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PARAHORIC FIXED SPACES IN UNRAMIFIED PRINCIPAL SERIES REPRESENTATIONS

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Let k be a non-archimedean locally compact field and let G be the set of k-points of a connected reductive group defined over k. Let W be the relative Weyl group of G, and let $\mathcal{H}(G, B)$ be the Hecke algebra of G with respect to an Iwahori subgroup B of G. We compute the effects of $\mathcal{H}(G, B)$ and W on the B-fixed vectors of an unramified principal series representation I of G. We use this computation to determine the dimension of the space of K-fixed vectors in I, where K is a parahoric subgroup of G.

1. Introduction.

Let $\mathbf{G}$ be a reductive group defined over a non-archimedean locally compact field k and let $G = \mathbf{G}(k)$. Let P be a minimal parabolic subgroup of G with Levi decomposition $P = MN$, and let $P^- = MN^-$ be the corresponding decomposition of the opposite parabolic P^-. Let B be an Iwahori subgroup of G with an Iwahori decomposition with respect to P and M, i.e.,

$$B = (B \cap P)(B \cap M)(B \cap P^-).$$

Denote by W the relative Weyl group of G. Let χ be an unramified character of M (i.e., χ is trivial on M_0). Since $M \cong P/N$, χ extends to a character of P which we will also denote by χ. Let δ be the modulus character of P. Define $I(\chi)$ to be the unramified principal series representation of G induced by χ, i.e., the space of all locally constant functions $G \to \mathbb{C}$ such that

$$f(pg) = \chi\delta^{1/2}(p)f(g) \text{ for all } p \text{ in } P,\ g \text{ in } G$$

on which G acts by right translation. It is well-known that the space $I(\chi)^B$ of B-fixed vectors in $I(\chi)$ has dimension $\dim I(\chi)^B = |W|$ [**3**, Prop. 2.1]. In this paper, we generalize this result to the fixed space $I(\chi)^K$ where K is a parahoric subgroup of G containing B.

Let A be a maximal split torus in M and let $\mathcal{N}$ be its normalizer in G. If M_0 is the maximal compact subgroup of M and $\widetilde{W} = \mathcal{N}/M_0$, then we have a surjection $\nu : \widetilde{W} \to W = \mathcal{N}/M$. Let K be a parahoric subgroup of G containing B and let W_K be the finite Coxeter subgroup of $\widetilde{W}$ such that $K = BW_KB$ (see [**4**, §1]). We will prove the following:

Theorem 1.1. *The dimension of* $I(\chi)^K$ *is* $|W/\nu(W_K)|$.

As a Coxeter group, W_K is generated by a canonical finite set S of reflections. Thus

$$I(\chi)^K = \bigcap_{s\in S} I(\chi)^{\langle B,s\rangle}.$$

In Section 3, we explicitly determine the effects of reflections $s \in S$ on $I(\chi)^B$ (Theorem 3.1) and as a corollary the actions of the generators of the Iwahori-Hecke algebra $\mathcal{H}(G,B)$ on $I(\chi)^B$ (Corollary 3.2). We then compute the subspaces $I(\chi)^{\langle B,s\rangle}$ in terms of the usual basis of $I(\chi)$ as given in [**3**, Prop. 2.1]. Then in Section 4, we complete the proof of Theorem 1.1 by showing that the dimension of the intersection of the $I(\chi)^{\langle B,s\rangle}$ is $|W/\nu(W_K)|$.

Let $\mathcal{H}(G,K)$ be the Hecke algebra of compactly supported functions $G \to \mathbb{C}$, bi-invariant by K. Let E be a simple $\mathcal{H}(G,K)$-module. It is known that there is an irreducible admissible representation V of G such that E is isomorphic as a $\mathcal{H}(G,K)$-module to the space V^K of K-fixed vectors [**1**, 2.10]. Since $V^B \supset V^K = E \neq 0$, it follows from a well-known result that V embeds inside some unramified principal series representation I of G so that $\dim E = \dim V^K \leq \dim I^K$. Thus Theorem 1.1 has the following corollary:

Corollary 1.2. *If* K *is a parahoric subgroup of* G *and* E *is a simple module over* $\mathcal{H}(G,K)$, *then*

$$\dim E \leq |W/\nu(W_K)|.$$

Moreover, this bound is sharp.

