PACIFIC JOURNAL OF MATHEMATICS
Vol. 209, No. 2, 2003

A ZETA FUNCTION FOR FLIP SYSTEMS

Young-One Kim, Jungseob Lee and Kyewon K. Park

In this paper, we investigate dynamical systems with flip maps, which can be regarded as infinite dihedral group actions. We introduce a zeta function for flip systems, and find its basic properties including a product formula. When the underlying $\mathbb{Z}$-action is conjugate to a topological Markov shift, the flip system is represented by a pair of matrices, and its zeta function is expressed explicitly in terms of the representation matrices.

1. Introduction.

Let (X, T) be a topological dynamical system, where X is a topological space and $T : X \to X$ a homeomorphism. A homeomorphism $F : X \to X$ is called a *flip map* or simply a flip for (X, T) if

$$TF = FT^{-1} \quad \text{and} \quad F^2 = \text{id}.$$

We call the triplet (X, T, F) a *flip system*. It is easy to see that if (X, T, F) is a flip system, then $(X, T^m, T^n F)$ is also a flip system for any $m, n \in \mathbb{Z}$. Since the infinite dihedral group D_∞ is generated by two elements a and b such that

$$ab = ba^{-1} \quad \text{and} \quad b^2 = 1, \tag{1.1}$$

a flip system can be regarded as a D_∞-action of homeomorphisms.

Two flip systems (X, T, F) and (X', T', F') are said to be *conjugate* if there is a homeomorphism $\Phi : X \to X'$ such that

$$\Phi T = T'\Phi \quad \text{and} \quad \Phi F = F'\Phi.$$

In this case, we write $(X, T, F) \cong (X', T', F')$, and Φ is called a *conjugacy* from (X, T, F) to (X', T', F'). For an arbitrary flip system (X, T, F), T is a conjugacy from (X, T, F) to (X, T, T^2F) and F is a conjugacy from (X, T, F) to (X, T^{-1}, F).

Since there is a dynamical system (X, T) which is not conjugate to its time reversal (X, T^{-1}), not every dynamical system has a flip. See [**3**, p. 104] and also Example 4.1. On the other hand, any topological Markov shift whose transition matrix is symmetric has a natural flip.

It is well-known that measurable D_∞-actions are isomorphic if the underlying $\mathbb{Z}$-actions are Bernoulli of the same entropy. In [**7**] it is shown that if the underlying $\mathbb{Z}$-actions are Kolmogorov and isomorphic, there are examples of non-isomorphic D_∞-actions. Unlike the measurable case, we can construct infinitely many non-conjugate flips for a full shift in the topological setting. See Example 4.2.

We establish a zeta function for flip systems which is a conjugacy invariant, and give a finite description of the function when the underlying $\mathbb{Z}$-action is conjugate to a topological Markov shift.

The Artin-Mazur zeta function ζ_T for a dynamical system (X, T), found in [1], is defined by

$$\zeta_T(t) = \exp\left(\sum_{n=1}^{\infty} \frac{p_n}{n} t^n\right), \tag{1.2}$$

where

$$p_n = |\{x \in X : T^n x = x\}| \qquad (n = 1, 2, \dots).$$

(We assume that the sequence $\{(p_n)^{1/n}\}$ is bounded.) The Artin-Mazur zeta function has the product formula

$$\zeta_T(t) = \prod_{\gamma} \frac{1}{1 - t^{|\gamma|}}, \tag{1.3}$$

where the product is taken over all finite orbits γ of T.

In [5], D. Lind introduced a zeta function for $\mathbb{Z}^d$-actions that generalizes the Artin-Mazur zeta function. It is straightforward to extend the notion to the case of general group actions. Let G be a group, X a set and $\alpha : G \times X \to X$ a G-action on X. Then the zeta function ζ_α of the action α is defined formally by

$$\zeta_\alpha(t) = \exp\left(\sum_{H} \frac{p_H}{|G/H|} t^{|G/H|}\right). \tag{1.4}$$

Here, the sum is taken over all finite-index subgroups H of G, that is, subgroups H such that $|G/H| < \infty$, and p_H is defined by

$$p_H = |\{x \in X : \forall\, h \in H \quad \alpha(h, x) = x\}|.$$

It is easy to see that this zeta function is *automorphism-invariant* in the following sense: If $\Psi : G \to G$ is an automorphism and two G-actions $\alpha : G \times X \to X$ and $\widetilde{\alpha} : G \times X \to X$ satisfy $\widetilde{\alpha}(g, x) = \alpha(\Psi(g), x)$ for all $(g, x) \in G \times X$, then $\zeta_\alpha = \zeta_{\widetilde{\alpha}}$.

We define the zeta function $\zeta_{T,F}$ of a flip system (X, T, F) to be the zeta function ζ_α of the D_∞-action $\alpha : D_\infty \times X \to X$ that is given by

$$\alpha(a, x) = Tx \quad \text{and} \quad \alpha(b, x) = Fx \qquad (x \in X), \tag{1.5}$$

where a and b are generators of D_∞ which satisfy (1.1). Since the zeta function is automorphism-invariant, our definition does not depend on the choice of the generators a and b. Moreover, it is clear that this zeta function is a conjugacy invariant. There are, however, non-conjugate flip systems with the same zeta function. See Examples 4.3 and 4.4.

