linear extension property for every compact Hausdorff space K; in the case when X is separable it suffices to consider metric K. We refer to [42]. We use the symbol $\mathcal{C}$ to represent an arbitrary $\mathcal{C}(K)$-space where K is compact metric. Thus we will say that X has the *linear* $(\lambda, \mathcal{C})$- *extension property* (linear $(\lambda, \mathcal{C})$-EP) if it has the $(\lambda, \mathcal{C}(K))$-linear extension property for every compact metric space K and the *linear* $\mathcal{C}$-*extension property* (linear $\mathcal{C}$-EP) if it has the linear $(\lambda, \mathcal{C})$-extension property for some $\lambda \geq 1$. As above we may also define the [*almost*] *isometric linear* $\mathcal{C}$-*extension property* ($\mathcal{C}$-[A]IEP).

If X, Y are separable Banach spaces we say that X has the *separable universal linear* (λ, Y) *extension property* if whenever Z is a separable Banach space containing X then (X, Z) has the (λ, Y)-extension property. By the above remarks every separable Banach space has the separable universal $(2, c_0)$-extension property; a result of McWilliams [30] shows that every separable Banach space has the separable universal $(3, c)$-EP. On the other hand, Lindenstrauss (Corollary to Theorem 7.5 and Theorem 7.6 of [27]) shows that a Banach space X has the separable universal c-IEP (or $(1, c)$-EP) if and only if X is finite-dimensional and polyhedral.

We shall be particularly interested in the *separable universal* $\mathcal{C}$-*AIEP* (i.e., the separable universal $(\lambda, \mathcal{C}(K))$-EP for every compact metric space K and every $\lambda > 1$.) This has been considered by Speegle [38].

In this paper we will exploit the connection between linear extension problems and nonlinear extension problems. Assume M is a metric space and E is any subset of M. If Y is an arbitrary Banach space then we say that (E, X) has the *Lipschitz* (λ, Y)-*extension property* (Lipschitz (λ, Y)-EP) if every Lipschitz map $F_0 : E \to Y$ has a Lipschitz extension $F : M \to Y$ with $\mathrm{Lip}(F) \leq \lambda \mathrm{Lip}(F_0)$ where

$$\mathrm{Lip}(F) = \sup \left\{ \frac{\|F(x) - F(y)\|}{d(x, y)} : \ x \neq y \in M, \ x \neq y \right\}.$$

As before we say (E, M) has the *Lipschitz Y-IEP* if it has the $(1, Y)$-EP and the *Lipschitz Y-AIEP* if it has the (λ, Y)-EP for every $\lambda > 1$. We also say that (E, M) has the *Lipschitz $(\lambda, \mathcal{C})$-extension property* if it has the Lipschitz $(\lambda, \mathcal{C}(K))$-EP for every compact metric K and define the corresponding Lipschitz $\mathcal{C}$-IEP and Lipschitz $\mathcal{C}$-AIEP.

In the nonlinear category the spaces $\mathcal{C}(K)$ for K compact metric are Lipschitz absolute retracts; this is due to Lindenstrauss [26]. The best constant is 2 as was shown in [19].

We say that M has the *universal Lipschitz (λ, Y)-extension property* if (M, M') has the Lipschitz (λ, Y)-EP for every metric space $M' \supset M$. In the case $Y = \mathcal{C}(K)$ where K is compact metric it is proved in [19] that M has the universal Lipschitz $\mathcal{C}$-IEP if and only if M has the collinearity property, which is equivalent when $M = X$ is a Banach space to the fact that X is finite-dimensional and polyhedral.

Proposition 2.1. *Let X, Y be Banach spaces. Then X has the universal Lipschitz (λ, Y)-EP if and only if (X, Z) has the Lipschitz (λ, Y)-EP for every Banach space Z containing X linearly isometrically.*

Proof. Simply embed X linearly isometrically into $\ell_\infty(I)$ for some index set I. It suffices that $(X, \ell_\infty(I))$ has the (λ, Y)-EP. $\square$

The following lemma is very well-known but will be useful later in the paper and we state it for reference.

Lemma 2.2. *Let X be a Banach space and suppose $(x_j)_{j\in\mathbb{J}}$ is a subset of X. Suppose we are given $(d_j)_{j\in\mathbb{J}}$ with $d_j > 0$ for all j. Suppose that*

$$\|x_j - x_k\| \le d_j + d_k \qquad j, k \in \mathbb{J}.$$

Then there is a Banach space $Y \supset X$ with $\dim Y/X \le 1$ and a point $y \in Y$ such that

$$\|y - x_j\| \le d_j \qquad j \in \mathbb{J}.$$

Proof. Embed X isometrically in the space $Z = \ell_\infty(B_{X^*})$ via the canonical embedding. Then the balls $x_j + d_j B_Z$ intersect pairwise and since Z has the binary intersection property there exists $y \in Z$ with $y \in x_j + d_j B_Z$ for every $j \in \mathbb{J}$. Let $Y = [X, y]$ and the proof is complete. $\square$

3. Some remarks about types

Let X be a Banach space. A *type* on X is a function $\tau : X \to [0, \infty)$ of the form

$$\tau(x) = \lim_d \|x + x_d\| \tag{3.1}$$

where $(x_d)_{d\in D}$ is a uniformly bounded net in X. If X is separable every type can written using a sequence i.e.,

$$\tau(x) = \lim_{n\to\infty} \|x + x_n\|. \tag{3.2}$$

In general let us say that τ is a sequential type if it is given in the form

$$\tau(x) = \lim_{n\in\mathcal{U}} \|x + x_n\| \tag{3.3}$$

for some nonprincipal ultrafilter $\mathcal{U}$.

A type τ is said to be nontrivial if

$$\tau(x) > 0 \qquad x \in X$$

and *strict* if

$$\tau(x) > \|x\| \qquad x \in X.$$

We shall say that τ is a *bidual* type if $\tau(x) = \|x + x^{**}\|$ for some $x^{**} \in X^{**}$. We also say that τ is *monotone* if

$$\tau(x) \geq \tau(0) \qquad x \in X$$

and τ is *symmetric* if

$$\tau(x) = \tau(-x) \qquad x \in X.$$

τ is said to be *weakly null* if it is of the form (3.3) with $(x_n)_{n=1}^{\infty}$ a weakly null sequence.

A type τ on X^* is said to be *weak*-null* if it is expressible in the form (3.3) with $(x_n)_{n=1}^{\infty}$ a weak* null sequence.

We recall that there is a canonical contractive projection π of X^{***} on its subspace X^* with kernel $X^{\perp}$. X is called a *strict u-ideal* ([12]) if $\|I - 2\pi\| = 1$.

Proposition 3.1. *Let X be a separable Banach space such that every weak*-null type on X^* is symmetric. Then X is a strict u-ideal and hence X^* is separable.*

Proof. Suppose $x^{***} \in X^{\perp} \subset X^{***}$. Then there is net $(x_d^*)_{d\in D}$ in X^* which converges weak* in X^{***} to x^{***} such that for any $x^* \in X^* \subset X^{***}$ we have

$$\|x^* + x^{***}\| = \lim_{d\in D} \|x^* + x_d^*\|.$$

Now since X is separable for any fixed $x^* \in X^*$ we can find a weak*-null sequence $(v_n^*)_{n=1}^{\infty}$ in $\{x_d^*\}_{d\in D}$ such that

$$\lim_{d\in D} \| \pm x^* + x_d^*\| = \lim_{n\to\infty} \| \pm x^* + v_n^*\|$$

and it follows that

$$\|x^* + x^{***}\| = \|x^* - x^{***}\|$$

or $\|I - 2\pi\| = 1$. The fact that this implies X^* is separable follows from Proposition 2.8 of [12]. □

Let us recall at this stage that a separable Banach space is said to have the *metric approximation property* (MAP) if there is a sequence of finite-rank operators $T_n : X \to X$ such that $\|T_n\| \leq 1$ and $\lim_{n\to\infty} T_n x = x$ for $x \in X$.

A separable Banach space X is said to have the *unconditional metric approximation property* (UMAP) if there is a sequence $(T_n)_{n=1}^{\infty}$ of finite-rank operators on X such that $\lim_{n\to\infty} T_n x = x$ for $x \in X$ and $\lim_{n\to\infty} \|I - 2T_n\| = 1$. This concept was introduced in [3]. See also [12] and [8].

Lemma 3.2. (i) *Suppose X is a Banach space and $(x_n)_{n=1}^{\infty}$ is a weakly null sequence in X. Suppose $u \in X$ is such that $\lim_{n\to\infty} \|u + x_n\|$ exists. Then there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$, $u^* \in X^*$ and a weak* null sequence $(x_n^*)_{n\in\mathbb{M}}$ so that $\|u^* + x_n^*\| = 1$ for $n \in \mathbb{M}$ and*

$$u^*(u) + \lim_{n\in\mathbb{M}} x_n^*(x_n) = \lim_{n\to\infty} \|u + x_n\|.$$

(ii) *Suppose X is a Banach space with UMAP not containing a copy of ℓ_1. Suppose $(x_n^*)_{n=1}^\infty$ is a weak* null sequence in X^*. Suppose $u^* \in X$ is such that $\lim_{n\to\infty} \|u^* + x_n^*\|$ exists. Then, given $\epsilon > 0$, there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$, $u \in X$ and a weakly null sequence $(x_n)_{n\in\mathbb{M}}$ so that $\|u+x_n\| \le 1$ for $n \in \mathbb{M}$ and*

$$u^*(u) + \liminf_{n\in\mathbb{M}} x_n^*(x_n) > (1-\epsilon)\lim_{n\to\infty} \|u^* + x_n^*\|.$$

Proof. For (i) pick v_n^* with $\|v_n^*\| = 1$ and $v_n^*(u + x_n) = \|u + x_n\|$. Pass to a subsequence $(v_n^*)_{n\in\mathbb{M}}$ which converges weak* to some u^* and then put $x_n = v_n^* - u^*$ for $n \in \mathbb{M}$.

(ii) First note that by Theorem 9.2 of [12], X has the shrinking UMAP, i.e., there is a sequence of finite-rank operators T_n with $\lim_{n\to\infty} \|I - 2T_n\| = 1$ and

$$\lim_{n\to\infty} \|x - T_n x\| = 0 \qquad x \in X$$

and

$$\lim_{n\to\infty} \|x^* - T_n^* x^*\| = 0 \qquad x^* \in X^*.$$

We can assume that for some sequence $\eta_n \downarrow 0$ we have

$$\|I - 2T_n\| < 1 + \eta_n,\ \|T_m T_n - T_n\| < \eta_n \qquad m > n.$$

Now pick $v_n \in X$ with $\|v_n\| = 1$ so that

$$\lim_{n\to\infty} (u^* + x_n^*)(v_n) = \lim_{n\to\infty} \|u^* + x_n^*\|.$$

We may now find an increasing sequence of natural numbers $(r_n)_{n=1}^\infty$ so that $\lim_{n\to\infty} \|T_{r_n}^* x_n^*\| = 0$. For $k \in \mathbb{N}$

$$\|(I - 2T_{r_n})(I - 2T_k) - (I - 2T_{r_n} + 2T_k)\| < 4\eta_k \qquad r_n > k$$

and so

$$\|I - T_{r_n} + T_k\| < (1 + \eta_n)(1 + \eta_k) + 4\eta_k.$$

Now note that $((I - T_{r_n})v_n)_{n=1}^\infty$ is weakly null; indeed if $x^* \in X^*$ then

$$|x^*((I - T_{r_n})v_n)| \le \|x^* - T_{r_n}^* x^*\|.$$

Let us now fix $k \in \mathbb{N}$. Select $\mathbb{M}$ so that $\lim_{n\in\mathbb{M}} T_k v_n = \widetilde{u}$ exists and let $\widetilde{x}_n = (I - T_{r_n})v_n$. Then

$$\limsup_{n\in\mathbb{M}} \|\widetilde{u} + \widetilde{x}_n\| \le 1 + 5\eta_k.$$

However

$$\lim_{n\to\infty} \|(T_k^* - T_{r_n}^*)(u^* + x_n^*) - T_k^* u^* + u^*\| = 0$$

and so

$$\limsup_{n\in\mathbb{M}} (u^* + x_n^*)(T_k v_n - T_{r_n} v_n) \le \|u^* - T_k^* u^*\|.$$

Hence

$$\begin{aligned} u^*(\widetilde{u}) + \liminf_{n\in\mathbb{M}} x_n^*(\widetilde{x}_n) &= \liminf_{n\in\mathbb{M}} (u^* + x_n^*)(v_n + T_k v_n - T_{r_n} v_n) \\ &\ge \lim_{n\to\infty} \|u^* + x_n^*\| - \|u^* - T_k^* u^*\|. \end{aligned}$$

If we take k large enough and let $u = (1 + 5\eta_k)^{-1}\widetilde{u}$ and $x_n = (1 + 5\eta_k)^{-1}\widetilde{x}_n$ we obtain (ii). $\square$

A Banach space X is said to have *property* (M) if

$$\lim_{n\to\infty}\|u+x_n\| = \lim_{n\to\infty}\|v+x_n\| \qquad u,v\in X,\ \|u\|=\|v\|$$

whenever (x_n) is weakly null and both limits exist. X has *property* (M^*) if

$$\lim_{n\to\infty}\|u^*+x_n^*\| = \lim_{n\to\infty}\|v^*+x_n^*\| \qquad u^*,v^*\in X^*,\ \|u^*\|=\|v^*\|$$

whenever (x_n^*) is weak* null in X^* and both limits exist. In the language of types X has property (M) (respectively (M^*)) if every weakly null (respectively weak* null) type on X (respectively X^*) is a function of the norm.

These properties were introduced in connection with the theory of M-ideals in [17]. It is known that for separable Banach spaces (M^*) implies (M) [17] and if X contains no copy of ℓ_1, (M) implies (M^*). If X is a separable Banach space with property (M^*) and with MAP then X has UMAP. In the language of types X has property (M) (respectively (M^*)) if every weakly null (respectively weak* null) type on X (respectively X^*) is a function of the norm.

We also want to introduce a reverse property. Let us say that a separable Banach space X has *property* (L) if

$$\lim_{n\to\infty}\|u+x_n\| = \lim_{n\to\infty}\|u+y_n\| \qquad u\in X$$

whenever (x_n) and (y_n) are weakly null and $\lim_{n\to\infty}\|x_n\| = \lim_{n\to\infty}\|y_n\|$. X has property (L^*) if

$$\lim_{n\to\infty}\|u^*+x_n^*\| = \lim_{n\to\infty}\|u^*+y_n^*\| \qquad u^*\in X^*$$

whenever (x_n^*) and (y_n^*) are weak* null in X^* and $\lim_{n\to\infty}\|x_n^*\| = \lim_{n\to\infty}\|y_n^*\|$.

Note that if X has property (L) or (M) it is clear that every weakly null type is symmetric: similarly if X has (L^*) or (M^*) then every weak*-null type on X^* is symmetric. From this we deduce via Proposition 3.1 that if X has (L^*) or (M^*) then X^* is automatically separable.

Proposition 3.3. (i) *X has property (M) if and only if whenever σ is a weakly null type on X then*

$$\|x\|=\|y\| \implies \sigma(x)=\sigma(y) \qquad x,y\in X.$$

(ii) *X has property (M^*) if and only if whenever σ is a weak* null type on X then*

$$\|x^*\|=\|y^*\| \implies \sigma(x^*)=\sigma(y^*) \qquad x^*,y^*\in X^*.$$

(iii) *X has property (L) if and only if, whenever σ,τ are weakly null types on X*

$$\sigma(0)=\tau(0) \implies \sigma(x)=\tau(x) \qquad x\in X.$$

(iv) *X has property (L^*) if and only if, whenever σ,τ are weak* null types on X*

$$\sigma(0)=\tau(0) \implies \sigma(x^*)=\tau(x^*) \qquad x^*\in X^*.$$

Note here that in each instance the equalities can be replaced by inequalities e.g., in (iii) X has property (L) if and only if, whenever σ,τ are weakly null types on X

$$\sigma(0)\le\tau(0) \implies \sigma(x)\le\tau(x) \qquad x\in X.$$

To see this suppose

$$\sigma(x)=\lim_{n\in\mathcal{U}}\|x+x_n\| \qquad x\in X$$

and

$$\tau(x) = \lim_{n\in\mathcal{U}} \|x + y_n\| \qquad x \in X$$

where $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}$ are weakly null sequences. If $\sigma(0) < \tau(0)$ we choose $\alpha = \sigma(0)/\tau(0)$ and note that

$$\sigma(x) = \lim_{n\in\mathcal{U}} \|x + \alpha y_n\|.$$

Since

$$\tau(x) = \lim_{n\in\mathcal{U}} \|x + y_n\|$$

convexity gives $\sigma(x) \le \tau(x)$.

Proposition 3.4. *Let X be a separable Banach space.*

(i) *If X has property (L^*) then X has property (L).*

(ii) *If X has property (L), contains no copy of ℓ_1 and has UMAP then X^* has property (L^*).*

Proof. These follow directly from Lemma 3.2. For (i) suppose $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}$ is weakly null and $\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\|$. Suppose further that $\lim_{n\to\infty} \|u + x_n\|$ and $\lim_{n\to\infty} \|u + y_n\|$ both exist. Then, using Lemma 3.2 (i), there is $u^* \in X^*$ and a weak*-null sequence (x_n^*) so that $\|u^* + x_n^*\| = 1$ and for some subsequence $\mathbb{M}$ we have

$$\lim_{n\in\mathbb{M}} u^*(u) + x_n^*(x_n) = \lim_{n\to\infty} \|u + x_n\|.$$

On the other hand we can also find a further subsequence $\mathbb{M}_0$, $v^* \in X^*$ and a weak* null sequence $(y_n^*)_{n\in\mathbb{M}_0}$ with $\|v^* + y_n^*\| = 1$ such that

$$\lim_{n\in\mathbb{M}_0} (v^* + y_n^*)(y_n) = \lim_{n\in\mathbb{M}_0} y_n^*(y_n) = \lim_{n\to\infty} \|y_n\|.$$

Since all weak*-null types on X^* are symmetric, it follows that

$$\limsup_{n\in\mathbb{M}_0} \|y_n^*\| \le \frac{1}{2} \limsup_{n\in\mathbb{M}_0} (\|v^* + y_n^*\| + \|v^* - y_n^*\|) \le 1.$$

Hence $\lim_{n\in\mathbb{M}_0} \|y_n^*\| = 1$. Consider the sequence $(u^* + \|x_n^*\| y_n^*)_{n\in\mathbb{M}_0}$. By property (L^*) we have

$$\lim_{n\in\mathbb{M}_0} \left\| u^* + \|x_n^*\| y_n^* \right\| = 1.$$

Thus

$$\begin{aligned}
\lim_{n\to\infty} \|u + x_n\| &\le u^*(u) + \lim_{n\in\mathbb{M}_0} x_n^*(x_n) \\
&\le u^*(u) + \lim_{n\in\mathbb{M}_0} \|x_n^*\| \|x_n\| \\
&\le u^*(u) + \limsup_{n\in\mathbb{M}_0} \|x_n^*\| \|y_n\| \\
&\le u^*(u) + \limsup_{n\in\mathbb{M}_0} \|x_n^*\| y_n^*(y_n) \\
&\le \limsup_{n\in\mathbb{M}_0} (u^* + \|x_n^*\| y_n^*)(u + y_n) \\
&\le \lim_{n\to\infty} \|u + y_n\|.
\end{aligned}$$

The reverse inequality also follows.

(ii) is similar. □

It is easy to create examples of space with properties (L) or (L^*). Curiously these examples are also isomorphic (not isometric) to spaces with property (M) or (M^*). It was first shown in [17] that if Φ is an Orlicz function then the space h_Φ (the closure of c_{00} in the Orlicz space ℓ_Φ) can be renormed to have a 1-unconditional basis and property (M); hence it has property (M^*) when it has separable dual. This construction was generalized in [1] to so-called Orlicz–Fenchel spaces, and it was noted that this class includes certain twisted sums such as the spaces Z_p for $1 < p < \infty$ constructed in [22].

We follow the construction first given in [1], but note that a small corretcion must be made. Let V_k be a sequence of finite-dimensional spaces and for each k let N_k be a norm on $\mathbb{R} \times V_k$ such that

$$N_k(\lambda, x) \le N_k(\mu, x) \qquad 0 \le \lambda \le \mu,\ x \in V_k, \tag{3.4}$$

$$N_k(\lambda, x) = N_k(-\lambda, x) \qquad \lambda \in \mathbb{R},\ x \in V_k, \tag{3.5}$$

and

$$N_k(1, 0) = 1. \tag{3.6}$$

It is clear that these conditions imply $N_k(\lambda, x) \ge |\lambda|$ and $N_k(\lambda, x) \ge N_k(0, x)$ for all λ, x. In [1] on p. 118 (3.5) is omitted. However in the subsequent application (Lemma 4.8 of [1]) it is easily verified that the constructed norms also satisfy (3.5).

Let $c_{00}(V_k)$ be the space of finitely nonzero sequences $v = (v_k)_{k=1}^\infty$ with $v_k \in V_k$. For $m < n$ we define a seminorm $N_{m,n}$ on $c_{00}(V)$ by

$$N_{n,n}(v) = N_n(0, , v_n)$$

and then, inductively define

$$N_{m,n}(v) = N_m(N_{m+1,n}(v), v_m) \qquad m < n.$$

Similarly define

$$N_{m,n}(v) = N_m(N_{m-1,n}(v), v_m) \qquad m > n.$$

Let

$$\|v\|_L = \sup_{m \le n} N_{m,n}(v).$$

Similarly let

$$\|v\|_M = \sup_{m \ge n} N_{m,n}(v).$$

Let Λ_L and Λ_M be the completions of $(c_{00}(V_k), \|\cdot\|_L)$ and $(c_{00}(V_k), \|\cdot\|_M)$. Note that both spaces have 1-unconditional FDD's and hence UMAP.

Let $\Phi_k(x) = N_k(1, x) - 1$. Thus $\Phi_k : V_k \to \mathbb{R}$ is convex and even.

Proposition 3.5. (i) *The space* $(\Lambda_L, \|\cdot\|_L)$ *has property* (L).

(ii) *The space* $(\Lambda_M, \|\cdot\|_M)$ *has property* (M).

(iii)

$$\Lambda_M = \Lambda_L = \left\{ v : \sum_{k=1}^\infty \Phi_k(t v_k) < \infty \ \forall\ t > 0 \right\}.$$

Proof. (i) and (ii) are trivial.

(iii) for Λ_M is essentially shown in [1] Theorem 4.1 for a fixed $N = N_k$. We will prove only the characterization of Λ_L which is in fact almost identical to the argument for Λ_M. First suppose $v \in \Lambda_L$ with $\|v\|_L = 1$. Let $v = (v_1, \ldots, v_n, 0, \cdots)$. Then for $1 \leq k \leq n-1$ we have, since $N_{k+1,n}(v) \leq 1$,

$$N_k(N_{k+1,n}(v), v_k) = N_{k+1,n}(v)(1 + \Phi_k(v_k/N_{k+1,n}(v))) \geq N_{k+1,n}(v) + \Phi_k(v_k).$$

$$\begin{aligned} 1 = N_{1,n}(v) &= \sum_{k=1}^{n-1} (N_k(N_{k+1,n}(v), v_k) - N_{k+1,n}(v)) \\ &\geq \sum_{k=1}^{n} (N_k(1, v_k) - 1) \\ &\geq \sum_{k=1}^{n} \Phi_k(v_k). \end{aligned}$$

Hence $\sum_{k=1}^\infty \Phi_k(v_k) \leq 2$.

It follows immediately that if $v \in \Lambda_L$ then $\sum_{k=1}^\infty \Phi_k(tv_k) < \infty$ for all t.

Conversely suppose $v \in c_{00}(V_k)$ and $\sum_{k=1}^\infty \Phi_k(v_k) < 1$. Suppose $m < n$ and that $N_{m,n}(v) > 2$. Note that $N_{n,n}(v) = N_n(0, v_n) \leq N_n(1, v_n) \leq 2$. Let r be the smallest index so that $N_{r,n}(v) \leq 2$. Then

$$N_{r-1,n}(v) = N_{r-1}(N_{r,n}(v), v_{r-1}) \leq N_{r-1}(2, v_{r-1}) \leq 2(1 + \Phi_{r-1}(v_{r-1}/2))$$

and then

$$N_{j-1,n}(v) = N_{j-1}(N_{j,n}(v), v_{j-1}) \leq N_{j,n}(v)(1+\Phi_{j-1}(v_{j-1}/2)) \qquad m+1 \leq j \leq r-1.$$

Thus

$$N_{m,n}(v) \leq 2 \prod_{j=m}^{r-1} (1 + \Phi_k(v_k)) \leq 2e.$$

It follows that $\|v\|_L \leq 2e$.

Now assume $\sum_{k=1}^\infty \Phi_k(tv_k) < \infty$ for every $t > 0$. Then for every $\epsilon > 0$ we can find r so that $\sum_{k=r+1}^\infty \Phi_k(2ev_k/\epsilon) < 1$. Thus

$$\|(0, \ldots, 0, v_{m+1}, \ldots, v_n, 0, \ldots)\|_L < \epsilon \qquad r \leq m < n < \infty.$$

This implies $v \in \Lambda_L$. $\square$

In particular if $(\phi_k)_{n=1}^\infty$ is a sequence of Orlicz functions on $\mathbb{R}$, normalized so that $\phi_k(1) = 1$, we can define N_k on $\mathbb{R}^2$ so that

$$N_k(t, 1) = \begin{cases} 1 + \phi_k(|t|) & 0 \leq t \leq 1 \\ 2 + c_k(t-1) & 1 < t < \infty \end{cases}$$

where c_k is chosen large enough to ensure convexity. Thus the Orlicz–Musielak space $h_{(\phi_k)}$ defined as the closure of c_{00} in space $\ell_{(\phi_k)}$ can be renormed to either have property (L) or property (M). For property (M) these results where first established in [17]. In the case when these spaces have separable dual one also deduces that they also enjoy (L^*) or (M^*). The class of spaces contained this way includes finite direct sums of spaces of type ℓ_p or c_0.