The sharpness of this bound is a result of the fact that there exist irreducible unramified principal series representations (see e.g., [**2**, Theorem 3.3]) and that for such a representation I, the $\mathcal{H}(G,K)$-module I^K is simple [**1**, 2.10] and, by Theorem 1.1, of dimension $|W/\nu(W_K)|$.

Remark 1.3. While Theorem 1.1 is needed to prove the sharpness in Corollary 1.2, the inequality itself can be proved by a simpler argument. Indeed, it is easily demonstrated that $\dim I(\chi)^K \leq |W/\nu(W_K)|$ by noting that

$$\dim I(\chi)^K \leq |P\backslash G/K|$$

and

$$|P\backslash G/K| = |W/\nu(W_K)|.$$

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2. Preliminaries.

See [**6**] or [**3**, §1] as a reference for much of the material in this section. In the following, we let k be a non-archimedean locally compact field. We denote by $\mathbf{G}$ a connected reductive algebraic group defined over k with group of

k-points G. Similarly, throughout this section, if $\mathbf{H}$ is any algebraic group defined over k, we will denote its k-points by the corresponding non-bold letter H.

Let $\mathbf{P}$ be a fixed minimal parabolic subgroup of $\mathbf{G}$ containing a maximal split torus $\mathbf{A}$ of $\mathbf{G}$. Denote by $\mathbf{N}$ the unipotent radical of $\mathbf{P}$, and by $\mathbf{M}$ the centralizer of $\mathbf{A}$. Then $\mathbf{P}$ has Levi decomposition $\mathbf{MN}$. Let Φ' denote the set of roots of $\mathbf{G}$ relative to $\mathbf{A}$ and Φ'_{nd} the subset of non-divisible roots. Also, let W be the relative Weyl group.

Denote by $\mathcal{B} = \mathcal{B}(\mathbf{G}, k)$ the Bruhat-Tits building of $\mathbf{G}$ over k and by $\mathcal{A}$ the apartment of $\mathcal{B}$ stabilized by A. The normalizer $\mathcal{N}$ of A in G is then the stabilizer of $\mathcal{A}$ and the maximal compact subgroup M_0 of M is the kernel of the map $\mathcal{N} \to \text{Aut}(\mathcal{A})$. Let $\widetilde{W} = \mathcal{N}/M_0$. Denote by Φ_{aff} the canonical affine root system on $\mathcal{A}$ and by W_{aff} the corresponding affine Weyl group. Then W_{aff} may be identified with a normal subgroup of $\widetilde{W}$.

Fix a special point x_0 in $\mathcal{B}$ and let Φ be the set of affine roots vanishing at x_0. Then Φ is a reduced root system, and we have a bijection between Φ and Φ'_{nd} corresponding to the choice of x_0 . We let Φ^+ be the subset of positive affine roots corresponding to P and Δ the subset of simple roots.

Let C be the unique chamber in $\mathcal{A}$ containing x_0 with the property that every α in Φ^+ takes positive values on C. Denote by B the Iwahori subgroup of G fixing C pointwise and by K_0 the special maximal compact subgroup fixing x_0. Then $W = \mathcal{N}/M \cong (\mathcal{N} \cap K_0)/M_0$, which is the stabilizer of x_0 in $\widetilde{W}$. We will identify these groups throughout. We denote by ν the surjection $\widetilde{W} \to W$. The kernel of ν is the group of translations in $\widetilde{W}$.

For each α in Φ_{aff}, denote by $N(\alpha)$ the pointwise stabilizer of the half-apartment $\{x \in \mathcal{A} \mid \alpha(x) \geq 0\}$. We note that

$$B = M_0 \cdot \prod_{\alpha \in \Phi^+} N(\alpha) \cdot \prod_{\alpha \in \Phi^-} N(\alpha + 1).$$

Let $P_0 \subset P$ be the compact subgroup

$$P \cap K_0 = M_0 \cdot \prod_{\alpha \in \Phi^+} N(\alpha).$$

Let $\Phi = \bigcup \Phi_i$ be the decomposition of Φ into irreducible root systems. Denote by $\widetilde{\Delta}$ the set containing the highest root $\widetilde{\alpha}_i$ of Φ_i for each i. Let

$$\Delta_{\text{aff}} = \{\alpha \in \Phi_{\text{aff}} \mid \alpha \in \Delta \text{ or } \alpha = \widetilde{\alpha} - 1 \text{ for some } \widetilde{\alpha} \in \widetilde{\Delta}\}.$$

For α in Δ_{aff}, let w_α be the reflection in $\text{Aut}(\mathcal{A})$ through the vanishing hyperplane of α. Then $S_{\text{aff}} = \{w_\alpha \mid \alpha \in \Delta_{\text{aff}}\}$ is a set of involutive generators for the Coxeter group W_{aff}.

