

ON THE ADDITIVITY OF THE THURSTON-BENNEQUIN INVARIANT OF LEGENDRIAN KNOTS

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In this article, we consider the maximal value of the Thurston-Bennequin invariant of Legendrian knots which topologically represent a fixed knot type in the standard contact 3-space and we prove a formula of the value under the connected sum operation of knots.

1. Introduction.

The *standard contact structure* ξ_0 on 3-space $\mathbb{R}^3 = \{(x, y, z)\}$ is the plane field on \mathbb{R}^3 given by the kernel of the 1-form $dz - ydx$. A *Legendrian knot* K in the contact manifold (\mathbb{R}^3, ξ_0) is a knot which is everywhere tangent to the contact structure ξ_0 . The *Thurston-Bennequin invariant* $\text{tb}(K)$ of a Legendrian knot K in (\mathbb{R}^3, ξ_0) is the linking number of K and a knot K' which is obtained by moving K slightly along the vector field $\frac{\partial}{\partial z}$. For a topological knot type k in \mathbb{R}^3 , the *maximal Thurston-Bennequin invariant* $\text{mtb}(k)$ is defined to be the maximal value of $\text{tb}(K)$, where K is a Legendrian knot which topologically represents k . For any k , by the *Bennequin's inequality* in [1], we know that $\text{mtb}(k)$ is an integer (i.e., not ∞). There are several computations of $\text{mtb}(k)$ (for example, see [3], [5], [8], [9], [10], [11]).

In this paper, we prove the following theorem:

Theorem 1.1. *Let $k_1 \# k_2$ be the connected sum of topological knots k_1 and k_2 in \mathbb{R}^3 . Then $\text{mtb}(k_1 \# k_2) = \text{mtb}(k_1) + \text{mtb}(k_2) + 1$.*

Remark 1.2. After writing this paper, the author was informed that J. Etnyre and K. Honda [4] have also obtained a result on connected sum of Legendrian knots which extensively includes Theorem 1.1 and that T. Tanaka [12] have partially proved Theorem 1.1 by using a technique of algebraic knot theory.

2. Fronts.

Let K be a Legendrian knot in $(\mathbb{R}^3, \xi_0 = \ker(dz - ydx))$. Then a diagram (i.e., projection) of K in xz -plane is called *front* as in Figure 1.

A front does not have vertical tangents; generically, its only singularities are transverse double points and semicubical cusps. Note that the number

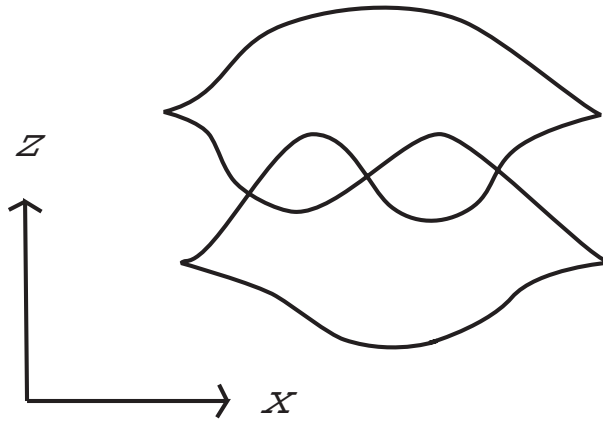


Figure 1.

of the cusps is even. Since $y = \frac{\partial z}{\partial x}$ along K , the missing y coordinate is the slope of the front. Therefore the front of K is free from selftangencies, and, at a double point, the branch with a greater slope is higher along the y axis. Conversely such a diagram uniquely determines K as its front. So, as usual in knot theory, we identify a Legendrian knot K with its front, also denoted by K .

The Thurston-Bennequin invariant $\text{tb}(K)$ is computed in terms of the double points and cusps of its front. See Figure 2, where K is oriented and the choice of the orientation is irrelevant for the value of $\text{tb}(K)$.

$$\text{tb} = \# \begin{array}{c} \nearrow \\ \times \\ \searrow \end{array} + \# \begin{array}{c} \nwarrow \\ \times \\ \swarrow \end{array} - \# \begin{array}{c} \nwarrow \\ \times \\ \nearrow \end{array} - \# \begin{array}{c} \nearrow \\ \times \\ \swarrow \end{array} - 1/2 \# \text{ of cusps}$$

Figure 2.