In [1] the same ideas are applied to Orlicz–Fenchel spaces. Here we take $V = V_k$ to be a fixed finite-dimensional space and $\Phi = \Phi_k$ to be a fixed Young's function on V i.e., an even continuous convex function $\Phi : V \to [0, \infty)$ with $\Phi(x) > 0$ if $x \neq 0$. It is shown in [1] that the space h_Φ of all V-valued sequences $(v_k)_{k=1}^\infty$ such that $\sum_{k=1}^\infty \Phi(tv_k) < \infty$ for all $t > 0$ can be equivalently normed to have property (M); however the same argument shows that they can also be renormed to have property (L). If these spaces have separable dual (e.g., if they have nontrivial type) then they can be renormed to have (L^*) or (M^*). This class of Orlicz–Fenchel spaces includes such spaces as the twisted sums Z_p for $1 < p < \infty$ originally introduced in [22]. Here we take $V = \mathbb{R}^2$ and define

$$\Psi(x, y) = \begin{cases} |y|^p + |x - y \log |y||^p & y \neq 0 \\ |x|^p & y = 0. \end{cases}$$

Then Ψ is not convex but is equivalent to a convex function (see [1]). The space Z_p has a subspace isomorphic to ℓ_p spanned by the vectors $(0, \dots, (0, 1), \dots)$ so that the quotient is also isomorphic to ℓ_p.

Let us now recall that a separable Banach space X is said to have property (m_p) for $1 < p \leq \infty$ if every weakly null type σ is of the form

$$\sigma(x) = \begin{cases} (a^p + \|x\|^p)^{\frac{1}{p}} & x \in X,\ 1 \leq p < \infty \\ \max(a, \|x\|) & x \in X,\ p = \infty. \end{cases}$$

Similarly X has property (m_p^*) for $1 \leq p < \infty$ if every weak*-null type σ on X^* has the form

$$\sigma(x^*) = \begin{cases} (a^p + \|x^*\|^p)^{\frac{1}{p}} & x^* \in X^*,\ 1 \leq p < \infty \\ \max(a, \|x^*\|) & x^* \in X,\ p = \infty. \end{cases}$$

It is clear that property (m_p) implies both property (M) and property (L); and similarly (m_p^*) implies both property (M^*) and property (L^*).

Proposition 3.6. *Suppose X is a separable Banach space.*

(i) *If X has both property (L) and property (M) and contains no copy of ℓ_1 then X has property (m_p) for some $1 < p \leq \infty$.*

(ii) *If X has both property (L^*) and property (M^*) then X has property (m_p^*) for some $1 \leq p < \infty$.*

Proof. (i) By Proposition 3.9 of [17] for every $a > 0$ there is a weakly null type σ on X of the form $\sigma(x) = (a^p + \|x\|^p)^{\frac{1}{p}}$ for some $1 < p \leq \infty$ (with obvious interpretation when $p = \infty$). But then property (L) implies that every weakly null type on X is of this form.

(ii) The proof is similar, but requires a couple of remarks. Let $(x_n^*)_{n=1}^\infty$ generate a nontrivial weak*-null type via the formula

$$\sigma(x^*) = \lim_{\mathcal{U}} \|x^* + x_n^*\|$$

where $\mathcal{U}$ is some nontrivial ultrafilter on $\mathbb{N}$. Then as in Lemma 3.6 of [17] one generates a weak*-null sequence (y_n^*) which is $(1 + \epsilon)$-equivalent to the unit vector basis of the space Λ_M corresponding to the fixed norm

$$N(\alpha, \beta) = \lim_{n \to \infty} \|\alpha u^* + \beta x_n^*\|$$

where $\|u^*\| = 1$. Arguing as in [17] one obtains from Krivine's theorem the existence of $1 \le p \le \infty$ so that

$$\tau(x^*) = (a^p + \|x^*\|^p)^{\frac{1}{p}}$$

is a weak*-null type on X^*. By property (L^*) every weak*-null type is of this form. However if $p = \infty$ then c_0 embeds in X^* and hence X^* is nonseparable by a classical result of Bessaga and Pełczyński [2]. As already observed (M^*) implies that X^* is separable. □

Finally let us recall that a Banach space X is called *stable* [25] if

$$\lim_{m\to\infty} \lim_{n\to\infty} \|x_m + y_n\| = \lim_{n\to\infty} \lim_{m\to\infty} \|x_m + y_n\|$$

whenever both limits exist. If X is stable it is possible to unambiguously define the convolution $\sigma * \tau$ of two types

$$\sigma(x) = \lim_{n\to\infty} \|x + x_n\|, \quad \tau(x) = \lim_{n\to\infty} \|x + y_n\| \qquad x \in X$$

by

$$\sigma * \tau(x) = \lim_{m\to\infty} \lim_{n\to\infty} \|x + x_m + y_n\|.$$

Then $\sigma * \tau = \tau * \sigma$.

We will need one elementary fact about stable spaces.

Proposition 3.7. *If X is a separable Banach space such that X^* is stable then every type on X^* is weak*-lower-semicontinuous.*

Proof. If X^* is stable and σ is a type defined by

$$\sigma(x^*) = \lim_{n\to\infty} \|x^* + y_n^*\| \qquad x^* \in X^*$$

then for any weak*-convergent sequence $(x_n^*)_{n=1}^\infty$ converging to x^*, we have for any nonprincipal ultrafilter $\mathcal{U}$,

$$\begin{aligned} \lim_{n\in\mathcal{U}} \sigma(x_n^*) &= \lim_{n\in\mathcal{U}} \lim_{m\to\infty} \|x_n^* + y_m^*\| \\ &= \lim_{m\in\mathcal{U}} \lim_{n\in\mathcal{U}} \|x_n^* + y_m^*\| \\ &\ge \lim_{m\in\mathcal{U}} \|x^* + y_m^*\| \\ &= \sigma(x^*). \end{aligned}$$

□

4. Types and Lipschitz extensions

Suppose X is a Banach space and C is a closed subset. Let us introduce two conditions on the pair (C, X). We will say that (C, X) satisfies the *condition* $\Sigma_0(\lambda)$ (or, more informally, C satisfies the condition $\Sigma_0(\lambda)$) if given a bounded sequence $(e_n)_{n=1}^\infty$ in C, $x \in X$ and $\epsilon > 0$ there exists $e \in C$ and an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that

$$\|e_n - e\| \le \lambda \|e_n - x\| + \epsilon \qquad n \in \mathbb{M}.$$

We will say that (C, X) satisfies the *condition* $\Sigma_1(\lambda)$ if given two bounded sequences $(e_n)_{n=1}^\infty$ and $(f_n)_{n=1}^\infty$ in C, $x \in X$ and $\epsilon > 0$ there exists $e \in C$ and an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that

$$\|e_n - e\| + \|f_n - e\| \le \lambda(\|e_n - x\| + \|f_n - x\|) + \epsilon \qquad n \in \mathbb{M}.$$

Notice that $\Sigma_1(\lambda)$ implies $\Sigma_0(\lambda)$ by taking $e_n = f_n$.

The following theorems were proved in [19].

Theorem 4.1. *Let X be a Banach space and suppose C is a separable subset. Suppose $1 < \lambda \leq 2$. The following conditions on (C, X) are equivalent:*

(i) *(C, X) has the Lipschitz (λ, c_0)-extension property.*
(ii) *(C, X) has the property $\Sigma_0(\lambda)$.*

In the case $\lambda = 1$ then (ii) *is equivalent to:*

(iii) *(C, X) has the Lipschitz c_0-AIEP.*

Theorem 4.2. *Let X be a Banach space and suppose C is a separable subset. Suppose $1 < \lambda \leq 2$. The following conditions on (C, X) are equivalent:*

(i) *(C, X) has the Lipschitz $(\lambda, \mathcal{C})$-extension property.*
(ii) *(C, X) has the property $\Sigma_1(\lambda)$.*

In the case $\lambda = 1$ then (ii) *is equivalent to:*

(iii) *(C, X) has the Lipschitz $\mathcal{C}$-AIEP.*

In the definition of $\Sigma_0(\lambda)$ and $\Sigma_1(\lambda)$, the word *bounded* is redundant when $\lambda > 1$ as the conditions are automatically satisfied by any unbounded sequence. However when $\lambda = 1$ we get a slightly stronger notion. We will say that (C, X) satisfies the *condition* $\Sigma_0^*(1)$ if given a sequence $(e_n)_{n=1}^\infty$ in C, $x \in X$ and $\epsilon > 0$ there exists $e \in C$ and an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that

$$\|e_n - e\| \leq \|e_n - x\| + \epsilon \qquad n \in \mathbb{M}.$$

We will say that (C, X) satisfies the *condition* $\Sigma_1^*(1)$ if given two sequences $(e_n)_{n=1}^\infty$ and $(f_n)_{n=1}^\infty$ in C, $x \in X$ and $\epsilon > 0$ there exists $e \in C$ and an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that

$$\|e_n - e\| + \|f_n - e\| \leq \|e_n - x\| + \|f_n - x\| + \epsilon \qquad n \in \mathbb{M}.$$

We then have [19]:

Theorem 4.3. (i) *(C, X) has the Lipschitz c_0-IEP if and only if (C, X) has property $\Sigma_0^*(1)$.*
(ii) *(C, X) has the Lipschitz $\mathcal{C}$-IEP if and only if (C, X) has property $\Sigma_1^*(1)$.*

From this the following is trivial:

Proposition 4.4. *If C is a bounded subset of X then (C, X) has the Lipschitz c_0-AIEP* (*respectively the Lipschitz $\mathcal{C}$-AIEP*) *if and only if it has the Lipschitz c_0-IEP* (*respectively the Lipschitz $\mathcal{C}$-IEP*).

Let us first point out the trivial connection between types and Lipschitz extension problems. Let C be a nonempty closed subset of a Banach space X. Then a type σ will be said to be supported on C if it is of the form

$$\sigma(x) = \lim_d \|x - x_d\| \qquad x \in X$$

where $x_d \in C$ for all d. Then we have the following elementary lemma, which is simply a rewriting of conditions $\Sigma_0(\lambda)$ and $\Sigma_1(\lambda)$ (see Theorems 4.1 and 4.2):

Lemma 4.5. *Let C be a nonempty separable closed subset of a Banach space X. Then, for $\lambda > 1$:*

(i) *(C, X) satisfies the condition $\Sigma_0(\lambda)$ (i.e., has the Lipschitz (λ, c_0)-EP) if and only if for every type σ supported on C we have*
$$\inf_{u\in C}\sigma(u) \le \lambda \inf_{x\in X}\sigma(x).$$
(ii) *(C, X) satisfies condition $\Sigma_1(\lambda)$ (i.e., has the Lipschitz $(\lambda, \mathcal{C})$-EP) if and only if for every pair of types σ, τ supported on C we have*
$$\inf_{u\in C}(\sigma(u) + \tau(u)) \le \lambda \inf_{x\in X}(\sigma(x) + \tau(x)).$$

We now establish some simple results to be used in the rest of the paper.

Lemma 4.6. *Let X be a separable Banach space and suppose $x^* \in X^*$ and $\lambda \ge 1$. Then the following conditions are equivalent:*

(i) *$(\ker x^*, X)$ satisfies condition $\Sigma_0(\lambda)$ (respectively $\Sigma_1(\lambda)$).*
(ii) *If $H = \{x : \ x^*(x) \le 0\}$ then (H, X) satisfies condition $\Sigma_0(\lambda)$ (respectively $\Sigma_1(\lambda)$).*

Proof. (i) $\implies$ (ii). It suffices to show that if $(\ker x^*, X)$ has the Lipschitz (λ, Y)-EP then so does (H, X), where $Y = c_0$ or $Y = \mathcal{C}(K)$. If $F_0 : H \to Y$ is a Lipschitz map, then there a Lipschitz map $G : X \to Y$ with $\mathrm{Lip}(G) \le \lambda \mathrm{Lip}(F_0)$ and $G|_{\ker x^*} = F_0$. Define
$$F(x) = \begin{cases} F_0(x) & x^*(x) \le 0 \\ G(x) & x^*(x) > 0. \end{cases}$$
Then F extends F_0 and $\mathrm{Lip}(F) \le \lambda \mathrm{Lip}(F_0)$.

(ii) $\implies$ (i). We do the case of $\Sigma_1(\lambda)$ since the arguments are similar. Suppose σ, τ are types supported on $\ker x^*$ and $\epsilon > 0$, Since H and $-H$ both satisfy $\Sigma_1(\lambda)$ we can find $u \in H$ and $v \in -H$ with
$$\sigma(u) + \tau(u) \le \lambda \inf_{x\in X}(\sigma(x) + \tau(x)) + \epsilon$$
$$\sigma(v) + \tau(v) \le \lambda \inf_{x\in X}(\sigma(x) + \tau(x)) + \epsilon.$$
Then a suitable convex combination $y = (1 - \theta)u + \theta v \in \ker x^*$ and by convexity of σ, τ,
$$\sigma(y) + \tau(y) \le \lambda \inf_{x\in X}(\sigma(x) + \tau(x)) + \epsilon. \qquad \square$$

Proposition 4.7. *Suppose X is a separable Banach space and $\lambda \ge 1$. Then the following conditions on X are equivalent:*

(i) *Every closed subspace E of codimension one satisfies condition $\Sigma_0(\lambda)$ (respectively $\Sigma_1(\lambda)$).*
(ii) *Every closed convex subset C of X satisfies condition $\Sigma_0(\lambda)$ (respectively $\Sigma_1(\lambda)$).*

Proof. We need only show (i) $\implies$ (ii). Suppose C is a convex set which fails $\Sigma_1(\lambda)$ (for example; the other case is similar). Then there exist types σ, τ, supported on C, such that for some $\epsilon > 0$, and $v \in X$
$$\sigma(x) + \tau(x) > \lambda(\sigma(v) + \tau(v)) + \epsilon \qquad x \in C.$$
Let
$$D = \{x : \ \sigma(x) + \tau(x) < \lambda(\sigma(v) + \tau(v)) + \tfrac{1}{2}\epsilon\}.$$

Then $D + \frac{1}{2}\epsilon B_X \cap C \neq \emptyset$ so by the Hahn–Banach theorem there exists $x^* \in X^*$ and $\alpha \in \mathbb{R}$ so that $x^*(x) \leq \alpha$ for $x \in C$ and $x^*(x) \geq \beta$ for $x \in D$ where $\beta > \alpha$. Thus $\{x : x^*(x) \leq \alpha\}$ also fails $\Sigma_1(\lambda)$ and thus by Lemma 4.6 so does $\ker x^*$. $\square$

The corresponding statement for weak*-closed convex subsets of a dual space does not appear to follow in general because a type need not be weak*-lower-semicontinuous. However we have, using Proposition 3.7:

Proposition 4.8. *Suppose X is a separable Banach space with X^* stable, and $\lambda \geq 1$. Then the following conditions on X^* are equivalent:*

(i) *Every weak*-closed subspace E of codimension one satisfies condition $\Sigma_0(\lambda)$ (respectively $\Sigma_1(\lambda)$).*
(ii) *Every weak*-closed convex subset C of X^* satisfies condition $\Sigma_0(\lambda)$ (respectively $\Sigma_1(\lambda)$).*

The following lemma connecting the linear and nonlinear theories is proved in [19]:

Lemma 4.9. *Suppose X and Y are Banach spaces and suppose E is a closed subspace of X of co-dimension one. Suppose (E, X) has the Lipschitz (λ, Y)-EP. Then (E, X) has the linear (λ, Y)-EP.*

This allows a simple deduction:

Proposition 4.10. *Suppose X and Y are Banach spaces so that for every closed subspace E of X, (E, X) has the Lipschitz Y-IEP then X has the linear Y-IEP.*

If X is separable, and for every closed subspace E of X, (E, X) has the Lipschitz Y-AIEP then X has the linear Y-AIEP.

If X^* is separable and for every weak-closed subspace E of X^* the pair (E, X^*) has the Lipschitz Y-AIEP then for weak*-closed subspace the pair (E, X^*) has the linear Y-AIEP.*

Proof. This is a trivial deduction from the previous lemma (the first part needs an argument involving Zorn's Lemma). $\square$

5. Extensions into c_0

Suppose X is a Banach space and suppose K_n is a uniformly bounded sequence of compact convex subsets of B_{X^*}. Then we may define the nonempty weak*-compact set $\limsup_{n\to\infty} K_n$ by

$$\limsup_{n\to\infty} K_n = \bigcap_{n=1}^{\infty} \overline{\bigcup_{m=n}^{\infty} K_m}$$

(where the closures are in the weak*-topology). If $\mathbb{M}$ is an infinite subset of $\mathbb{N}$ we can define

$$\limsup_{n\in\mathbb{M}} K_n = \bigcap_{n\in\mathbb{M}} \overline{\bigcup_{\substack{m\in\mathbb{M}\\ m\geq n}} K_m}$$

and it is clear that $\limsup_{n\in\mathbb{M}} K_n \subset \limsup_{n\to\infty} K_n$.

Lemma 5.1. *Suppose X is a Banach space and suppose K_n is a uniformly bounded sequence of compact convex subsets of X^*. Suppose $K = \limsup_{n\to\infty} K_n$ is weak*-metrizable. Then:*

(i) *If $x^* \in K$, there is a sequence $x_n^* \in K_n$ converging weak* to x^* if and only if $x^* \in \limsup_{n\in\mathbb{M}} K_n$ for every infinite subset $\mathbb{M}$ of $\mathbb{N}$.*

(ii) *There is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ so that with the property that*

$$\limsup_{n\in\mathbb{M}'} K_n = \limsup_{n\in\mathbb{M}} K_n$$

for every infinite subset $\mathbb{M}'$ of $\mathbb{M}$ and this set is convex.

Proof. (i) Assume $x^* \in \limsup_{n\in\mathbb{M}} K_n$ for every infinite subset $\mathbb{M}$ of $\mathbb{N}$. Let $(V_k)_{k=1}^\infty$ be a decreasing sequence of closed weak*-neighborhoods of x^* with the property that $K \cap \cap_{n=1}^\infty V_n = \{x^*\}$. The the set $\{n : V_k \cap K_n \neq \emptyset\}$ is cofinite for every k and we may pick $x_n^* \in K_n$ so that for each k, $x_n^* \in V_k$ eventually. It is clear that this sequence has exactly one accumulation point x^* and hence converges to x^*. The converse is trivial.

(ii) Let $(W_k)_{k=1}^\infty$ be a sequence of closed symmetric weak*-neighborhoods of 0 such that $W_{k+1} + W_{k+1} \subset W_k$ for $k \geq 1$ and if $x^*, y^* \in \limsup_{n\to\infty} K_n$ with $x^* - y^* \in W_k$ for all k then $x^* = y^*$. Let $(u_j^*)_{j=1}^\infty$ be a dense sequence in $\limsup_{n\to\infty} K_n$.

Arrange the countable family of sets $(u_j^* + W_k)$ as a single sequence $(U_n)_{n=1}^\infty$. Then we construct inductively subsequences $\mathbb{M}_1 \supset \mathbb{M}_2 \supset \cdots$ so that for each k the set $\{n \in \mathbb{M}_k : K_n \cap U_k \neq \emptyset\}$ is either finite or cofinite in $\mathbb{M}_k$. Let $\mathbb{M}$ be obtained from $\mathbb{M}_k$ by a diagonal procedure so that $\mathbb{M} \subset \mathbb{M}_k \cup A_k$ for a finite set A_k for each k. Then $\{n \in \mathbb{M} : K_n \cap U_k \neq \emptyset\}$ is either finite or cofinite for every k.

Now if $x^* \in \limsup_{n\in\mathbb{M}} K_n$ then for each k there exists j so that $x^* - u_j^* \in W_{k+2}$ and so that $u_j^* + W_{k+1}$ is a neighborhood of x^*. Thus $(u_j^* + W_{k+1}) \cap K_n \neq \emptyset$ for infinitely many $n \in \mathbb{M}$ and hence for a co-finite subset. It follows that, for every k, $x^* + W_k$ meets K_n for a cofinite set of $n \in \mathbb{M}$. Hence $x^* \in \limsup_{n\in\mathbb{M}'} K_n$ for every infinite subset $\mathbb{M}'$ of $\mathbb{M}$. By (i) it follows that $x^* \in \limsup_{n\in\mathbb{M}} K_n$ if and only if there is a sequence $(x_n^*)_{n\in\mathbb{M}}$ with $x_n^* \in K_n$ and $(x_n^*)_{n\in\mathbb{M}}$ converging to x^*. The set of such x^* is clearly convex. $\square$

Theorem 5.2. *Let X be a Banach space and suppose E is a closed linear subspace of X such that X/E is separable. Suppose $\lambda \geq 1$. Then the following are equivalent:*

(i) *(E, X) has the linear (λ, c_0)-EP.*

(ii) *(E, F) has the linear (λ, c_0)-EP whenever F is a linear subspace of X with $E \subset F$ and $\dim F/E = 1$.*

Remark. It follows from Sobczyk's theorem [37] that (i) and (ii) automatically hold if $\lambda \geq 2$.

Proof. We only need prove (ii) implies (i). Suppose

$$Tx = (e_n^*(x))_{n=1}^\infty$$

defines an operator $T : E \to c_0$ with $\|T\| = 1$ which has no extension $\widetilde{T}$ to X with $\|\widetilde{T}\| \leq \lambda$. Let K_n denote the set of all $x^* \in X^*$ with $x^*|_E = e_n^*$ and $\|x^*\| \leq \lambda$. Then $\limsup_{n\to\infty} K_n \subset E^\perp$ is weak*-metrizable.

By Lemma 5.1 (i) we conclude that there exists an infinite subset $\mathbb{M}$ of $\mathbb{N}$ so that $0 \notin K_{\mathbb{M}} = \limsup_{n\in\mathbb{M}} K_n$. By (ii) of the same lemma, we can suppose $K_{\mathbb{M}}$ is convex. Hence by the Hahn–Banach theorem we can find $x \in X$ so that $x^*(x) \geq 1$ for $x^* \in K_{\mathbb{M}}$. Let F be the space $E + [x]$. By assumption T can be extended to an operator $\widetilde{T} : F \to c_0$ with $\|\widetilde{T}\| \leq \lambda$. It follows from the Hahn–Banach theorem that we can find $x_n^* \in K_n$ such that $\|x_n^*\| \le \lambda$ and $\lim_{n\to\infty} x_n^*(x) = 0$. If x^* is any accumulation point of $(x_i^*)_{i\in\mathbb{M}}$ then $x^* \in K$ but $x^*(x) = 0$. This contradiction proves the implication. $\square$

Proposition 5.3. *Let X be a separable Banach space and suppose $x^* \in X^*$. Then for any $\lambda > 1$, the following conditions are equivalent:*

(i) $(\ker x^*, X)$ *has the linear* (λ, c_0)*-EP*
(ii) $(\ker x^*, X)$ *has the Lipschitz* (λ, c_0)*-EP.*
(iii) *If σ is any weak*-null type on X^* then $\lambda\sigma(x^*) \ge \sigma(0)$.*

Proof. (i) $\implies$ (iii). Suppose $\epsilon > 0$. Suppose x_n^* is a weak*-null sequence such that

$$\sigma(\alpha x^*) = \lim_{n\to\infty} \|\alpha x^* + x_n^*\| \qquad \alpha \in \mathbb{R}$$

and

$$\|x^* + x_n^*\| < \sigma(x^*) + \epsilon \quad n = 1, 2, \dots.$$

Define $T_0 : \ker x^* \to c_0$ by $T_0 x = (x_n^*(x))_{n=1}^\infty$. Then $\|T_0\| \le \sigma(x^*) + \epsilon$. Let T be any extension with $\|T\| \le \lambda\|T_0\|$. Then $Tx = (y_n^*(x))_{n=1}^\infty$ where $y_n^* = \alpha_n x^* + x_n^*$ with $\alpha_n \in \mathbb{R}$. Since $(y_n^*)_{n=1}^\infty$ is weak*-null, $\lim_{n\to\infty} \alpha_n = 0$. Thus

$$\lim_{n\to\infty} \|x_n^*\| \le \lambda(\sigma(x^*) + \epsilon)$$

and (iii) follows.

(iii) $\implies$ (ii). Let us assume that $\ker x^*$ fails condition $\Sigma_0(\lambda)$. Then there is an $x \in X$, $\epsilon > 0$ and a bounded sequence $(e_n)_{n=1}^\infty$ in E so that for every $e \in \ker x^*$ there exists $N = N(e)$ so that

$$\|e - e_n\| \ge \lambda\|x - e_n\| + 2\epsilon \qquad n \ge N.$$

It follows that for any compact subset K of $\ker x^*$ there exists $N = N(K)$ with

$$\|e - e_n\| > \lambda\|x - e_n\| + \epsilon \qquad e \in K,\ n \ge N.$$

Now let (F_n) be an increasing sequence of finite-dimensional subspaces of $\ker x^*$ whose union is dense in $\ker x^*$. Let $K_n = F_n \cap nB_X$. We may pass to a subsequence of $(e_n)_{n=1}^\infty$ (still denoted $(e_n)_{n=1}^\infty$) so that

$$\|e - e_n\| > \lambda\|x - e_n\| + \epsilon \qquad e \in K_n,\ n \ge 1.$$

It follows from the Hahn–Banach theorem that we can, for $n \ge 1$, pick $x_n^* \in X^*$ with $\|x_n^*\| = 1$ and

$$x_n^*(e - e_n) \ge \lambda\|x - e_n\| + \epsilon \qquad e \in K_n, n \ge 1.$$

In particular, since K_n is symmetric,

$$0 \le |x_n^*(e)| \le x_n^*(e_n) - \lambda\|x - e_n\| - \epsilon \qquad e \in K_n$$

which from the boundedness of (e_n) implies that

$$|x_n^*(e)| \le C/n \qquad e \in B_X \cap F_n$$

where C is some constant. Thus $\lim_{n\to\infty} x_n^*(e) = 0$ for $e \in \ker x^*$ so that every weak*-cluster point of $(x_n^*)_{n=1}^\infty$ is a multiple of x^*. Suppose $(x_n^*)_{n\in\mathbb{M}}$ is weak*-convergent to some αx^*. Let $y_n^* = x_n^* - \alpha x^*$. Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$ containing $\mathbb{M}$. Then by hypothesis

$$\lim_{n\in\mathcal{U}} \|y_n^*\| \le \lambda \lim_{n\in\mathcal{U}} \|x_n^*\| = \lambda.$$

Thus

$$\begin{aligned} 0 &= \lim_{n\in\mathcal{U}} y_n^*(x) \\ &\geq \lim_{n\in\mathcal{U}} y_n^*(x-e_n) + \lim_{n\in\mathcal{U}} y_n^*(e_n)) \\ &\geq -\lambda \lim_{n\in\mathcal{U}} \|x-e_n\| + \lim_{n\in\mathcal{U}} x_n^*(e_n) \\ &\geq \epsilon. \end{aligned}$$

This contradiction establishes (ii).

(ii) $\implies$ (i) follows from Lemma 4.9. □

Theorem 5.4. *Let X be a separable Banach space and suppose E is a closed linear subspace. Suppose $1 < \lambda \leq 2$. Then (E, X) has the Lipschitz (λ, c_0)-EP if and only if (E, X) has the linear (λ, c_0)-EP.*

Hence, (E, X) has the linear c_0-AIEP if and only if (E, X) has the Lipschitz c_0-AIEP.