In Section 2, we express the zeta function of flip systems in a more tractable form, and establish some of its basic properties including the product formula. In Section 3, we consider the flip systems (X, T, F) such that (X, T) is conjugate to a topological Markov shift. We prove that such a system can be represented by a pair of matrices (Representation Theorem), and express its zeta function in terms of those matrices. Finally, in Section 4, we conclude this paper with some examples.

2. The zeta function of a flip system.

Let (X, T, F) be a flip system, and suppose that D_∞ is generated by a and b satisfying (1.1). Let $\alpha : D_\infty \times X \to X$ denote the D_∞-action defined by (1.5). For a finite-index subgroup H of D_∞ set $p_H = |\{x \in X : \forall\, h \in H,\, \alpha(h, x) = x\}|$ and suppose that $p_H < \infty$ for all finite-index subgroups H of D_∞.

In order to express ζ_α explicitly, we need to identify all the finite-index subgroups of D_∞. Suppose that H is a finite-index subgroup of D_∞. Then there is an integer $k \neq 0$ such that $a^k \in H$, since otherwise we must have $|H| \leq 2$. Hence, either H is generated by a^i for some integer $i \neq 0$ or by a^i and $a^j b$ for some integers i and j with $i \neq 0$.

Let $H(i)$ denote the subgroup generated by a^i, and $H(i, j)$ the one generated by a^i and $a^j b$. Then it is clear that $H(i) = H(k)$ if and only if $|i| = |k|$, and that $H(i, j) = H(k, l)$ if and only if $|i| = |k|$ and $j - l$ is a multiple of i. Moreover, $|D_\infty/H(i)| = 2|i|$ and $|D_\infty/H(i, j)| = |i|$ for $i \neq 0$. Therefore we obtain the following:

Lemma 2.1. *Let n be a positive integer. If n is odd, then*

$$H(n, 0), H(n, 1), \ldots, H(n, n-1)$$

are all the subgroups of D_∞ with index n. In addition to these, there is one more such subgroup $H(n/2)$ if n is even.

For convenience, we set $p_i = p_{H(i)}$ and $p_{i,j} = p_{H(i,j)}$. Then we have

$$p_i = |\{x \in X : T^i x = x\}| \qquad \text{and} \qquad p_{i,j} = |\{x \in X : T^i x = T^j F x = x\}|. \tag{2.1}$$

Hence (1.4) and Lemma 2.1 imply that

$$\zeta_{T,F}(t) = \exp\left(\sum_{n=1}^{\infty} \frac{p_n}{2n} t^{2n} + \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{p_{n,k}}{n} t^n\right). \tag{2.2}$$

Now, observe that $aH(i, j)a^{-1} = H(i, j + 2)$. From this, we see that $p_{i,j} = p_{i,j+2}$. Moreover, it is clear that $p_{i,j} = p_{i,i+j}$. Hence we obtain the following:

$$\sum_{k=0}^{n-1}\frac{p_{n,k}}{n}=\begin{cases}p_{n,0} & \text{if } n \text{ is odd,}\\ (p_{n,0}+p_{n,1})/2 & \text{if } n \text{ is even.}\end{cases} \tag{2.3}$$

By (1.2) we have

$$\exp\left(\sum_{n=1}^{\infty}\frac{p_n}{2n}t^{2n}\right)=\sqrt{\zeta_T(t^2)}. \tag{2.4}$$

Theorem 2.2. *The zeta function $\zeta_{T,F}$ of the flip system (X,T,F) is given by*

$$\zeta_{T,F}(t)=\sqrt{\zeta_T(t^2)}\exp\left(G_{T,F}(t)\right),$$

where ζ_T is the Artin-Mazur zeta function of (X,T), and

$$G_{T,F}(t)=\sum_{m=1}^{\infty}\left(p_{2m-1,0}t^{2m-1}+\frac{p_{2m,0}+p_{2m,1}}{2}t^{2m}\right).$$

Proof. The theorem is an immediate consequence of (2.2), (2.3) and (2.4). □

Corollary 2.3. *Let R_T and $R_{T,F}$ denote the radii of convergence of the Maclaurin series of $\zeta_T(t)$ and $\zeta_{T,F}(t)$, respectively. If $p_n>0$ for some n, then we have*

$$0\le R_T\le R_{T,F}\le\sqrt{R_T}\le 1.$$

Remark 2.4. If (X,T) is conjugate to a subshift, then it is easy to see that the radius of convergence of $G_{T,F}$ is at least $\exp(h_T/2)$, where h_T is the topological entropy of (X,T). Moreover, if (X,T) is conjugate to a sofic shift, then $h_T=\log R_T$ (see [**6**, Chapter 4]), and hence $R_{T,F}=\sqrt{R_T}$.