For α in Φ, let a_α be the translation $w_\alpha w_{\alpha-1}$ on $\mathcal{A}$. We note that

$$a_{-\alpha} = a_\alpha^{-1} \text{ for any } \alpha \text{ in } \Phi.$$

We let K be a fixed parahoric subgroup of G containing B. Since the triple $(G, B, \mathcal{N})$ is a generalized Tits system (see [4, §1]), there exists a special subgroup W_K of W_{aff} such that $K = BW_KB$; W_K is finite as K is compact. We denote by S the subset of S_{aff} generating W_K.

For any w in $\widetilde{W}$, we denote by $q(w)$ the index $[BwB : B]$. Also for α in Φ_{aff}, we let q_α be the index $[N(\alpha - 1) : N(\alpha)]$. We note that $q_{\alpha+2} = q_\alpha$. Since (cf. [5, Cor. 2.7])

$$Bw_\alpha B = N(\alpha)w_\alpha B \text{ for } \alpha \text{ in } \Delta, \tag{1}$$

$$Bw_{\widetilde{\alpha}-1}B = N(-\widetilde{\alpha}+1)w_{\widetilde{\alpha}-1}B \text{ for } \widetilde{\alpha} \text{ in } \widetilde{\Delta}, \tag{2}$$

it follows that

$$q(w_\alpha) = q_{\alpha+1} \text{ for } \alpha \text{ in } \Delta, \qquad q(w_{\widetilde{\alpha}-1}) = q_{\widetilde{\alpha}+2} = q_{\widetilde{\alpha}} \text{ for } \widetilde{\alpha} \text{ in } \widetilde{\Delta}.$$

If $\alpha \in \Delta$, we denote by B_α the group $B \cap w_\alpha B w_\alpha$, and if $\widetilde{\alpha} \in \widetilde{\Delta}$, $B_{\widetilde{\alpha}-1}$ denotes the group $B \cap w_{\widetilde{\alpha}-1}Bw_{\widetilde{\alpha}-1}$.

Let dx be the Haar measure on G for which B has volume 1. We denote by $\mathcal{H}(G, B)$ the Iwahori-Hecke algebra of compactly supported functions $G \to \mathbb{C}$ bi-invariant by B. The product on $\mathcal{H}(G, B)$ is given by convolution with respect to dx. Fix an unramified character χ of M and let δ be the modulus character of P. Denote by $I(\chi)$ the induced representation $\mathrm{Ind}_P^G(\chi\delta^{1/2})$, i.e., the unramified principal series representation induced by χ as described in Section 1. If x is an element of G, we will denote the action of x on $u \in I(\chi)$ by $u \mapsto x \cdot u$. Note that if $w \in \widetilde{W}$ then the expression $w \cdot u$ is well-defined for $u \in I(\chi)^B$ as w is determined modulo $M_0 \subset B$. A function $h \in \mathcal{H}(G, B)$ acts on $I(\chi)^B$ by the formula

$$h \cdot u = \int_G (x \cdot u)\, h(x)\, dx,$$

where $v \in I(\chi)^B$.

Let $C_c^\infty(G)$ be the space of locally constant, compactly supported functions $G \to \mathbb{C}$. The map $\mathcal{P}_\chi : C_c^\infty(G) \to I(\chi)$ defined by

$$\mathcal{P}_\chi(f)(g) = \int_P \chi^{-1}\delta^{1/2}(p)\, f(pg)\, dp$$

(where dp is the left Haar measure on P giving P_0 measure 1) is a G-equivariant surjection. The functions $\phi_{w,\chi} = \mathcal{P}_\chi(\mathrm{ch}_{BwB})$ (w in W) form a basis of the subspace of B-fixed vectors $I(\chi)^B$ [3, Prop. 2.1]. Concretely, for $p \in P$,$w' \in W$ and $b \in B$, $\phi_{w,\chi}(pw'b)$ equals $\chi\delta^{1/2}(p)$ if $w' = w$ and is zero otherwise.

