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Total cofibres of diagrams of spectra

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Abstract. If Y is a diagram of spectra indexed by an arbitrary poset $\mathcal{C}$ together with a specified sub-poset $\mathcal{D}$, we define the *total cofibre* $\Gamma(Y)$ of Y as $\mathrm{cofibre}(\mathrm{hocolim}_{\mathcal{D}}(Y) \longrightarrow \mathrm{hocolim}_{\mathcal{C}}(Y))$. We construct a comparison map $\hat{\Gamma}_Y \colon \mathrm{holim}_{\mathcal{C}} Y \longrightarrow \hom(Z, \hat{\Gamma}(Y))$ to a mapping spectrum of a fibrant replacement of $\Gamma(Y)$ where Z is a simplicial set obtained from $\mathcal{C}$ and $\mathcal{D}$, and characterise those poset pairs $\mathcal{D} \subset \mathcal{C}$ for which $\hat{\Gamma}_Y$ is a stable equivalence. The characterisation is given in terms of stable cohomotopy of spaces related to Z. For example, if $\mathcal{C}$ is a finite polytopal complex with $|\mathcal{C}| \cong B^m$ a ball with boundary sphere $|\mathcal{D}|$, then $|Z| \cong_{PL} S^m$, and $\hat{\Gamma}(Y)$ and $\mathrm{holim}_{\mathcal{C}}(Y)$ agree up to m-fold looping and up to stable equivalence. As an application of the general result we give a spectral sequence for $\pi_*(\Gamma(Y))$ with E_2-term involving higher derived inverse limits of $\pi_*(Y)$, generalising earlier constructions for space-valued diagrams indexed by the face lattice of a polytope.

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1. Introduction

Let $\mathcal{D} \subseteq \mathcal{C}$ be a pair of posets, considered as categories (with arrows starting at the smaller element). The motivating example will be the poset $\mathcal{C}(P)$ of nonempty faces of a polytope P, and its sub-poset $\mathcal{D}(P)$ of proper faces. More generally, we will consider polytopal complexes $\mathcal{D} \subset \mathcal{C}$ with underlying spaces PL-homeomorphic to $S^{m-1} \subset B^m$.

Definition 1. Let $X \colon \mathcal{C} \longrightarrow \mathrm{sSet}_*,\ F \mapsto X^F$ be a diagram of pointed simplicial sets. The *total cofibre* $\Gamma(X)$ of the diagram X is the (strict) cofibre of the map $\mathrm{hocolim}_{\mathcal{D}}(X) \longrightarrow \mathrm{hocolim}_{\mathcal{C}}(X)$.

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The homotopy colimit is the coend $\operatorname{hocolim}_{\mathcal{C}}(X) = N(- \downarrow \mathcal{C})_+ \otimes_{\mathcal{C}} X$ formed with respect to the smash product of pointed simplicial sets. In a similar way, the total cofibre functor can be defined for diagrams of pointed topological spaces (or, more generally, for diagrams with values in a pointed simplicial model category).

For diagrams with values in pointed topological spaces, indexed by $\mathcal{C} = \mathcal{C}(P)$ and $\mathcal{D} = \mathcal{D}(P)$, there is a convergent first quadrant spectral sequence

$$E^2_{p,q} = \lim{}^{n-p} \widetilde{H}_q(X;\mathbb{Z}) \implies \widetilde{H}_{p+q}(\Gamma(X);\mathbb{Z})$$

where $n = \dim P$, cf. [Hüt04, Theorem 2.23] (note that the definition of Γ given there is homotopy equivalent to the one given here since the natural map

$$\operatorname{hocolim}_{\mathcal{D}} Y \longrightarrow \operatorname{hocolim}_{\mathcal{C}} Y$$

is a cofibration, hence cofibre and homotopy cofibre are homotopy equivalent). The appearance of higher derived *inverse* limits in the E^2-term might be unexpected and results from an argument using ČECH cohomology. This paper offers a different and more satisfying explanation: we prove that, stably, total cofibres and homotopy limits agree. More precisely, we construct a comparison map from homotopy limits to a loop space of the total cofibre (Definition 5), and prove that under certain conditions on $\mathcal{D}$ and $\mathcal{C}$ this map is a weak equivalence in stable homotopy (i.e., for diagrams of spectra instead of simplicial sets; Theorem 6). Then the usual BOUSFIELD–KAN spectral sequence for homotopy limits can be used to construct a spectral sequence of the type above (Theorem 10). The conditions on $\mathcal{C}$ and $\mathcal{D}$ are phrased in terms of stable cohomotopy and completely characterise those poset pairs for which total cofibres and homotopy limits agree.

The total cofibre functor has been used by the author to examine categories of space-valued quasi-coherent sheaves on projective toric varieties [Hüt04, Hüt]. It replaces the global sections functor and its higher deriveds, the sheaf cohomology functors, in algebraic geometry. The spectral sequence mentioned above provides a means to relate the topological construction to the algebraic context [Hüt04, Theorem 3.8] since computation of $\lim^p$ amounts to taking pth sheaf cohomology. Note also that computing sheaf cohomology can be thought of as a homotopy limit process in a category of chain complexes, so the results of this paper show that in fact the total cofibre functor is completely analogous to sheaf cohomology.

2. Spectra and stable cohomotopy

We denote the category of BOUSFIELD–FRIEDLANDER spectra [BF78] by $\mathfrak{Sp}$. Thus a spectrum X is a sequence of pointed simplicial sets $X_0, X_1, \ldots$ and structure maps $\Sigma X_n \longrightarrow X_{n+1}$. The category $\mathfrak{Sp}$ has two simplicial closed model structures, the *level structure* with levelwise weak equivalences and levelwise fibrations, and the *stable structure* with stable equivalences and with cofibrations as in the level structure [BF78, §2].