In the remainder of this section, we establish the product formula of the zeta function. Suppose that γ is a finite orbit of (X,T,F). Then there is a point x such that $\gamma=\{x,Tx,\dots,T^{|\gamma|-1}x\}$, or there is a point x such that $\gamma=\{x,Tx,\dots,T^{k-1}x\}\cup\{Fx,TFx,\dots,T^{k-1}Fx\}$ with $|\gamma|=2k$. In the first case, we write $\gamma\in\mathcal{O}_1$, and in the second case, $\gamma\in\mathcal{O}_2$. It is obvious that $\mathcal{O}_1\cap\mathcal{O}_2=\emptyset$. We denote by $\zeta_{(\gamma)}$ the zeta function of the flip system $(\gamma,T|_\gamma,F|_\gamma)$.

Lemma 2.5. *If $\gamma\in\mathcal{O}_1$,*

$$\zeta_{(\gamma)}(t)=\sqrt{\frac{1}{1-t^{2|\gamma|}}}\exp\left(\frac{t^{|\gamma|}}{1-t^{|\gamma|}}\right),$$

and if $\gamma\in\mathcal{O}_2$,

$$\zeta_{(\gamma)}(t)=\frac{1}{1-t^{|\gamma|}}.$$

Proof. Let

$$\widetilde{p}_i = |\{x \in \gamma : T^i x = x\}| \quad \text{and}$$
$$\widetilde{p}_{i,j} = |\{x \in \gamma : T^i x = T^j F x = x\}|.$$

Assume that $\gamma \in \mathcal{O}_1$ and n is a positive integer. If n is not a multiple of $|\gamma|$, then no elements of γ are fixed by T^n, and hence $\widetilde{p}_n = 0$ and $\widetilde{p}_{n,k} = 0$ for all k. Now suppose n is a multiple of $|\gamma|$. Then every element of γ is fixed by T^n, so that $\widetilde{p}_n = |\gamma|$. We can see that if $|\gamma|$ is odd, $\widetilde{p}_{n,0} = 1$; if $|\gamma|$ even, either $\widetilde{p}_{n,0} = 2$, $\widetilde{p}_{n,1} = 0$ or $\widetilde{p}_{n,0} = 0$, $\widetilde{p}_{n,1} = 2$. Using (2.2) and (2.3) with $\widetilde{p}_n$ and $\widetilde{p}_{n,k}$ in place of p_n and $p_{n,k}$ respectively, we have

$$\zeta_{(\gamma)}(t) = \exp\left(\sum_{m=1}^{\infty} \frac{1}{2m} t^{2m|\gamma|} + \sum_{m=1}^{\infty} t^{m|\gamma|}\right),$$

from which the first assertion follows.

Next, assume that $\gamma \in \mathcal{O}_2$. Then for each integer j no elements of γ are fixed by $T^j F$. Hence $\widetilde{p}_{n,0} = \widetilde{p}_{n,1} = 0$ for all n. Moreover, it is easy to see that $\widetilde{p}_n = |\gamma|$ if n is a multiple of $|\gamma|/2$, and $\widetilde{p}_n = 0$ otherwise. Again using (2.2) and (2.3) we have

$$\zeta_{(\gamma)}(t) = \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} t^{m|\gamma|}\right),$$

from which the second assertion follows. □

Theorem 2.6. *Let $R_{T,F}$ be the radius of convergence of the Maclaurin series of $\zeta_{T,F}$, and suppose that $R_{T,F} > 0$. Then we have*

$$\zeta_{T,F}(t) = \prod_{\gamma \in \mathcal{O}_1} \sqrt{\frac{1}{1 - t^{2|\gamma|}}} \exp\left(\frac{t^{|\gamma|}}{1 - t^{|\gamma|}}\right) \prod_{\gamma \in \mathcal{O}_2} \frac{1}{1 - t^{|\gamma|}} \qquad (|t| < R_{T,F}).$$

Proof. It is clear from the definition that

$$\zeta_{T,F}(t) = \prod_{\gamma} \zeta_{(\gamma)}(t) \qquad (|t| < R_{T,F}),$$

where the product is taken over all finite orbits γ. Now, the result follows from Lemma 2.5. □

Let $\mathcal{O}_T$ denote the set of all periodic T-orbits. It is clear that $\mathcal{O}_1 \subset \mathcal{O}_T$, but a periodic T-orbit may not be an orbit of the flip system (X, T, F). We restate Theorem 2.6 as follows:

Theorem 2.7. *Let $R_{T,F}$ be the radius of convergence of the Maclaurin series of $\zeta_{T,F}$, and suppose that $R_{T,F} > 0$. Then we have*

$$\zeta_{T,F}(t) = \prod_{\beta \in \mathcal{O}_T} \sqrt{\frac{1}{1 - t^{2|\beta|}}} \prod_{\gamma \in \mathcal{O}_1} \exp\left(\frac{t^{|\gamma|}}{1 - t^{|\gamma|}}\right) \qquad (|t| < R_{T,F}). \tag{2.5}$$

Proof. Since $\mathcal{O}_1 \subset \mathcal{O}_T$, we have

$$\prod_{\beta\in\mathcal{O}_T}\sqrt{\frac{1}{1-t^{2|\beta|}}}=\prod_{\beta\in\mathcal{O}_1}\sqrt{\frac{1}{1-t^{2|\beta|}}}\prod_{\beta\in\mathcal{O}_T\setminus\mathcal{O}_1}\sqrt{\frac{1}{1-t^{2|\beta|}}}.$$