3. The effect of $W_{\mathbf{aff}}$ on $I(\chi)^B$.

The goal of this section is to compute the effect of $s \in S_{\text{aff}}$ on $I(\chi)^B$. This will be important for the proof in the following section since we will need to determine the space $I(\chi)^{\langle B,s\rangle}$ of vectors in $I(\chi)^B$ fixed by s.

Theorem 3.1. *Suppose that $w \in W$, $\alpha \in \Delta$ and $\widetilde{\alpha} \in \widetilde{\Delta}$. Then*

$$w_\alpha \cdot \phi_{w,\chi} = \begin{cases} \text{ch}_{Pww_\alpha B_\alpha} \phi_{ww_\alpha,\chi} & \text{if } w\alpha \in \Phi^+ \\ \phi_{ww_\alpha,\chi} + \text{ch}_{Pw(B-B_\alpha)} \phi_{w,\chi} & \text{if } w\alpha \in \Phi^-, \end{cases}$$

$$w_{\widetilde{\alpha}-1} \cdot \phi_{w,\chi} = \begin{cases} \chi\delta^{1/2}(a_{w\widetilde{\alpha}}) \text{ch}_{Pww_{\widetilde{\alpha}} B_{\widetilde{\alpha}-1}} \phi_{ww_{\widetilde{\alpha}},\chi} & \text{if } w\widetilde{\alpha} \in \Phi^- \\ \chi\delta^{1/2}(a_{w\widetilde{\alpha}}) \phi_{ww_{\widetilde{\alpha}},\chi} + \text{ch}_{Pw(B-B_{\widetilde{\alpha}-1})} \phi_{w,\chi} & \text{if } w\widetilde{\alpha} \in \Phi^+. \end{cases}$$

Proof. For any s in S_{aff}, $g \in G$,

$$(s \cdot \phi_{w,\chi})(g) = \phi_{w,\chi}(gs).$$

The Iwasawa decomposition enables us to write $g = p'w'b'$ for some p' in P, w' in W, and b' in B. We will evaluate $\phi_{w,\chi}(gs) = \phi_{w,\chi}(p'w'b's)$ by determining the double coset in which $p'w'b's$ lies.

We first consider $s = w_\alpha$ for $\alpha \in \Delta$. Now if $w'\alpha \in \Phi^+$ then by (1)

$$\begin{aligned} p'w'b'w_\alpha &\in p'w'Bw_\alpha B \\ &= p'w'N(\alpha)w_\alpha B \\ &= p'N(w'\alpha)w'w_\alpha B \\ &\subset (p'N)w'w_\alpha B. \end{aligned}$$

Since $\chi\delta^{1/2}$ is trivial on N, it follows that $\phi_{w,\chi}(p'w'b'w_\alpha)$ equals $\chi\delta^{1/2}(p')$ if $w = w'w_\alpha$ and 0 otherwise.

If, on the other hand, $w'\alpha \in \Phi^-$ then suppose first that $b' \in B_\alpha$. Then

$$p'w'b'w_\alpha \in p'w'b'w_\alpha B = p'w'w_\alpha B$$

since $w_\alpha B_\alpha w_\alpha \subset B$. Thus $\phi_{w,\chi}(p'w'b'w_\alpha)$ equals $\chi\delta^{1/2}(p')$ if $w = w'w_\alpha$ and 0 otherwise.

Lastly, suppose that $w'\alpha \in \Phi^-$ and $b' \in B - B_\alpha$. It is easily deduced from $w'\alpha \in \Phi^-$ that

$$Pw'Bw_\alpha B = Pw'w_\alpha B \cup Pw'B.$$

Moreover, one can show that $p'w'b'w_\alpha \in Pw'B$ if and only if b' is an element of $B - B_\alpha$. Thus $p'w'b'w_\alpha = pw'b$ for some $p \in P$, $b \in B$. Since

$$p^{-1}p' = w'bw_\alpha {b'}^{-1}{w'}^{-1} \in P \cap K_0 = P_0$$

and since $\chi\delta^{1/2}$ is trivial on P_0, we have that $\chi\delta^{1/2}(p) = \chi\delta^{1/2}(p')$. Therefore, $\phi_{w,\chi}(p'w'b'w_\alpha)$ equals $\chi\delta^{1/2}(p')$ if $w = w'$ and 0 otherwise.