Both model structures of $\mathfrak{Sp}$ are cofibrantly generated [Hir03, §11.1] with generating cofibrations the maps $\alpha_{j,k}\colon F_j(\partial\Delta^k_+) \longrightarrow F_j(\Delta^k_+)$ for $j,k \in \mathbb{N}$. Here $F_j(A)$ is the free spectrum generated by $A \in \mathrm{sSet}_*$ in degree j; in other words, $F_j(A)$ is the suspension spectrum of A shifted j times. The cofibre of $\alpha_{j,k}$ is the spectrum $\widetilde{K}_{j,k} := F_j(\Delta^k/\partial\Delta^k)$. We let $\widetilde{K}_{j,k} \xrightarrow{\simeq} K_{j,k}$ denote a stably fibrant replacement.

For $W \in \mathfrak{Sp}$ and $A \in \mathrm{sSet}_*$ we denote by $\hom_{\mathrm{sSet}_*}(A, W)$ the spectrum which has the mapping space $\hom_{\mathrm{sSet}_*}(A, W_n)$ in level n; since the hom functor commutes with looping, we can use the structure maps $W_n \longrightarrow \Omega W_{n+1}$ to make $\hom_{\mathrm{sSet}_*}(A, W)$ into a spectrum.

Definition 2. For $n \in \mathbb{Z}$ the nth stable cohomotopy group of $A \in \mathrm{sSet}_*$ is defined as $\pi_s^n(A) := \pi_m \hom_{\mathrm{sSet}_*}(A, K_{j,k})$ where $m \in \mathbb{Z}$ and $j, k \in \mathbb{N}$ are numbers satisfying $k - j - m = n$.

The definition of $\pi_s^n(A)$ does not depend on the choices of m, j and k: There are isomorphisms $\pi_m \hom_{\mathrm{sSet}_*}(A, K_{j,k}) \cong \pi_{m-1} \hom_{\mathrm{sSet}_*}(S^1, \hom_{\mathrm{sSet}_*}(A, K_{j,k})) \cong \pi_{m-1} \hom_{\mathrm{sSet}_*}(A, \hom_{\mathrm{sSet}_*}(S^1, K_{j,k}))$, and the two spectra $\hom_{\mathrm{sSet}_*}(S^1, K_{j,k})$ and $K_{j+1,k}$ are weakly equivalent. Similarly, $\hom_{\mathrm{sSet}_*}(S^1, K_{j,k}) \simeq K_{j,k-1}$. Similar arguments work for suspension instead of looping.

For a $\mathcal{C}$-diagram X in sSet_* recall that $\mathrm{holim}_{\mathcal{C}}(X) = \hom_{\mathcal{C}}(N(\mathcal{C} \downarrow \text{-})_+, X)$, the space of natural transformations $N(\mathcal{C} \downarrow \text{-})_+ \longrightarrow X$. If Y is a diagram of spectra we can define $\mathrm{holim}_{\mathcal{C}}(Y) = \hom_{\mathcal{C}}(N(\mathcal{C} \downarrow \text{-})_+, Y)$ by forming the homotopy limit in each level; since holim commutes with looping, we can use the structure maps $Y_n \longrightarrow \Omega Y_{n+1}$ to make the result into a spectrum.

3. Comparing homotopy limits and total cofibres

Let $Y \colon \mathcal{C} \longrightarrow \mathfrak{Sp}$ be a diagram of spectra. Since the functor Γ (Definition 1) commutes with suspensions, we can apply Γ in each level of Y and obtain a spectrum $\Gamma(Y)$. Thus we can and will consider Γ as a functor from diagrams of spectra to spectra.

Lemma 3. *The functor Γ commutes with homotopy colimits. If $Y \longrightarrow Z$ is a natural transformation of diagrams of spectra consisting of stable equivalences, then $\Gamma(Y) \longrightarrow \Gamma(Z)$ is a stable equivalence.*

Proof. The first assertion is immediate from the definition. For the second, note that homotopy colimits preserve stable equivalences [Tho85, Lemma 5.18]. The claim then follows by comparing the long exact sequence of stable homotopy groups associated to the injection $\iota \colon \mathrm{hocolim}_{\mathcal{D}} Y \longrightarrow \mathrm{hocolim}_{\mathcal{C}} Y$ and the corresponding sequence for Z in place of Y. (Note that the homotopy cofibre of ι is levelwise weakly equivalent to the strict cofibre, and similarly for Z instead of Y.) □

Lemma 4. *The functor Γ is continuous (i.e., induces a map of* hom*-spaces).*

Proof. Let $Y, Z \colon \mathcal{C} \longrightarrow \mathfrak{Sp}$ be two diagrams. An n-simplex on the mapping space $\hom_{\mathcal{C}}(Y, Z)$ is a natural transformation of the form $Y \wedge \Delta^n_+ \longrightarrow Z$. Since Γ commutes with the functor $\text{-} \wedge \Delta^n_+$, application of the total cofibre functor yields a map $\Gamma(Y) \wedge \Delta^n_+ \longrightarrow \Gamma(Z)$ which is a simplex in the mapping space $\hom_{\mathfrak{Sp}}(\Gamma(Y), \Gamma(Z))$. □

Definition 5. Let $Y \colon \mathcal{C} \longrightarrow \mathfrak{Sp}$ be a diagram of spectra. By Lemma 4 the functor Γ induces a natural map of spectra

$$\Gamma_Y \colon \mathrm{holim}_{\mathcal{C}}(Y) = \hom_{\mathcal{C}}\big(N(\mathcal{C} \downarrow \text{-})_+, Y\big) \longrightarrow \hom_{\mathrm{sSet}_*}\big(\Gamma(N(\mathcal{C} \downarrow \text{-})_+), \Gamma(Y)\big).$$

Let $j_Y\colon \Gamma(Y) \xrightarrow{\simeq} \hat{\Gamma}(Y)$ denote a functorial level fibrant replacement; we denote by $\hat{\Gamma}_Y$ the composition

$$\hat{\Gamma}_Y = (j_Y)_* \circ \Gamma_Y \colon \operatorname{holim}_{\mathcal{C}}(Y) \longrightarrow \hom_{\mathrm{sSet}_*}\big(\Gamma(N(\mathcal{C}\downarrow \text{-})_+), \hat{\Gamma}(Y)\big).$$

We need another bit of notation. Suppose that $\mathcal{C}$ is a (possibly infinite) poset, considered as a category; for any $F \in \mathcal{C}$ we write $\mathcal{C}^F := \{G \in \mathcal{C} \mid F \not\leq G\}$. If $\mathcal{C}$ happens to be a polytopal or simplicial complex, then $\mathcal{C}^F$ is $\mathcal{C}$ with the open star of F removed (see [Hüt, §1.1] for more on this terminology).