Then the right-hand side of (2.5) is equal to

$$\prod_{\gamma\in\mathcal{O}_1}\sqrt{\frac{1}{1-t^{2|\gamma|}}}\exp\left(\frac{t^{|\gamma|}}{1-t^{|\gamma|}}\right)\prod_{\beta\in\mathcal{O}_T\setminus\mathcal{O}_1}\sqrt{\frac{1}{1-t^{2|\beta|}}}.$$

In view of Theorem 2.6, we need only to prove the following:

$$\prod_{\beta\in\mathcal{O}_T\setminus\mathcal{O}_1}\sqrt{\frac{1}{1-t^{2|\beta|}}}=\prod_{\gamma\in\mathcal{O}_2}\frac{1}{1-t^{|\gamma|}}. \tag{2.6}$$

We note that if $\beta \in \mathcal{O}_T \setminus \mathcal{O}_1$, then $F\beta \in \mathcal{O}_T \setminus \mathcal{O}_1$, $\beta \cap F\beta = \emptyset$ and $\beta \cup F\beta \in \mathcal{O}_2$. Conversely, if $\gamma \in \mathcal{O}_2$, then there is an element $\beta_\gamma \in \mathcal{O}_T \setminus \mathcal{O}_1$ such that $\gamma = \beta_\gamma \cup F\beta_\gamma$. In this case, we have $|\gamma| = 2|\beta_\gamma| = 2|F\beta_\gamma|$. Thus

$$\begin{aligned}\prod_{\beta\in\mathcal{O}_T\setminus\mathcal{O}_1}\sqrt{\frac{1}{1-t^{2|\beta|}}}&=\prod_{\gamma\in\mathcal{O}_2}\sqrt{\frac{1}{1-t^{2|\beta_\gamma|}}}\sqrt{\frac{1}{1-t^{2|F\beta_\gamma|}}}\\&=\prod_{\gamma\in\mathcal{O}_2}\sqrt{\frac{1}{1-t^{|\gamma|}}}\sqrt{\frac{1}{1-t^{|\gamma|}}}\\&=\prod_{\gamma\in\mathcal{O}_2}\frac{1}{1-t^{|\gamma|}}.\end{aligned}$$

This proves (2.6). □

Corollary 2.8. *Let $G_{T,F}$ be as in Theorem 2.2. Then*

$$G_{T,F}(t)=\sum_{\gamma\in\mathcal{O}_1}\frac{t^{|\gamma|}}{1-t^{|\gamma|}}.$$

Proof. The result is an immediate consequence of (1.3), Theorem 2.2 and the above theorem. □

3. Flips for topological Markov shifts.

Let $\mathcal{A}$ be a finite discrete topological space. For $x \in \mathcal{A}^{\mathbb{Z}}$ and $i \in \mathbb{Z}$ the i-th coordinate of x is denoted by x_i, and if $i, j \in \mathbb{Z}$ with $i < j$, the block $x_i x_{i+1} \dots x_j$ is denoted by $x_{[i,j]}$. For $x \in \mathcal{A}^{\mathbb{Z}}$, we define σx and ρx by

$$(\sigma x)_i = x_{i+1} \quad \text{and} \quad (\rho x)_i = x_{-i} \qquad (i \in \mathbb{Z}).$$

Then σ and ρ are homeomorphisms of $\mathcal{A}^{\mathbb{Z}}$ onto itself, and satisfy

$$\sigma\rho = \rho\sigma^{-1} \quad \text{and} \quad \rho^2 = \text{id},$$

that is, ρ is a flip for the dynamical system $(\mathcal{A}^{\mathbb{Z}}, \sigma)$. This dynamical system is called the *full $\mathcal{A}$-shift*. The map σ is called the *shift map*, and ρ the *reverse map*. When we express a point as a bi-infinite sequence, we will underline the 0-th coordinate. For instance, if $x = \dots x_{-2}x_{-1}\underline{x_0}x_1x_2\dots$, then $\sigma x = \dots x_{-2}x_{-1}x_0\underline{x_1}x_2\dots$ and $\rho x = \dots x_2x_1\underline{x_0}x_{-1}x_{-2}\dots$.

Let A be a 0-1, $\mathcal{A}\times\mathcal{A}$ matrix, and (X_A, σ_A) denote the topological Markov shift whose transition matrix is A. If $A = A^T$, then X_A is ρ-invariant, and hence $\rho|_{\mathsf{X}_A}$ is a flip for (X_A, σ_A). More generally, if there is a 0-1, $\mathcal{A}\times\mathcal{A}$ matrix P such that

$$AP = PA^T \quad \text{and} \quad P^2 = I, \tag{3.1}$$

then there is a flip, denoted by $\phi_{A,P}$, for (X_A, σ_A) that is defined as follows: Since P is 0-1 and $P^2 = I$, it is a symmetric permutation matrix, that is, $P = P^T$ and for each $a \in \mathcal{A}$ there is a unique $a^* \in \mathcal{A}$ such that $P(a, a^*) = 1$. Then it is easy to see that

$$(a^*)^* = a \qquad (a \in \mathcal{A}) \tag{3.2}$$

and

$$A(a,b) = 1 \iff A(b^*, a^*) = 1 \qquad (a, b \in \mathcal{A}). \tag{3.3}$$

For $x \in \mathsf{X}_A$ we define $\phi_{A,P}x$ by

$$(\phi_{A,P}x)_i = (x_{-i})^* \qquad (i \in \mathbb{Z}).$$

Then from (3.2) and (3.3) it follows that $\phi_{A,P}$ is a flip for (X_A, σ_A).