Proof. This is an immediate deduction from Theorem 5.2 and Proposition 5.3 above. The only necessary observation is that (E, X) satisfies condition $\Sigma_0(\lambda)$ if and only if (E, F) satisfies condition $\Sigma_0(\lambda)$ for every linear space $F \supset E$ with $\dim F/E = 1$. □

The following theorem is now a trivial deduction.

Theorem 5.5. *Let X be a separable Banach space. Suppose $1 < \lambda \leq 2$. The following conditions are equivalent:*

(i) *(E, X) has the linear (λ, c_0)-EP for every closed subspace E.*
(ii) *(E, X) has the Lipschitz (λ, c_0)-EP for every closed subspace E.*
(iii) *(C, X) has the Lipschitz (λ, c_0)-EP for every closed convex subset C of X.*
(iv) *(C, X) has the Lipschitz (λ, c_0)-EP for every closed bounded convex subset C of X.*
(v) *For any weak*-null sequence (x_n^*) and $x^* \in X^*$ we have*

$$\limsup_{n\to\infty} \|x_n^*\| \leq \lambda \limsup_{n\to\infty} \|x^* + x_n^*\|. \tag{5.1}$$

This yields an immediate conclusion for the almost isometric case:

Theorem 5.6. *Let X be a separable Banach space. The following conditions are equivalent:*

(i) *(E, X) has the linear c_0-AIEP for every closed subspace E.*
(ii) *(E, X) has the Lipschitz c_0-AIEP for every closed subspace E.*
(iii) *(C, X) has the Lipschitz c_0-AIEP for every convex subset C of X.*
(iv) *(C, X) has the Lipschitz c_0-IEP for every closed bounded convex subset C of X.*
(v) *If $x^* \in X^*$ and $(x_n^*)_{n=1}^\infty \in X^*$ is a weak*-null sequence, then,*

$$\limsup_{n\to\infty} \|x_n^*\| \leq \limsup_{n\to\infty} \|x^* + x_n^*\|. \tag{5.2}$$

Proof. The only point necessary to observe here is that for bounded sets the c_0-AIEP and the c_0-IEP are equivalent (Proposition 4.4). □

Note here that (v) simply says that every weak*-null type on X^* is monotone.

6. Extensions into c

We now replace c_0 by c. First note that:

Proposition 6.1. *If (E, X) has the (λ, c)-EP then (E, X) has the (λ, c_0)-EP.*

Proof. This is due to Johnson (see [38]). Suppose $T_0 : E \to c_0$ with $\|T_0\| = 1$, is defined by $T_0 e = (e_n^*(e))_{n=1}^\infty$ where $e_n^* \in E^*$. Define $S_0 : E \to c$ by $S_0 x = (e_1^*(x), -e_1^*(x), e_2^*(x), -e_2^*(x), \ldots)$ and let $S : X \to c$ be an extension with $\|S\| \le \lambda$. Let $Sx = (u_1^*(x), -v_1^*(x), u_2^*(x), -v_2^*(x), \ldots)$. Then

$$Tx = (\tfrac{1}{2}(u_1^* + v_1^*)(x), \tfrac{1}{2}(u_2^* + v_2^*)(x), \ldots)$$

defines an extension of T_0 with $\|T\| \le \lambda$. $\square$

Suppose X is a separable Banach space and E is a closed linear subspace of X. Then if $n \in \mathbb{N}$ and $\lambda \ge 1$ we say that (E, X) *satisfies condition* $(\Gamma_n(\lambda))$ if given, $\epsilon > 0$, a family of $(n+1)$ bounded sequences $(e_{jk})_{j=1,k=1}^{n+1,\infty}$ in E and any $x_1, \ldots, x_{n+1} \in X$ with

$$x_1 + \cdots + x_{n+1} = 0$$

then there exist $u_1, \ldots, u_{n+1} \in E$ with

$$u_1 + \cdots + u_{n+1} = 0$$

and an infinite subset $\mathbb{M}$ of $\mathbb{N}$ so that

$$\sum_{j=1}^{n+1} \|u_j + e_{jk}\| < \lambda \sum_{j=1}^{n+1} \|x_j + e_{jk}\| + \epsilon \qquad k \in \mathbb{M}.$$

We note immediately that $\Gamma_n(\lambda) \implies \Gamma_{n-1}(\lambda)$ when $n \ge 2$. We also observe that every subspace E satisfies the conditions $\Gamma_n(3)$. Indeed we have:

Proposition 6.2. *For any subspace E, (E, X) satisfies condition $\Gamma_n(\lambda)$ with*

$$\lambda = \frac{3n+1}{n+1}.$$

Proof. We may choose $v_j \in E$ with $\|x_j - v_j\| < \epsilon/2(n+1) + d(x_j, E)$ and then put $u_j = v_j - \overline{v}$ where

$$\overline{v} = \frac{1}{n+1}(v_1 + \cdots + v_{n+1}).$$

Then

$$\|u_j + e_{jk}\| \le \|x_j + e_{jk}\| + \|x_j - v_j + \overline{v}\|.$$

Now

$$\|x_j - v_j + \overline{v}\| \le \frac{n}{n+1}\|x_j - v_j\| + \sum_{i \ne j} \|x_i - v_i\|$$

so that

$$\begin{aligned}\sum_{j=1}^{n+1} \|u_j + e_{jk}\| &\le \sum_{j=1}^{n+1} \|x_j + e_{jk}\| + \frac{2n}{n+1} \sum_{j=1}^{n+1} \|x_j - v_j\| \\ &\le \sum_{j=1}^{n+1} \|x_j + e_{jk}\| + \frac{2n}{n+1} \sum_{j=1}^{n+1} d(x_j, E) + \epsilon \\ &\le \frac{3n+1}{n+1} \sum_{j=1}^{n+1} \|x_j + e_{jk}\| + \epsilon. \end{aligned}$$

$\square$

It will also be useful to translate the condition $\Gamma_n(\lambda)$ in terms of types:

Proposition 6.3. *In order that (E,X) has property $\Gamma_n(\lambda)$ it is necessary and sufficient that for every $(n+1)$ types $\sigma_1,\ldots,\sigma_{n+1}$ supported on E and every $e\in E$ then*

$$\inf_{\substack{e_1,\ldots,e_{n+1}\in E\\ e_1+\cdots+e_{n+1}=e}}\sum_{j=1}^{n+1}\sigma_j(e_j)\le\lambda\inf_{\substack{x_1,\ldots,x_{n+1}\in X\\ x_1+\cdots+x_{n+1}=e}}\sum_{j=1}^{n+1}\sigma_j(x_j). \tag{6.1}$$

The proof of this is trivial once one observes that the condition (6.1) is equivalent to the same statement with $e=0$. Indeed simply replace σ_1 with $\sigma_1'(x):=\sigma_1(x-e)$.

We also observe:

Proposition 6.4. *For any closed linear subspace E of X and $\lambda\ge 1$, properties $\Sigma_1(\lambda)$ and $\Gamma_1(\lambda)$ are equivalent for (E,X).*

This, again, is trivial from the definition, exploiting only the fact that $e\in E\implies -e\in E$.

Proposition 6.5. *Let X be a Banach space and suppose E is a closed subspace such that X/E is separable. Suppose $T_0:E\to c$ is an operator with $\|T_0\|=1$. Then:*

(i) *If $\lambda\ge 1$, in order that there exist a linear extension $T:X\to c$ of T_0 with $\|T\|\le\lambda$ it is necessary and sufficient that for every subspace $E\subset F\subset X$ with $\dim F/E<\infty$ there is an extension $T_F:F\to c$ with $\|T_F\|\le\lambda$.*

(ii) *If $\lambda>1$ in order that there exist a linear extension $T:X\to c$ of T_0 with $\|T\|\le\lambda$ it is necessary and sufficient that for every $\mu>\lambda$ there exists a linear extension $T_\mu:X\to c$ with $\|T_\mu\|\le\mu$.*

Proof. Fix some sequence $(x_n)_{n=1}^\infty$ so that $[E\cup\{x_n\}_{n=1}^\infty]=X$. Let $T_0(e)=\{e_n^*(e)\}_{n=1}^\infty$ where $e_n^*\in E^*$ and $\|e_n^*\|\le 1$.

If (i) holds then for every n, using the Hahn–Banach theorem we can find extensions x_{nk}^* of e_k^* with $\|x_{nk}^*\|\le\lambda$ and such that $\lim_{k\to\infty}x_{nk}^*(x_j)=a_{nj}$ exists for $j\le n$. We can then pick an infinite subset $\mathbb{M}$ of $\mathbb{N}$ so that $\lim_{n\in\mathbb{M}}a_{nj}=a_j$ exists for every j. Now by a diagonal argument we find a nondecreasing map $k\to m_k$ from $\mathbb{N}$ into $\mathbb{M}$ with $\lim_{k\to\infty}m_k=\infty$ so that $\lim_{k\to\infty}x_{m_k,k}^*(x_j)=a_j$ for every j. Now define $Tx=(x_{m_k,k}^*(x))_{k=1}^\infty$ and T is the desired extension.

Now suppose (ii) holds. Proceeding in a similar way we produce extensions x_{nk}^* of e_k^* so that $\|x_{nk}^*\|\le\lambda+1/n$ and $\lim_{k\to\infty}x_{nk}^*=x_n^*$ exists (weak*). We can pass to a subsequence $\mathbb{M}$ so that $\lim_{n\in\mathbb{M}}x_{nk}^*(x_j)=a_j$ exists for every j. As before we can then define $m_k\in\mathbb{M}$ (a nondecreasing sequence with $\lim_{k\to\infty}m_k=\infty$) so that $\lim_{k\to\infty}x_{m_k,k}^*(x_j)=a_j$ for all j. Let $y_k^*=x_{m_k,k}^*$ so that $\limsup_{k\to\infty}\|y_k^*\|\le\lambda$ and $(y_k^*)_{k=1}^\infty$ is weak*-convergent.

Finally let u_k^* be any norm-preserving extension of e_k^*. Then, since $\lambda>1$, we may pick c_k with $0\le c_k\le 1$ and $\lim c_k=0$ so that $\|c_ku_k^*+(1-c_k)y_k^*\|\le\lambda$. Letting $Tx=(c_ku_k^*(x)+(1-c_k)y_k^*(x))_{k=1}^\infty$ gives the result. □

Theorem 6.6. *Let X be a Banach space and suppose E is a closed subspace of X with the linear (λ,c)-extension property. Then for every $n\in\mathbb{N}$, (E,X) satisfies the condition $\Gamma_n(\lambda)$.*

Proof. Suppose on the contrary that E fails the condition $\Gamma_n(\lambda)$ for some n. Then we may find $\epsilon > 0$, $x_1, \dots, x_{n+1} \in X$ with $\sum_{j=1}^{n+1} x_j = 0$ and bounded sequences $(e_{jk})_{k=1}^\infty$ in E for $j = 1, \dots, n+1$ with the property that for every choice of $u_1, \dots, u_{n+1}$ in E with $\sum_{j=1}^{n+1} u_j = 0$ the set of k such that

$$\sum_{j=1}^{n+1} \|u_j + e_{jk}\| < \lambda \sum_{j=1}^{n+1} \|x_j + e_{jk}\| + 2\epsilon$$

is finite.

Consider the space $\ell_1^{n+1}(X)$ and the subspace F of all $(u_1, \dots, u_{n+1})$ with $u_j \in E$ and $\sum_{j=1}^{n+1} u_j = 0$. It follows that for every compact subset K of F there exists $m = m(K)$ so that:

$$\sum_{j=1}^{n+1} \|u_j + e_{jk}\| \ge \lambda \sum_{j=1}^{n+1} \|x_j + e_{jk}\| + \epsilon \qquad (u_1, \dots, u_{n+1}) \in K,\ k \ge m.$$

Since F is separable we can find an increasing sequence of compact convex sets K_m, each containing the origin, so that $\cup_{m=1}^\infty K_m$ is dense in F. It then follows that we can choose a subsequence $\mathbb{M}_0 = \{k_1, k_2, \dots\}$ so that

$$\sum_{j=1}^{n+1} \|u_j + e_{jk_m}\| \ge \lambda \sum_{j=1}^{n+1} \|x_j + e_{jk_m}\| + \epsilon \qquad (u_1, \dots, u_{n+1}) \in K_m.$$

Now by the Hahn–Banach theorem we can find $x^*_{1k_m}, \dots, x^*_{n+1,k_m} \in X^*$ so that $\|x^*_{jk_m}\| \le 1$ and

$$\sum_{j=1}^{n+1} x^*_{jk_m}(u_j + e_{jk_m}) \ge \lambda \sum_{j=1}^{n+1} \|x_j + e_{jk_m}\| + \epsilon \qquad (u_1, \dots, u_{n+1}) \in K_m. \tag{6.2}$$

In particular

$$\sum_{j=1}^{n+1} x^*_{jk_m}(u_j) \ge \sum_{j=1}^{n+1} x^*_{jk_m}(x_j) + \epsilon \qquad (u_1, \dots, u_{n+1}) \in K_m.$$

At this point we pass to a further subsequence $\mathbb{M}$ of $\mathbb{M}_0$ where

$$\lim_{k \in \mathbb{M}} x^*_{jk} = x^*_j$$

exists for $1 \le j \le n+1$. We then have

$$\sum_{j=1}^{n+1} x^*_j(u_j) \ge \sum_{j=1}^{n+1} x^*_j(x_j) + \epsilon \qquad (u_1, \dots, u_{n+1}) \in \cup_{m=1}^\infty K_m.$$

By a density argument we can conclude that

$$\sum_{j=1}^{n+1} x^*_j(u_j) \ge \sum_{j=1}^{n+1} x^*_j(x_j) \qquad (u_1, \dots, u_{n+1}) \in F.$$

Since F is a linear space, this implies that

$$\sum_{j=1}^{n+1} x^*_j(u_j) = 0 \qquad (u_1, \dots, u_{n+1}) \in F.$$

Hence $x_1^*|_E = x_2^*|_E = \cdots = x_{n+1}^*|_E = e^*$, say. Now we can consider the map

$$Tx = (x_{jk}^*(x))_{1\le j\le n+1,\ k\in\mathbb{M}}$$

as a linear operator from X into $\ell_\infty(\mathbb{A})$ where $\mathbb{A} = [1, n+1] \times \mathbb{M}$ is a countable set. Then T maps E into $c(\mathbb{A})$ (the subspace of converging sequences) and so has an extension $S : X \to c(\mathbb{A})$ with $\|S\| \le \lambda$, given by

$$Sx = (y_{jk}^*(x))_{1\le j\le n+1,\ k\in\mathbb{M}}.$$

Then

$$\sum_{j=1}^{n+1} \|S(x_j + e_{jk})\| \le \lambda \sum_{j=1}^{n+1} \|x_j + e_{jk}\| \qquad k \in \mathbb{M}.$$

This means that

$$\sum_{j=1}^{n+1} y_{jk}^*(x_j + e_{jk}) \le \lambda \sum_{j=1}^{n+1} \|x_j + e_{jk}\|.$$

Now $\lim_{k\in\mathbb{M}} y_{jk}^* = y^*$ in the weak*-topology (independent of j), where $y^*|_E = e^*$. Hence

$$\lim_{k\in\mathbb{M}} \sum_{j=1}^{n+1} y_{jk}^*(x_j) = y^*(\sum_{j=1}^{n+1} x_j) = 0.$$

Thus

$$\limsup_{k\in\mathbb{M}} \left(\sum_{j=1}^{n+1} x_{jk}^*(e_{jk}) - \lambda \sum_{j=1}^{n+1} \|x_j + e_{jk}\| \right) \le 0.$$

This contradicts (6.2) when $u_j = 0$ for all j. □

Theorem 6.7. *Let X be a Banach space and suppose E is a closed subspace of codimension n. Suppose $\lambda > 1$. Then (E, X) has the linear (λ, c)-extension property if and only if (E, X) satisfies the condition $\Gamma_n(\lambda)$.*

Proof. One direction follows from Theorem 6.6. Let us therefore suppose that E satisfies $\Gamma_n(\lambda)$ and $T : E \to c$ is an operator with $\|T\| \le 1$. By Proposition 6.5 we need only show the existence of an extension $T : X \to c$ with $\|T\| \le \mu$ when $\mu > \lambda$. Assume therefore $\mu > \lambda$ and that $Tx = (e_m^*(x))_{m=1}^\infty$ where $e_m^* \in E^*$. We suppose that $(e_m^*)_{m=1}^\infty$ is weak* convergent in E^* to $e^* \in E^*$. Let K_m be the set of all $x^* \in X^*$ so that $x^*|_E = e_m^*$ and $\|x^*\| \le \mu$. Note that $K = \limsup_{m\to\infty} K_m \subset E^\perp$ which is an n-dimensional linear subspace. For $\mathbb{M}$ an infinite subset of $\mathbb{N}$ we define $K_\mathbb{M} = \limsup_{m\in\mathbb{M}} K_n$.

Suppose $\mathbb{M}_1, \ldots, \mathbb{M}_{n+1}$ are any $n+1$ infinite subsets of $\mathbb{N}$. We will show that

$$\cap_{j=1}^{n+1} K_{\mathbb{M}_j} \neq \emptyset. \tag{6.3}$$

Let $\mathbb{M}_j = \{m_{jk}\}_{k=1}^\infty$. For convenience we define $f_{jk}^* = e_{m_{jk}}^*$.

We consider the space $\ell_1^{n+1}(X)$ with its usual norm denoted by $\|\cdot\|$ and the unit ball by B. Let G be the subspace of all $(x_1, \ldots, x_{n+1})$ such that $x_1 + \cdot + x_{n+1} = 0$. On $\ell_1^{n+1}(X)$for each $m \in \mathbb{N}$ we define an auxiliary norm $\|\cdot\|_m$ as the Minkowski functional of $B + m(B \cap G)$.

We define h_k^* on $\ell_1^{n+1}(X)$ by

$$h_k^*(e_1,\ldots,e_{n+1}) = \sum_{j=1}^{n+1} f_{jk}^*(e_j).$$

We claim that

$$\limsup_{k\to\infty} \|h_k^*\|_m \le \lambda. \tag{6.4}$$

Indeed if not we can find $\lambda' > \lambda$ and an infinite subset $\mathbb{A}$ of $\mathbb{N}$ so that $\|h_k^*\|_m > \lambda'$. Thus there are $(e_{jk})_{j=1}^{n+1}$ and $(x_{jk})_{j=1}^{n+1}$ for $k \in \mathbb{A}$ so that

$$e_{jk} \in E \qquad j = 1,2,\ldots,n+1,\ k \in \mathbb{A}, \tag{6.5}$$

$$\sum_{j=1}^{n+1} x_{jk} = 0 \qquad k \in \mathbb{A}, \tag{6.6}$$

$$\sum_{j=1}^{n+1} \|x_{jk}\| \le m \qquad k \in \mathbb{A}, \tag{6.7}$$

$$\sum_{j=1}^{n+1} \|e_{jk} - x_{jk}\| \le 1 \qquad k \in \mathbb{A}, \tag{6.8}$$

$$\sum_{j=1}^{n+1} f_{jk}^*(e_{jk}) > \lambda' \qquad k \in \mathbb{A}. \tag{6.9}$$

We fix $\epsilon > 0$ so that $\lambda + (\lambda+2)\epsilon < \lambda'$. Let $Q : X \to X/E$ be the quotient map. The sequences $(Q(x_{jk}))_{k\in\mathbb{A}}$ are bounded and X/E is finite-dimensional. Hence by passing to another subsequence $\mathbb{A}_0$ so that each $(Q(x_{jk})_{k\in\mathbb{A}_0}$ is norm-convergent and picking suitable representatives in X we may find $y_1,\ldots,y_{n+1} \in X$ and $f_{jk} \in E$ for $1 \le j \le n+1,\ k \in \mathbb{A}_0$ so that

$$\sum_{j=1}^{n+1} y_j = 0, \tag{6.10}$$

$$\|x_{jk} - y_j - f_{jk}\| \le \epsilon/(n+1), \qquad 1 \le j \le n+1,\ k \in \mathbb{A}_0. \tag{6.11}$$

Now applying the condition $\Gamma_n(\lambda)$ we can find a further infinite subset $\mathbb{A}_1$ of $\mathbb{A}_0$ and $u_j \in E$ for $1 \le j \le n+1$ with

$$u_1 + \cdots + u_{n+1} = 0 \tag{6.12}$$

and with the property that for $k \in \mathbb{A}_1$,

$$\sum_{j=1}^{n} \|e_{jk} - f_{jk} - u_j\| \le \epsilon + \lambda \sum_{j=1}^{n+1} \|e_{jk} - f_{jk} - y_j\|. \tag{6.13}$$

Combining (6.8) and (6.11) we have

$$\begin{aligned}\epsilon + \lambda \sum_{j=1}^{n+1} \|e_{jk} - f_{jk} - y_j\| &\le (\lambda+1)\epsilon + \sum_{j=1}^{n+1} \|e_{jk} - x_{jk}\| \\ &\le \lambda + (\lambda+1)\epsilon.\end{aligned}$$

Thus we have by (6.13),

$$\sum_{j=1}^{n} \|e_{jk} - f_{jk} - u_j\| \le \lambda + (\lambda+1)\epsilon, \qquad k \in \mathbb{A}_1. \tag{6.14}$$

Note also that, since $f_{jk} \in E$, we have by (6.11),

$$\begin{aligned}\left|\sum_{j=1}^{n+1} f^*_{jk}(f_{jk})\right| &\le \left\|\sum_{j=1}^{n+1} f_{jk}\right\| \\ &\le \epsilon + \left\|\sum_{j=1}^{n+1}(x_{jk} - y_j)\right\| = \epsilon, \qquad k \in \mathbb{A}_1.\end{aligned}$$

Combining this with (6.13) gives:

$$\sum_{j=1}^{n+1} f^*_{jk}(e_{jk}) \le \lambda + (\lambda+2)\epsilon + \sum_{j=1}^{n+1} f^*_{jk}(u_j) \qquad k \in \mathbb{A}_1.$$

At this point we note that the sequences $(f_{jk})_{k\in\mathbb{A}_1}$ converge weak* to a common limit e^*. By (6.9), and using (6.12),

$$\begin{aligned}\lambda' &\le \limsup_{k\in\mathbb{A}_1} \sum_{j=1}^{n+1} f^*_{jk}(e_{jk}) \\ &\le \lambda + (\lambda+2)\epsilon + e^*\left(\sum_{j=1}^{n+1} u_j\right) \\ &= \lambda + (\lambda+2)\epsilon.\end{aligned}$$

This contradicts the choice of ϵ and shows that (6.4) holds.

Now it also follows that we can find a sequence of natural numbers $m_k \uparrow \infty$ so that

$$\limsup_{k\to\infty} \|h^*_k\|_{m_k} = \lambda.$$

In particular we can find k_0 so that $\|h^*_k\|_{m_k} \le \mu$ for $k \ge k_0$. Using the Hahn–Banach theorem we can find norm-preserving extensions to $\ell_1^{n+1}(X)$ for the norm $\|\cdot\|_{m_k}$. This means that we can find $x^*_{jk} \in X^*$ for $k \ge k_0$ and $1 \le j \le n+1$ so that $x^*_{jk}|_E = f^*_{jk}$, $\|x^*_{jk}\| \le \mu$ and

$$\sum_{j=1}^{n+1} x^*_{jk}(x_j) \le \mu m_k^{-1} \sum_{j=1}^{n+1} \|x_j\|$$

whenever $\sum_{j=1}^{n+1} x_j = 0$. Again by the Hahn–Banach theorem we can find $y^*_{jk} \in X^*$ with $\|y^*_{jk}\| \le \mu m_k^{-1}$ and

$$\sum_{j=1}^{n+1} x^*_{jk}(x_j) = \sum_{j=1}^{n+1} y^*_{jk}(x_j)$$

whenever $\sum_{j=1}^{n+1} x_j = 0$. This means that for each $k \ge k_0$ we can find $z^*_k \in X^*$ so that $x^*_{jk} = y^*_{jk} + z^*_k$. Now $\lim_{k\to\infty} \|x^*_{jk} - z^*_k\| = 0$ for $1 \le j \le n+1$ and it follows that any weak*-cluster point z^* of $(z^*_k)_{k=1}^\infty$ is a cluster point of each sequence $(x^*_{jk})_{k\ge k_0}$. Hence $z^* \in K_{\mathbb{M}_j}$ for all $1 \le j \le n+1$. This establishes (6.3).

We now deduce that $\cap_{\mathbb{M}} K_{\mathbb{M}} \neq \emptyset$. Let $\mathcal{C} = \{\mathbb{M} : \ K_{\mathbb{M}} \text{ is convex}\}$. Since $\dim E^{\perp} = n$ it follows from (6.3) and Helly's theorem (see e.g., [39]) that $\cap_{\mathbb{M}\in\mathcal{C}} K_{\mathbb{M}}$ is nonempty; however, Lemma 5.1 (ii) implies this means that the entire intersection is nonempty. Now from Lemma 5.1 (i) we can pick $x_n^* \in K_n$ so that $(x_n^*)_{n=1}^{\infty}$ is weak*-convergent. Thus $\widetilde{T} : X \to c$ given by $\widetilde{T}(x) = (x_n^*(x))_{n=1}^{\infty}$ is the desired extension with $\|T\| \leq \mu$. □

Corollary 6.8. *Suppose $x^* \in X^*$. Then for $1 < \lambda \leq 2$, $(\ker x^*, X)$ has the linear (λ, c)-EP if and only $(\ker x^*, X)$ has the Lipschitz $(\lambda, \mathcal{C})$-EP.*

Proof. This follows from Proposition 6.4. See Theorem 4.2 (v) of [19] and Theorem 4.2 above. □

Combining Proposition 6.5 with Theorems 6.6 and 6.7 gives:

Theorem 6.9. *Let X be a separable Banach space and suppose E is a closed subspace of X. Suppose $1 < \lambda \leq 3$. Then the following conditions are equivalent:*

(i) *(E, X) has the linear (λ, c)-EP.*
(ii) *(E, X) satisfies the condition $\Gamma_n(\lambda)$ for every $n \in \mathbb{N}$.*

When $\lambda = 1$ this reduces to:

Theorem 6.10. *Let X be a separable Banach space and suppose E is a closed subspace of X. Then the following conditions are equivalent:*

(i) *(E, X) has the linear c-AIEP.*
(ii) *(E, X) satisfies the condition $\Gamma_n(1)$ for every $n \in \mathbb{N}$.*

7. Separable Banach spaces with the almost isometric linear $\mathcal{C}$-extension property

If we look for conditions on X so that every closed subspace E has the linear almost isometric extension property, different techniques can be used, using a nonlinear approach. Thus we have:

Theorem 7.1. *Let X be a separable Banach space. The following conditions on X are equivalent:*

(i) *X has the linear $\mathcal{C}$–AIEP.*
(ii) *(E, X) has the linear $\mathcal{C}$-AIEP for every closed subspace E of codimension one.*
(iii) *(E, X) has the Lipschitz $\mathcal{C}$-AIEP for every closed subspace E.*
(iv) *(C, X) has the Lipschitz $\mathcal{C}$-AIEP for every closed convex subset C of X.*
(v) *(C, X) has the Lipschitz $\mathcal{C}$-IEP for every closed bounded convex subset C of X.*
(vi) *Let σ and τ be weak*-null types on X^*. If $u^*, v^* \in X^*$ then there exists $0 \leq \theta \leq 1$ so that if $w^* = (1-\theta)u^* + \theta v^*$*

$$\max(\sigma(w^*), \tau(w^*)) \leq \max(\sigma(u^*), \tau(v^*)). \tag{7.1}$$

Proof. Let us start by observing that (ii) and (vi) are equivalent.