Note that $w'\alpha \in \Phi^{\pm}$ if and only if $w'w_\alpha\alpha = -w'\alpha \in \Phi^{\mp}$. Using this, we assemble the preceding cases to obtain that

$$(w_\alpha \cdot \phi_{w,\chi})(p'w'b') = \begin{cases} \chi\delta^{1/2}(p') & \text{if } w\alpha \in \Phi^+,\ w' = ww_\alpha,\ b' \in B_\alpha \\ \chi\delta^{1/2}(p') & \text{if } w\alpha \in \Phi^-,\ w' = ww_\alpha \\ \chi\delta^{1/2}(p') & \text{if } w\alpha \in \Phi^-,\ w' = w,\ b' \in B - B_\alpha \\ 0 & \text{otherwise.} \end{cases}$$

This immediately implies the first result of the theorem.

We now prove the second formula by calculating $w_{\widetilde{\alpha}-1} \cdot \phi_{w,\chi}$ for $\widetilde{\alpha} \in \widetilde{\Delta}$. Assume first that $w'\widetilde{\alpha} \in \Phi^-$. Then by (2)

$$\begin{array}{rcl} p'w'b'w_{\widetilde{\alpha}-1} & \in & p'w'Bw_{\widetilde{\alpha}-1}B \\ & = & p'w'N(-\widetilde{\alpha}+1)w_{\widetilde{\alpha}-1}B \\ & = & p'N(-w'\widetilde{\alpha}+1)w'w_{\widetilde{\alpha}}a_{\widetilde{\alpha}}B \\ & \subset & (p'a_{-w'\widetilde{\alpha}}N)w'w_{\widetilde{\alpha}}B. \end{array}$$

Since χ is trivial on N, it follows that $\phi_{w,\chi}(p'w'b'w_{\widetilde{\alpha}-1})$ equals $\chi\delta^{1/2}(p'a_{-w'\widetilde{\alpha}})$ if $w = w'w_{\widetilde{\alpha}}$ and 0 otherwise.

Now suppose that $w'\widetilde{\alpha} \in \Phi^+$ and that $b' \in B_{\widetilde{\alpha}-1}$. Then

$$p'w'b'w_{\widetilde{\alpha}-1} \in p'w'b'w_{\widetilde{\alpha}-1}B = p'w'w_{\widetilde{\alpha}-1}B = (p'a_{-w'\widetilde{\alpha}})w'w_{\widetilde{\alpha}}B$$

since $w_{\widetilde{\alpha}-1}B_{\widetilde{\alpha}-1}w_{\widetilde{\alpha}-1} \subset B$. It follows that $\phi_{w,\chi}(p'w'b'w_{\widetilde{\alpha}-1})$ is equal to $\chi\delta^{1/2}(p'a_{-w'\widetilde{\alpha}})$ if $w = w'w_{\widetilde{\alpha}}$ and 0 otherwise.

Finally, suppose that $b' \in B - B_{\widetilde{\alpha}-1}$. As before, it can be shown that

$$Pw'Bw_{\widetilde{\alpha}-1}B = Pw'w_{\widetilde{\alpha}}B \cup Pw'B,$$

and furthermore that $p'w'b'w_{\widetilde{\alpha}-1} \in Pw'B$ if and only if b' is an element of $B - B_{\widetilde{\alpha}-1}$. Hence $p'w'b'w_{\widetilde{\alpha}-1} = pw'b$ for some $p \in P$, $b \in B$. It is easily shown that this forces $p^{-1}p' \in NP_0$ so that $\chi\delta^{1/2}(p) = \chi\delta^{1/2}(p')$. Thus $\phi_{w,\chi}(p'w'b'w_{\widetilde{\alpha}-1})$ equals $\chi\delta^{1/2}(p')$ if $w = w'$ and 0 otherwise.