The following theorem states that every flip for a topological Markov shift can be represented in this way:

Theorem 3.1 (Representation Theorem). *Let (X, T, F) be a flip system, and suppose that (X, T) is conjugate to a topological Markov shift. Then there are* 0-1 *square matrices A and P satisfying* (3.1) *such that (X, T, F) is conjugate to $(\mathsf{X}_A, \sigma_A, \phi_{A,P})$.*

Proof. We suppose that (X, T) is conjugate to a topological Markov shift (X_M, σ_M) through a conjugacy Ψ. Set $\phi = \Psi F \Psi^{-1}$. Then this is a flip for (X_M, σ_M), and $(\mathsf{X}_M, \sigma_M, \phi)$ is conjugate to (X, T, F). We will construct a finite set $\mathcal{A}$ and two 0-1, $\mathcal{A}\times\mathcal{A}$ matrices A and P satisfying (3.1) such that $(\mathsf{X}_M, \sigma_M, \phi) \cong (\mathsf{X}_A, \sigma_A, \phi_{A,P})$.

Since ϕ is continuous, there is a positive integer N such that

$$x_{[-N,N]} = y_{[-N,N]} \ \Rightarrow \ (\phi x)_0 = (\phi y)_0 \qquad (x, y \in \mathsf{X}_M). \tag{3.4}$$

For $x \in \mathsf{X}_M$ let $\widetilde{x}$ denote the bi-infinite sequence defined by

$$\widetilde{x} = \dots(\phi x)_2(\phi x)_1\underline{(\phi x)_0}(\phi x)_{-1}(\phi x)_{-2}\dots,$$

that is, $\widetilde{x} = \rho\phi x$. It should be noted that if M is symmetric, then $\widetilde{x} \in \mathsf{X}_M$ for all $x \in \mathsf{X}_M$, but in general, this is not the case. For $x \in \mathsf{X}_M$ let $[x]$

denote the ordered pair of the $(2N+1)$-blocks $x_{[-N,N]}$ and $\widetilde{x}_{[-N,N]}$, and we express $[x]$ as

$$[x] = \begin{bmatrix} x_{-N} \dots x_0 \dots x_N \\ \widetilde{x}_{-N} \dots \widetilde{x}_0 \dots \widetilde{x}_N \end{bmatrix}.$$

Note that if $x, y \in \mathsf{X}_M$ and $x_{[-2N,2N]} = y_{[-2N,2N]}$, then $[x] = [y]$.

Now we define $\mathcal{A}$. An ordered pair $\mathbf{a} = \begin{bmatrix} a_{-N} \dots a_0 \dots a_N \\ \widetilde{a}_{-N} \dots \widetilde{a}_0 \dots \widetilde{a}_N \end{bmatrix}$ of $(2N+1)$-blocks is an element of $\mathcal{A}$ if and only if $\mathbf{a} = [x]$ for some $x \in \mathsf{X}_M$. It is clear that $\mathcal{A}$ is a finite set. For $\mathbf{a} = \begin{bmatrix} a_{-N} \dots a_0 \dots a_N \\ \widetilde{a}_{-N} \dots \widetilde{a}_0 \dots \widetilde{a}_N \end{bmatrix} \in \mathcal{A}$ we define

$$\begin{aligned} \mathbf{a}^* &= \begin{bmatrix} \widetilde{a}_N \dots \widetilde{a}_0 \dots \widetilde{a}_{-N} \\ a_N \dots a_0 \dots a_{-N} \end{bmatrix}, \\ l(\mathbf{a}) &= \begin{bmatrix} a_{-N} \dots a_0 \dots a_{N-1} \\ \widetilde{a}_{-N} \dots \widetilde{a}_0 \dots \widetilde{a}_{N-1} \end{bmatrix}, \\ r(\mathbf{a}) &= \begin{bmatrix} a_{-N+1} \dots a_0 \dots a_N \\ \widetilde{a}_{-N+1} \dots \widetilde{a}_0 \dots \widetilde{a}_N \end{bmatrix}, \\ c(\mathbf{a}) &= a_{-N} \dots a_0 \dots a_N \quad \text{and} \\ b_0(\mathbf{a}) &= \widetilde{a}_0. \end{aligned}$$

Obviously $[x]^* = [\phi x]$ for all $x \in \mathsf{X}_M$. Hence $\mathbf{a}^* \in \mathcal{A}$ and $(\mathbf{a}^*)^* = \mathbf{a}$ for all $\mathbf{a} \in \mathcal{A}$. Moreover, (3.4) implies that

$$c(\mathbf{a}) = c(\mathbf{b}) \;\Rightarrow\; b_0(\mathbf{a}) = b_0(\mathbf{b}) \qquad (\mathbf{a}, \mathbf{b} \in \mathcal{A}). \tag{3.5}$$

Next, define the matrices A and P by

$$A(\mathbf{a}, \mathbf{b}) = \delta(r(\mathbf{a}), l(\mathbf{b})) \qquad (\mathbf{a}, \mathbf{b} \in \mathcal{A})$$

and

$$P(\mathbf{a}, \mathbf{b}) = \delta(\mathbf{a}^*, \mathbf{b}) \qquad (\mathbf{a}, \mathbf{b} \in \mathcal{A}),$$

where δ denotes the Kronecker delta. Then it is straightforward to check that A and P satisfy (3.1).