(ii) $\implies$ (vi). Pick $(x_n^*)_{n=1}^{\infty}$ and $(y_n^*)_{n=1}^{\infty}$ be weak*-null sequences determining the weak*-null σ, τ respectively on the span of u^* and v^*, i.e., so that

$$\sigma(x^*) = \lim_{n\to\infty} \|x^* + x_n^*\|, \ \tau(x^*) = \lim_{n\to\infty} \|x^* + y_n^*\|, \qquad x^* \in [u^*, v^*].$$

Let E be the kernel of $v^* - u^*$. For $m \geq 1$ define $T_m : E \to c$ by

$$(T_m e)_j = \begin{cases} 0 & j \leq m \\ u^*(e) + x_j^*(e) & j > m,\ j \text{ even} \\ u^*(e) + y_j^*(e) & j > m,\ j \text{ odd.} \end{cases}$$

Then

$$\limsup_{m\to\infty} \|T_m\| \leq \max(\sigma(u^*), \tau(v^*)).$$

Now suppose we extend each T_m to define $\widetilde{T}_m : X \to C$ with

$$\limsup_{m\to\infty} \|\widetilde{T}_m\| \leq \max(\sigma(u^*), \tau(v^*)).$$

This implies we can find θ_{jm} for $j > m$ so that $\lim_{j\to\infty} \theta_{jm} = \theta_m$ exists and

$$\|u^* + \theta_{jm}(v^* - u^*) + x_j^*\| \leq \|\widetilde{T}_m\| \qquad j \text{ even}$$
$$\|u^* + \theta_{jm}(v^* - u^*) + y_j^*\| \leq \|\widetilde{T}_m\| \qquad j \text{ odd.}$$

Hence

$$\sigma((1-\theta_m)u^* + \theta_m v^*), \tau((1-\theta_m)u^* + \theta_m v^*) \leq \|\widetilde{T}_m\|.$$

The sequence $(\theta_m)_{m=1}^\infty$ is bounded and hence has a cluster point θ so that if $w^* = (1-\theta)u^* + \theta v^*$ we have (7.1). It remains only to add that if $\theta \leq 0$ we can take $\theta = 0$ and if $\theta \geq 1$ we can take $\theta = 1$ by simple convexity arguments.

(vi) $\implies$ (ii). Suppose $T : E \to c$ is given by $Te = (e_n^*(e))_{n=1}^\infty$. If $\lambda > 1$ we define K_n to be the set of $x^* \in X^*$ with $\|x^*\| \leq \lambda$ and $x^*|_E = 0$. Let $K_{\mathbb{M}} = \limsup_{n\in\mathbb{M}} K_n$. In order to prove the existence of an extension $\widetilde{T} : X \to c$ with $\|\widetilde{T}\| \leq \lambda$, it is, as in the argument for Theorem 6.7, sufficient to show that $\cap_{\mathbb{M}} K_{\mathbb{M}} \neq \emptyset$. In this case $\dim E^\perp = 1$ and so it suffices to prove (using Lemma 5.1 that $K_{\mathbb{M}_1} \cap K_{\mathbb{M}_2} \neq \emptyset$ for any two infinite sets $\mathbb{M}_1, \mathbb{M}_2$. Pick any $v_n^* \in K_n$ with $\|v_n^*\| \leq 1$. We can pass to infinite subsets $\mathbb{M}_1'$ and $\mathbb{M}_2'$ of $\mathbb{M}_1$ and $\mathbb{M}_2$ respectively so that $\lim_{n\in\mathbb{M}_1'} v_n^* = x^*$ and $\lim_{n\in\mathbb{M}_2'} v_n^* = y^*$ exist weak*. Note that $y^* - x^* \in E^\perp$. We can further suppose that

$$\lim_{n\in\mathbb{M}_1'} \|z^* + v_n^* - x^*\|, \quad \lim_{n\in\mathbb{M}_2'} \|z^* + v_n^* - y^*\|$$

exist for every $z^* \in [x^*, y^*]$. Now by hypothesis we can find $u^* = (1-\theta)x^* + \theta y^*$ so that

$$\lim_{n\in\mathbb{M}_1'} \|u^* + v_n^* - x^*\|, \lim_{n\in\mathbb{M}_2'} \|u^* + v_n^* - y^*\| \leq 1.$$

For large enough $n \in \mathbb{M}_1'$ we have $u^* + v_n^* - x^* \in K_n$ since $u^* - x^* \in E^\perp$; similarly for large enough $n \in \mathbb{M}_2'$, $u^* + v_n^* - y^* \in K_n$. It follows that $u^* \in K_{\mathbb{M}_1} \cap K_{\mathbb{M}_2}$ and we are done.

It remains to show that conditions (i)–(v) are equivalent. In fact we observe from Proposition 4.7 that if every closed subspace of codimension one has property $\Sigma_1(1)$ then so does every closed convex subset. From Corollary 6.8 it is clear that (ii) implies (iii), (iv) and (v). We also observe from Proposition 4.10 that once every subspace of X has the Lipschitz $\mathcal{C}$-AIEP then in fact X has the linear $\mathcal{C}$-AIEP. This implies that (iii) $\implies$ (i). The other equivalences are easy. $\square$

Remark 7.2. Of course we can use Theorem 5.6 and Proposition 6.1 to deduce that any weak*-null type σ on X^* must be monotone i.e.,

$$\sigma(x^*) \geq \sigma(0) \qquad x^* \in X^*$$

under the hypotheses of the theorem. This implies

$$\sigma(\alpha x^*) \leq \sigma(x^*) \qquad x^* \in X^*,\ 0 \leq \alpha \leq 1,$$

i.e., $\alpha \to \sigma(\alpha x^*)$ is monotone on $(0, \infty)$. These facts will be used repeatedly in this section.

In practice we only know of examples where (7.1) holds with $\theta = 0$ or 1 (i.e., $w^* \in \{u^*, v^*\}$). In this case (7.1) reduces to the condition

$$\min(\sigma(v^*), \tau(u^*)) \leq \max(\sigma(u^*), \tau(v^*)) \qquad u^*, v^* \in X^*$$

whenever σ, τ are weak*-null types. We will now show that this condition is necessary in most cases (i.e., with a quite mild additional condition on X). We start by considering an equivalent form of this stronger version of (7.1) (which is equivalent to (7.2) below).

Proposition 7.3. *Suppose X is a separable Banach space with the linear c_0-AIEP. Let E be a closed subspace of X^*. Then the following conditions are equivalent:*

(i) *For any pair σ, τ of weak*-null types on X^* we have*

$$\min(\sigma(v^*), \tau(u^*)) \leq \max(\sigma(u^*), \tau(v^*)) \qquad u^*, v^* \in E. \tag{7.2}$$

(ii) *For any pair of weak*-null types σ, τ and any $u^*, v^* \in E$,*

$$\sigma(u^*) = \sigma(v^*) = \tau(u^*) > \max(\sigma(0), \|u^*\|) \Rightarrow \tau(v^*) \leq \sigma(v^*). \tag{7.3}$$

Proof. (i) $\Rightarrow$ (ii). Suppose E satisfies (i) but fails (ii). We suppose σ, τ are weak*-null types and $u^*, v^* \in E$ are such that

$$\tau(v^*) > \sigma(u^*) = \sigma(v^*) = \tau(u^*) > \max(\sigma(0), \|u^*\|).$$

Now pick any $0 < \theta < 1$ so that $\tau_\theta(v^*) > \sigma(v^*)$, where τ_θ denotes the type $\tau_\theta(x^*) = \theta\tau(\theta^{-1}x^*)$. Then $\tau_\theta(u^*) < \tau(u^*)$ since $\tau(u^*) > \|u^*\|$, and $\theta \to \tau_\theta(u^*)$ is a convex function with $\lim_{\theta\to 0} \tau_\theta(u^*) = \|u^*\|$. Pick $\phi < 1$ so that $\tau_\theta(\phi v^*) > \sigma(v^*)$. Then $\sigma(\phi v^*) < \sigma(v^*)$ since $\sigma(v^*) > \sigma(0)$. Now

$$\min(\tau_\theta(\phi v^*), \sigma(u^*)) > \max(\tau_\theta(u^*), \sigma(\phi v^*))$$

contradicting (7.2).

(ii) $\Rightarrow$ (i). Conversely suppose E satisfies (7.3) but there exist weak*-null types σ, τ and $u^*, v^* \in E$ such that

$$\min(\sigma(v^*), \tau(u^*)) > \max(\sigma(u^*), \tau(v^*)).$$

Let us suppose $\sigma(u^*) \geq \tau(v^*)$. As in Remark 7.2 all weak*-null types on X^* are monotone. It is then clear that by replacing u^* with a suitable positive multiple we can suppose, without loss of generality, that $\sigma(u^*) > \sigma(0)$. Then there exists $0 < \phi < 1$ so that $\sigma(\phi v^*) = \sigma(u^*)$. Now τ cannot coincide with the norm since that implies $\sigma \geq \tau$ and gives a contradiction. Thus there also exists $\theta \geq 1$ so that $\tau_\theta(\phi v^*) = \sigma(u^*)$.

Thus we have $\tau_\theta(\phi v^*) = \sigma(u^*) = \sigma(\phi v^*) > \sigma(0)$. We now apply (7.3). Since $\tau_\theta(u^*) \geq \tau(u^*) > \sigma(u^*)$ we must have $\sigma(\phi v^*) = \|\phi v^*\|$. But then we have

$$\sigma(v^*) \leq \|\phi v^*\| + \|(1-\phi)v^*\| = \|v^*\| \leq \tau(v^*),$$

contradicting our assumption. Thus we have (7.2). $\square$

We will need the notion of the subdifferential of a convex function. If Y is a Banach space and $F : Y \to \mathbb{R}$ is a Lipschitz convex function we define the *subdifferential* $\nabla F(u)$ at $u \in Y$ by

$$\nabla F(u)(y) = \lim_{t\to 0+} \frac{F(u+ty) - F(u)}{t}.$$

Then $\nabla F(u)$ is a continuous sublinear functional and indeed

$$|\nabla F(u)(y)| \le \operatorname{Lip}(F)\|y\|.$$

Furthermore if Y is finite-dimensional we have an estimate

$$|F(y+u) - F(u) - \nabla F(u)(y)| \le \|y\|\rho_u(y) \tag{7.4}$$

where $\lim_{y\to 0} \rho_u(y) = 0$. The following lemma will be used later:

Lemma 7.4. *Suppose $F : Y \to \mathbb{R}$ is a Lipschitz convex function and $f : [a,b] \to Y$ is a continuous function. Suppose f is differentiable at some $a < s < b$. Then $F \circ f$ is left- and right-differentiable at s with*

$$(F \circ f)'_+(s) = \nabla F(f(s))(f(s)),$$
$$(F \circ f)'_-(s) = -\nabla F(f(s))(-f(s)).$$

Proof. Let $f(t) = f(s) + (t-s)f'(s) + r(t)$ where

$$\lim_{t\to s} \frac{\|r(t)\|}{|t-s|} = 0.$$

Then

$$|F(f(t)) - F(f(s) + (t-s)f'(s))| \le \operatorname{Lip}(F)\|r(t)\|$$

and the conclusion follows easily. $\square$

Lemma 7.5. *Suppose X is a separable Banach space with the linear $\mathcal{C}$-AIEP. Suppose σ and τ are two weak*-null types on X^* and that for some $x^* \in X^*$ we have $\sigma(0) < \sigma(x^*) = \tau(x^*)$. Then for any $y^* \in X^*$ such that $\nabla\sigma(x^*)(y^*) \le 0$ we have $\nabla\tau(x^*)(-y^*) \ge 0$.*

Proof. First suppose $\nabla\sigma(x^*)(y^*) < 0$ and $\nabla\tau(x^*)(-y^*) < 0$. Then (using the fact that σ, τ are convex functions) we can choose $s, t > 0$ so that

$$\sigma(x^* + sy^*) = \tau(x^* - ty^*) < \sigma(x^*) = \tau(x^*).$$

It is then clear that the choices $u^* = x^* + sy^*,\ v^* = x^* - ty^*$ violate (7.1) because

$$\begin{aligned} \sigma(x^* + ay^*) &\ge \sigma(x^*) \qquad & -t \le a \le 0,\\ \tau(x^* + ay^*) &\ge \tau(x^*) \qquad & 0 \le a \le s. \end{aligned}$$

Next suppose $\nabla\sigma(x^*)(y^*) = 0$ and $\nabla\tau(x^*)(-y^*) < 0$. The condition $\sigma(x^*) > \sigma(0)$ implies that $\nabla\sigma(x^*)(-x^*) < 0$. Hence for small enough ϵ we have

$$\nabla\sigma(x^*)(y^* - \epsilon x^*) \le \epsilon\nabla\sigma(x^*)(-x^*) < 0,$$
$$\nabla\tau(x^*)(-y^* + \epsilon x^*) \le \nabla\tau(x^*)(-y^*) + \epsilon\nabla\tau(x^*)(x^*) < 0.$$

This gives a contradiction, by the first part of the proof. $\square$

Lemma 7.6. *Let X be a Banach space with the linear $\mathcal{C}$-AIEP. Suppose σ and τ are two weak*-null types on X^*. Suppose E is a closed subspace of X^* with* $\dim E > 1$ *and that*

$$\sigma(e^*) > \|e^*\| \qquad e^* \in E^*.$$

Suppose further that $u^*, v^* \in E$ are such that:

$$\sigma(u^*) = \sigma(v^*) = \tau(u^*) = \theta > \sigma(0).$$

Then $\tau(v^*) = \theta$.

Proof. It will suffice to consider the case when E is finite-dimensional (indeed even of dimension two). Let us assume that

$$\tau(x^*) = \lim_{n\in\mathcal{V}} \|x^* + v_n^*\| \qquad x^* \in X^*$$

where $(v_n^*)_{n=1}^\infty$ is a weakly null sequences and $\mathcal{V}$ is a nonprincipal ultrafilters. For $\lambda \in \mathbb{R}$ we define

$$\tau_\lambda(x^*) = \lim_{n\in\mathcal{V}} \|x^* + \lambda v_n^*\| \qquad x^* \in X^*.$$

We define a norm on $E \oplus \mathbb{R}$ by

$$F(e^*, \lambda) = \tau_\lambda(e^*) \qquad e^* \in E^*.$$

We next construct a Lipschitz map $f : [0,1] \to E$ with $f(0) = u^*,\ f(1) = v^*$ and $\sigma(f(t)) = \theta$ for $0 \le t \le 1$. In order to do this we first find any Lipschitz map $f_0 : [0,1] \to E$ with $f_0(0) = u^*,\ f_0(1) = v^*$ and such that $f_0(t) \ne 0$ for all $0 \le t \le 1$. This is possible since $\dim E \ge 2$. Let us denote the Lipschitz constant of f_0 by K_0. We then define $\varphi(t)$ to be the unique choice of $\lambda > 0$ so that so that $\sigma(\lambda f_0(t)) = \theta$. The uniqueness of the choice follows from the fact that $\theta > \sigma(0)$. We will let $f(t) = \varphi(t) f_0(t)$. To show that f is Lipschitz we need only show that φ is Lipschitz. Note first that f is bounded. It follows, since f_0 has a lower estimate $\|f_0(t)\| \ge c > 0$ for $0 \le t \le 1$ that there exists constant M so that

$$|\varphi(t)| \le M,\ 0 \le t \le 1.$$

Of course $\varphi(0) = \varphi(1) = 1$.

Assume $0 \le s, t < 1$ and that $\varphi(t) > \varphi(s)$. Then

$$\theta < \sigma(\varphi(t) f_0(s)) \le \sigma(\varphi(t) f_0(t)) + K_0 M|t-s| = \theta + K_0 M|t-s|.$$

Now, by convexity of the map $\lambda \to \sigma(\lambda f_0(s))$ we have

$$\sigma(\varphi(t) f_0(s)) - \sigma(\varphi(s) f_0(s)) \ge \frac{\varphi(t) - \varphi(s)}{\varphi(s)}(\theta - \sigma(0)).$$

Hence

$$\frac{\varphi(t) - \varphi(s)}{\varphi(s)} \le \frac{K_0 M|t-s|}{\theta - \sigma(0)}$$

or

$$\varphi(t) - \varphi(s) \le \frac{K_0 M^2|t-s|}{\theta - \sigma(0)}$$

and so φ is Lipschitz. It follows that f is Lipschitz with some constant K.

By assumption $\theta = \sigma(f(t)) > \|f(t)\| = F(f(t), 0)$ for $0 \le t \le 1$. It follows that there is a unique choice of $g(t) > 0$ so that

$$F(f(t), g(t)) = \theta, \qquad 0 \le t \le 1.$$

Note that $g(0) = 1$ by our assumptions.

We next show that g is also Lipschitz. The argument is similar to that for φ above; of course g is bounded by some constant M'. Let

$$\theta_0 = \sup\{\|f(t)\| : 0 \le t \le 1\} < \theta.$$

Suppose $g(t) > g(s)$. Then

$$\theta < F(f(s), g(t)) \le F(f(t), g(t)) + K|t-s| = \theta + K|t-s|.$$

However

$$F(f(s), g(t)) - F(f(s), g(s)) \ge \frac{g(t)-g(s)}{g(s)}(\theta - \theta_0)$$

so that

$$g(t) - g(s) \le \frac{KM|t-s|}{\theta - \theta_0}.$$

It follows that the maps $t \to f(t)$ and $t \to g(t)$ are differentiable almost everywhere. Let $\delta(t)$ be chosen so that $\delta(t) = \pm 1$ and $\delta(t)g'(t) = |g'(t)|$ almost everywhere.

Since $F(f(t), g(t)) = \theta > F(f(t), 0) = \|f(t)\|$ we have that

$$\nabla F(f(t), g(t))(0, -1) = -h(t) \qquad 0 \le t \le 1$$

where $h(t) > 0$ for all $0 \le t \le 1$. Thus

$$\nabla F(f(t), g(t))(0, -|g'(t)|) = -h(t)|g'(t)|, \qquad 0 \le t \le 1, \text{ a.e.}$$

Next since $\sigma(f(t))$ is constant we have by Lemma 7.4,

$$\nabla\sigma(f(t))(f'(t)) = \nabla\sigma(f(t))(-f'(t)) = 0, \qquad 0 < t < 1, \text{ a.e.}$$

Thus by Lemma 7.5

$$\nabla\tau_{g(t)}(f'(t)), \nabla\tau_{g(t)}(-f'(t)) \ge 0, \qquad 0 < t < 1, \text{ a.e.}$$

In particular

$$\nabla F(f(t), g(t))(\delta(t)f'(t), 0) \ge 0, \qquad 0 < t < 1, \text{ a.e.}$$

Hence

$$\begin{aligned} 0 &\le \nabla F(f(t), g(t))(\delta(t)f'(t), 0) \\ &\le \nabla F(f(t), g(t))(\delta(t)f'(t), \delta(t)g'(t)) + \nabla F(f(t), g(t))(0, -\delta(t)g'(t)) \\ &= -h(t)|g'(t)|, \qquad 0 < t < 1, \text{ a.e.} \end{aligned}$$

Here we use again Lemma 7.4 to deduce that

$$\nabla F(f(t), g(t))(\delta(t)f'(t), \delta(t)g'(t)) = 0, \qquad 0 < t < 1.$$

Since $h(t) > 0$ everywhere we deduce that $g'(t) = 0$ almost everywhere and so g is constant, i.e., $g(t) = 1$ for $0 \le t \le 1$. This implies that

$$\theta = F(f(1), g(1)) = \tau(v^*). \qquad \square$$

Lemma 7.7. *Let X be a separable Banach space with the linear $\mathcal{C}$-AIEP. Suppose E is a linear subspace of X^* with $\dim E > 1$. Suppose there exists a weak*-null type ρ on X^* such that*

$$\rho(x^*) > \|x^*\| \qquad x^* \in E.$$

Then, if σ and τ are two weak*-null types on X^* and $u^*, v^* \in E$,

$$\min(\sigma(v^*), \tau(u^*)) \le \max(\sigma(u^*), \tau(v^*)). \tag{7.5}$$

In particular every weak*-null type is symmetric on E i.e., $\sigma(x^*) = \sigma(-x^*)$ for $x^* \in E$.

Proof. Let $\rho(x^*) = \lim_{n\in\mathcal{V}} \|x^* + v_n^*\|$ for some weakly null sequence $(v_n^*)_{n=1}^\infty$ and some nonprincipal ultrafilter $\mathcal{V}$. For any type σ we can define

$$\sigma_\epsilon(x^*) = \lim_{n\in\mathcal{U}} \sigma(x^* + \epsilon v_n^*).$$

Then σ_ϵ is a weak*-null type and $|\sigma_\epsilon(x^*) - \sigma(x^*)| \le \epsilon\|x^*\|$. Furthermore if $e^* \in E$ we have

$$\sigma_\epsilon(e^*) \ge \epsilon\rho(\epsilon^{-1}e^*) > \|e^*\|.$$

It therefore suffices to show (7.5) holds when both σ and τ verify the additional conditions

$$\sigma(e^*), \tau(e^*) > \|e^*\|, \qquad e^* \in E.$$

Let us assume with these additional conditions that (7.5) fails to hold i.e.,

$$\min(\sigma(v^*), \tau(u^*)) > \max(\sigma(u^*), \tau(v^*)).$$

We assume without loss of generality that $\sigma(v^*) = \theta \le \tau(u^*)$. Then $\theta > \sigma(u^*) \ge \sigma(0)$ and $\theta > \tau(v^*) \ge \|v\|$. Hence there exist $\mu > 1$ so that $\sigma(\mu u^*) = \theta$ and $\lambda > 1$ so that $\tau_\lambda(v^*) = \theta$. By Lemma 7.6 applied to σ and τ_λ we deduce that $\tau_\lambda(\mu u^*) = \theta \le \tau(u^*)$. If $\alpha = \min(\lambda, \mu) > 1$ this means that $\tau_\alpha(\alpha u^*) = \alpha\tau(u^*) \le \tau(u^*)$ and we obtain a contradiction. □

The following theorem is a special case of the preceding lemma. We recall that a type ρ on X^* is called *strict* if $\rho(x^*) > \|x^*\|$ for all $x^* \in X^*$.

Theorem 7.8. *Suppose X is a separable Banach space such that X^* admits a strict weak*-null type ρ. Then X has the linear $\mathcal{C}$-AIEP if and only if for every pair of weak*-null types σ, τ on X^* we have*

$$\min(\sigma(v^*), \tau(u^*)) \le \max(\sigma(u^*), \tau(v^*)) \qquad u^*, v^* \in X^*.$$

In particular every weak*-null type on X^* is symmetric and hence X^* is separable.

We have not been able to eliminate the condition that X^* admits a strict weak*-null type. In practice, this is a quite a weak condition which is implied by asymptotic uniform smoothness. We recall [13] that if X is a Banach space we define its *modulus of asymptotic smoothness* by

$$\overline{\rho}_X(t) = \sup_{\|x\|=1} \inf_{\dim X/Y<\infty} \sup_{\substack{y\in Y\\ \|y\|=t}} (\|x + ty\| - 1).$$

X is called *asymptotically uniformly smooth* if $\lim_{t\to 0} \overline{\rho}_X(t)/t = 0$.

Proposition 7.9. *If X is a separable asymptotically uniformly smooth Banach space then X^* admits a strict weak*-null type.*

Proof. In this case X^* is separable (see [13], Proposition 2.4, where even $\overline{\rho}_X(t) < t$ for some t is shown to be sufficient for this conclusion) and it follows from Proposition 2.8 of [10] that for any normalized weak*-null sequence $(x_n^*)_{n=1}^\infty$ in X^* and any x^* with $\|x^*\| = 1$ we have

$$\liminf_{n\to\infty} \|x^* + tx_n^*\| > 1.$$

Thus any nontrivial weak*-null type on X^* is strict. □

Let us give some examples of spaces which satisfy the equivalent conditions of Theorem 7.1. It is well-known that it holds for $X = \ell_p$ where $1 < p < \infty$ (Zippin [41]) or $X = c_0$ (Lindenstrauss–Pełczynński [28]).

Theorem 7.10. *Let X be a separable Banach space with properties (L^*) or (M^*). Then X has the linear $\mathcal{C}$-AIEP.*

Proof. Suppose σ, τ are weak*-null types on X^* and suppose $u^*, v^* \in X^*$.

If X has property (M^*), let us assume without loss of generality that $\|u^*\| \leq \|v^*\|$. Then (Proposition 3.3) $\tau(u^*) \leq \tau(v^*)$ so that (7.2) and hence (7.1) hold.

If X has property (L^*) let us assume that $\sigma(0) \leq \tau(0)$. Then $\sigma(v^*) \leq \tau(v^*)$ and again (7.2) holds. □

Corollary 7.11. *Let X be an Orlicz–Musielak sequence space $h_{(\phi_k)}$ not containing ℓ_1. Then if E is a closed subspace of X and $T : E \to \mathcal{C}(K)$ is a bounded operator, there is a bounded linear extension $\widetilde{T} : X \to \mathcal{C}(K)$ i.e., X has the $\mathcal{C}$-LEP. In particular if $X = \ell_{p_1} \oplus \cdots \oplus \ell_{p_n}$ or $X = \ell_{p_1} \oplus \cdots \oplus \ell_{p_n} \oplus c_0$ where $p_1, \ldots, p_n > 1$ then X has the linear $\mathcal{C}$-EP.*

Proof. We need only remark that X has an equivalent norm with respect to which X has property (M) and hence also (M^*). □

The same argument applies to Orlicz–Fenchel spaces as discussed in §3 and hence to such spaces as the twisted sums Z_p for $1 < p < \infty$.