Suppose $\mathcal{D}$ is an order ideal in the poset $\mathcal{C}$ (i.e., $\mathcal{D} \subseteq \mathcal{C}$, and for all $x, y \in \mathcal{C}$, if $x \leq y \in \mathcal{D}$ then $x \in \mathcal{D}$). We will consider the following conditions on the pair $(\mathcal{D}, \mathcal{C})$:

(P1) The space $N(\mathcal{C})/N(\mathcal{C}^F)$ has trivial cohomotopy for all $F \in \mathcal{D}$, i.e., for all $F \in \mathcal{D}$ and all $m \in \mathbb{Z}$ we have $\pi_s^m(N(\mathcal{C})/N(\mathcal{C}^F)) = 0$.

(P2) The map $\beta\colon N(\mathcal{C})/N(\mathcal{D}) \longrightarrow N(\mathcal{C})/N(\mathcal{C}^F)$ is a stable cohomotopy equivalence for all $F \in \mathcal{C} \setminus \mathcal{D}$; equivalently, the homotopy cofibre of β has trivial stable cohomotopy for all $F \in \mathcal{C} \setminus \mathcal{D}$.

Theorem 6. *Suppose $\mathcal{D}$ is an order ideal in the poset $\mathcal{C}$. The following statements are equivalent:*

(1) *Given any diagram $Y\colon \mathcal{C} \longrightarrow \mathfrak{Sp}$, $F \mapsto Y^F$ of spectra with Y^F levelwise* Kan *for all $F \in \mathcal{C}$, the comparison map*

$$\hat{\Gamma}_Y \colon \operatorname{holim}_{\mathcal{C}}(Y) \longrightarrow \hom_{\mathrm{sSet}_*}\big(\Gamma(N(\mathcal{C}\downarrow \text{-})_+), \hat{\Gamma}(Y)\big)$$

is a stable weak equivalence of spectra.

(2) *The pair $(\mathcal{D}, \mathcal{C})$ satisfies the conditions* (P1) *and* (P2) *specified above.*

Lemma 7. *Suppose that the posets $\mathcal{D}$ and $\mathcal{C}$ satisfy the conditions* (P1) *and* (P2). *Fix $j, k \in \mathbb{N}$. Given $F \in \mathcal{C}$ we define a diagram of spectra*

$$Q := \mathcal{C}(F, \text{-})_+ \wedge K_{j,k} \colon \mathcal{C} \longrightarrow \mathfrak{Sp}, \quad G \mapsto \mathcal{C}(F, G)_+ \wedge K_{j,k}$$

where $K_{j,k}$ is as defined in §2. Then Γ_Q and $\hat{\Gamma}_Q$ are weak equivalences.

Proof. *Case* 1: $F \in \mathcal{D}$. Recall that $\Gamma(Q)$ is the (strict) cofibre of the map

$$\operatorname{hocolim}_{\mathcal{D}}(Q) \xrightarrow{\kappa} \operatorname{hocolim}_{\mathcal{C}}(Q).$$

Since $Q = \mathcal{C}(F, \text{-})_+ \wedge K_{j,k}$, we have isomorphisms

$$\operatorname{hocolim}_{\mathcal{D}} Q \cong N(F \downarrow \mathcal{D})_+ \wedge K_{j,k} \quad \text{and} \quad \operatorname{hocolim}_{\mathcal{C}} Q \cong N(F \downarrow \mathcal{C})_+ \wedge K_{j,k},$$

the map κ being induced by the inclusion $\mathcal{D} \subset \mathcal{C}$. Both $F \downarrow \mathcal{D}$ and $F \downarrow \mathcal{C}$ have the initial object F, hence their nerves are simplicially contractible, with contraction induced by the functor sending every object to F. Thus $\Gamma(Q)$ is simplicially contractible as well. Since mapping spaces respect simplicial homotopies we conclude

$$\hom_{\mathrm{sSet}_*}(\Gamma(N(\mathcal{C}\downarrow \text{-})_+), \Gamma(Q)) \simeq *.$$

Now $\Gamma(Q) \simeq *$ implies $\hat{\Gamma}(Q) \simeq *$, hence $\hom_{\mathrm{sSet}_*}(\Gamma(N(\mathcal{C}\downarrow \text{-})_+), \hat{\Gamma}(Q)) \simeq *$ since $\hat{\Gamma}(Q)$ is level fibrant.

On the other hand, by definition

$$\operatorname{holim}_{\mathcal{C}}(Q) = \hom_{\mathcal{C}}\left(N(\mathcal{C}\downarrow \text{-})_+, Q\right).$$

Since Q is the constant diagram with value $K_{j,k}$ on $F \downarrow \mathcal{C}$, and since $Q(G) = *$ if $G \in \mathcal{C}^F$ we have an isomorphism

$$\operatorname{holim}_{\mathcal{C}}(Q) \cong \hom_{\mathrm{sSet}_*}(N(\mathcal{C})/N(\mathcal{C}^F), K_{j,k}).$$

The target spectrum has homotopy groups the stable cohomotopy of the space $N(\mathcal{C})/N(\mathcal{C}^F)$ (Definition 2). By hypothesis (P1) this means that both sides of the comparison map

$$\Gamma_Q \colon \operatorname{holim}_{\mathcal{C}}(Q) \longrightarrow \hom_{\mathrm{sSet}_*}(\Gamma(N(\mathcal{C} \downarrow \text{-})_+), \Gamma(Q))$$

are weakly contractible, and the same is true for $\hat{\Gamma}_Q$. Hence Γ_Q and $\hat{\Gamma}_Q$ are weak equivalences as claimed.