Noting that $w'\widetilde{\alpha} \in \Phi^{\pm}$ if and only if $w'w_{\widetilde{\alpha}}\widetilde{\alpha} = -w'\widetilde{\alpha} \in \Phi^{\mp}$, we obtain

$$\begin{array}{l} (w_\alpha \cdot \phi_{w,\chi})(p'w'b') \\ \quad = \begin{cases} \chi\delta^{1/2}(a_{w\widetilde{\alpha}})\chi\delta^{1/2}(p') & \text{if } w\widetilde{\alpha} \in \Phi^-,\ w' = ww_{\widetilde{\alpha}},\ b' \in B_{\widetilde{\alpha}-1} \\ \chi\delta^{1/2}(a_{w\widetilde{\alpha}})\chi\delta^{1/2}(p') & \text{if } w\widetilde{\alpha} \in \Phi^+,\ w' = ww_{\widetilde{\alpha}} \\ \chi\delta^{1/2}(p') & \text{if } w\widetilde{\alpha} \in \Phi^+,\ w' = w,\ b' \in B - B_{\widetilde{\alpha}-1} \\ 0 & \text{otherwise.} \end{cases} \end{array}$$

The second result follows. □

Theorem 3.1 has the following corollary giving the action of ch_{BsB} for s in S_{aff}.

Corollary 3.2. *Suppose that $w \in W$, $\alpha \in \Delta$ and $\widetilde{\alpha} \in \widetilde{\Delta}$. Then*

$$\mathrm{ch}_{Bw_\alpha B} \cdot \phi_{w,\chi} = \begin{cases} \phi_{ww_\alpha,\chi} & \text{if } w\alpha \in \Phi^+ \\ q_{\alpha+1}\phi_{ww_\alpha,\chi} + (q_{\alpha+1}-1)\phi_{w,\chi} & \text{if } w\alpha \in \Phi^-, \end{cases}$$

$$\mathrm{ch}_{Bw_{\widetilde{\alpha}-1}B} \cdot \phi_{w,\chi} = \begin{cases} \chi\delta^{1/2}(a_{w\widetilde{\alpha}})\phi_{ww_{\widetilde{\alpha}},\chi} & \text{if } w\widetilde{\alpha} \in \Phi^- \\ \chi\delta^{1/2}(a_{w\widetilde{\alpha}})q_{\widetilde{\alpha}}\phi_{ww_{\widetilde{\alpha}},\chi} + (q_{\widetilde{\alpha}}-1)\phi_{w,\chi} & \text{if } w\widetilde{\alpha} \in \Phi^+. \end{cases}$$

Proof. We prove the first formula in the case $w\alpha \in \Phi^-$. The other cases are handled similarly. For $g \in G$ we have

$$\begin{aligned} (\mathrm{ch}_{Bw_\alpha B} \cdot \phi_{w,\chi})(g) &= \int_G \phi_{w,\chi}(gx)\mathrm{ch}_{Bw_\alpha B}(x)dx \\ &= \int_{Bw_\alpha B} \phi_{w,\chi}(gx)dx \\ &= \sum_n \phi_{w,\chi}(gnw_\alpha) \\ &= \sum_n (w_\alpha \cdot \phi_{w,\chi})(gn), \end{aligned}$$

where n ranges over a set of representatives in $N(\alpha)$ for $N(\alpha)/N(\alpha+1)$.

If $g \in Pww_\alpha B$ then so is gn for each of the $q_{w_\alpha} = q_{\alpha+1}$ representatives n. On the other hand, if $g \in PwB$, then $gn \in Pw(B - B_\alpha)$ for precisely $q_{\alpha+1} - 1$ of the representatives n. Thus

$$\begin{aligned} (\mathrm{ch}_{Bw_\alpha B} \cdot \phi_{w,\chi})(g) &= \sum_n (w_\alpha \cdot \phi_{w,\chi})(gn) \\ &= \sum_n \left[\phi_{ww_\alpha,\chi}(gn) + \mathrm{ch}_{Pw(B-B_\alpha)}(gn)\phi_{w,\chi}(gn)\right] \\ &= q_{\alpha+1}\phi_{ww_\alpha,\chi}(g) + (q_{\alpha+1}-1)\phi_{w,\chi}(g). \end{aligned}$$

□

The following corollary of Theorem 3.1 gives a basis for $I(\chi)^{\langle B,s\rangle}$, $s \in S_{\mathrm{aff}}$.

Corollary 3.3. *Suppose $\alpha \in \Delta$ and $\widetilde{\alpha} \in \widetilde{\Delta}$. Then*

(i) $\{\phi_{w,\chi} + \phi_{ww_\alpha,\chi} \mid w \in W, w\alpha \in \Phi^+\}$ *is a basis for the fixed space* $I(\chi)^{\langle B,w_\alpha\rangle}$.