Corollary 7.12. *The spaces Z_p for $1 < p < \infty$ have the linear $\mathcal{C}$-EP.*

Proposition 7.13. *Let X be a separable Banach space with linear $\mathcal{C}$-AIEP and suppose E is a finite-dimensional normed space.*

(i) *If $1 < p \leq \infty$ and X^* admits a strict weak*-null type, then $X \oplus_p E$ has the linear $\mathcal{C}$-AIEP if and only if X has property (L^*).*
(ii) *$X \oplus_1 E$ always has the linear $\mathcal{C}$-AIEP.*

Proof. (i) In this case $(X \oplus_p E)^* = X^* \oplus_q E^*$ (where q is the conjugate index) also admits a strict weak*-null type given by

$$\widetilde{\rho}(x^*, e^*) = (\rho(x^*)^q + \|e^*\|^q)^{\frac{1}{q}} \qquad x^* \in X^*,\ e^* \in E^*.$$

Theorem 7.8 thus applies. Suppose $X \oplus_p E$ has the linear $\mathcal{C}$-AIEP. Let σ, τ be two weak*-null types on X^* and assume $\sigma(0) < \tau(0)$. We will show that $\sigma \leq \tau$ and this suffices to show that X has property (L^*). Suppose there exists $u^* \in X^*$ such that $\tau(u^*) < \sigma(u^*)$. Pick $a, b > 0$ so that

$$a^q + \sigma(u^*)^q = b^q + \tau(0)^q.$$

Then there is a weak*-null type $\widetilde{\sigma}$ on $X^* \oplus_q E^*$ so that

$$\widetilde{\sigma}(x^*, e^*) = (\sigma(x^*)^q + \|e^*\|^q)^{\frac{1}{q}} \qquad x^* \in X^*,\ e^* \in E^*$$

and a weak*-null type $\widetilde{\tau}$ given by

$$\widetilde{\tau}(x^*, e^*) = (\tau(x^*)^q + \|e^*\|^q)^{\frac{1}{q}} \qquad x^* \in X^*,\ e^* \in E^*.$$

Pick $f^* \in E^*$ with $\|f^*\| = 1$ and then

$$\widetilde{\sigma}(u^*, af^*) = \widetilde{\tau}(0, bf^*)$$

but

$$\widetilde{\sigma}(0,bf^*)<\widetilde{\tau}(0,bf^*),\ \widetilde{\tau}(u^*,af^*)<\widetilde{\sigma}(u^*,af^*).$$

This contradicts Theorem 7.8. Hence we have $\sigma\le\tau$, i.e., X has property (L^*).

For the converse it is easy to see that if X has (L^*) then so does $X\oplus_p E$ since every weak*-null type on $X^*\oplus_q E^*$ is of the form $\widetilde{\sigma}$ for some σ. We can then use Theorem 7.10.

(ii) In this case $q=\infty$. Every weak*-null type on $X^*\oplus_\infty E^*$ is of the form

$$\widetilde{\sigma}(x^*,e^*)=\max(\sigma(x^*),e^*)\qquad x^*\in X^*,\ e^*\in E^*.$$

Now suppose σ,τ are two weak*-null types on X^*. If $(u^*,e^*),(v^*,f^*)\in X^*\oplus_\infty E^*$ we may pick $0\le\lambda\le 1$ so that

$$\sigma(\lambda u^*+(1-\lambda)v^*),\tau(\lambda u^*+(1-\lambda)v^*)\le\max(\sigma(u^*),\tau(v^*))$$

and then

$$\widetilde{\sigma}(\lambda u^*+(1-\lambda)v^*,\lambda e^*+(1-\lambda)f^*)\le\max(\widetilde{\sigma}(u^*,e^*),\widetilde{\tau}(v^*,f^*))$$

and

$$\widetilde{\tau}(\lambda u^*+(1-\lambda)v^*,\lambda e^*+(1-\lambda)f^*)\le\max(\widetilde{\sigma}(u^*,e^*),\widetilde{\tau}(v^*,f^*)),$$

so that $X\oplus_\infty E$ has the linear $\mathcal{C}$-AIEP by Theorem 7.1. □

Example. Now if X is a reflexive space with property (M) but not (m_p) for any p, then $X\oplus_1\mathbb{R}$ has the linear $\mathcal{C}$-AIEP but $(X\oplus_1\mathbb{R})^*$ does not have the linear $\mathcal{C}$-AIEP. Note that X must admit a strict weakly null type of the form $(a^p+\|x\|^p)^{\frac{1}{p}}$ where $1<p<\infty$ and similarly X^* must admit a strict weak*-null type. If $X^*\oplus_\infty\mathbb{R}$ has the linear $\mathcal{C}$-AIEP then X^* would have property (L^*) i.e., X would have property (L) and Proposition 3.6 would apply to give a contradiction.

Theorem 7.14. *Suppose X and Y are two infinite-dimensional Banach spaces. Then $X\oplus_1 Y$ cannot have the linear $\mathcal{C}$-AIEP.*

Proof. Observe that $(X\oplus_1 Y)^*=X^*\oplus_\infty Y^*$. Assume $X\oplus_1 Y$ has the linear $\mathcal{C}$-AIEP. If σ and τ are weak*-null types on X^* and Y^* respectively then

$$\sigma\oplus_\infty\tau(x^*,y^*)=\max(\sigma(x^*),\tau(y^*))$$

defines a typical weak*-null type on $(X\oplus_1 Y)^*$. Then for any pair of weak*-null types σ_1,σ_2 on X^* and any x_1^*,x_2^* with $\sigma_1(x_1^*),\sigma_2(x_2^*)\le a$ let $I(\sigma_1,\sigma_2,x_1^*,x_2^*)$ be the nonempty closed subinterval of $[0,1]$ of all λ such that

$$\sigma_1((1-\lambda)x_1^*+\lambda x_2^*)\le 1,\ \sigma_2((1-\lambda)x_1^*+\lambda x_2^*)\le 1.$$

This is nonempty since X has the linear $\mathcal{C}$-AIEP by Theorem 7.1. Similarly for any pair of weak*-null types τ_1,τ_2 on Y^* and any y_1^*,y_2^* with $\tau_1(y_1^*),\tau_2(y_2^*)\le a$ let $J(\tau_1,\tau_2,y_1^*,y_2^*)$ be the nonempty closed subinterval of $[0,1]$ of all λ such that

$$\tau_1((1-\lambda)y_1^*+\lambda y_2^*)\le 1,\ \sigma_2((1-\lambda)y_1^*+\lambda y_2^*)\le 1.$$

By applying Theorem 7.1 to the pair of types $(\sigma_1\oplus_\infty\tau_1,\sigma_2\oplus_\infty\tau_2)$ and the pair of points $(x_1^*y_1^*),(x_2^*,y_2^*)$ we must have $I(\sigma_1,\sigma_2,x_1^*,x_2^*)\cap J(\tau_1,\tau_2,y_1^*,y_2^*)\neq\emptyset$.

We will argue that either $\frac{1}{2}\in I(\sigma_1,\sigma_2,x_1^*,x_2^*)$ for every permissible choice of $\sigma_1,\sigma_2,x_1^*,x_2^*$ or $\frac{1}{2}\in J(\tau_1,\tau_2,y_1^*,y_2^*)$ for every permissible choice of $J(\tau_1,\tau_2,y_1^*,y_2^*)$. Indeed suppose the latter condition fails so there exists $J(\tau_1,\tau_2,y_1^*,y_2^*)$ not containing $\frac{1}{2}$. Thus $J(\tau_1,\tau_2,y_1^*,y_2^*)$ fails to meet $J(\tau_2,\tau_1,y_2^*,y_1^*)$, so every $I(\sigma_1,\sigma_2,x_1^*,x_2^*)$

must contain the points μ and $1-\mu$ where μ is the endpoint of $J(\tau_1,\tau_2,y_1^*,y_2^*)$ closest to $\frac12$. Hence $\frac12\in I(\sigma_1,\sigma_2,x_1^*,x_2^*)$.

We therefore assume $\frac12\in I(\sigma_1,\sigma_2,x_1^*,x_2^*)$ for every permissible choice. Let us assume σ is any weak*-null type with $\sigma(0)>0$. Let $\sigma'(x^*)=(\sigma(0))^{-1}\sigma(\sigma(0)x^*)$ so that $\sigma'(0)=1$. If $\|x^*\|\le 1$ let $x_1^*=\alpha x^*$ where $\alpha\ge 0$ is the largest t so that $\sigma'(tx^*)\le 1$. Then if $\alpha<1$ we deduce using $\sigma_1=\sigma'$, $x_2^*=x^*$ and $\sigma_2(x^*)=\|x^*\|$ that $\sigma'(\frac12(1+\alpha)x_2^*)\le 1$ which is a contradiction. Hence $\sigma'(x^*)\le 1$ if $\|x^*\|\le 1$ or, equivalently, $\sigma(x^*)\le\sigma(0)$ if $\|x^*\|\le\sigma(0)$. This quickly reduces to

$$\sigma(x^*)=\max(\sigma(0),\|x^*\|)\qquad x^*\in X^*.$$

This implies that X has property (m_∞^*) which is impossible for an infinite-dimensional space. Although this is not explicit in [23] or [17] it can be seen very easily. For example, X cannot contain a copy of ℓ_1 by the linear $\mathcal{C}$-AIEP yet the arguments of [17] imply that X^* contains a copy of c_0. □

Theorem 7.15. *Suppose $1<p<\infty$ and X and Y are infinite-dimensional separable Banach spaces such that $X\oplus_p Y$ has the linear $\mathcal{C}$-AIEP Then $X\oplus_p Y$ has property (m_p).*

Proof. Note that $(X\oplus_p Y)^*=X^*\oplus_q Y^*$ where $q<\infty$ is the conjugate index.

It follows from Proposition 7.13 that both X and Y have property (L^*). We will also deduce that X and Y have property (M^*). Indeed regarding X^* as a subspace of $X^*\oplus_q Y^*$ it is clear that there is a weak*-null type (for $X^*\oplus_q Y^*$) on X^* of the form

$$\sigma_a(x^*)=(\|x^*\|^q+a^q)^{\frac1q}\qquad x^*\in X^*$$

for an arbitrary choice of $a>0$. Note that $\sigma_a(x^*)>\|x^*\|$ for $x^*\in X^*$. Hence if τ is any weak*-null type on X^* we can use Lemma 7.6 to deduce that as if $x_1^*,x_2^*\in X^*$ and $\|x_1^*\|=\|x_2^*\|$ then $\tau(x_1^*)=\tau(x_2^*)$. Indeed if $\tau(x_1^*)>\|x_1^*\|$ then pick $a>0$ so that $\tau(x_1^*)=\sigma_a(x_1^*)=\sigma_a(x_2^*)$; by Lemma 7.6 we have $\tau(x_2^*)=\tau(x_1^*)$. If $\tau(x_1^*)=\|x_1^*\|$ then $\tau'(x^*)=(\tau(x^*)^q+\epsilon^q)^{1/q}$ is a weak*-null type restricted to X^* for every $\epsilon>0$ and so we can deduce the same conclusion. Thus X has property (M^*) and by Proposition 3.6 we conclude that X has property (m_r^*) for some $1\le r<\infty$. Similarly Y has property (m_s^*) for some $1\le s<\infty$.

Let us fix weak*-null types σ and τ defined by normalized weak*-null sequences (x_n^*) in X^* and (y_n^*) in Y^*. Fix $u^*\in X^*$ and $v^*\in Y^*$ of norm one.

For $a>0$ define the weak*-null types,

$$\begin{aligned}\sigma_a(z^*)&=a\sigma(a^{-1}z^*)&\qquad& z^*\in X^*\oplus_q Y^*,\\ \tau_a(z^*)&=a\sigma(a^{-1}z^*)&& z^*\in X^*\oplus_q Y^*.\end{aligned}$$

For $\epsilon>0$ we can choose $\eta=\eta(\epsilon)$ so that

$$\sigma_\epsilon(v^*)=\tau_\eta(v^*).$$

Indeed we require that

$$(1+\epsilon^q)^{\frac1q}=(1+\eta^s)^{\frac1s}. \tag{7.6}$$

Now pick $\nu=\nu(\epsilon)>-1$ so that

$$\sigma_\epsilon((1+\nu)u^*)=\sigma_\epsilon(v^*)$$

i.e., we require

$$((1+\nu)^r+\epsilon^r)^{\frac{1}{r}}=(1+\epsilon^q)^{\frac{1}{q}}. \tag{7.7}$$

Then we deduce from Lemma 7.6 that $\tau_\eta((1+\nu)u^*)=\sigma_\epsilon(v^*)$, i.e.,

$$(1+\epsilon^q)^{\frac{1}{q}}=((1+\nu)^q+\eta^q)^{\frac{1}{q}}. \tag{7.8}$$

Treating ν,η as functions of ϵ we see that from (7.6) that

$$\lim_{\epsilon\to 0}\frac{\eta^s}{\epsilon^q}=\frac{s}{q}.$$

We also have from (7.7) and (7.8),

$$\lim_{\epsilon\to 0}\frac{((1+\nu)^q+\eta^q)^{\frac{1}{q}}-1-\nu}{((1+\nu)^r+\epsilon^r)^{\frac{1}{r}}-1-\nu}=1$$

which reduces to

$$\lim_{\epsilon\to 0}\frac{\eta^q}{\epsilon^r}=\frac{q}{r}.$$

Thus $q/s=r/q=\theta$, say, and then

$$\theta^\theta=\theta$$

which implies $\theta=1$ and we are done. $\square$

Remark. Johnson and Zippin ([16] Corollary 4.2) proved that $L_p(0,1)$ for $1<p\neq 2<\infty$ fails to have the linear $\mathcal{C}$-IEP and hence by Proposition 4.1 of [16] it also fails the linear $\mathcal{C}$-AIEP. The same result can be obtained from Theorem 7.15 since L_p fails to have (m_p) unless $p=2$.

8. Isometric extension properties

We now turn to the question of when we can obtain the linear $\mathcal{C}$-IEP. Johnson and Zippin [16] show that if X is uniformly smooth then the linear $\mathcal{C}$-AIEP implies the linear $\mathcal{C}$-IEP. They also ask in Problem 4.3 if this conclusion holds when X is assumed to be smooth and reflexive. In this section, we answer this question negatively. First, however, we give an extension of the Johnson–Zippin theorem.

We recall that a Banach space X is called *uniformly rotund (or convex) in every direction* (URED) if whenever $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ are sequences in the unit ball such that $\lim_{n\to\infty}\|x_n+y_n\|=2$ and $\lim_{n\to\infty}(x_n-y_n)=z$ exists then $z=0$. See [7] p. 61. Equivalently, given $z\in X$ and $\epsilon>0$ there exists $1<\lambda=\lambda(\epsilon,z)<2$ so that

$$\|x\|,\|y\|\le 1,\ \|x+y\|>\lambda,\ x-y=\mu z\implies |\mu|<\epsilon.$$

The following theorem was proved by Johnson and Zippin under the stronger hypothesis that X is uniformly smooth, i.e., X^* is uniformly convex.

Theorem 8.1. *Let X be a Banach space such that X^* is URED. Then if X has the linear $\mathcal{C}$-AIEP, then X also has the linear $\mathcal{C}$-IEP.*

Proof. Since X is necessarily smooth we can reduce the problem using Proposition 4.2 and Corollary 4.1 of [16]. It will suffice to consider the case when $E=\ker z^*$ is a subspace of codimension one, with $\|z^*\|=1$, and $(x_d^*)_{d\in D},(y_d^*)_{d\in D}$ are weak*-convergent nets to x^*,y^* respectively satisfying $\|x_d^*\|=\|y_d^*\|=\|x_d^*|_E\|=\|y_d^*|_E\|=1$ for $d\in D$. and $x^*-y^*=\mu z^*\in[z^*]$. We then must show that $x^*=y^*$ i.e., $\mu=0$.

Suppose $\epsilon > 0$ and choose $\lambda = \lambda(\epsilon/4, z^*)$ as in the definition. Pick $1 < \alpha < (2/\lambda)$. Then the canonical injection $j : E \to \mathcal{C}(B_{E^*})$ has an extension $T : X \to \mathcal{C}(B_{E^*})$ with $\|T\| < \alpha$. Let $\Phi : B_{X^*} \to B_{X^*}$ be defined by $\Phi(u^*) = T^*(\delta_{u^*|_E})$ (where δ_a is the point-mass at a). Then $\Phi : B_{X^*} \to \alpha B_{X^*}$, Φ is weak*-continuous, $\Phi(u^*) = \Phi(v^*)$ if $u^* - v^* \in [z^*]$ and $\Phi(u^*) - u^* \in [z^*]$ for $u^* \in B_{X^*}$.

Observe that $\|\alpha^{-1}(x_d^* + \Phi(x_d^*))\| \geq 2\alpha^{-1} > \lambda$ and so $\|x_d^* - \Phi(x_d^*)\| < \alpha(\epsilon/4) < \epsilon/2$ for $d \in D$. Hence $\|x^* - \Phi(x^*)\| \leq \epsilon/2$ and similarly $\|y^* - \Phi(y^*)\| \leq \epsilon/2$. Hence $\|x^* - y^*\| \leq \epsilon$.

Since $\epsilon > 0$ is arbitrary, we obtain $x^* = y^*$ and the theorem is proved. □

It would be natural to expect that one could also obtain an isometric Lipschitz result for mappings defined on convex sets under the same hypotheses; however we have only been able to achieve this under the assumption that Theorem 7.8 also holds.

Theorem 8.2. *Suppose X is a separable Banach space such that X^* is URED and admits a strict weak*-null type. If X has the linear $\mathcal{C}$-AIEP, then (C, X) has the Lipschitz IEP for every closed convex subset C of X.*

Remark. These hyptheses hold if X is uniformly smooth.

Proof. Of course Theorem 7.8 applies and in particular every weak*-null type on X^* is symmetric.

We next argue that if σ is any type on X^* then

$$\sigma(\tfrac{1}{2}(x^* + y^*)) < \max(\sigma(x^*), \sigma(y^*)), \qquad x^* \neq y^*. \tag{8.1}$$

Indeed suppose $\sigma(x^*) = \lim_{n\in\mathcal{U}} \|x^* + u_n^*\|$ where u_n^* is a sequence in X^*. Assume $\sigma(\frac{1}{2}(x^* + y^*)) = \sigma(x^*) = \sigma(y^*)$ (which we may assume nonzero). Let $\alpha_n = \max(\|x^* + u_n^*\|, \|y^* + u_n^*\|)$. Then $\lim_{n\in\mathcal{U}} \alpha_n^{-1}\|x^* + y^* + 2u_n^*\| = 2$ and so $\|x^* - y^*\| = 0$. Thus (8.1) holds.

By Theorem 7.1 (C, X) has the Lipschitz IEP for bounded closed convex sets C. Suppose (C, X) fails Lipschitz IEP for some (unbounded) closed convex subset C of X. Then for some $\epsilon > 0, a \in X$ it follows from Theorem 4.3 above or Theorem 4.2 of [19] that we can therefore select two sequences $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ in C such that

$$\|x_n\| + \|y_n\| + \epsilon < \min_{v\in K_n} (\|x_n - v\| + \|y_n - v\|) \qquad n = 2, 3, \ldots$$

where $K_n = \text{co}\,\{x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}\}$. This means that the set

$$\{(x_n - v, y_n - v);\ v \in K_n\}$$

in $X \oplus_1 X$ does not meet the closed ball of radius $\|x_n\| + \|y_n\| + \epsilon$. By the Hahn–Banach Theorem we can therefore find $x_n^*, y_n^* \in X^*$ with $\|x_n^*\|, \|y_n^*\| \leq 1$ and

$$x_n^*(x_n - v) + y_n^*(y_n - v) \geq \|x_n\| + \|y_n\| + \epsilon, \qquad v \in K_n,\ n \geq 2.$$

This implies that, in particular,

$$x_n^*(x_n - x_k) + y_n^*(y_n - x_k) \geq \|x_n\| + \|y_n\| + \epsilon, \qquad 1 \leq k \leq n-1, \tag{8.2}$$

$$x_n^*(x_n - y_k) + y_n^*(y_n - y_k) \geq \|x_n\| + \|y_n\| + \epsilon, \qquad 1 \leq k \leq n-1. \tag{8.3}$$

By passing to a suitable subsequence we can assume that (8.2) and (8.3) hold and, in addition, $(x_n^*)_{n=1}^\infty$ and $(y_n^*)_{n=1}^\infty$ are weak*-convergent to u^* and v^* respectively.

It follows from (8.2) and (8.3) that

$$(x_n^*+y_n^*)(x_k) \le (x_n^*(x_n)-\|x_n\|)+(y_n^*(y_n)-\|y_n\|)-\epsilon, \tag{8.4}$$
$$(x_n^*+y_n^*)(y_k) \le (x_n^*(x_n)-\|x_n\|)+(y_n^*(y_n)-\|y_n\|)-\epsilon, \qquad n>k. \tag{8.5}$$

From these we deduce that

$$(u^*+v^*)(x_k) \le -\epsilon,\ (u^*+v^*)(y_k) \le -\epsilon. \tag{8.6}$$

In particular $u^*+v^* \neq 0$. We also may deduce (keeping k fixed and letting n vary that the nonnegative sequences $(\|x_n\|-x_n^*(x_n))_{n=1}^\infty$ and $(\|y_n\|-y_n^*(y_n))_{n=1}^\infty$ are each bounded above (say by some constant M)

Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$. Define σ, τ to be the weak*-null types on X^* given by

$$\sigma(x^*) = \lim_{n\in\mathcal{U}} \|x^*+x_n^*-u^*\| \qquad x^* \in X^*$$
$$\tau(x^*) = \lim_{n\in\mathcal{U}} \|x^*+y_n^*-v^*\| \qquad x^* \in X^*.$$

We recall that these types are necessarily symmetric. Observe that $\sigma(u^*), \tau(v^*) \le 1$.

Assume $\tau(u^*) \le 1$. Then $\tau(-u^*) \le 1$ and so by (8.1), $\tau(\frac{1}{2}(v^*-u^*)) < \theta < 1$, i.e.,

$$\lim_{n\in\mathcal{U}} \|y_n^* - \frac{1}{2}(u^*+v^*)\| < \theta.$$

Thus there exists a set $\mathbb{M} \in \mathcal{U}$ so that

$$y_n^*(y_n) - \frac{1}{2}(u^*+v^*)(y_n) \le \theta\|y_n\| \qquad n \in \mathbb{M}$$

which implies

$$(u^*+v^*)(y_n) > 2(1-\theta)\|y_n\| - 2M \qquad n \in \mathbb{M}.$$

By (8.6) this gives

$$2(1-\theta)\|y_n\| \le M - \epsilon$$

and so the sequence $(y_n)_{n\in\mathbb{M}}$ is bounded. Similarly $\sigma(v^*) \le 1$ would imply that $(x_n)_{n\in\mathbb{M}'}$ is bounded for some $\mathbb{M}' \in \mathcal{U}$. .

If both $(x_n)_{n\in\mathbb{M}}$ and $(y_n)_{n\in\mathbb{M}'}$, with $\mathbb{M}, \mathbb{M}' \in \mathcal{U}$, are bounded then we reach a contradiction, because the closed convex hull of the set $\{x_n, y_n : n \in \mathbb{M}\cap\mathbb{M}'\}$ then fails the Lipschitz IEP, contradicting Theorem 7.1. We therefore consider, without losing generality, the case when

$$\lim_{n\in\mathcal{U}} \|y_n\| = \infty. \tag{8.7}$$

Thus $\tau(u^*) > 1$. However by Theorem 7.8 we have

$$\min(\sigma(v^*), \tau(u^*)) \le 1$$

so that $\sigma(v^*) \le 1$ and $\lim_{n\in\mathcal{U}} \|x_n\| < \infty$.

Now, by (8.2),

$$\begin{aligned}\epsilon &\le \lim_{k\in\mathcal{U}}\lim_{n\in\mathcal{U}}\left((x_n^*(x_n)-\|x_n\|)+(y_n^*(y_n)-\|y_n\|)-(x_n^*+y_n^*)(x_k)\right)\\ &=\lim_{n\in\mathcal{U}}\left((x_n^*(x_n)-\|x_n\|)+(y_n^*(y_n)-\|y_n\|)\right)-\lim_{k\in\mathcal{U}}(u^*+v^*)(x_k)\\ &=\lim_{n\in\mathcal{U}}\left((x_n^*(x_n)-(u^*+v^*)(x_n))-\|x_n\|+(y_n^*(y_n)-\|y_n\|)\right)\\ &\le(\sigma(v^*)-1)\lim_{n\in\mathcal{U}}\|x_n\|+\lim_{n\in\mathcal{U}}(y_n^*(y_n)-\|y_n\|)\\ &\le 0,\end{aligned}$$

which gives a contradiction. □

The remainder of this section will be devoted to the construction of an example of a smooth Banach space X isomorphic to ℓ_2, having the linear $\mathcal{C}$-AIEP but failing the linear $\mathcal{C}$-IEP. This answers problem 4.3 of [16].

Let N be the absolute norm on $\mathbb{R}^2$ given by

$$N(\xi_1,\xi_2)=\max((|\xi_1|^2+\tfrac14|\xi_2|^2)^{1/2},|\xi_2|)\qquad(\xi_1,\xi_2)\in\mathbb{R}^2.$$

Let c_{00} denote the space of finitely nonzero sequences with the canonical basis $(e_n)_{n=1}^\infty$. Let $E_n=[e_1,\ldots,e_n]$ be the span of the first n basis vectors,and let $F_n=[e_{n=1},\ldots]\cap c_{00}$. Associated to N we may construct a norm $\|\cdot\|_\Lambda$ on c_{00} with $(e_n)_{n=1}^\infty$ a normalized unconditional basis such that

$$\|x+\xi e_n\|_\Lambda=N(\|x\|_\Lambda,|\xi|)\qquad x\in E_{n-1},\xi\in\mathbb{R},\ n\ge 2.$$

Let Λ denote the completion of $(c_{00},\|\cdot\|_\Lambda)$. Then Λ has property (M) Also Λ is isomorphic to the Orlicz sequence space ℓ_φ where $\varphi(t)=\max((1+\frac14t^2)^{1/2},t)-1$. Thus Λ is isomorphic to ℓ_2 and $\|\cdot\|_\Lambda$ is equivalent to the standard ℓ_2-norm.

Let $(\epsilon_n)_{n\ge2}$ be any sequence of positive numbers such that $\sum_{k=2}^\infty\epsilon_k<\infty$.