For example, $\text{tb}(K) = 1$ for the front in Figure 1.

Proposition 2.1. *For two topological knots k_1 and k_2 , we have $\text{mtb}(k_1 \# k_2) \geq \text{mtb}(k_1) + \text{mtb}(k_2) + 1$.*

Proof. Let K_1 and K_2 be Legendrian knots whose topological types are k_1 and k_2 , respectively and $\text{mtb}(k_1) = \text{tb}(K_1)$ and $\text{mtb}(k_2) = \text{tb}(K_2)$. We also regard K_1 and K_2 as fronts. Further we can assume that $K_1 \cap K_2 = \emptyset$ and K_1

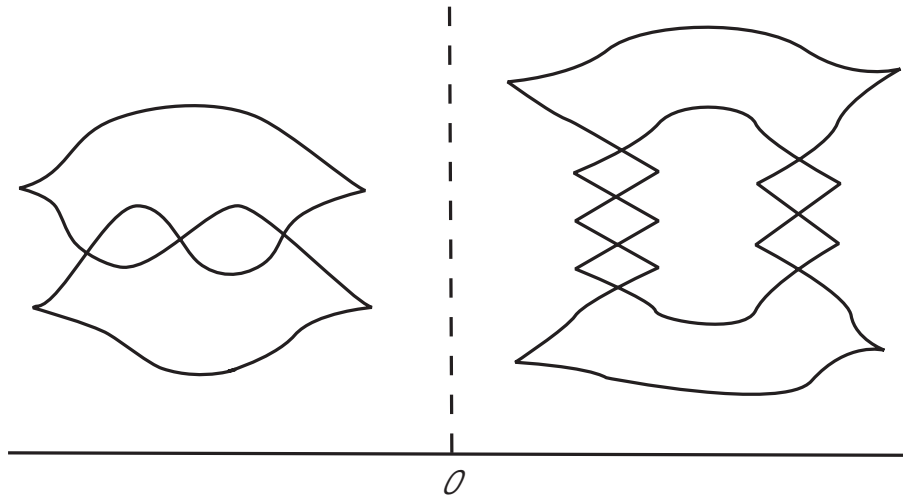


Figure 3.

(resp. K_2) lies in the left (resp. right) region of xz -plane, i.e., $\{(x, z)|x < 0\}$ (resp. $\{(x, z)|x > 0\}$) as in Figure 3.

Then we connect K_1 and K_2 by joining a right cusp of K_1 and a left cusp of K_2 as in Figure 4.

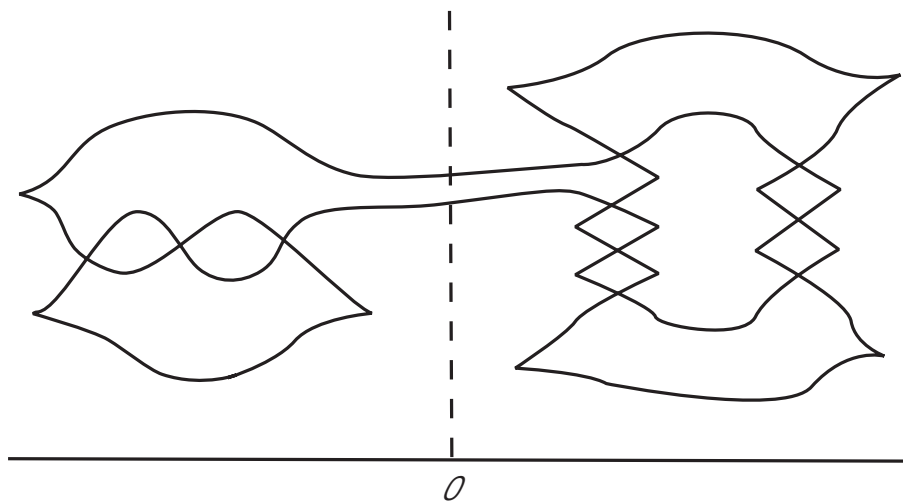


Figure 4.