Case 2: $F \in \mathcal{C} \setminus \mathcal{D}$. Then Q is trivial on $\mathcal{D}$ and hence $\operatorname{hocolim}_{\mathcal{D}}(Q) = *$. The natural map α from homotopy colimit to colimit

$$\alpha \colon \operatorname{hocolim}_{\mathcal{C}}(Q) \cong \Gamma(Q) \cong N(F \downarrow \mathcal{C})_+ \wedge K_{j,k} \longrightarrow \operatorname{colim}_{\mathcal{C}} Q \cong S^0 \wedge K_{j,k} \cong K_{j,k}$$

is induced by sending all nonbasepoints in $N(F \downarrow \mathcal{C})_+$ to the nonbasepoint in S^0. Note that $N(F \downarrow \mathcal{C})$ is simplicially contractible, hence α is a simplicial homotopy equivalence.

Arguing as in Case 1 we see that $\operatorname{holim}_{\mathcal{C}} Q \cong \hom_{\mathrm{sSet}_*}(N(\mathcal{C})/N(\mathcal{C}^F), K_{j,k})$. Moreover, since $\mathcal{D}$ is an order ideal in $\mathcal{C}$ we have $\mathcal{D} \subseteq \mathcal{C}^F$. Thus we obtain the following commutative diagram:

$$\begin{array}{ccc}
\operatorname{holim}_{\mathcal{C}} Q = \hom_{\mathcal{C}}(N(\mathcal{C} \downarrow \text{-})_+, Q) & \xrightarrow{\Gamma_Q} & \hom_{\mathrm{sSet}_*}(\Gamma(N(\mathcal{C} \downarrow \text{-})_+), \Gamma(Q)) \\
\cong \downarrow & & \\
\hom_{\mathrm{sSet}_*}\big(N(\mathcal{C})/N(\mathcal{C}^F), K_{j,k}\big) & & \alpha_* \downarrow \simeq \\
f \downarrow & & \\
\hom_{\mathrm{sSet}_*}(N(\mathcal{C})/N(\mathcal{D}), K_{j,k}) & \xrightarrow[g]{} & \hom_{\mathrm{sSet}_*}(\Gamma(N(\mathcal{C} \downarrow \text{-})_+), K_{j,k}).
\end{array}$$

Here α_* is a weak equivalence since α is a simplicial homotopy equivalence. The map f is a weak equivalence by (P2). The map g is induced by the natural weak equivalences from homotopy colimits to colimits

$$(*) \qquad \operatorname{hocolim}_{\mathcal{D}} N(\mathcal{D} \downarrow \text{-})_+ \xrightarrow{\simeq} N\mathcal{D}_+ \quad \text{and} \quad \operatorname{hocolim}_{\mathcal{C}} N(\mathcal{C} \downarrow \text{-})_+ \xrightarrow{\simeq} N\mathcal{C}_+,$$

hence is a weak equivalence as well; recall that since $K_{j,k}$ is fibrant, the hom-functor preserves weak equivalences. (To see why the maps $(*)$ are weak equivalences, one can argue as follows: for any category $\mathcal{E}$ we have (using *unpointed topological spaces* for ease of notation, and writing $\operatorname{hocolim}'$ for unpointed homotopy colimits)

$$\begin{aligned}
\operatorname{hocolim}'_{\mathcal{E}} |N(\mathcal{E} \downarrow \text{-})| = |N(\text{-} \downarrow \mathcal{E})| \otimes_{\mathcal{E}} |N(\mathcal{E} \downarrow \text{-})| &\xrightarrow{(a)} \operatorname{hocolim}'_{\mathcal{E}^{\mathrm{op}}} |N(\text{-} \downarrow \mathcal{E})| \\
&\xrightarrow{\simeq} \operatorname{hocolim}'_{\mathcal{E}^{\mathrm{op}}} |*| \xrightarrow{(a)} |*| \otimes_{\mathcal{E}} |N(\mathcal{E} \downarrow \text{-})| = \operatorname{colim}_{\mathcal{E}} |N(\mathcal{E} \downarrow \text{-})| = |N\mathcal{E}|,
\end{aligned}$$

the arrow marked with "$\simeq$" being a weak equivalence by homotopy invariance of homotopy colimits. The two arrows marked (a) use the *anti-simplicial* identification

$N(\mathcal{C} \downarrow \text{-}) \cong N(\text{-} \downarrow \mathcal{C}^{\mathrm{op}})$, inducing homeomorphisms after geometric realisation. The composite map is induced by the canonical map from homotopy colimits to colimits, as can be checked by a direct calculation.)

Since four of the maps in the diagram are weak equivalences, the fifth map Γ_Q is necessarily a weak equivalence as well.