Lemma 8.3. *There is a norm $\|\cdot\|_X$ on c_{00} with the following properties:*

$$(1-\epsilon_n)N(\|x\|_X,|\xi|)\le\|x+\xi e_n\|_X\le(1+\epsilon_n)N(\|x\|_X,|\xi|),\qquad x\in E_{n-1},\ \xi\in\mathbb{R},\ n\ge2. \tag{8.8}$$

$$\|x+\xi e_n\|_X=(\|x\|_X^2+\tfrac14|\xi|^2)^{1/2},\qquad |\xi|\le\|x\|_X. \tag{8.9}$$

$$\|x+y\|_X=\|x-y\|_X=1\implies\|x\|_X<1\qquad x,y\in c_{00} \tag{8.10}$$

and for each $n\ge2$ there exists $v_n\in[e_1,\ldots,e_{n-1}]$ with $\|v_n-\frac12(-1)^ne_1\|_X<\epsilon_n$ and

$$\|v_n+e_n\|_X=d(x,E_{n-1})=\min\{\|x+e_n\|_X;\ x\in E_{n-1}]\}. \tag{8.11}$$

Proof. We define the norm on E_n by induction on n. To start the induction we define $\|e_1\|_X=1$. Now suppose $n\ge2$ and that $\|\cdot\|_X$ is well-defined on E_{n-1} and satisfies the conditions (8.8), (8.9), (8.10) and (8.11) where applicable. In particular $\|\cdot\|_X$ is assumed stricly convex on E_{n-1}.

We start by defining a norm $\|\cdot\|_0$ on E_n by the formula

$$\|x+\xi e_n\|_0=N(\|x\|_X,|\xi|)\qquad x\in E_{n-1},\ \xi\in\mathbb{R}.$$

Next we define $\|\cdot\|_1$ on E_n by setting

$$\|x+\xi e_n\|_1 = \inf\left\{(1+\epsilon_n)\|y+\alpha e_n\|_0 + |\beta| + \sum_{j=1}^k \|z_j+\gamma_j e_n\|_0\right\}.$$

where the infimum is taken over all representations

$$x+\xi e_n = (y+\alpha e_n) + \beta(\tfrac12(-1)^n e_1 + e_n) + \sum_{j=1}^k (z_j+\gamma_j e_n)$$

with $\alpha,\beta,\gamma_1,\dots,\gamma_n \in \mathbb{R}$, $y,z_1,\dots,z_k \in E_{n-1}$ and such that $|\gamma_j| \le \|z_j\|_X$ for $1\le j\le k$. It follows from the construction that

$$\|x+\xi e_n\|_1 \le (1+\epsilon_n)\|x+\xi e_n\|_0. \tag{8.12}$$

We will now argue that

$$\|\tfrac12(-1)^n e_1+e_n\|_1 = 1 < \|x+e_n\|_1, \qquad x\in E_{n-1},\ x\ne \frac12(-1)^n e_1. \tag{8.13}$$

Indeed it is clear that $\|\frac12(-1)^n e_1 + e_n\|_1 \le 1$. Now suppose $0<\eta<\epsilon_n$ and $x\in E_n$ is such that $\|x+e_n\|_1 < 1+\eta$. Then $x+e_n$ has a representation

$$x+e_n = (y+\alpha e_n) + \beta(\tfrac12(-1)^n e_1 + e_n) + \sum_{j=1}^k (z_j+\gamma_j e_n)$$

with $|\gamma_j|\le \|z_j\|_X$ for $1\le j\le k$.

$$(1+\epsilon_n)\|y+\alpha e_n\|_0 + |\beta| + \sum_{j=1}^k \|z_j+\gamma_j e_n\|_0 < 1+\eta.$$

Then

$$\alpha+\beta+\sum_{j=1}^k \gamma_j = 1.$$

Note that

$$\|y+\alpha e_n\|_0 \ge |\alpha|$$

and

$$\|z_j+\gamma_j e_n\|_0 = (\|z_j\|_X^2 + \tfrac14|\gamma_j|^2)^{1/2} \ge \tfrac{\sqrt5}{2}|\gamma_j|, \qquad 1\le j\le k.$$

Thus

$$1+\eta > \left(|\alpha|+|\beta|+\sum_{j=1}^k|\gamma_j|\right) + \eta_n|\alpha| + \frac{1}{10}\sum_{j=1}^k|\gamma_j|.$$

Thus

$$\sum_{j=1}^k |\gamma_j| < 10\eta$$

and

$$|\alpha| \le \eta/\epsilon_n.$$

We deduce that

$$|1-\beta| \le 10\eta + \epsilon_n^{-1}\eta.$$

Thus

$$\|y+\alpha e_n\|_X + \sum_{j=1}^k \|z_j+\gamma_j e_n\|_X < (11+\epsilon_n^{-1})\eta$$

and so

$$\Big\|y+\sum_{j=1}^k z_j + (1-\beta)e_n\Big\|_0 < (11+\epsilon_n^{-1})\eta.$$

Hence

$$\Big\|y+\sum_{j=1}^k z_j\Big\|_X < (21+2\epsilon_n^{-1})\eta.$$

Finally we deduce

$$\|x-\tfrac{1}{2}(-1)^n e_1\|_X < (31+3\epsilon_n^{-1})\eta.$$

This establishes (8.13).

For $0<\theta<1$ we now define

$$\|x+\xi e_n\|_\theta = \big((1-\theta)(\max(\|x+\xi e_n\|_0, \|x+\xi e_n\|_1)^2 + \theta(\|x\|_X^2 + \tfrac{1}{4}|\xi|^2)\big)^{1/2}.$$

We will show that taking $\|\cdot\|_X = \|\cdot\|_\theta$ for some choice of θ we can satisfy (8.8), (8.9),(8.10) and (8.11). First observe that

$$(1-\theta)^{1/2}\|x+\xi e_n\|_0 \le \|x+\xi e_n\|_X \le (1+\epsilon_n)\|x+\xi e_n\|_0$$

so that for θ small enough we will have (8.8).

If $|\xi| \le \|x\|_X$ then $\|x+\xi e_n\|_1 \le \|x+\xi e_n\|_0$ and so (8.9) follows immediately for any choice of θ.

For (8.10) we need only observe that $\|\cdot\|_\theta$ is strictly convex on E_n by construction, since $\|\cdot\|_X$ is strictly convex on E_{n-1}.

Finally let $v_\theta \in E_{n-1}$ be (uniquely) defined by

$$\|v_\theta + e_n\|_\theta = \min\{\|x+e_n\|_\theta;\ x\in E_{n-1}\}.$$

Then by an elementary compactness argument and (8.13) we have $\lim_{\theta\to 0} v_\theta = \frac{1}{2}(-1)^n e_1$. Thus for θ small enough we satisfy (8.11). □

Lemma 8.4. *If $x\in E_n$ and $y=\sum_{j=n+1}^\infty \xi_j e_j \in F_n$ then*

$$a_n\Big\|\|x\|_X e_1 + y\Big\|_\Lambda \le \|x+y\|_X \le b_n\Big\|\|x\|_X e_1+y\Big\|_\Lambda \tag{8.14}$$

where

$$a_n = \prod_{k=n+1}^\infty (1-\epsilon_k),\ b_n = \prod_{k=n+1}^\infty (1+\epsilon_k).$$

In particular $\|\cdot\|_X$ is equivalent to $\|\cdot\|_\Lambda$.

Proof. This follows easily by induction from (8.8). Indeed we show that if $y\in F_n\cap E_r$, where $r>n$, then

$$a_{nr}\Big\|\|x\|_X e_1+y\Big\|_\Lambda \le \|x+y\|_X \le b_{nr}\Big\|\|x\|_X e_1 + y\Big\|_\Lambda$$

where

$$a_{nr} = \prod_{k=n+1}^r (1-\epsilon_k), \quad b_n = \prod_{k=n+1}^r (1+\epsilon_k).$$

For $r = n+1$ this is given by (8.13). Assume it is now proved for $r-1 \geq n+1$. Suppose $y \in F_n \cap E_r$ is given by $y = y_0 + \xi e_r$ where $y_0 \in E_{r-1}$. Then

$$(1-\epsilon_r)N(\|x+y_0\|_X, |\xi|) \leq \|x+y\|_X \leq (1+\epsilon_r)N(\|x+y_0\|_X, |\xi|).$$

However

$$\begin{aligned} N(\|x+y_0\|_X, |\xi|) &\leq b_{n,r-1}N(\|\|x\|_X e_1 + y_0\|_\Lambda, |\xi|) \\ &= b_{n,r-1}\Big\|\|x\|_X e_1 + y\Big\|_\Lambda \end{aligned}$$

and similarly

$$N(\|x+y_0\|_X, |\xi|) \geq a_{n,r-1}\Big\|\|x\|_X e_1 + y\Big\|_\Lambda$$

and the induction step follows easily.

In particular taking $n=1$ we see that

$$a_1\|x\|_\Lambda \leq \|x\|_X \leq b_1\|x\|_\Lambda \qquad x \in c_{00}. \qquad \square$$

We now define X to be the completion of $(c_{00}, \|\cdot\|_X)$; clearly X coincides with ℓ_2 and $\|\cdot\|_X$ is equivalent to the ℓ_2-norm.

Lemma 8.5. *X is strictly convex and satisfies property (M).*

Proof. Suppose $x, y \in X$ with $\|x+y\|_X = \|x\|_X = \|x-y\|_X = 1$. Let $x = \sum_{j=1}^\infty \xi_j e_j$ and $y = \sum_{j=1}^\infty \eta_j e_j$. Then there exists an integer m so that if $k \geq m$ we have

$$|\xi_{k+1} + \eta_{k+1}| \leq \Big\|\sum_{j=1}^k (\xi_j + \eta_j)e_j\Big\|_X,$$

$$|\xi_{k+1} - \eta_{k+1}| \leq \Big\|\sum_{j=1}^k (\xi_j - \eta_j)e_j\Big\|_X,$$

$$|\xi_{k+1}| \leq \Big\|\sum_{j=1}^k \xi_j e_j\Big\|_X.$$

It follows that

$$\begin{aligned} \|x+y\|^2 &= \Big\|\sum_{j=1}^m (\xi_j + \eta_j)e_j\Big\|_X^2 + \frac{1}{4}\sum_{j=m+1}^\infty (\xi_j + \eta_j)^2, \\ \|x-y\|^2 &= \Big\|\sum_{j=1}^m (\xi_j - \eta_j)e_j\Big\|_X^2 + \frac{1}{4}\sum_{j=m+1}^\infty (\xi_j - \eta_j)^2, \\ \|x\|^2 &= \Big\|\sum_{j=1}^m \xi_j e_j\Big\|_X^2 + \frac{1}{4}\sum_{j=m+1}^\infty \xi_j^2. \end{aligned}$$

Thus

$$0 = \Big\|\sum_{j=1}^m (\xi_j + \eta_j)e_j\Big\|_X^2 + \Big\|\sum_{j=1}^m (\xi_j - \eta_j)e_j\Big\|_X^2 - 2\Big\|\sum_{j=1}^m \xi_j e_j\Big\|_X^2 + \frac{1}{2}\sum_{j=m+1}^\infty \eta_j^2.$$

Hence $\eta_j = 0$ for $j \geq m+1$ and by the strict convexity of c_{00} in $\|\cdot\|_X$ (i.e., (8.10)) we have $\eta_j = 0$ for $1 \leq j \leq m$, i.e., $y = 0$.

Next suppose $u, v \in c_{00}$ with $\|u\| \le \|v\|$ and than $(x_n)_{n=1}^\infty$ is a bounded block basic sequence such that $\lim_{n\to\infty} \|u + x_n\|_X$ and $\lim_{n\to\infty} \|v + x_n\|_X$ exist. Then it follows from (8.14) that

$$\lim_{n\to\infty} \|u + x_n\|_X = \lim_{n\to\infty} \Big\| \|u\|_X e_1 + x_n \Big\|_\Lambda$$

and

$$\lim_{n\to\infty} \|v + x_n\|_X = \lim_{n\to\infty} \Big\| \|v\|_X e_1 + x_n \Big\|_\Lambda$$

so that

$$\lim_{n\to\infty} \|u + x_n\|_X \le \lim_{n\to\infty} \|v + x_n\|_X. \qquad \square$$

The following proposition gives a counterexample to Problem 4.3 of Johnson and Zippin [16].

Proposition 8.6. *The space X^* is isomorphic to a Hilbert space, has a smooth norm and satisfies the linear $\mathcal{C}$-AIEP. However it does not satisfy the linear $\mathcal{C}$-IEP (and indeed fails the linear c_0-IEP).*

Proof. We have seen that X is isomorphic to ℓ_2 and hence so is X^*. The dual norm is smooth since X is strictly convex. X has property (M) (and is reflexive) so that X^* has property (M^*) and hence the linear $\mathcal{C}$-AIEP. Finally let us take the sequence $(v_n)_{n=2}^\infty$ given by (8.11) and define $x_n = (v_n + e_n)/\|v_n + e_n\|_X$. Then $d(x_n, E_n) = 1 = d(x_n, E_1)$ for all n. By (8.8) we have $(1 - \epsilon_n) \le \|v_n + e_n\|_X \le (1 + \epsilon_n)$. Hence

$$\lim_{n\to\infty} x_{2n} = \tfrac{1}{2} e_1, \quad \lim_{n\to\infty} x_{2n-1} = -\tfrac{1}{2} e_1 \qquad \text{weakly.}$$

Define the map $T : E_1^\perp \to c_0$ by $Tx^* = (0, x^*(x_2), x^*(x_3), \ldots)$. The linear functionals $x^* \to x^*(x_n)$ for $n \ge 2$ have norm one on $E_1^\perp$ and by smoothness have unique norm-preserving extensions namely $x^* \to x^*(x_n)$ on X^*. But $(x_n)_{n=2}^\infty$ is not a weakly convergent sequence and so T has no norm-preserving extension $\widetilde{T} : X^* \to c$. $\square$

9. Extension from weak*-closed subspaces

We now consider the extensions of operators on dual spaces, restricting ourselves to weak*-closed subspaces. Our results are somewhat different from the preceding section in that our criteria are in the form of conditions on types on the original space not its dual. This gives different characterizations of the linear $\mathcal{C}$-AIEP for reflexive spaces.

Theorem 9.1. *Let X be a Banach space with separable dual. Consider the following conditions on X:*

(i) *(E, X^*) has the linear AIEP for every weak*-closed subspace E of X^*.*
(ii) *(C, X^*) has the Lipschitz AIEP for every weak*-closed sonvex subset C of X^*.*
(iii) *For every pair σ, τ of weak*-null types on X^* and $u^*, v^* \in X^*$ there exists $0 \le \theta \le 1$ such that*

$$\sigma(\theta(u^* - v^*)) + \tau((1-\theta)(v^* - u^*)) \le \sigma(u^*) + \tau(v^*).$$

Then (iii) $\implies$ (i) *and* (ii). *Conversely if* X *is reflexive or* X^* *is stable* (i), (ii) *and* (iii) *are equivalent.*

Proof. Note that (ii) $\implies$ (i) by Proposition 4.10. Now suppose (iii) holds. We use Lemma 4.5. Let σ, τ be types supported on C. We must show that

$$\inf_{u^*\in C}(\sigma(u^*)+\tau(u^*)) = \inf_{x^*\in X^*}(\sigma(x^*)+\tau(x^*)). \tag{9.1}$$

Then since C is weak*-compact we may find $e^*, f^* \in C$ so that $\sigma_0(x^*) := \sigma(x^*+e^*)$ and $\tau_0(x^*) := \tau(x^*+f^*)$ are weak*-null types.

Fix $x^* \in X^*$. Then letting $u^* = x^*-e^*$ and $v^* = x^*-f^*$ we can find $0 \le \theta \le 1$ so that

$$\sigma_0(\theta(f^*-e^*)) + \tau_0((1-\theta)(e^*-f^*)) \le \sigma_0(x^*-e^*)+\tau_0(x^*-f^*).$$

This simplifies to

$$\sigma(w^*)+\tau(w^*) \le \sigma(x^*)+\tau(x^*)$$

where $w^* = (1-\theta)e^* + \theta f^* \in C$. Thus (9.1) holds. Thus (C, X) has the Lipschitz $\mathcal{C}$-AIEP.

Now let us suppose that either X is reflexive or X^* is stable. In this case Propositions 4.7 and 4.8 will imply that (i), (ii) are equivalent.

Now suppose (i) is satisfied. Let σ, τ be weak*-null types on X^*. Then σ, τ are weak*-lower-semicontinuous by Proposition 3.7. Let $(F_n)_{n=1}^\infty$ be an increasing sequence of finite-dimensional subspaces of X so that $\cup_n F_n$ is dense in X. Then we can suppose σ, τ are given by weak*-null sequences

$$\sigma(x^*) = \lim_{n\to\infty}\|x^*+e_n^*\| \qquad x^* \in X^*$$
$$\tau(x^*) = \lim_{n\to\infty}\|x^*+f_n^*\| \qquad x^* \in X^*,$$

where $e_n^*, f_n^* \in F_n^\perp$ for all n.

Fix $u^*, v^* \in X^*$. Since for each weak*-closed subspace E the pair (E, X^*) has the linear $\mathcal{C}$-AIEP, it follows that the condition $\Gamma_1(1)$, or equivalently $\Sigma_1(1)$ holds for any weak*-closed subspace. Hence $\Sigma_1(1)$ holds for any translate of a weak*-closed subspace.

Let E_n be the linear span of u^*-v^* and $F_n^\perp$. Then the types $\sigma'(x^*) := \sigma(x^*-v^*)$ and $\tau'(x^*) := \tau(x^*-u^*)$ are both supported on $\frac{1}{2}(u+v)+E_n$. By Lemma 4.5 for each n we can find $w_n^* \in \frac{1}{2}(u^*+v^*)+E_n$ so that

$$\sigma'(w_n^*)+\tau'(w_n^*) \le \sigma'(u^*+v^*)+\tau'(u^*+v^*)+\tfrac{1}{n} \qquad n = 1, 2, \ldots.$$

Thus

$$\sigma(w_n^*-v^*)+\tau(w_n^*-u^*) \le \sigma(u^*)+\tau(v^*)+\tfrac{1}{n} \qquad n = 1, 2, \ldots.$$

The sequence $(w_n^*)_{n=1}^\infty$ is clearly bounded and has a weak*-cluster point $w^* = \lambda u^* + (1-\lambda)v^*$ where $-\infty < \lambda < \infty$. Thus

$$\sigma(\lambda(u^*-v^*)) + \tau((1-\lambda)(v^*-u^*)) \le \sigma(u^*)+\tau(v^*).$$

This inequality implies in the case $\sigma = \tau$ and $u^* = v^*$ that every weak*-null type is monotone. Hence

$$\sigma(u^*-v^*)+\tau(0) \le \sigma(\lambda(u^*-v^*)) + \tau((1-\lambda)(v^*-u^*)) \qquad \lambda \ge 1$$

and

$$\sigma(0)+\tau(v^*-u^*) \le \sigma(\lambda(u^*-v^*)) + \tau((1-\lambda)(v^*-u^*)) \qquad \lambda \le 0.$$

Hence we may take $\lambda = \theta$ where $0 \leq \theta \leq 1$. $\square$

Theorem 9.2. *Let X be a Banach space with either property (L^*) or (M^*).*

(i) *If C is a convex weak*-closed subset of X^* then (C, X) has the Lipschitz $\mathcal{C}$-AIEP.*

(ii) *If E is a weak*-closed subspace of X^* then (E, X^*) has the linear $\mathcal{C}$-AIEP.*

Proof. We establish that both (L^*) and (M^*) imply (iii) of Theorem 9.1.

(i) Assume X has property (M^*). Suppose σ and τ are weak*-null types defined by weak*-null sequences $(x_n^*)_{n=1}^\infty$ and $(y_n^*)_{n=1}^\infty$. Suppose $u^*, v^* \in X^*$. Pick θ with $0 \leq \theta \leq 1$ so that $(1-\theta)\|u^*\| = \theta\|v^*\|$. Then

$$\begin{aligned}
\sigma(\theta(u^* - v^*)) &= \lim_{n\to\infty} \|\theta u^* - \theta v^* + x_n^*\| \\
&\leq \lim_{n\to\infty} \|\theta u^* + \theta x_n^*\| + \lim_{n\to\infty} \|-\theta v^* + (1-\theta)x_n^*\| \\
&= \lim_{n\to\infty} \|\theta u^* + \theta x_n^*\| + \lim_{n\to\infty} \|(1-\theta)u^* + (1-\theta)x_n^*\| \\
&= \sigma(u^*).
\end{aligned}$$

Similarly $\tau((1-\theta)(v^* - u^*)) \leq \tau(v^*)$. This verifies (iii) of Theorem 9.1 and gives the conclusion.

Now assume X has property (L^*) and keep the same notation. In this case pick θ so that

$$(1-\theta)\lim_{n\to\infty} \|x_n^*\| = \theta \lim_{n\to\infty} \|y_n^*\|.$$

As before

$$\begin{aligned}
\sigma(\theta(u^* - v^*)) &= \lim_{n\to\infty} \|\theta u^* - \theta v^* + x_n^*\| \\
&\leq \lim_{n\to\infty} \|\theta u^* + \theta x_n^*\| + \lim_{n\to\infty} \|-\theta v^* + (1-\theta)x_n^*\| \\
&= \lim_{n\to\infty} \|\theta u^* + \theta x_n^*\| + \lim_{n\to\infty} \|\theta v^* + \theta y_n^*\| \\
&= \theta(\sigma(u^*) + \tau(v^*)).
\end{aligned}$$

Similarly

$$\tau((1-\theta)(v^* - u^*)) \leq (1-\theta)(\sigma(u^*) + \tau(v^*))$$

and again Theorem 9.1 can be applied. $\square$

Corollary 9.3. *Let E be a closed subspace of ℓ_1. Then (E, ℓ_1) has the linear $\mathcal{C}$-AIEP if and only if E is weak*-closed (with respect to c_0).*

This corollary answers a question of Johnson and Zippin [16], Problem 4.5, who proved that (E, ℓ_1) has the linear $(3+\epsilon, \mathcal{C})$-EP when E is a weak*-closed subspace. We remark here that the classification of subspaces E so that (E, ℓ_1) has the linear $\mathcal{C}$-EP (not necessarily almost isometric) is open. The author showed in [18] that if ℓ_1/E has a UFDD and (E, ℓ_1) has the linear $\mathcal{C}$-EP then there is an automorphism $S : \ell_1 \to \ell_1$ so that $S(E)$ is weak*-closed.

Proof. If E is weak*-closed this is an immediate conclusion from Theorem 9.2. Conversely suppose E has the linear $\mathcal{C}$-AIEP. Then E certainly has the linear c_0-AIEP and satisfies $\Gamma_0(1)$. Let $(e_n^*)_{n=1}^\infty$ be a weak*-converging sequence in E and let

e^* be the weak*-limit. By passing to a subsequence we may suppose that $(e_n^*)_{n=1}^\infty$ defines a weak*-null type σ which (by property (m_1^*)) must be of the form

$$\sigma(x^*) = \|x^* - e^*\| + \sigma(e^*) \qquad x^* \in X^*.$$

By Lemma 4.5 we have

$$\inf_{u^* \in E} \sigma(u^*) \leq \sigma(e^*)$$

so that we have $e^* \in E$. By the Banach–Dieudonné Theorem this means that E is weak*-closed. $\square$

Remark. Of course the above argument uses only the fact that c_0 has property (m_1^*) and so applies to any dual of a space with property (m_1^*).

Proposition 9.4. *Suppose X is a separable Banach space and that E is a closed subspace of X. Suppose:*

(i) *E is isometric to the dual of a space Y.*
(ii) *If σ, τ are types on X induced by weak*-null sequences in $E = Y^*$ then*

$$\sigma(0) = \tau(0) \quad \Longrightarrow \quad \sigma(x) = \tau(x) \qquad x \in X.$$

Then (E, X) has the linear $\mathcal{C}$-AIEP. If X is uniformly smooth, then (E, X) has the linear $\mathcal{C}$-IEP.

Proof. Let $E_0 = E$ and then E_n be an increasing sequence of subspaces of X so that $\dim E_n/E_{n-1} = 1$ for $n \geq 1$ and $\cup_n E_n$ is dense in X. We note that each E_n is isometrically a dual space in such a way that the weak*-topology on E_n restricts to the weak*-topology on E_{n-1}, since E_n/E_{n-1} is finite-dimensional. We claim that $E_n = Y_n^*$ where Y_n has property (L^*). Indeed if $(u_n)_{n=1}^\infty$ is a weak*-null sequence in E_n defining a type on X then since E_n/E is finite-dimensional, $\lim_{n\to\infty} d(u_n, E) = 0$ and so σ is supported on E. Our assumption quickly ensures that Y_n has property (L^*). By Theorem 9.2 if $\epsilon > 0$, we can now extend an operator $T : E \to \mathcal{C}(K)$ inductively to $\widetilde{T} : \cup_n E_n \to \mathcal{C}(K)$ with $\|\widetilde{T}\| < \|T\| + \epsilon$.

If X is uniformly smooth we can apply Proposition 4.1 of [16]. $\square$

Proposition 9.5. *Suppose $1 < p < \infty$ and that E is a closed subspace of $L_p(0,1)$. Then:*

(i) *If E is almost isometric to a subspace of ℓ_p then (E, L_p) has the linear $\mathcal{C}$-IEP.*
(ii) *If $1 < p < q \leq 2$ and E is isometric to a subspace of ℓ_q then (E, L_p) has the linear $\mathcal{C}$-IEP.*

Proof. (i) By Theorem 4.4 of [23] B_E is compact in L_1. Thus if F/E is finite-dimensional then B_F is also compact in L_1. Hence for any type σ induced by a weakly null sequence in E we have

$$\sigma(f) = (\|f\|^p + \sigma(0)^p)^{1/p} \qquad f \in L_p.$$

By Proposition 9.4 and Theorem 8.2 we are done.

(ii) It follows from the Plotkin–Rudin Theorem [34],[36],[24] Theorem 4 that it suffices to prove this result for any isometric embedding of E into $L_p(0,1)$. We therefore take E as a subspace of a space E_1 defined as the closed linear span of a sequence of independent q-stable random variables $(h_n)_{n=1}^\infty$ with

$$\int e^{ith_n(s)} ds = e^{-|t|^q}.$$

Since E_1 has the $\mathcal{C}$-IEP it is enough to prove the result for $E = E_1$.

Suppose σ is any type induced by a weakly null sequence in E then there is a sequence of independent q-stable random variables $(g_n)_{n=1}^\infty$ (with the same distribution as h_1) and a constant $\alpha > 0$ so that

$$\sigma(f) = \lim_{n\to\infty} \|f + \alpha g_n\| \qquad f \in L_p.$$

But by Lemma 2.5 of [21] we have

$$\sigma(f) = \left(\int_0^1 \int_0^1 |f(s) + \alpha h_1(t)|^p \, ds\, dt\right)^{1/p}.$$

Now an appeal to Proposition 9.4 concludes the proof. □

10. The universal linear $\mathcal{C}$-AIEP

We shall say that a separable Banach space X has the *separable universal linear $\mathcal{C}$-AIEP* if whenever X is isometric to a subspace of a separable Banach space Y then (X, Y) has the linear $\mathcal{C}$-AIEP. This property has been investigated by Speegle [38] who showed that a uniformly smooth space cannot have the universal linear $\mathcal{C}$-AIEP. However, no examples were previously known of spaces with the universal $\mathcal{C}$-AIEP.