This procedure produces a Legendrian knot whose topological type is $k_1 \# k_2$ and Thurston-Bennequin invariant is $\text{mtb}(k_1) + \text{mtb}(k_2) + 1$. This completes the proof. \square

3. Preliminaries from contact topology.

In this section, we recall some basic notions and theorems from recent 3-dimensional contact topology. In fact, the proof of Theorem 1.1 essentially relies on the previous foundational work of E. Giroux, Honda and Y. Eliashberg-M. Fraser. In particular, we assume the reader is familiar with convex surface theory started by Giroux in [6]. For details and proofs, see [2], [3], [6], [7], [8]. Let $\xi_n = \ker(\sin(2\pi n z)dx + \cos(2\pi n z)dy)$ be the contact structure on a solid torus $V = \{(x, y, z) \in \mathbb{R}_z^3 \mid x^2 + y^2 \leq \epsilon\}$, where $n \in \mathbb{N}$ and \mathbb{R}_z^3 is \mathbb{R}^3 modulo $z \mapsto z + 1$. The *characteristic foliation* on an embedded surface in a contact 3-manifold is the singular foliation defined by the intersection of the contact structure and the surface. The set of tangents of ξ_n to ∂V forms a disjoint union of two simple closed curves on ∂V , which are called *Legendrian divides*.

The next lemma is proved by a standard Darboux-type argument.

Lemma 3.1. *For any Legendrian knot K in (\mathbb{R}^3, ξ_0) , there exists a sufficiently small neighborhood $N(K)$ such that $(N(K), K, \xi_0)$ is isomorphic to $(V, \{(0, 0, z)\}, \xi_n)$ for some n .*

As ∂V is a *convex surface* (i.e., has a contact vector field transverse to ∂V), the following lemma can be proved by convex surface theory:

Lemma 3.2. *Let T be any embedded torus in (\mathbb{R}^3, ξ_0) and W a solid torus bounded by T . Suppose the characteristic foliation on T is diffeomorphic to that on ∂V and identifying these, the Legendrian divides on T are isotopic to the core curve of W through an isotopy in W . Then (W, ξ_0) is isomorphic to (V, ξ_n) for some n .*

The following theorem on the classification of topologically trivial Legendrian knots due to Eliashberg-Fraser [2] is also needed for the proof of Theorem 1.1:

Theorem 3.3. *Any topologically trivial Legendrian knot is Legendrian isotopic to one of standard forms expressed as fronts in Figure 5.*

4. Proof of Theorem 1.1.

By Proposition 2.1, it is sufficient to show the converse inequality.

Suppose \hat{K} is a Legendrian knot in (\mathbb{R}^3, ξ_0) whose topological type is the connected sum of k_1 and k_2 and its Thurston-Bennequin invariant is maximal. By Lemma 3.1, there exists a neighbourhood $N(\hat{K})$ of \hat{K} such

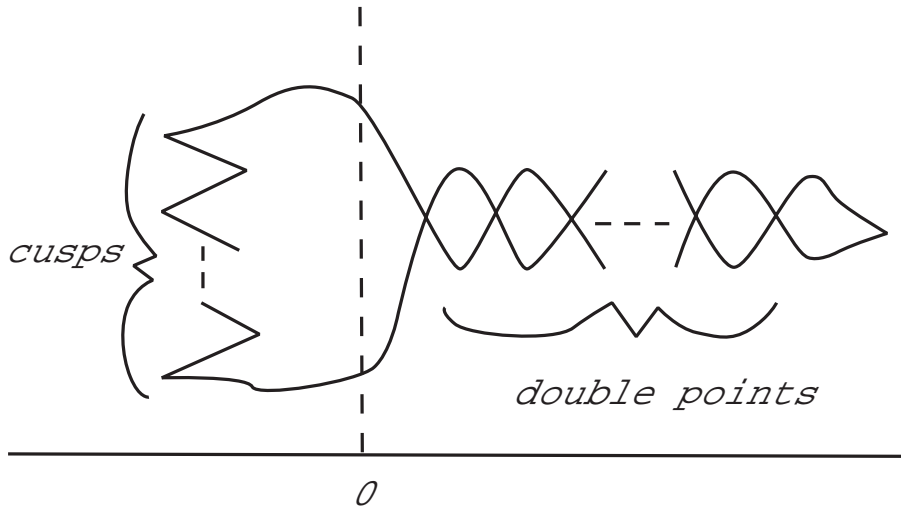


Figure 5.

that $(N(\hat{K}), \xi_0)$ is isomorphic to (V, ξ_n) for some n . Let B_1 and B_2 be 3-balls in \mathbb{R}^3 such that B_1 (resp. B_2) splits \hat{K} into the component corresponding to k_1 (resp. k_2) and $B_1 \cap B_2 = \emptyset$ (Figure 6).