Finally, define $\Phi : \mathsf{X}_M \to \mathsf{X}_A$ by

$$(\Phi x)_i = [(\sigma_M)^i x] \qquad (x \in \mathsf{X}_M,\ i \in \mathbb{Z}).$$

Then Φ is an injective sliding block code of memory and anticipation $2N$. Moreover a direct calculation shows that $\Phi\phi = \phi_{A,P}\Phi$. It remains only to show that Φ is surjective. Let $y = \dots \mathbf{a}_{-2}\mathbf{a}_{-1}\underline{\mathbf{a}_0}\mathbf{a}_1\mathbf{a}_2 \dots$ be any point in X_A. Then there is a point $x \in \mathsf{X}_M$ such that

$$x_{[-N+i,N+i]} = c(\mathbf{a}_i) \qquad (i \in \mathbb{Z}). \tag{3.6}$$

Let $z = \Phi x$, and write $z = \dots \mathbf{b}_{-2}\mathbf{b}_{-1}\underline{\mathbf{b}_0}\mathbf{b}_1\mathbf{b}_2\dots$. Then from the definition of Φ, we have

$$x_{[-N+i,N+i]} = c(\mathbf{b}_i) \qquad (i \in \mathbb{Z}). \tag{3.7}$$

Hence $b_0(\mathbf{a}_i) = b_0(\mathbf{b}_i)$ for all $i \in \mathbb{Z}$ by (3.5), (3.6) and (3.7). This implies $y = z$. □

Let $\zeta_{A,P}$ be the zeta function of the flip system $(\mathsf{X}_A, \sigma_A, \phi_{A,P})$. In Theorem 3.2 below, we express $\zeta_{A,P}$ in terms of the matrices A and P. It is well-known that the Artin-Mazur zeta function ζ_A of the topological Markov shift (X_A, σ_A) satisfies

$$\zeta_A(t) = \frac{1}{\det(I - tA)}. \tag{3.8}$$

See Theorem 6.4.6 in [6].

We need some notations. For an $\mathcal{A} \times \mathcal{A}$ matrix B, the adjugate of B is denoted by $B^\star$, so that $BB^\star = (\det B)I$, the entry sum $\mathcal{S}[B]$ of B is defined by

$$\mathcal{S}[B] = \sum_{(a,b)\in\mathcal{A}\times\mathcal{A}} B(a,b),$$

and the diagonal projection B^Δ of B is defined by

$$B^\Delta(a,b) = B(a,b)\delta(a,b) \qquad (a, b \in \mathcal{A}).$$

Theorem 3.2. *If A and P are 0-1, square matrices which satisfy (3.1), then*

$$\zeta_{A,P}(t) = \sqrt{\zeta_A(t^2)}\exp\left(\zeta_A(t^2)H_{A,P}(t)\right),$$

where $H_{A,P}$ is the polynomial defined by

$$H_{A,P}(t) = \mathcal{S}\Big[tP^\Delta(I - t^2A)^\star(AP)^\Delta + \frac{t^2}{2}\left\{P^\Delta A(I - t^2A)^\star P^\Delta + (PA)^\Delta(I - t^2A)^\star(AP)^\Delta\right\}\Big].$$

Proof. For $i, j \in \mathbb{Z}$ let $p_{i,j}$ denote the number of points in X_A that are fixed by $(\sigma_A)^i$ and $(\sigma_A)^j\phi_{A,P}$. Set

$$G_{A,P}(t) = \sum_{m=1}^{\infty}\left(p_{2m-1,0}t^{2m-1} + \frac{p_{2m,0} + p_{2m,1}}{2}t^{2m}\right). \tag{3.9}$$

Then, in view of Theorem 2.2 and (3.8), we need only to prove the following:

$$G_{A,P}(t) = \frac{H_{A,P}(t)}{\det(I - t^2A)}. \tag{3.10}$$

Let $\mathcal{B}_n$ denote the set of all n-blocks that occur in points in X_A. Then it is easy to see that

$$\begin{aligned} p_{2m+1,0} &= |\{x_0 \dots x_m \in \mathcal{B}_{m+1} : x_0^* = x_0,\ A(x_m, x_m^*) = 1\}|, \\ p_{2m,0} &= |\{x_0 \dots x_m \in \mathcal{B}_{m+1} : x_0^* = x_0,\ x_m^* = x_m\}|, \quad \text{and} \\ p_{2m,1} &= |\{x_1 \dots x_m \in \mathcal{B}_m : A(x_1^*, x_1) = A(x_m, x_m^*) = 1\}|. \end{aligned}$$