(ii) $\{\phi_{w,\chi} + \chi\delta^{1/2}(a_{w\widetilde{\alpha}})\phi_{ww_{\widetilde{\alpha}},\chi} \mid w \in W, w\widetilde{\alpha} \in \Phi^+\}$ *is a basis for the fixed space* $I(\chi)^{\langle B,w_{\widetilde{\alpha}-1}\rangle}$.

Proof. Let $s \in S_{\mathrm{aff}}$. Note that

$$s \cdot I(\chi)^B \cap I(\chi)^B = I(\chi)^{sBs} \cap I(\chi)^B = I(\chi)^{\langle sBs,B\rangle} = I(\chi)^{\langle B,s\rangle}.$$

Thus $I(\chi)^{\langle B,s\rangle}$ is precisely the set of vectors in $I(\chi)^B$ sent to $I(\chi)^B$ by s. It is clear from Theorem 3.1 that if $s = w_\alpha$ this set is spanned by

$$\{\phi_{w,\chi} + \phi_{ww_\alpha,\chi} \mid w \in W, w\alpha \in \Phi^+\},$$

and if $s = w_{\widetilde{\alpha}-1}$ this set is spanned by

$$\{\phi_{w,\chi} + \chi\delta^{1/2}(a_{w\widetilde{\alpha}})\phi_{ww_{\widetilde{\alpha}},\chi} \mid w \in W, w\widetilde{\alpha} \in \Phi^+\}.$$

□

4. Proof of Theorem 1.1.

We now prove that the dimension of

$$I(\chi)^K = I(\chi)^{BW_KB} = \bigcap_{s\in S} I(\chi)^{\langle B,s\rangle}$$

is equal to $|W/\nu(W_K)|$.

Suppose that $f = \sum_{w\in W} c(w)\phi_{w,\chi}$ is a vector in $I(\chi)^B$ with the $c(w) \in \mathbb{C}$. Then it is easily deduced from Corollary 3.3 that $f \in \bigcap_{s\in S} I(\chi)^{\langle B,s\rangle}$ if and only if for all $w \in W$,

$$c(ww_\alpha) = c(w) \text{ for all } \alpha \in \Delta \text{ with } w_\alpha \in S \tag{3}$$

$$c(ww_{\widetilde{\alpha}}) = \chi\delta^{1/2}(a_{w\widetilde{\alpha}})c(w) \text{ for all } \widetilde{\alpha} \in \widetilde{\Delta} \text{ with } w_{\widetilde{\alpha}-1} \in S. \tag{4}$$

Let V be the space of functions $c : W \to \mathbb{C}$ satisfying (3) and (4). Then $\dim I(\chi)^K = \dim V$. Since $\nu(w_{\beta-1}) = \nu(w_\beta) = w_\beta$ for all $\beta \in \Phi$, it follows that $c(w)$ determines $c(ww')$ for all $w' \in \langle \nu(s) \mid s \in S\rangle = \nu(W_K)$ so

$$\dim V \leq |W/\nu(W_K)|.$$

We will prove that $\dim V = |W/\nu(W_K)|$.

Remark 4.1. We note that if $W_K \subset W$ (i.e., if $K \subset K_0$) then it is clear that $\dim V = \dim I(\chi)^K = |W/\nu(W_K)|$ since in this case only the relations in (3) appear.

Since W_K is finite, it contains no non-trivial translations so ν is injective on W_K. Thus $\nu(W_K) \cong W_K$, and $\nu(W_K)$ is generated as a Coxeter group by $\nu(S)$. We will denote the element of W_K corresponding to $t \in \nu(S)$ by $\nu^{-1}(t)$. Define recursively a function [ ] from the set of finite sequences of elements of $\nu(S)$ to W_{aff}. Let $t_1, \ldots, t_n \in \nu(S)$. For the empty sequence $\emptyset$, let $[\emptyset] = e$. Define

$$[t_1] = \begin{cases} e & \text{if } \nu^{-1}(t_1) = w_\alpha,\ \alpha \in \Delta \\ a_{\widetilde{\alpha}} & \text{if } \nu^{-1}(t_1) = w_{\widetilde{\alpha}-1},\ \widetilde{\alpha} \in \widetilde{\Delta}, \end{cases}$$

and then set

$$[t_1, \ldots, t_n] = \begin{cases} [t_1, \ldots, t_{n-1}] & \text{if } \nu^{-1}(t_n) = w_\alpha,\ \alpha \in \Delta \\ [t_1, \ldots, t_{n-1}]\, a_{t_1\cdots t_{n-1}\widetilde{\alpha}} & \text{if } \nu^{-1}(t_n) = w_{\widetilde{\alpha}-1},\ \widetilde{\alpha} \in \widetilde{\Delta}. \end{cases}$$