Finally, since K_{j_k} is level fibrant we can choose a map $\ell\colon \hat{\Gamma}(Q) \longrightarrow K_{j,k}$ with $\ell \circ j_Q = \alpha$ (cf. Definition 5). Then ℓ is a level equivalence between level fibrant spectra, hence ℓ_* is a weak equivalence of hom spectra. From $\ell_* \circ (j_Q)_* = \alpha_*$ we conclude that $(j_Q)_*$ is a weak equivalence, hence so is $\hat{\Gamma}_Q = (j_Q)_* \circ \Gamma_Q$. □

Proof of Theorem 6. The category $\mathrm{Fun}\,(\mathcal{C}, \mathfrak{Sp})$ has a model structure where a morphism $Y \longrightarrow X$ is a weak equivalence (fibration) if and only if for all $F \in \mathcal{C}$ the map $Y^F \longrightarrow X^F$ is a levelwise weak equivalence (levelwise fibration). We call the resulting model structure on $\mathrm{Fun}\,(\mathcal{C}, \mathfrak{Sp})$ the *ℓ-structure* and will speak of ℓ-equivalences, ℓ-fibrations, *etc.* We will use the symbol $\simeq$ to denote ℓ-equivalences. Similarly, the category $\mathrm{Fun}\,(\mathcal{C}, \mathfrak{Sp})$ has a model structure where a morphism $Y \longrightarrow X$ is a weak equivalence (fibration) if and only if for all $F \in \mathcal{C}$ the map $Y^F \longrightarrow X^F$ is a stable weak equivalence (stable fibration). We call the resulting model structure on $\mathrm{Fun}\,(\mathcal{C}, \mathfrak{Sp})$ the *s-structure* and will speak of s-equivalences, s-fibrations, *etc.* We will use the symbol $\sim$ to denote s-equivalences. Cofibrations are the same in both cases.

Suppose that $\mathcal{D}$ and $\mathcal{C}$ satisfy conditions (P1) and (P2). Let $Y\colon \mathcal{C} \longrightarrow \mathfrak{Sp}$ be ℓ-fibrant, and let $Y' \xrightarrow{\simeq} Y$ be an ℓ-cofibrant replacement of Y. Then Y' is ℓ-fibrant. Homotopy colimits map ℓ-equivalences to level equivalences since homotopy colimits of diagrams of simplicial sets are weakly homotopy invariant. Homotopy limits map ℓ-equivalences of ℓ-fibrant objects to ℓ-equivalences since homotopy limits of diagrams of KAN sets are weakly homotopy invariant. From the commutative square below we infer that we may assume Y to be ℓ-cofibrant and ℓ-fibrant:

$$\begin{array}{ccc} \mathrm{holim}(Y') & \xrightarrow{\hat{\Gamma}_{Y'}} & \hom_{\mathrm{sSet}_*}(\Gamma(N(\mathcal{C} \downarrow \text{-})_+), \hat{\Gamma}(Y')) \\ \simeq \downarrow & & \downarrow \simeq \\ \mathrm{holim}(Y) & \xrightarrow{\hat{\Gamma}_{Y}} & \hom_{\mathrm{sSet}_*}(\Gamma(N(\mathcal{C} \downarrow \text{-})_+), \hat{\Gamma}(Y)). \end{array}$$

Since the level structure of $\mathfrak{Sp}$ is cofibrantly generated, so is the ℓ-structure of $\mathrm{Fun}\,(\mathcal{C}, \mathfrak{Sp})$: for each $F \in \mathcal{C}$ and each generating cofibration $i\colon A \longrightarrow B$ of $\mathfrak{Sp}$ we have one generating cofibration

$$\mathcal{C}(F, \text{-})_+ \wedge A \xrightarrow{\mathrm{id} \wedge i} \mathcal{C}(F, \text{-})_+ \wedge B$$

for the ℓ-structure. Since Y is ℓ-cofibrant, it is a retract of an ℓ-cellular object. By naturality of $\hat{\Gamma}_Y$ it is thus enough to prove that $\hat{\Gamma}_Y$ is a stable equivalence for Y an ℓ-cellular object (which might not be ℓ-fibrant).

Since Y is cellular we may filter Y by a (possibly transfinite) sequence

$$(\dagger) \qquad \cdots \longrightarrow Y_i \xrightarrow{\alpha_i} Y_{i+1} \xrightarrow{\alpha_{i+1}} \cdots$$

starting with the trivial diagram $Y_0 = *$, where α_i is a pushout of a generating cofibration. (This is a special case of the small object argument, cf. [Hir03, Corollary 11.2.2].) In particular α_i is a cofibration and Y_i is cofibrant. (If $\mathcal{C}$ is countable the sequence (†) is indexed by the natural numbers. In general, it will be indexed by a regular cardinal λ, and at each limit ordinal $\kappa < \lambda$ the natural map $\operatorname{colim}_{\mu<\kappa} Y_\mu \longrightarrow Y_\kappa$ is an isomorphism.)

We can replace this sequence by an s-equivalent one consisting of objects which are s-cofibrant and s-fibrant. This can be done by fibrant replacement in the REEDY model structure of sequences of diagrams, formed with respect to the s-structure [Hir03, §15.3]. More explicitly, set $Y'_0 := Y_0 = *$ and $\rho_0 := \mathrm{id}_{Y_0}$. Given $\rho_i \colon Y_i \xrightarrow{\sim} Y'_i$ we define $\rho_{i+1} \colon Y_{i+1} \xrightarrow{\sim} Y'_{i+1}$ as the pushout of ρ_i along α_i composed with an s-fibrant replacement $Y'_i \cup_{Y_i} Y_{i+1} \xrightarrow{\sim} Y'_{i+1}$: see Figure 1. (For a limit ordinal κ we let ρ_κ denote the composition of $\operatorname{colim}_{\mu<\kappa} \rho_\mu$ composed with an s-fibrant replacement $\operatorname{colim}_{\mu<\kappa} Y'_\mu \xrightarrow{\sim} Y'_\kappa$.) Note that by construction Y'_i is fibrant and cofibrant in both the s-structure and the ℓ-structure, and the natural maps $\beta_i \colon Y'_i \longrightarrow Y'_{i+1}$ are cofibrations.