Let us say that a Banach space X has the *m_1-type property* if every type σ on X has the form

$$\sigma(x) = \|x - u\| + \sigma(u) \qquad x \in X$$

for some $u \in X$.

Theorem 10.1. *Let X be a separable Banach space with the m_1-type property. Then X has the separable universal linear $\mathcal{C}$-AIEP.*

Proof. Let us suppose that X is embedded in a separable Banach space Y. We start by considering on Y a minimal function $\varphi : Y \to \mathbb{R}$ which respect to the following conditions:

- $\varphi(y) \le \|y\|$ for $y \in Y$.
- φ is convex.
- $\varphi(y) = \varphi(-y)$ for $y \in Y$.
- $\varphi(x) = \|x\|$ for $x \in X$.

The existence of such a minimal function is guaranteed by Zorn's Lemma. It is clear that $\varphi(y) \ge 0$ for all $y \in Y$ since $\varphi(y) + \varphi(-y) \ge 2\varphi(0) = 0$.

We first claim that φ is a seminorm. Indeed note that if $0 < \alpha < 1$ then $\psi(y) = \alpha^{-1}\varphi(\alpha y)$ defines another function in the class with $\psi \le \varphi$. Hence $\psi = \varphi$ and this shows that $\varphi(tx) = |t|\varphi(x)$ for t real.

Let Z be the completion of $Y/\varphi^{-1}\{0\}$ which respect to the induced seminorm. Then X is isometrically embedded into Z and it is enough to show that (X, Z) has the $\mathcal{C}$-AIEP.

Note that Z has the property that if $\psi : Z \to \mathbb{R}$ is a function such that

- $\psi(z) \le \|z\|$ for $z \in Z$,
- ψ is convex,
- $\psi(z) = \psi(-z)$ for $z \in Z$,
- $\psi(x) = \|x\|$ for $x \in X$,

then $\psi(z) = \|z\|$ for all $z \in Z$.

Clearly Z is separable and we may construct an increasing sequence $(Z_n)_{n=1}^{\infty}$ of subspaces of Z so that $Z_0 = X$, $\dim Z_n/Z_{n-1} = 1$ (for $n \geq 1$) and $\cup_n Z_n$ is dense in Z. It will suffice to show that (Z_{n-1}, Z_n) has the linear $\mathcal{C}$-AIEP for all $n \geq 1$. To do this it suffices to show that (Z_{n-1}, Z) (and hence (Z_{n-1}, Z_n)) satisfies condition $\Gamma_1(1)$ or $\Sigma_1(1)$.

For fixed $n \geq 1$ let σ be any type on Z supported on Z_{n-1}. Since Z_{n-1}/X is finite-dimensional we can write σ in the form

$$\sigma(z) = \sigma'(z - u_0) \qquad z \in Z$$

where σ' is supported on X and $u_0 \in Z_{n-1}$. By assumption there exists $u_1 \in X$ so that for $x \in X$

$$\sigma'(x) = \|x - u_1\| + \sigma'(u_1) \qquad x \in X.$$

Let

$$\sigma_0(z) = \sigma(z + u) \qquad z \in Z$$

where $u = u_0 + u_1 \in Z_{n-1}$. We have

$$\sigma_0(x) = \|x\| + \sigma(u) \qquad x \in X.$$

Note that in general

$$\sigma_0(z) \leq \sigma_0(0) + \|z\| = \sigma(u) + \|z\|.$$

Now define

$$\psi(z) = \frac{1}{2}(\sigma_0(z) + \sigma_0(-z) - 2\sigma(u)).$$

Clearly ψ is convex and symmetric. Also $\psi(0) = 0$ and ψ has Lipschitz constant one so that $\psi(z) \leq \|z\|$. If $x \in X$ then $\psi(x) = \|x\|$. Thus by our assumptions on Z, $\psi(z) = \|z\|$ for every $z \in Z$. It follows that we also have, since $\sigma_0(-z) - \sigma(u) \leq \|z\|$,

$$\sigma_0(z) - \sigma(u) = \|z\| \qquad z \in Z$$

or

$$\sigma(z) = \|z - u\| + \sigma(u) \qquad z \in Z.$$

Now suppose σ and τ are two types supported on Z_{n-1}. Then we can find $u, v \in Z_{n-1}$ with

$$\sigma(z) = \|z - u\| + \sigma(u), \quad \tau(z) = \|z - v\| + \tau(v) \qquad z \in Z.$$

Thus if $z \in Z$,

$$\begin{aligned} \sigma(u) + \tau(u) &= \sigma(u) + \tau(v) + \|u - v\| \\ &\leq \sigma(u) + \tau(v) + \|z - u\| + \|z - v\| \\ &\leq \sigma(z) + \tau(z). \end{aligned}$$

Thus (Z_{n-1}, Z) satisfies $\Sigma_1(1)$. □

Corollary 10.2. *Let X be the dual of a subspace of c_0. Then X has the separable universal $\mathcal{C}$-AIEP. In particular every weak*-closed subspace of ℓ_1 has the separable universal $\mathcal{C}$-AIEP.*

Proof. $X = Y^*$ where Y has property (m_1^*) which implies that X has the m_1-type property. □

Remark. Of course this applies to any space $X = Y^*$ where Y is a separable Banach space with property (m_∞) and not containing ℓ_1. Such spaces are characterized in [23] by the fact that Y is almost isometric to a subspace of c_0 (i.e., for every $\epsilon > 0$, there exists $Y_\epsilon \subset c_0$ with $d(Y, Y_\epsilon) < 1 + \epsilon$).

Corollary 10.3. *If X is a subspace of ℓ_1 then X has the separable universal $\mathcal{C}$-AIEP if and only if X is weak*-closed (with respect to c_0).*

Proof. This follows from Corollary 9.3. □

This corollary suggests that there might be a converse to Corollary 10.2. This is false by two examples in [11] (one due to Talagrand) and the following corollary:

Corollary 10.4. *Let X be a subspace of $L_1[0,1]$ such that B_X is compact for convergence in measure. Then X has the separable universal linear $\mathcal{C}$-AIEP.*

Proof. Suppose $\sigma(f) = \lim_{n\to\infty} \|f + f_n\|$ for $f \in X$. Then we can pass to a subsequence and suppose that $f_n \to -g$ a.e. where $g \in X$. Thus

$$\sigma(f) = \|f - g\| + \sigma(g) \qquad f \in X$$

and Theorem 10.1 applies. □

The examples in [11] §4 show that it is possible for X to satisfy the conditions of this corollary and not be almost isometric to the dual of a subspace of c_0. To see this note that if X^* has MAP where X is a subspace of c_0 then X^* has UMAP and use Theorem 4.2. It should be noted, however, that both spaces in [11] are isomorphic (but not isometric) to ℓ_1-sums of finite-dimensional spaces.

We shall now give further slightly more general examples; unfortunately all the examples we know seem to be at least isomorphic to weak*-closed subspaces of an ℓ_1-sum of finite-dimensional spaces.

Proposition 10.5. *Let X be a separable Banach space with the m_1-type property. Let F be a finite-dimensional normed space and suppose N is a norm on $F \times \mathbb{R}$ such that:*

(i) $N(f, 0) = \|f\| \qquad f \in F$.
(ii) $N(f, \xi) = N(f, -\xi) \qquad f \in F, \xi \in \mathbb{R}$.
(iii) $N(0, 1) = 1$.

Then the space $W = F \oplus_N X$ also has the universal linear $\mathcal{C}$-AIEP. Here $F \oplus_N X$ is the direct sum $F \oplus X$ normed by

$$\|(f, x)\| = N(f, \|x\|).$$

Proof. Suppose $\epsilon > 0$ and let $\theta = (1+\epsilon)^{1/2}$ and that W is isometrically embedded in a Banach space Y. Suppose $T : W \to \mathcal{C}(K)$ is a bounded linear operator with $\|T\| \le 1$. We denote by P_F and P_X the canonical projections of W onto F and X respectively.

We start by considering the restriction of T to F. This may be represented in the form

$$Tf(t) = \langle f, f^*(t)\rangle \qquad t \in K$$

where $t \to f^*(t)$ is a norm continuous map $K \to F^*$.

Suppose N^* is the dual norm on $F^* \oplus \mathbb{R}$. Let us define $v^*(t) \in F^* \oplus_{N^*} R$ by $v^*(t) = (f^*(t), \psi(t))$, with $\psi(t)$ the unique positive solution of $N^*(f^*(t), \psi(t)) = \theta$

(note that $\|f^*(t)\| \leq 1 < \theta$). It is easy to verify that ψ is continuous and so the map $S : F \oplus_N \mathbb{R} \to \mathcal{C}(K)$ given by

$$S(f,\lambda) = \langle f, f^*(t)\rangle + \lambda\psi(t)$$

satisfies $\|S\| = \theta$.

Let us observe that if $0 \neq x \in X$ then so any $f \in F$ we have

$$|\langle f, f^*(t)\rangle| + |\lambda||Tx(t)| \leq \|x + f\| = N(f, \lambda\|x\|).$$

Thus

$$N^*(f^*(t), |Tx(t)|/\|x\|) \leq 1$$

and so

$$|Tx(t)| \leq \psi(t)\|x\| \qquad t \in K.$$

Now, for m large enough we may find a linear map $V : F \oplus_N \mathbb{R} \to \ell_\infty^m$ such that $\|V\| < \theta$ and $\|g\| \leq \|Vg\|$ for $g \in F \oplus_N \mathbb{R}$. Then using the extension properties for finite-dimensional polyhedral spaces we can find an operator $S_0 : \ell_\infty^m \to \mathcal{C}(K)$ with $\|S_0\| \leq \theta$ and $S_0V = S$. Let

$$S_0(\xi_1, \ldots, \xi_m) = \sum_{i=1}^m \xi_i h_i$$

where $h_i \in \mathcal{C}(K)$. Then

$$\sum_{i=1}^m |h_i(t)| \leq \theta \qquad t \in K.$$

Let

$$V(f,\lambda) = (f_i^*(f) + \beta_i\lambda)_{i=1}^m$$

where

$$N^*(f_i, \beta_i) \leq \theta.$$

Thus

$$S(f,\lambda) = \sum_{i=1}^m (f_i^*(f) + \beta_i\lambda)h_i$$

and so

$$f^*(t) = \sum_{i=1}^m h_i(t)f_i^* \qquad t \in K$$

and

$$\psi(t) = \sum_{i=1}^m \beta_i h_i(t) \qquad t \in K.$$

At this point we define the operators $T_i : W \to \mathcal{C}(K)$ by

$$T_i(w) = f_i^*(P_F w) + \beta_i\psi^{-1}TP_X w.$$

Let π_i be the seminorm $\pi_i(w) = |f_i^*(P_F w)| + |\beta_i|\|P_X w\|$. Then

$$\|T_i w\| \leq \pi_i(w) \qquad w \in W.$$

Now

$$\pi_i(w) \leq N^*(f_i^*, |\beta_i|)N(P_F w, \|TP_X w\|) \leq \theta\|w\| \qquad w \in W.$$

Define

$$\rho_i(y) = \inf_{w \in W}(\pi_i(w) + \theta\|y - w\|) \qquad y \in Y.$$

Then ρ_i is a seminorm on Y such that $\rho_i|_W = \pi_i$. The normed space $W/\rho_i^{-1}(0)$ is (except when $\beta_i = 0$, when it reduces to a space of dimension one) isometric to $\mathbb{R} \oplus_1 X$ and so has the m_1-type property; thus we can apply Theorem 10.1 to produce an extension $\widetilde{T}_i : Y \to \mathcal{C}(K)$ with

$$\|\widetilde{T}_i y\| \le \rho_i(y) \le \theta \|y\| \qquad y \in Y.$$

Now define

$$\widetilde{T} y = \sum_{i=1}^{m} h_i \widetilde{T}_i y \qquad y \in Y.$$

Then

$$\|\widetilde{T}\| \le \theta \sup_{t\in K} \sum_{i=1}^{m} |h_i(t)| \le \theta^2 = 1 + \epsilon.$$

If $w \in W$ then

$$\begin{aligned} \widetilde{T} y &= \sum_{i=1}^{m} (f_i^*(P_F w) + \beta_i \psi^{-1} T P_X w(t)) h_i \\ &= T P_F w + \psi^{-1} \left(\sum_{i=1}^{m} \beta_i h_i \right) T P_X w \\ &= Tw. \end{aligned}$$

□

Theorem 10.6. *The Nakano space ℓ_{p_n} where $p_n > 1$ and $\lim_{n\to\infty} p_n = 1$ has the separable universal linear $\mathcal{C}$-AIEP.*

Proof. The canonical basis of $X = \ell_{p_n}$ is clearly boundedly complete and so X is isometric to the dual of a space Z with a 1-unconditional basis. Then Z has property (L^*). If fact if $\xi = (\xi_k)_{k=1}^{\infty} \in X$ and $(u_n)_{n=1}^{\infty}$ is a normalized block basic sequence and $\lambda \ge 0$, then

$$\lim_{n\to\infty} \|\xi + \lambda u_n\| = N(\xi, \lambda)$$

where $\alpha = N(\xi, \lambda)$ is the unique solution of

$$\sum_{k=1}^{\infty} \alpha^{-p_k} |\xi_k|^{p_k} + \alpha^{-1}\lambda = 1.$$

If $\|\xi\| = 1$ then

$$\frac{\partial N(x,\lambda)}{\partial \lambda}\Big|_{\lambda=0} = \left(\sum_{k=1}^{\infty} p_k |\xi_k|^{p_k} \right)^{-1}$$

so that

$$N(\xi, \lambda) \ge 1 + \lambda \left(\sum_{k=1}^{\infty} p_k |\xi_k|^{p_k} \right)^{-1}.$$

Now fix m and let $F = [e_1, \ldots, e_m]$ and $Z = [e_{m+1}, e_{m+2}, \ldots]$ where $(e_n)_{n=1}^{\infty}$ denotes the canonical basis. Let $p = \max_{n>m} p_n > 1$. Let $q = p'$ the conjugate index. Then for $\xi \in Z$ with $\|\xi\| = 1$ we have

$$N(\xi, \lambda) \ge 1 + \frac{1}{p} \lambda.$$

Hence, using Theorem 2.4 of [9] for every $\epsilon > 0$ there is a closed subspace W of c_0 so that $d(Z, W^*) \le p^2 + \epsilon$.

If $\xi \in F$ and $\eta \in Z$ with $\|\xi\| = \|\eta\| = 1$ then, for $\lambda \ge 0$, $\|\xi + \lambda\eta\| \le N(\xi, \lambda)$. On the other hand $\|\xi + \lambda\eta\| = \alpha$ is the solution of

$$\sum_{k=1}^{m} \alpha^{-p_k} |\xi_k|^{p_k} + \sum_{k=m+1}^{\infty} \alpha^{-p_k} \lambda^{p_k} |\eta_k|^{p_k} = 1.$$

Thus, since $\alpha \ge \max(\lambda, 1)$,

$$\sum_{k=1}^{m} \alpha^{-pp_k} |\xi|^{p_k} + \lambda^p \alpha^{-p} \le 1.$$

This implies that

$$\alpha^p \ge N(\xi, \lambda)$$

i.e.,

$$N(\xi, \lambda)^{1/p} \le \|\xi + \lambda\eta\|.$$

Now if $\lambda \ge q$ we have

$$N(\xi, \lambda) \le \lambda + 1 \le (1 + \tfrac{1}{q})\|\xi + \lambda\eta\|.$$

If $\lambda \le q$ then

$$N(\xi, \lambda) \le \|\xi + \lambda\eta\|^p \le (q+1)^{p-1}\|\xi + \lambda\eta\|.$$

It follows that X has Banach–Mazur distance from $F \oplus_N W^*$ at most

$$\mu = (p^2 + \epsilon)\max((q+1)^{p-1}, (1 + 1/q)).$$

Hence X has the universal linear $(\mu, \mathcal{C})$-EP with μ as above. Now $(q+1)^{p-1} = (q+1)^{-p/q}$. By taking m large enough we can make p arbitrarily close to 1 and so X has the separable universal linear $\mathcal{C}$-AIEP. □

Let us remark that Lindenstrauss and Pełczyński showed that every subspace of c_0 has the separable universal linear $(2 + \epsilon, \mathcal{C})$-extension property for for every $\epsilon > 0$. The class of spaces with the separable universal linear $\mathcal{C}$-extension property is important in the study of automorphisms of $\mathcal{C}(K)$-spaces (see [4]). In a separate publication [20] we will study these spaces in more detail; we will show for example that ℓ_p fails to have the separable universal linear $\mathcal{C}$-extension property when $1 < p < \infty$.

We conclude this section by observing that ℓ_1 also has a separable universal extension property with respect to a wider class of spaces. We refer to two recent papers [5] and [6] for results in a similar spirit.

Theorem 10.7. *Let Y be a separable Banach space. The following are equivalent:*

(i) *ℓ_1 has the separable universal Y-AIEP.*

(ii) *For every $\epsilon > 0$ there is a quotient Z of $\mathcal{C}[0,1]$ so that $d(Y, Z) \le 1 + \epsilon$.*

Proof. (i) $\implies$ (ii). Let $Q : \ell_1 \to Y$ be a quotient map. Embed ℓ_1 isometrically in $\mathcal{C}[0,1]$ and extend Q to an operator $T : \mathcal{C}[0,1] \to Y$ with $\|T\| \le 1 + \epsilon$. Then let $Z = \mathcal{C}[0,1]/\ker T$.

(ii) $\implies$ (i). It is enough to assume Y is isometric to a quotient of $\mathcal{C}[0,1]$. Suppose $\ell_1 \subset X$ where X is a separable Banach space and $T : \ell_1 \to Y$ is a bounded operator. Then $T : \ell_1 \to Y$ can be lifted to an operator $S : \ell_1 \to \mathcal{C}[0,1]$ with $\|S\| =$

$\|T\|$, and T_1 has an extension $\widetilde{S} : X \to \mathcal{C}[0,1]$ with $\|\widetilde{S}\| < (1+\epsilon)\|S\|$. Composing with the quotient map gives an extension $\widetilde{T} : X \to Y$ of T with $\|\widetilde{T}\| < (1+\epsilon)\|T\|$.

$$\begin{tikzcd} X \arrow[r, "\widetilde{S}"] & \mathcal{C}[0,1] \arrow[d, "Q"] \\ \ell_1 \arrow[u] \arrow[ur, "S"] \arrow[r, "T"] & Y. \end{tikzcd}$$

□

Remark. In particular Theorem 10.7 applies when Y is a Lindenstrauss space (i.e., an L_1-predual) by [15].

Theorem 10.8. *For any $n \in \mathbb{N}$ the space $\ell_\infty^n(\ell_1) = \ell_1 \oplus_\infty \cdots \oplus_\infty \ell_1$ has the separable universal $\mathcal{C}$-AIEP.*

Proof. Let $X = X_1 \oplus_\infty \cdots \oplus_\infty X_n$ where each X_j is isometric to ℓ_1. Suppose $Z \supset X$ is a separable Banach space. Let $\theta = (\theta_1, \dots, \theta_n) \in \{-1,1\}^n = D_n$ and define an isometry $U_\theta : X \to X$ by

$$U_\theta(x_1 + \cdots + x_n) = \theta_1 x_1 + \cdots + \theta_n x_n \qquad x_j \in X_j.$$

We first embed Z in a larger Z_1 so that there is homomorphism $\theta \to V_\theta$ from D_n into the group of isometries of Z_1 with $V_\theta|_X = U_\theta$. This is a standard construction. To do this consider the space $\ell_1(D_n; Z)$ and form the quotient $Z_1 = \ell_1(D_n; Z)/W$ where W is the set of elements $(x_\theta)_{\theta \in D_n}$ where $\sum_{\theta \in D_n} U_\theta x_\theta = 0$. Let $Q : \ell_1(D_n; Z) \to Z_1$ be the quotient map. Let $J : Z \to \ell_1(D_n; Z)$ be the injection $z \to \widetilde{z}$ where $\widetilde{z}_\theta = z$ if $\theta = e$ (the identity) and 0 otherwise. Then QJ is an isometric embedding of Z into Z_1.

Define $\widetilde{V}_\theta : \ell_1(D_n; Z) \to \ell_1(D_n; Z)$ by $\widetilde{V}_\theta(z) = (z_{\theta\phi})_{\phi \in D_n}$. Then $\theta \to \widetilde{V}_\theta$ is a homomorphism into the group of isometries of $\ell_1(D_n; Z)$. Since $\widetilde{V}_\theta(W) \subset W$ for each θ this factors to $V_\theta : Z_1 \to Z_1$ and $V_\theta QJ = QJU_\theta$ as required. We can therefore regard Z as embedded in Z_1 and V_θ as an extension of U_θ.

Let $T : X \to \mathcal{C}(K)$ be a bounded linear operator with $\|T\| < 1$. Consider $T|_{X_j} = T_j$. Let $\varphi_j(s) = \sup_{\|x\| \le 1} |T_j x(s)|$ so that φ_j is lower semicontinuous and $\sup_{s \in K} \varphi_j < 1$. More generally since $\|T\| < 1$ we have

$$\varphi_1(s) + \cdots + \varphi_n(s) \le \|T\|, \qquad s \in K.$$

Let E_j be the subspace of $\mathcal{C}(K)$ of all functions f satisfying an estimate

$$|f(s)| \le M\varphi_j(s) \qquad s \in K$$

under the norm

$$\|f\|_{E_j} = \sup_{\varphi_j(s) > 0} |f(s)|\varphi_j(s)^{-1}.$$

Then E_j is an abstract M-space (cf. [29] pp. 15-18). If we consider the smallest closed sublattice $\widetilde{E}_j$ of E_j containing $T_j(X_j)$ then $\widetilde{E}_j$ is a separable M-space and hence an isometric quotient of $C[0,1]$ by the result of [15] cited above. Clearly $\|T_j\|_{X_j \to \widetilde{E}_j} \le 1$ and so by Theorem 10.7 we have an extension $S_j : Z_1 \to \widetilde{E}_j \subset \mathcal{C}(K)$

with $\|S_j\|_{Z_1\to\widetilde{E}_j} < \|T\|^{-1}$. Now let

$$R_j = \frac{1}{2^{n-1}} \sum_{\substack{\theta\in D_n\\ \theta_j=1}} S_j V_j.$$

Then $\|R_j\|_{Z_1\to\widetilde{E}_j} < \|T\|^{-1}$, $R_j|_{X_j} = T_j$ and $R_j|_{X_i} = 0$ for $i \neq j$.

Finally consider $R = \sum_{j=1}^n R_j$. Then $R|_X = T$ and if $z \in Z_1$ with $\|z\| \leq 1$ we have

$$|Rz(s)| \leq \sum_{j=1}^n |R_j z(s)| \leq \|T\|^{-1} \sum_{j=1}^n \varphi_j(s) \leq 1$$

so that $\|R\|_{Z_1\to\mathcal{C}(K)} \leq 1$.

Hence X has the separable universal $\mathcal{C}$-AIEP. □

11. Necessary conditions for universal extension properties

In this last section, we will attempt to classify spaces with the universal Lipschitz $\mathcal{C}$-AIEP. Our conditions will suggest that spaces must be close to ℓ_1.

Proposition 11.1. *Let X be a separable Banach space. The following are equivalent:*

(i) *X has the separable universal linear c_0-AIEP.*
(ii) *(X, Y) satisfies $\Sigma_0(1)$ for every Banach space $Y \supset X$ with $\dim Y/X = 1$.*
(iii) *X has the universal Lipschitz c_0-AIEP.*

Proposition 11.2. *Let X be a separable Banach space. The following statements are equivalent:*

(i) *X has the separable universal linear c-AIEP.*
(ii) *For every $n \in \mathbb{N}$, (X, Y) satisfies $\Sigma_n(1)$ for every Banach space $Y \supset X$ with $\dim Y/X = n$.*

These statements simply reword Theorems 5.4 and 6.9 above. There is a corresponding statement for Lipschitz extensions:

Proposition 11.3. *Let X be a separable Banach space. The following statements are equivalent:*

(i) *X has the universal Lipschitz $\mathcal{C}$-AIEP.*
(ii) *(X, Y) satisfies $\Sigma_1(1)$ for every Banach space $Y \supset X$ with $\dim Y/X = 1$.*

Proof. This follows from Proposition 2.1 and Theorem 4.2 of [19]. □

Notice that these propositions imply that the separable universal linear $\mathcal{C}$-AIEP (or c-AIEP) implies the universal Lipschitz $\mathcal{C}$-AIEP.

If σ is a type on X it will be convenient to introduce the notation

$$\mu_\sigma = \inf_{x\in X} \sigma(x).$$

We also introduce the parameter

$$\delta_\sigma = \inf\{\text{diam } C : \ \sigma \text{ is supported on } C\}.$$

Note that

$$\delta_\sigma \leq 2\mu_\sigma.$$

Indeed if $\sigma(x) = \lim_{n\to\infty} \|x + x_n\|$ for $x \in X$ then for any fixed y

$$\|x_n - x_m\| \le \|y + x_n\| + \|y + x_m\|$$

which implies

$$\limsup_{m,n\to\infty} \|x_m - x_n\| \le 2\sigma(y).$$

Proposition 11.4. *Let X be a separable Banach space. Then X has the separable universal linear c_0-AIEP if and only if for every type σ on X, $\delta_\sigma = 2\mu_\sigma$.*

Proof. Assume $\delta_\sigma = 2\mu_\sigma$ for every type σ on X. Suppose $X \subset Y$ where $\dim Y/X = 1$ and σ is a type on Y supported on X. Let

$$\sigma(y) = \lim_{n\to\infty} \|y + x_n\| \qquad y \in Y.$$

Then

$$\delta_{\sigma|X} \le c := \limsup_{m,n\to\infty} \|x_m - x_n\|.$$

Now there exists $u \in X$ with

$$\sigma(u) \le \tfrac{1}{2}\delta_{\sigma|X} + \epsilon \le \frac{1}{2}c + \epsilon.$$

Now for any $m, n \in \mathbb{N}$,

$$\|x_m - x_n\| \le \|y + x_n\| + \|y + x_m\| \qquad y \in Y$$

so that

$$c \le 2\sigma(y) \qquad y \in Y.$$

Hence

$$\sigma(u) \le \sigma(y) + \epsilon \qquad y \in Y$$

so that (X, Y) satisfies $\Sigma_0(1)$

Conversely suppose X has the separable universal linear c_0-AIEP and suppose σ is any type on X. Suppose $\lambda > \delta_\sigma$. Then we can find a sequence $(x_n)_{n=1}^\infty$ with $\|x_m - x_n\| \le \lambda$ for all m, n so that

$$\sigma(x) = \lim_{n\to\infty} \|x + x_n\| \qquad x \in X.$$

By Lemma 2.2 we can embed X into a space Y with $\dim Y/X \le 1$ containing a point y so that $\|y - x_j\| \le \frac{1}{2}\lambda$. We can (by passing to a subsequence of (x_n)) therefore extend σ to a type $\widetilde{\sigma}$ on Y with $\widetilde{\sigma}(y) \le \frac{1}{2}\lambda$. Since (X, Y) satisfies $\Sigma_0(1)$ we have

$$\mu_\sigma = \mu_{\widetilde{\sigma}} \le \tfrac{1}{2}\lambda.$$

Hence $\mu_\sigma \le \frac{1}{2}\delta_\sigma$.