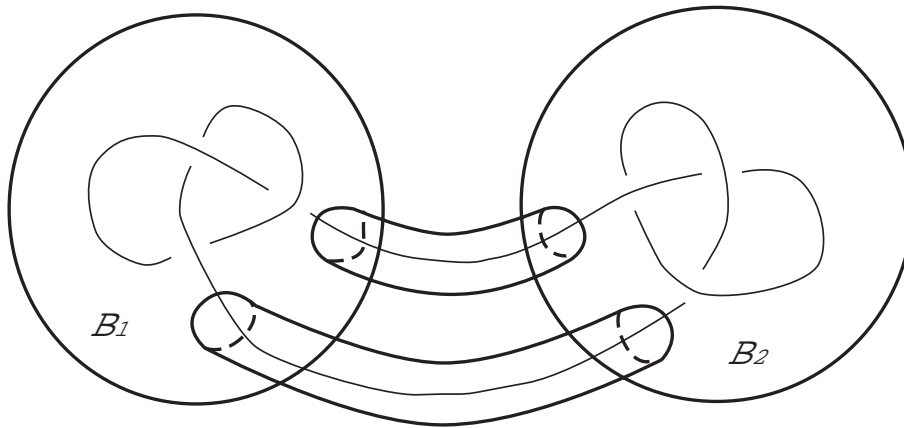


Figure 6.

Further, by convex surface theory, we can assume that (i) ∂B_1 and ∂B_2 are convex and (ii) $\partial B_1 \cap \partial N(\hat{K})$ and $\partial B_2 \cap \partial N(\hat{K})$ are Legendrian knots on ∂B_1 and ∂B_2 , respectively and (iii) each *dividing set* on ∂B_i (i.e., the subset of ∂B_i consisting of tangents of ξ_0 and a contact vector field defining the convex surface) intersects $\partial B_i \cap N(\hat{K})$ as a diameter of the disk.

Then by Edge-Rounding Lemma due to Honda in [7], we have a solid torus W such that (i) W equals $B_1 \cup B_2 \cup N(\hat{K})$ except small neighbourhoods of $\partial B_1 \cap \partial N(\hat{K})$ and $\partial B_2 \cap \partial N(\hat{K})$ and (ii) ∂W is a convex surface whose characteristic foliation is diffeomorphic to that of ∂V . By Lemma 3.2, it follows that (W, ξ_0) is isomorphic to (V, ξ_n) for some n . And notice that W is unknotted in \mathbb{R}^3 and hence the core curve K of W which is Legendrian is also unknotted. Further, by a standard argument, we can assume that K agrees with \hat{K} in the region of $N(\hat{K}) - (B_1 \cup B_2)$. So by Theorem 3.3, K is Legendrian isotopic to one of standard forms in Figure 5. Therefore W is also identified with a small neighbourhood of that of the standard form. Further, by a homogeneous property of V and a parallel translation of W , we can assume that a region of W corresponding to B_1 (resp. B_2) lies in $\{(x, y, z) | x < 0\}$ (resp. $\{(x, y, z) | x > 0\}$). Then, identifying \hat{K} with its front, we can divide \hat{K} along a vertical line into Legendrian knots K_1 and K_2 corresponding to k_1 and k_2 , respectively as the converse procedure in the proof of Proposition 2.1.

Counting the Thurston-Bennequin invariant of K_1 and K_2 , we have $\text{tb}(\hat{K}) = \text{mtb}(k_1 \# k_2) = \text{tb}(K_1) + \text{tb}(K_2) + 1$. Therefore $\text{mtb}(k_1 \# k_2) \leq \text{mtb}(k_1) + \text{mtb}(k_2) + 1$.

This completes the proof of the main theorem.

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