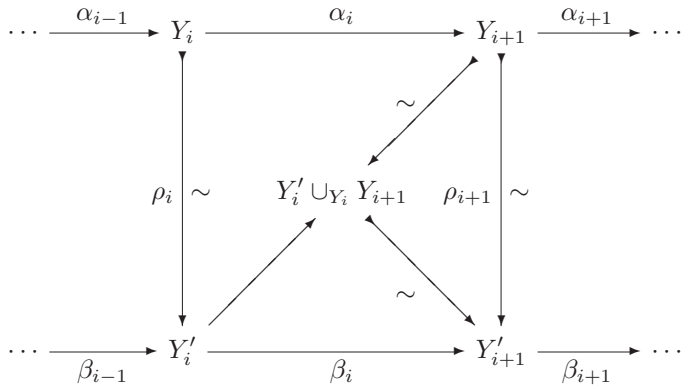

FIGURE 1. Replacement in the REEDY model structure

We then certainly have a chain of weak equivalences

$$\operatorname{hocofibre}(\beta_i) \xleftarrow{\sim} \operatorname{hocofibre}(\alpha_i) \xrightarrow{\simeq} \operatorname{cofibre}(\alpha_i).$$

Since α_i is a generating cofibration, its cofibre is of the form $\mathcal{C}(F_i, \text{-})_+ \wedge \widetilde{K}_{j,k}$ for some element $F_i \in \mathcal{C}$ and numbers $j, k \in \mathbb{N}$ (see §2 for the definition of $\widetilde{K}_{j,k}$ and $K_{j,k}$). The stably fibrant replacement $\widetilde{K}_{j,k} \xrightarrow{\simeq} K_{j,k}$ induces an s-equivalence

$$\operatorname{cofibre}(\alpha_i) \xrightarrow{\sim} \mathcal{C}(F_i, \text{-})_+ \wedge K_{j,k} =: Q_i$$

to an s-fibrant diagram Q_i. Since $\operatorname{hocofibre}(\beta_i)$ is s-cofibrant, we can find a single s-equivalence

$$\omega_i \colon \operatorname{hocofibre}(\beta_i) \xrightarrow{\simeq} Q_i$$

representing the zig-zag constructed above. By composing this with the canonical map $Y'_{i+1} \longrightarrow \operatorname{hocofibre}(\beta_i)$ we get a sequence

$$Y'_i \xrightarrow{\beta_i} Y'_{i+1} \longrightarrow Q_i.$$

By construction this is a cofibration sequence in the sense that the composite map is weakly null homotopic (in the homotopy category the composite factors as $Y'_i \longrightarrow Y_i \longrightarrow \operatorname{cofibre}\alpha_i \longrightarrow Q_i$ which is the trivial map), and the map $\operatorname{hocofibre}(\beta_i) \xrightarrow{\omega_i} Q_i$ is an s-equivalence. Since Y'_i and Q_i are s-cofibrant and s-fibrant, the composite map is actually simplicially null homotopic and factors over the simplicial reduced cone $CY'_i := Y'_i \wedge \Delta^1$ of Y'_i (here Δ^1 has basepoint 1). Let E_i denote an s-fibrant replacement of the cone CY'_i. By the lifting axioms, we can extend the map $CY'_i \longrightarrow Q_i$ to E_i, so we get a commutative diagram of s-fibrant and s-cofibrant objects

$$\begin{array}{ccc} Y'_i & \xrightarrow{\beta_i} & Y'_{i+1} \\ \downarrow & & \downarrow \\ E_i & \longrightarrow & Q_i \end{array}$$

which is homotopy cocartesian, i.e., for each $F \in \mathcal{C}$ the canonical map

$$\operatorname{hocolim}\left(E_i^F \longleftarrow Y_i^F \longrightarrow Y_{i+1}^F\right) \longrightarrow Q_i^F$$

is a stable equivalence. By the properties of spectra, the square is then homotopy cartesian as well, i.e., for each $F \in \mathcal{C}$ the canonical map

$$Y_i^F \longrightarrow \operatorname{holim}\left(E_i^F \longrightarrow Q_i^F \longleftarrow Y_{i+1}^F\right)$$

is a stable equivalence.

We now use that if $\hat{\Gamma}_{Y'_i}$ and $\hat{\Gamma}_{Q_i}$ are stable equivalences, then so is $\hat{\Gamma}_{Y'_{i+1}}$. This follows from the fact that holim preserves homotopy cartesian squares, and that $\hat{\Gamma}$ preserves homotopy cocartesian squares (the latter follows immediately from Lemma 3, the former follows from a formally dual argument). Moreover, the hom spectrum functor preserves homotopy cartesian squares of level fibrant spectra. Hence the comparison maps $\hat{\Gamma}_?$ assemble to a map of square diagrams of spectra (writing $Z = \Gamma(N(\mathcal{C} \downarrow \text{-})_+)$ for compact notation)

$$\begin{array}{ccc} \operatorname{holim}(Y'_i) & \longrightarrow & \operatorname{holim}(Y'_{i+1}) \\ \downarrow & & \downarrow \\ \operatorname{holim}(E_i) & \longrightarrow & \operatorname{holim}(Q_i) \end{array} \xrightarrow{\hat{\Gamma}_?} \begin{array}{ccc} \hom_{\mathrm{sSet}_*}(Z, \hat{\Gamma}(Y'_i)) & \longrightarrow & \hom_{\mathrm{sSet}_*}(Z, \hat{\Gamma}(Y'_{i+1})) \\ \downarrow & & \downarrow \\ \hom_{\mathrm{sSet}_*}(Z, \hat{\Gamma}(E_i)) & \longrightarrow & \hom_{\mathrm{sSet}_*}(Z, \hat{\Gamma}(Q_i)) \end{array}$$

where source and target are homotopy cartesian. The map is a weak equivalence on three corners: this is true for $\hat{\Gamma}_{Y'_i}$ and $\hat{\Gamma}_{Q_i}$ by assumption, and since E_i is simplicially contractible, we have $\operatorname{holim}(E_i) \simeq * \simeq \hat{\Gamma}(E_i)$. Hence the map of diagrams is a weak equivalence everywhere.

Consequently, since $\hat{\Gamma}_{Y_0}$ is a weak equivalence, it is sufficient to check that $\hat{\Gamma}_{Q_i}$ is a weak equivalence. But this follows from Lemma 7: the diagram Q defined there is now called Q_i.