Recall that for $a \in \mathcal{B}_1$, a^* is the unique element of $\mathcal{B}_1$ such that $P(a, a^*) = 1$. Moreover, for $a, b \in \mathcal{B}_1$ the following are obvious:

$$\begin{aligned} a^* = b &\Leftrightarrow P(a,b) = 1, \\ A(a, b^*) = 1 &\Leftrightarrow AP(a,b) = 1, \quad \text{and} \\ A(a^*, b) = 1 &\Leftrightarrow PA(a,b) = 1. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} p_{2m+1,0} &= \mathcal{S}\left[P^\Delta A^m (AP)^\Delta\right], \\ p_{2m,0} &= \mathcal{S}\left[P^\Delta A^m P^\Delta\right], \qquad \text{and} \\ p_{2m,1} &= \mathcal{S}\left[(PA)^\Delta A^{m-1} (AP)^\Delta\right]. \end{aligned} \tag{3.11}$$

On the other hand, we have

$$\sum_{m=0}^{\infty} s^m A^m = (I - sA)^{-1} = \frac{1}{\det(I - sA)} (I - sA)^\star \qquad (s \in \mathbb{C},\ \Lambda|s| < 1), \tag{3.12}$$

where Λ denotes the spectral radius of A. Finally, put (3.11) into (3.9), and use (3.12) to obtain (3.10). □

4. Examples.

In order for a dynamical system (X, T) to have a flip, it is necessary that (X, T) is conjugate to its time reversal (X, T^{-1}). However, it is not known whether the condition is sufficient. The first example shows that there is a dynamical system with no flips.

Example 4.1. Let

$$A = \begin{bmatrix} 19 & 5 \\ 4 & 1 \end{bmatrix},$$

and (X_A, σ_A) denote the edge shift of A. It is known that A is not shift equivalent to its transpose A^T [**3**, p. 104]. Hence (X_A, σ_A) is not conjugate to its time reversal $(\mathsf{X}_A, \sigma_A^{-1}) \cong (\mathsf{X}_{A^T}, \sigma_{A^T})$. Consequently, (X_A, σ_A) does not admit a flip.

In the remainder of this section, we consider various flips on full shifts. We show that some of them are not conjugate by calculating their zeta functions or counting the number of fixed points.

Example 4.2. Let (X,σ) be the full 2-shift. We will show that there are infinitely many non-conjugate flips for (X,σ). For each positive integer n we define the $(2n+5)$-block map K_n by $K_n(110^{2n+1}11)=1$, $K_n(110^n10^n11)=0$, and $K_n(x_{-n-2}\dots x_0\dots x_{n+2})=x_0$ when the block is not equal to any of the above two. Let κ_n denote the sliding block code on X induced by the block map K_n. Then clearly κ_n is an automorphism of order 2. Let $\omega_n=\rho\kappa_n$, where ρ is the reverse map. It is easy to see that ω_n is a flip map for (X,σ). The flip systems (X,σ,ω_n), $n\geq 1$, are not conjugate to each other. In fact, for $1\leq n<m$,

$$|\{x\in X:\sigma^{2m+5}x=x,\ \omega_n x=x\}|=2^{m+3}-2^{m-n+1},$$

and

$$|\{x\in X:\sigma^{2m+5}x=x,\ \omega_m x=x\}|=2^{m+3}-2.$$

From this and Theorem 2.2, it also follows that ζ_{σ,ω_n}, $n\geq 1$, are all distinct. A long but straightforward calculation using Theorem 3.2 yields that the zeta function for (X,σ,ω_1) is equal to

$$\sqrt{\frac{1}{1-2t^2}}\exp\left(\frac{2t+3t^2-2t^5-2t^6+2t^7+2t^{10}-2t^{12}-2t^{14}}{1-2t^2}\right).$$

Example 4.3. Let $n\geq 2$ be an integer, (X,σ) the full n-shift, and $\rho:X\to X$ the reverse map. As the zeta function is automorphism-invariant, the flip systems (X,σ,ρ) and $(X,\sigma,\sigma\rho)$ have the same zeta function, which is

$$\zeta_{\sigma,\rho}(t)=\sqrt{\frac{1}{1-nt^2}}\exp\left(\frac{nt+(n+n^2)t^2/2}{1-nt^2}\right).$$

They are, however, not conjugate. In fact, we have

$$|\{x\in X:\sigma^2x=x,\ \rho x=x\}|=n^2,$$

whereas

$$|\{x\in X:\sigma^2x=x,\ \sigma\rho x=x\}|=n.$$

As we have seen in the above examples, a dynamical system may have many non-conjugate flip maps. However the following question still remains to be answered: Let A and B be symmetric 0-1 matrices such that $(\mathsf{X}_A,\sigma_A)\cong(\mathsf{X}_B,\sigma_B)$. Does it follow that $(\mathsf{X}_A,\sigma_A,\rho_A)\cong(\mathsf{X}_B,\sigma_B,\rho_B)$?