The converse inequality $\delta_\sigma \le 2\mu_\sigma$ is trivial as observed above. $\square$

Theorem 11.5. *Let X be a closed subspace of $L_1(0,1)$. The following conditions on X are equivalent:*

(i) *X has the separable universal linear $\mathcal{C}$-AIEP.*
(ii) *X has the separable universal linear c_0-AIEP.*
(iii) *B_X is compact for the topology of convergence in measure.*

Proof. (i) $\implies$ (ii) was observed in [38].

(ii) $\implies$ (iii). Let $(f_n)_{n=1}^\infty$ be a sequence in B_X. We will prove that there is a subsequence which converges in $L_{1/2}$ to an element of B_X. By passing to a subsequence we assume that $(f_n)_{n=1}^\infty$ induces a type on L_1

$$\sigma(f) = \lim_{n\to\infty} \|f - f_n\| \qquad f \in L_1,$$

and that $\lim_{n\to\infty} \sigma(f_n) = 2\theta$ where $0 \le \theta \le 1$. By passing to a further subsequence we can suppose that

$$\|f_j - f_k\| \ge 2\theta - \nu_n \qquad j, k > n$$

where $\nu_n \downarrow 0$.

For $\epsilon > 0$ pick $h \in X$, using Proposition 11.4, so that

$$\sigma(h) < \theta + \epsilon.$$

Then there exists N so that

$$\|f_n - h\| < \theta + \epsilon, \qquad n \ge N$$

and

$$\|f_j - f_k\| > 2\theta - \epsilon \qquad j, k \ge N.$$

For any n, let $u_n = \max(f_n - h, 0)$ and $v_n = \max(h - f_n, 0)$. Then for $k > j$ if $\min(u_j(t), u_k(t)) > 0$ we have

$$\min(u_j(t), u_k(t)) = |f_j(t) - h(t)| + |f_k(t) - h(t)| - |f_j(t) - f_k(t)|.$$

Hence

$$\min(u_j, u_k) \le |f_j - h| + |f_k - h| - |f_j - f_k|$$

and so

$$\|\min(u_j, u_k)\| \le \|f_j - h\| + \|f_k - h\| - \|f_j - f_k\| < 4\epsilon \qquad j, k > N.$$

Since $\|\max(u_j, u_k)\| \le 2$ this implies that

$$\begin{aligned} \||u_j|^{1/2}|u_k|^{1/2}\| &\le \|\min(u_j, u_k)\|^{1/2}\|\max(u_j, u_k)\|^{1/2} \\ &\le 2\sqrt{2}\sqrt{\epsilon} \\ &< 3\sqrt{\epsilon}. \end{aligned}$$

Let $\mathbb{F}$ be any finite subset of $\{N+1, \ldots\}$ with $|\mathbb{F}| = r$. Then

$$\int \left(\sum_{j\in\mathbb{F}} |u_j|^{1/2} \right)^2 dt \le 2r + 3(r^2 - r)\sqrt{\epsilon}$$

and this implies that

$$\sum_{j\in\mathbb{F}} \int |u_j|^{1/2} dt \le r \left(\frac{2}{r} + 3\sqrt{\epsilon} \right)^{1/2}.$$

Since this is true for every such $\mathbb{F}$ we conclude that

$$\limsup_{j\to\infty} \int |u_j|^{1/2} dt \le \sqrt{3}\epsilon^{1/4}.$$

We can make an exactly similar calculation with v_j and we conclude that

$$\limsup_{j\to\infty} \int |v_j|^{1/2} dt \le \sqrt{3}\epsilon^{1/4}.$$

Now $f_j - h = u_j - v_j$ so that

$$\limsup_{j\to\infty} \int |f_j - h|^{1/2} \le 2\sqrt{3}\epsilon^{1/4}$$

and so

$$\limsup_{j,k\to\infty} \int |f_j - f_k|^{1/2} \le 4\sqrt{3}\epsilon^{1/4}.$$

As this holds for all ϵ we have that $(f_n)_{n=1}^\infty$ converges in $L_{1/2}(0,1)$ to some $g \in B_{L_1}$.

To conclude the proof we need to show that $g \in X$. Since $(f_n)_{n=1}^\infty$ is now convergent in measure, the type σ takes the form

$$\sigma(f) = \|f - g\| + \sigma(g) \qquad f \in L_1.$$

Furthermore $\sigma(g) = \inf_{f\in L_1} \sigma(f) = \inf_{f\in X} \sigma(f) = \theta$ using the fact that (X, L_1) has the linear c_0-AIEP and so satisfies condition $\Sigma_0(1)$ (then use Lemma 4.5). Now there is a sequence $(h_n)_{n=1}^\infty$ in X with $\lim_{n\to\infty} \sigma(h_n) = \theta$ and clearly

$$\lim_{n\to\infty} \|h_n - g\| = 0$$

so that $g \in X$.

(iii) $\implies$ (i). Corollary 10.4. □

The following result improves the result of Speegle [38] who showed that no uniformly smooth space can have the separable universal linear c_0-AIEP (actually Speegle proved this but only stated the result for the separable universal linear $\mathcal{C}$-AIEP).

Theorem 11.6. *Let X be a separable Banach space with the universal linear c_0-EP. Then no subspace of X is uniformly nonsquare.*

Proof. We use a result of Milman [31] that there exists in every subspace E of X a normalized sequence $(x_n)_{n=1}^\infty$ defining a symmetric type σ. Since σ is symmetric $\sigma(0) = \mu_\sigma = 1$ and so $\delta_\sigma = 2$. It follows that

$$\lim_{m\to\infty} \sigma(x_m) = 2$$

and hence by symmetry

$$\lim_{m\to\infty} \lim_{n\to\infty} \|x_m + x_n\| = \lim_{m\to\infty} \lim_{n\to\infty} \|x_m - x_n\| = 2.$$

Thus E cannot be uniformly nonsquare. □

Let us recall that a Banach space X has the 1-*strong Schur property* if for every bounded sequence $(x_n)_{n=1}^\infty$ with $\|x_m - x_n\| \ge 2$ for all m, n and for every $\epsilon > 0$ there is a subsequence $(x_n)_{n\in\mathbb{M}}$ such that

$$\left\|\sum_{j\in\mathbb{M}} \alpha_j x_j\right\| \ge (1-\epsilon) \sum_{j\in\mathbb{M}} |\alpha_j|$$

for every finitely nonzero sequence $(\alpha_j)_{j\in\mathbb{M}}$. This concept was introduced by Johnson and Odell [14] and studied by Rosenthal [35].

Let us say that X has the 1-*positive Schur property* if for every normalized sequence $(x_n)_{n=1}^\infty$ and for every $\epsilon > 0$ there is a subsequence $(x_n)_{n\in\mathbb{M}}$ such that

$$\left\|\sum_{j\in\mathbb{M}} \alpha_j x_j\right\| \ge (1-\epsilon) \sum_{j\in\mathbb{M}} \alpha_j$$

for every finitely nonzero sequence $(\alpha_j)_{j\in\mathbb{M}}$ with $\alpha_j \geq 0$. It is clear that the 1-positive Schur property implies the Schur property.

Theorem 11.7. *Let X be a separable Banach space. The following conditions on X are equivalent:*

(i) *X has the 1-positive Schur property.*
(ii) *For every normalized sequence $(x_n)_{n=1}^\infty$ we have $\sup_{m,n}\|x_m + x_n\| = 2$.*
(iii) *For every normalized sequence $(x_n)_{n=1}^\infty$ and $\epsilon > 0$, there exists $x^* \in X^*$ with $\|x^*\| = 1$ and $\limsup_{n\to\infty} x^*(x_n) > 1-\epsilon$.*
(iv) *For every type σ supported on a subset C there exists x^{**} in the weak*-closure of C with*

$$\sigma(x) = \|x - x^{**}\| \qquad x \in X.$$

In particular if the equivalent hypotheses (i)–(iv) *hold, X is stable.*

Proof. (i) $\implies$ (ii) is trivial.

(ii) $\implies$ (iii). This is essentially a deep result of Odell and Schlumprecht [33] (Theorem 2.1). Indeed if (ii) holds then every spreading model $(s_n)_{n=1}^\infty$ of a normalized sequence satisfies $\|s_1 + s_2\| = 2$.

(iii) $\implies$ (iv). If $\sigma(x) = \lim_{n\to\infty}\|x - x_n\|$ where $x_n \in C$ we may by passing to a subsequence assume that for every $x \in X$ and $\epsilon > 0$ there exists $x^* \in X^*$ with $\|x^*\| = 1$ and $\liminf_{n\to\infty} x^*(x - x_n) \geq (1-\epsilon)\sigma(x)$. Let x^{**} be any weak*-cluster point of this sequence $(x_n)_{n=1}^\infty$. Then $\|x - x^{**}\| = \sigma(x)$ for $x \in X$.

(iv) $\implies$ (ii). Let $(x_n)_{n=1}^\infty$ be a normalized sequence; by passing to a subsequence we can assume it induces a type $\sigma(x) = \lim_{n\to\infty}\|x - x_n\| = \|x - x^{**}\|$ for some weak*-cluster point of $(x_n)_{n=1}^\infty$. Then $\|x^{**}\| = 1$ and

$$\liminf_{m\to\infty}\lim_{n\to\infty}\|x_m + x_n\| \geq \liminf_{m\to\infty}\|x_m + x^{**}\| \geq 2.$$

(iii) $\implies$ (i). This is trivial.

Now suppose (i)–(iv) hold. We show first that X is stable. Indeed suppose $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ are two bounded sequences so that the limits

$$\lim_{m\to\infty}\lim_{n\to\infty}\|x_m + y_n\|, \lim_{n\to\infty}\lim_{m\to\infty}\|x_m + y_n\|$$

exist and are unequal. We can also suppose that the types

$$\varphi(x) = \lim_{m\to\infty}\lim_{n\to\infty}\|x + x_m + y_n\|$$

and

$$\varphi'(x) = \lim_{n\to\infty}\lim_{m\to\infty}\|x + x_m + y_n\|$$

are well-defined.

If $x \in X$ and $\epsilon > 0$ we can find sequences $n_k, m_k \to \infty$ so that

$$\varphi(x) = \lim_{k\to\infty}\|x + x_{m_k} + y_{n_k}\|$$

and there is $x^* \in X^*$ with $\|x^*\| = 1$ so that

$$\liminf_{k\to\infty} x^*(x + x_{m_k} + y_{n_k}) \geq \varphi(x) - \epsilon.$$

Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$. Then if $x^{**} = \lim_{k\in\mathcal{U}} x_{m_k}$ and $y^{**} = \lim_{k\in\mathcal{U}} y_{n_k}$ (weak*-limits in X^{**}), we have

$$\|x + x^{**} + y^{**}\| \geq \varphi(x) - \epsilon.$$

On the other hand

$$\varphi'(x) \geq \lim_{m\in\mathcal{U}} \|x + x_m + y^{**}\| \geq \|x + x^{**} + y^{**}\|.$$

Thus

$$\varphi(x) \leq \varphi'(x) + \epsilon.$$

Letting $\epsilon \to 0$ and reversing the roles of φ, φ' gives the conclusion that $\varphi = \varphi'$ and so X is stable. $\square$

Let us define the inf-convolution of two nonnegative convex functions ϕ, ψ on a Banach space by

$$\phi \square \psi(x) = \inf_{y\in X} (\phi(x-y) + \psi(y)).$$

Let us note for future reference that if σ is any type on X,

$$\sigma \square \sigma(x) = 2\sigma(x/2). \tag{11.1}$$

To see this observe that if $x, y \in X$

$$\sigma(x-y) + \sigma(y) \geq 2\sigma(x/2)$$

by the convexity of σ. We recall that if X is stable then we can define the standard convolution of two type σ, τ by

$$\sigma * \tau(x) = \lim_{m\in\mathcal{U}} \lim_{n\in\mathcal{V}} \|x - u_m - v_n\|$$

where

$$\sigma(x) = \lim_{m\in\mathcal{U}} \|x - u_m\|, \ \tau(x) = \lim_{n\in\mathcal{V}} \|x - v_n\|.$$

This convolution is unambiguous and $\sigma * \tau = \tau * \sigma$. Observe that

$$\sigma * \tau \leq \sigma \square \tau. \tag{11.2}$$

Indeed suppose $y \in X$. Then

$$\sigma(x-y) + \tau(y) = \lim_{m\in\mathcal{U}} \lim_{n\in\mathcal{V}} (\|x - y - u_m\| + \|y - v_n\|) \geq \sigma * \tau(x).$$

Let us note the following simple proposition:

Proposition 11.8. *Let X be a separable Banach space. Then X has the 1-positive Schur property if and only if X is stable and for every type σ we have*

$$\sigma * \sigma(x) = 2\sigma(x/2) = \sigma \square \sigma(x) \qquad x \in X.$$

Proof. If X has the 1-positive Schur property then X is stable. If $(x_n)_{n\in\mathbb{N}}$ is any sequence and $\mathcal{U}$ is a nonprincipal ultrafilter defining a type $\sigma(x) = \lim_{n\in\mathcal{U}} \|x + x_n\|$ then

$$\sigma * \sigma(x) = \lim_{m\in\mathcal{U}} \lim_{n\in\mathcal{U}} \|(\tfrac{1}{2}x + x_m) + (\tfrac{1}{2}x + x_n)\| = 2\sigma(x/2).$$

The converse is similar. $\square$

Theorem 11.9. *Let X be a separable Banach space. Then the following conditions on X are equivalent:*

(i) *X has the universal Lipschitz $\mathcal{C}$-AIEP.*
(ii) *The following three conditions hold:*
  (a) *X has the 1-positive Schur property (and hence is stable).*
  (b) *For any type σ, we have $\delta_\sigma = 2\mu_\sigma$.*

(c) *For any pair of types σ,τ we have*

$$\sigma\Box\tau(x)=\max(\sigma*\tau(x),\mu_\sigma+\mu_\tau)\qquad x\in X. \tag{11.3}$$

Proof. (ii) $\implies$ (i). Let Y be any Banach space containing X. We show that (X,Y) satisfies condition $\Sigma_1(1)$. Suppose $y\in Y\setminus X$. By Lemma 4.5 we need only show that if σ and τ are two types on $[X,y]$, supported on X, then we have

$$\sigma|_X\Box\tau|_X(0)=\sigma\Box\tau(0).$$

(apply the criterion of Lemma 4.5 to σ and $\widetilde{\tau}(x)=\tau(-x)$.

It is clear from (b) that (X,Y) satisfies condition $\Sigma_0(1)$ so that $\mu_\sigma=\mu_{\sigma|_X}$ and $\mu_\tau=\mu_{\tau|_X}$. Thus applying (c), (11.3), we have (since $(\sigma*\tau)|_X=\sigma|_X*\tau|_X$)

$$\sigma|_X\Box\tau|_X(0)=\max(\sigma*\tau(0),\mu_\sigma+\mu_\tau)\le\sigma\Box\tau(0).$$

This implies (i).

(i) $\implies$ (ii). X certainly must have the universal linear c_0-AIEP by Proposition 11.1; this implies that for any type σ we have $\delta_\sigma=2\mu_\sigma$ (Proposition 11.4). We next make the following claim:

Claim. If $(u_n)_{n=1}^\infty$ and $(v_n)_{n=1}^\infty$ are two bounded sequences such that the types $\sigma(x)=\lim_{n\to\infty}\|x-u_n\|$, $\tau(x)=\lim_{n\to\infty}\|y-v_n\|$ and $\varphi(x)=\lim_{n\to\infty}\sigma(x-v_n)$ are well-defined then

$$\sigma\Box\tau(x)=\max(\varphi(x),\mu_\sigma+\mu_\tau)\qquad x\in X. \tag{11.4}$$

Let us suppose by way of contradiction that there exists $w\in X$ with

$$\lambda:=\max(\varphi(w),\mu_\sigma+\mu_\tau)<\sigma\Box\tau(w).$$

We will choose

$$\nu=\tfrac{1}{3}(\sigma\Box\tau(w)-\lambda).$$

By passing to subsequences, since $\delta_\sigma=2\mu_\sigma$, we can suppose that

$$\|u_m-u_n\|<2\mu_\sigma+\nu,\quad \|v_m-v_n\|<2\mu_\tau+\nu\qquad m,n\in\mathbb{N}. \tag{11.5}$$

We may further suppose (by passing to subsequences) that

$$\|w-u_m-v_n\|<\varphi(w)+\nu\le\lambda+\nu\qquad m,n\in\mathbb{N}. \tag{11.6}$$

We may now choose $\theta_1,\theta_2\ge 0$ so that

$$\begin{aligned}2\mu_\sigma+\nu&\le 2\theta_1\\ 2\mu_\tau+\nu&\le 2\theta_2\\ \lambda+\nu&=\theta_1+\theta_2.\end{aligned}$$

This is permissible since

$$\mu_\sigma+\mu_\tau+\nu\le\lambda+\nu.$$

It follows from (11.5), (11.6) and Lemma 2.2 above that we can find a Banach space Y containing X with $\dim Y/X\le 1$ and $y\in Y$ such that

$$\|y+w-u_n\|\le\theta_1,\quad \|y+v_n\|\le\theta_2\qquad n\in\mathbb{N}.$$

Now (X,Y) satisfies $\Sigma_1(1)$. It follows that there exists a point $x\in X$ with

$$\|x+w-u_m\|+\|x+v_n\|<\theta_1+\theta_2+\nu$$

for infinitely many n so that

$$\sigma(x+w)+\tau(-x)\leq\theta_1+\theta_2+\nu$$

i.e.,

$$\sigma\Box\tau(w)\leq\theta_1+\theta_2+2\nu<\sigma\Box\tau(w).$$

This contradiction establishes (11.4) and the claim.

Now suppose $(u_n)_{n=1}^{\infty}$ is any normalized sequence in X. Passing to a subsequence we can suppose that for any finite sequence of nonzero reals $(\alpha_1,\ldots,\alpha_n)$ and any $x\in X$ the limit

$$\sigma_{\alpha_1,\ldots,\alpha_n}(x):=\lim_{\substack{i_1\to\infty\\ i_1<i_2<\cdots<i_n}}\|x+\alpha_1u_{i_1}+\cdots+\alpha_nu_{i_n}\|\ \text{ exists.}$$

Then $\sigma_{1,-1,1,-1}\leq\sigma_{1,1}\Box\sigma_{-1,-1}$ so that

$$\sigma_{1,-1,1,-1}(0)\leq 2\sigma_{1,1}(0).$$

Similarly

$$\sigma_{1,-1,-1,1}(0)\leq 2\sigma_{1,1}(0).$$

However

$$\delta_{\sigma_{1,-1}}\leq\sigma_{1,-1,-1,1}(0)$$

so that

$$\mu_{\sigma_{1,-1}}\leq\sigma_{1,1}(0).$$

Now

$$2\sigma_{1,-1}(0)=\sigma_{1,-1}\Box\sigma_{1,-1}(0)=\max(2\mu_{\sigma_{1,-1}},\sigma_{1,-1,1-1}(0))$$

by (11.4). Hence

$$\sigma_{1,-1}(0)\leq\sigma_{1,1}(0).$$

Now

$$\begin{aligned}2&=2\sigma_1(0)\\&=\sigma_1\Box\sigma_1(0)\\&=\max(2\mu_{\sigma_1},\sigma_{1,1}(0))\\&=\max(\delta_{\sigma_1},\sigma_{1,1}(0))\\&\leq\max(\sigma_{1,-1}(0),\sigma_{1,1}(0))\\&\leq\sigma_{1,1}(0).\end{aligned}$$

Hence

$$\lim_{m,n\to\infty}\|u_m+u_n\|=2$$

and X has the 1-positive Schur property by Theorem 11.7. In particular X is stable and we may rewrite (11.4) as (11.3). □

Note that in (11.3) the term $\mu_\sigma+\mu_\tau$ cannot in general be eliminated. Indeed consider the space $\ell_1\oplus_\infty\ell_1$ which has the Lipschitz $\mathcal{C}$-AIEP by Theorem 10.8. We now invoke a hypothesis which eliminates this type of example.

We recall [13] that if X is a Banach space we define its modulus of asymtotic convexity by

$$\bar{\delta}_X(t)=\inf_{\|x\|=1}\sup_{\dim X/Y<\infty}\inf_{\substack{y\in Y\\ \|y\|=t}}\left(\|x+ty\|-1\right).$$

X is called *asymptotically uniformly convex* if $\bar{\delta}_X(t)>0$ whenever $t>0$.

Lemma 11.10. *Suppose X is a separable asymptotically uniformly convex Banach space with the universal Lipschitz $\mathcal{C}$-AIEP. Then for any type σ the set $K_\sigma = \{x : \sigma(x) = \mu_\sigma\}$ is nonempty and compact and if $\lim_{n\to\infty} \sigma(y_n) = \mu_\sigma$ then $\lim_{n\to\infty} d(y_n, K_\sigma) = 0$.*

Proof. It suffices to consider the case when $\mu_\sigma > 0$. It suffices to show that if $(y_n)_{n=1}^\infty$ is any sequence such that $\lim_{n\to\infty} \sigma(y_n) = 0$ then $\inf_{m\neq n} \|y_m - y_n\| = 0$. Indeed let us suppose that $\|y_m - y_n\| \geq \nu > 0$ if $m \neq n$. Let $\sigma(x) = \lim_{n\to\infty} \|x - x_n\|$ for some sequence $(x_n)_{n=1}^\infty$. Since the family $\{x_m - x_n : m \neq n\}$ is bounded above and below there exists $\epsilon > 0$ and a finite co-dimensional subspace $Z(m,n)$ for each $m \neq n$ so that if $z \in Z(m,n)$ and $\|z\| \geq \nu/2$,

$$\|x_m - x_n + z\| \geq \|x_m - x_n\| + 3\epsilon.$$

Fix a nonprincipal ultrafilter $\mathcal{U}$. Then

$$\lim_{r\in\mathcal{U}} \lim_{s\in\mathcal{U}} d(y_r - y_s, Z(m,n)) = 0$$

and so

$$\lim_{r\in\mathcal{U}} \lim_{s\in\mathcal{U}} \|x_m - x_n + y_r - y_s\| \geq \|x_m - x_n\| + 3\epsilon, \qquad m \neq n.$$

This implies that

$$\lim_{m\in\mathcal{U}} \lim_{n\in\mathcal{U}} \lim_{r\in\mathcal{U}} \lim_{s\in\mathcal{U}} \|x_m - x_n + y_r - y_s\| \geq \delta_\sigma + 2\epsilon.$$

Since $\delta_\sigma = 2\mu_\sigma$ we deduce that

$$\lim_{n\in\mathcal{U}} \lim_{r\in\mathcal{U}} \|y_r - x_n\| \geq \mu_\sigma + \epsilon$$

and since X is stable we have

$$\lim_{r\in\mathcal{U}} \sigma(y_r) \geq \mu_\sigma + \epsilon$$

which is a contradiction. $\square$

Theorem 11.11. *Let X be a separable Banach space which is asymptotically uniformly convex. Then the following are equivalent:*

(i) *X has the universal Lipschitz $\mathcal{C}$-AIEP.*
(ii) *For any pair of types σ and τ we have $\sigma * \tau = \sigma \square \tau$.*
(iii) *X has the separable universal linear c-AIEP.*

Proof. (i) $\implies$ (ii). Let us suppose that

$$\sigma * \tau(x) \leq \mu_\sigma + \mu_\tau.$$

Suppose $y \neq 0$. Then there exist $-\infty < \lambda_1 \leq 0 \leq \lambda_2 < \infty$ so that

$$\sigma * \tau(x + \lambda_j y) = \mu_\sigma + \tau_\sigma = \sigma \square \tau(x + \lambda_j y) \quad j = 1, 2.$$

Thus for $j = 1, 2$ we can find sequences $(u_n)_{n=1}^\infty$ and $(v_n)_{n=1}^\infty$ with $u_{j,n} + v_{j,n} = x + \lambda_j y$ and $\lim_{n\to\infty} \sigma(u_n) + \tau(v_n) = \mu_\sigma + \mu_\tau$. By Lemma 11.10 we can find $u_j \in K_\sigma$, $v_j \in K_\tau$ so that $u_j + v_j = x + \lambda_j y$. Since the sets K_σ and K_τ are both convex this implies that $x \in K_\sigma + K_\tau$. However $K_\sigma + K_\tau$ is also compact and thus has no interior. This implies that $\sigma * \tau$ is constant on $K_\sigma + K_\tau$. Since $\sigma * \tau$ is continuous it follows that $\sigma * \tau(x) = \mu_\sigma + \mu_\tau$ on $K_\sigma + K_\tau$. By (11.3) we obtain (ii).

(ii) $\implies$ (iii). By induction we deduce that for any n types $\sigma_1, \ldots, \sigma_n$ we have

$$\sigma_1 * \sigma_2 * \cdots * \sigma_n = \sigma_1 \square \sigma_2 \square \cdots \square \sigma_n.$$

Now suppose X is embedded in a separable Banach space Y. We use Proposition 6.3 to show that (X, Y) satisfies $\Gamma_n(1)$ for all n. Indeed if $\sigma_1, \dots, \sigma_n$ are types on Y supported on X, we have for $x \in X$

$$\sigma_1|_X \Box \sigma_2|_X \Box \cdots \Box \sigma_n|_X(x) = \sigma_1 * \sigma_2 * \cdots * \sigma_n(x) \leq \sigma_1 \Box \sigma_2 \Box \cdots \Box \sigma_n(x).$$

Finally we use Theorem 6.9.

(iii) $\implies$ (i). This follows trivially from Corollary 6.8. □

We note here that we do not know if the universal Lipschitz $\mathcal{C}$-AIEP is in general equivalent to the separable universal linear $\mathcal{C}$-AIEP even under the conditions of Theorem 11.11.

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