To prove the other implication, suppose first that condition (P1) is not satisfied. Then we can find $F \in \mathcal{D}$ and $j, k \in \mathbb{N}$ such that $\hom_{\mathrm{sSet}_*}(N(\mathcal{C})/N(\mathcal{C}^F), K_{j,k})$ is not contractible, and the the proof of Lemma 7 shows that $Q = \mathcal{C}(F, \text{-})_+ \wedge K_{j,k}$ is a diagram of spectra such that $\hat{\Gamma}_Q$ is not an equivalence of spectra. A similar argument works if (P2) is not satisfied. □

Remark 8. Conditions (P1) and (P2) are implied by the following stronger conditions:

(P1′) For every $F \in \mathcal{D}$ the inclusion $N\mathcal{C}^F \longrightarrow N\mathcal{C}$ is a weak equivalence.
(P2′) For every $F \in \mathcal{C} \setminus \mathcal{D}$ the inclusion $N\mathcal{D} \longrightarrow N\mathcal{C}^F$ is a weak equivalence.

For (P2′) implies that the the map $N(\mathcal{C})/N(D) \longrightarrow N(\mathcal{C})/N(\mathcal{C}^F)$ is a weak equivalence, and (P1′) implies $N(\mathcal{C})/N(\mathcal{C}^F) \simeq *$. Conditions (P1) and (P2) now follow directly from weak homotopy invariance of the mapping spectra $\hom_{\mathrm{sSet}_*}(\text{-}, K_{j,k})$.

4. The Bousfield–Kan spectral sequence

Diagrams defined on finite posets.

Remark 9. For finite posets, conditions (P1) and (P2) can be simplified by applying the following fact to the geometric realisation of hocofibre(β) and the simplicial set $N\mathcal{C}/N\mathcal{C}^F$: *A finite pointed CW complex K has trivial stable cohomotopy if and only if $\Sigma^2 K$ is contractible.* Indeed, $\Sigma^2 K \simeq *$ clearly implies $\pi_s^n(K) = 0$ for all n. Conversely, if K has trivial stable cohomotopy, then its SPANIER–WHITEHEAD dual DK has trivial stable homotopy, hence is stably contractible. Thus the double dual DDK is stably contractible, hence has trivial reduced homology. Since K is finite there is a homotopy equivalence $K \simeq DDK$. So K has trivial homology, hence so has $\Sigma^2 K$. Consequently $\Sigma^2 K$, being simply connected, is contractible.

Theorem 10. *Let $\mathcal{D} \subseteq \mathcal{C}$ be a pair of finite posets satisfying conditions* (P1) *and* (P2)*, and let $Y\colon \mathcal{C} \longrightarrow \mathfrak{Sp}$ be a diagram with Y^F levelwise* KAN *for all $F \in \mathcal{C}$. There is a strongly convergent right-halfplane spectral sequence*

$$E_2^{p,q} = \lim{}^p(\pi_q Y) \implies \pi_{q-p} \hom_{\mathrm{sSet}_*} \Big(\Gamma(N(\mathcal{C} \downarrow \text{-})_+), \hat{\Gamma}(Y)\Big);$$

the r-differential is a map $d_r\colon E_r^{p,q} \longrightarrow E_r^{p+r,q+r-1}$.

Proof. If $X \in \mathfrak{Sp}$ is levelwise KAN, we can define a new spectrum QX by setting $(QX)_n := \mathrm{colim}_{k\to\infty} \Omega^k X_{n+k}$. The spectrum QX depends functorially on X; it is stably fibrant, and the natural map $X \longrightarrow QX$ is a stable equivalence.

By forming QY^F for all $F \in \mathcal{C}$ we obtain a diagram QY consisting entirely of stably fibrant spectra, and an s-equivalence $Y \longrightarrow QY$. Now since $\mathcal{C}$ is a finite poset, all the over-categories $\mathcal{C} \downarrow F$ have finite nerve. Since $\hom_{\mathrm{sSet}_*}(K, \text{-})$ commutes with filtered colimits for any finite simplicial set K, the functor Q commutes with holim and the composite map

$$(**) \qquad \mathrm{holim}_{\mathcal{C}}(Y) \longrightarrow Q\mathrm{holim}_{\mathcal{C}}(Y) \cong \mathrm{holim}_{C}(QY)$$

is a stable equivalence. Now by [Tho85, Proposition 5.13] we have a spectral sequence

$$E_2^{p,q} = \lim{}^p(\pi_q Y) \implies \pi_{q-p}(\mathrm{holim}_{\mathcal{C}}(QY))$$

which converges strongly since $N\mathcal{C}$ is finite-dimensional. Using the stable equivalence $(**)$ and Theorem 6 we obtain a natural isomorphism

$$\pi_{q-p}(\mathrm{holim}_{\mathcal{C}}(QY)) \cong \pi_{q-p}\,\mathrm{hom}_{\mathrm{sSet}_*}\Big(\Gamma(N(\mathcal{C}\downarrow\text{-})_+), \hat{\Gamma}(Y)\Big). \qquad \square$$