Example 4.4. Let (X,σ) be the full 2-shift, and $\psi:X\to X$ defined by

$$\psi(x)=\dots x_2^*\,x_1^*\,\underline{x_0^*}\,x_{-1}^*\,x_{-2}^*\dots,$$

where $0^* = 1$ and $1^* = 0$. Then ψ is a flip for (X, σ). The flips ψ and $\sigma\psi$ are not conjugate since ψ has no fixed points but $\sigma\psi$ has fixed points. But they have the same zeta function

$$\zeta_{\sigma,\psi}(t) = \sqrt{\frac{1}{1-2t^2}} \exp\left(\frac{t^2}{1-2t^2}\right).$$

On taking $n = 2$ in Example 4.3, we know that ρ and $\sigma\rho$ are not conjugate, and have the same zeta function

$$\zeta_{\sigma,\rho}(t) = \sqrt{\frac{1}{1-2t^2}} \exp\left(\frac{2t+3t^2}{1-2t^2}\right).$$

Therefore the four flips ρ, $\sigma\rho$, ψ and $\sigma\psi$ for (X, σ) are not conjugate to each other.

Example 4.5. Let $\mathcal{A} = \{0, 1, 2, 3\}$. Let A and P be 0-1, $\mathcal{A} \times \mathcal{A}$ matrices defined by $A(i, j) = 1$ for all (i, j), and $P(i, j) = 1$ if and only if $(i, j) \in \{(0, 0), (1, 1), (2, 3), (3, 2)\}$. Then we find that the flip system $(\mathsf{X}_A, \sigma, \phi_{A,P})$ has the zeta function

$$\zeta_{A,P} = \sqrt{\frac{1}{1-4t^2}} \exp\left(\frac{2t+4t^2}{1-4t^2}\right).$$

Now, we will show that the flips $\phi_{A,P}$ and $\sigma_A\phi_{A,P}$ for the full 4-shift (X_A, σ_A) are conjugate. Let $X = \{0, 1\}^{\mathbb{Z}}$, $\sigma : X \to X$ the shift map, and $\rho : X \to X$ the reverse map. Let $\pi_1 : \{0, 1\}^2 \ni ab \mapsto a \in \{0, 1\}$, and $\pi_2 : \{0, 1\}^2 \ni ab \mapsto b \in \{0, 1\}$. Define $f : \mathcal{A} \to \{0, 1\}^2$ by $f(0) = 00$, $f(1) = 11$, $f(2) = 01$ and $f(3) = 10$, and $\Phi : \mathsf{X}_A \to X$ by

$$\Phi(x) = \ldots \pi_1 f(x_{-1})\, \pi_2 f(x_{-1})\, \pi_1 f(x_0)\, \underline{\pi_2 f(x_0)}\, \pi_1 f(x_1)\, \pi_2 f(x_1) \ldots.$$

We can easily check that Φ is a conjugacy from $(\mathsf{X}_A, \sigma_A, \phi_{A,P})$ to $(X, \sigma^2, \sigma\rho)$, and so one from $(\mathsf{X}_A, \sigma_A, \sigma_A\phi_{A,P})$ to $(X, \sigma^2, \sigma^3\rho)$. Trivially σ is a conjugacy from $(X, \sigma^2, \sigma\rho)$ to $(X, \sigma^2, \sigma^3\rho)$. Therefore $\Phi^{-1}\sigma\Phi$ is a conjugacy from $(\mathsf{X}_A, \sigma_A, \phi_{A,P})$ to $(\mathsf{X}_A, \sigma_A, \sigma_A\phi_{A,P})$. This proves the assertion.

Acknowledgment. The authors would like to thank Professors R. Burton, K. H. Kim and F. Roush for their helpful comments in the preparation of this paper.

References

[1] M. Artin and B. Mazur, *On periodic points*, Ann. of Math., **81** (1965), 82-99, MR 31 #754, Zbl 0127.13401.

[2] R. Bowen and O. Lanford, *Zeta functions of restrictions of the shift transformation*, Proc. Sympos. Pure Math., **14** (1970), 43-49, MR 42 #6284, Zbl 0211.56501.

[3] M. Boyle, B. Marcus and P. Trow, *Resolving maps and the dimension group for shifts of finite type*, Mem. Amer. Math. Soc., **377** (1987), MR 89c:28019, Zbl 0651.54018.

[4] R. Burton, *Private communications.*

[5] D. Lind, *A zeta function for* $\mathbb{Z}^d$*-actions*, London Math. Soc. Lecture Note Ser., **228** (M. Pollicott and K. Schmidt, ed.), 1996, 433-450, MR 97e:58185, Zbl 0881.58052.

[6] D. Lind and B. Marcus, *An Introduction to Symbolic Dynamics and Coding*, Cambridge University Press, 1995, MR 97a:58050.

[7] K.K. Park, *On ergodic foliations*, Ergodic Theory Dynam. Systems, **8** (1988), 437-457, MR 90b:28021, Zbl 0627.28015.

Received December 26, 2001. This research was supported by the Brain Korea 21 Project.

School of Mathematical Sciences
Seoul National University
Seoul 151-747, Korea
E-mail address: kimyo@math.snu.ac.kr

Department of Mathematics
Ajou University
Suwon 442-749, Korea
E-mail address: jslee@madang.ajou.ac.kr

Department of Mathematics
Ajou University
Suwon 442-749, Korea
E-mail address: kkpark@madang.ajou.ac.kr

This paper is available via http://www.pacjmath.org/2003/209-2-6.html.