Topological spectra. In all we did so far we could replace the category $\mathfrak{Sp}$ of BOUSFIELD–FRIEDLANDER spectra with the category $\mathfrak{Tsp}$ of topological spectra (sequences of pointed topological spaces $(X_i)_{i\in\mathbb{N}}$ and structure maps $\Sigma X_i \longrightarrow X_{i+1}$). Indeed, the category $\mathfrak{Tsp}$ has a (topological) model structure similar to the one of $\mathfrak{Sp}$ [BF78, §2.5]; a set of generating cofibrations consists of the maps

$$|\alpha_{j,k}|\colon F_j(|\partial\Delta^k_+|) \longrightarrow F_j(|\Delta^k_+|)$$

where $F_j(T)$ is the suspension spectrum of T shifted by j. We can also define the functor Γ and the map

$$\Gamma_Y\colon \mathrm{holim}_{\mathcal{C}}(Y) = \mathrm{hom}_{\mathcal{C}}\big(|N(\mathcal{C}\downarrow\text{-})_+|, Y\big) \longrightarrow \mathrm{hom}\big(\Gamma(|N(\mathcal{C}\downarrow\text{-})_+|), \Gamma(Y)\big)$$

in this context. The statements of Theorem 6, Lemma 7 and Theorem 10 with $\hat{\Gamma}$ replaced by Γ remain valid *mutatis mutandis*; in fact their proofs could be simplified slightly since all topological spectra are automatically levelwise fibrant. This latter observation implies that *given $K \in \mathrm{sSet}_*$ and $Y \in \mathfrak{Tsp}$, if $|K|$ is homeomorphic to an m-sphere there is a natural isomorphism $\pi_p\,\mathrm{hom}(|K|, Y) \cong \pi_{p+m}Y$.* Here $\mathrm{hom}(|K|, Y)$ is the topological spectrum which has the topological mapping space $\mathrm{hom}(|K|, Y_n)$ in level n.

Diagrams defined on *PL* balls. A *polytopal complex* $\mathcal{C}$ is a collection of nonempty polytopes in $\mathbb{R}^n$ which is closed under taking nonempty faces, such that the intersection of $F, G \in \mathcal{C}$ is a (possibly empty) face of F and G. For any subset $\mathcal{K} \subseteq \mathcal{C}$ we write $|\mathcal{K}| := \bigcup_{K\in\mathcal{K}} K$ for the underlying space of $\mathcal{K}$.

Let $\mathcal{C}$ be (the underlying poset of) a finite polytopal complex with $|\mathcal{C}|$ a PL ball of dimension m, and let $\mathcal{D}$ be the subcomplex with $|\mathcal{D}| = \partial|\mathcal{C}|$. Then $N\mathcal{C}$ and $N\mathcal{D}$ are the barycentric subdivisions of $\mathcal{C}$ and $\mathcal{D}$, respectively. For $F \in \mathcal{C}\setminus\mathcal{D}$ the poset $\mathcal{C}^F$ is obtained from $\mathcal{C}$ by removing the open star $st_{\mathcal{C}}(F)$ of F in $\mathcal{C}$, i.e., removing all polytopes containing F. Thus $|N\mathcal{C}^F|$ is the m-ball $|N\mathcal{C}|$ with an open m-ball in the interior removed, hence condition (P2′) is satisfied. Similarly, condition (P1′) holds since for $F \in \mathcal{D}$ the open star $st_{\mathcal{C}}(F)$ of F in $\mathcal{C}$ is an m-ball, and the open star of F in $\mathcal{D}$ is an $(m-1)$-ball with underlying space $|st_{\mathcal{C}}(F)| \cap |\mathcal{D}|$.

By Remark 8, Theorem 6 applies to diagrams $\mathcal{C} \longrightarrow \mathfrak{Tsp}$. Since there is a PL homeomorphism of pairs

$$(|\mathrm{hocolim}_{\mathcal{C}}(N(\mathcal{C}\downarrow\text{-})_+)|,\ |\mathrm{hocolim}_{\mathcal{D}}(N(\mathcal{D}\downarrow\text{-})_+)|) \cong (|N\mathcal{C}_+|,\ |N\mathcal{D}_+|),$$

and since $|\mathcal{C}|/|\mathcal{D}| \cong S^m$, this means that total cofibres and homotopy limits agree up to m-fold looping and up to stable weak equivalences:

Theorem 11. *Let $\mathcal{C}$ be a finite polytopal complex with $|\mathcal{C}|$ a PL ball of dimension m, and let $\mathcal{D}$ be the subcomplex with $|\mathcal{D}| = \partial|\mathcal{C}|$. If $Y\colon \mathcal{C} \longrightarrow \mathfrak{Tsp}$ is a diagram of topological spectra, the natural map*

$$\Gamma_Y\colon \mathrm{holim}_{\mathcal{C}}(Y) \longrightarrow \mathrm{hom}\big(\Gamma(|N(\mathcal{C}\downarrow\text{-})|_+), \Gamma(Y)\big)$$

is a stable equivalence of topological spectra. Moreover, Theorem 10 *yields a strongly convergent right half-plane spectral sequence*

$$E_2^{p,q} = \lim{}^p(\pi_q Y) \implies \pi_{m+q-p}(\Gamma(Y)). \qquad \square$$

If $X\colon \mathcal{C} \longrightarrow \mathrm{Top}_*$ is a diagram of pointed topological spaces, we can consider the diagram of topological spectra $X \wedge H\mathbb{Z}$ where $H\mathbb{Z}$ denotes the EILENBERG–MACLANE spectrum of $\mathbb{Z}$. Then $\pi_q(X \wedge H\mathbb{Z}) \cong \widetilde{H}_q(X;\mathbb{Z})$ (reduced integral homology), and we obtain the following theorem generalising [Hüt04, Theorem 2.23] (where the result is stated only for the special case $\mathcal{C} = \mathcal{C}(P)$ and $\mathcal{D} = \mathcal{D}(P)$, and a different indexing convention is used for the spectral sequence):

Theorem 12. *Let $\mathcal{C}$ be a finite polytopal complex with $|\mathcal{C}|$ a PL ball of dimension m, and let $\mathcal{D}$ be the subcomplex with $|\mathcal{D}| = \partial|\mathcal{C}|$. For any diagram*

$$X\colon \mathcal{C} \longrightarrow \mathrm{Top}_*$$

there is a strongly convergent spectral sequence

$$E_2^{p,q} = \lim{}^p(\widetilde{H}_q(X;\mathbb{Z})) \implies \widetilde{H}_{m+q-p}(\Gamma(X);\mathbb{Z}). \qquad \square$$

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