It follows easily from the definition of [ ] that

$$[t_1,\dots,t_k](t_1\cdots t_k)[t_{k+1},\dots,t_n](t_1\cdots t_k)^{-1}=[t_1,\dots,t_n]. \tag{5}$$

We claim that the element $[t_1,\dots,t_n]$ of W_{aff} depends only on the product $t_1\cdots t_n$ and not on the particular sequence $t_1,\dots,t_n$.

Lemma 4.2. *Let $t_1,\dots,t_n,u_1,\dots,u_m$ be elements of $\nu(S)$ such that*

$$t_1\cdots t_n=u_1\cdots u_m.$$

Then $[t_1,\dots,t_n]=[u_1,\dots,u_m]$.

Proof. Since $(\nu(W_K),\nu(S))$ is a Coxeter group, the word $t_1\cdots t_n$ is obtainable from $u_1\cdots u_m$ via the basic Coxeter group relations among the elements of $\nu(S)$, i.e., those of the form $(tu)^{m(t,u)}=e$, where $t,u\in\nu(S)$ and $m(t,u)$ is some number in $\{1,2,3,4,6\}$ (see e.g. [**5**, 1.6]). Therefore, it suffices to show that [ ] remains unchanged when a subsequence of consecutive terms in a sequence $t_1,\dots,t_n$ is deleted according to such a relation. In fact, due to (5) one need only show that

$$[\underbrace{t,u,t,u,\dots,t,u}_{m(t,u)}]=[\emptyset]=e \tag{6}$$

for each basic relation $(tu)^{m(t,u)}=e$ among the elements of $\nu(S)$.

It is clear that (6) holds if $\nu^{-1}(t),\nu^{-1}(u)\in W$. Therefore we shall consider only those relations which involve some reflection $t\in\nu(S)$ such that $\nu^{-1}(t)\notin W$. Such a t is necessarily of the form $w_{\widetilde{\alpha}}=\nu(w_{\widetilde{\alpha}-1})$ for some $\widetilde{\alpha}\in\widetilde{\Delta}$. The basic relations involving $w_{\widetilde{\alpha}}$ are of the form

$$(w_{\widetilde{\alpha}}u)^m=e \tag{7}$$

where $u\in\nu(S)$ and $m\in\{1,2,3,4\}$. (It is never the case that $m=6$.)

First consider the case $m=1$. Here u must equal $w_{\widetilde{\alpha}}$ so (6) holds as

$$[w_{\widetilde{\alpha}},w_{\widetilde{\alpha}}]=a_{\widetilde{\alpha}}a_{w_{\widetilde{\alpha}}\widetilde{\alpha}}=a_{\widetilde{\alpha}}a_{-\widetilde{\alpha}}=e.$$

Now suppose that $m>1$ and $\nu^{-1}(u)\in W$ in (7). Then

$$[\underbrace{w_{\widetilde{\alpha}},u,\dots,w_{\widetilde{\alpha}},u}_{m}]=a_{\widetilde{\alpha}}\dots a_{(w_{\widetilde{\alpha}}u)^{m-1}\widetilde{\alpha}}.$$

Since $w_{\widetilde{\alpha}}u$ is a rotation of order m, $\widetilde{\alpha}+\cdots+(w_{\widetilde{\alpha}}u)^{m-1}\widetilde{\alpha}=0$ so (6) holds as

$$a_{\widetilde{\alpha}}\dots a_{(w_{\widetilde{\alpha}}u)^{m-1}\widetilde{\alpha}}=e.$$

Finally, suppose $m>1$ and $\nu^{-1}(u)\notin W$ in (7). In this case, it follows that $m=2$ and $u=w_{\widetilde{\beta}}$ for some $\widetilde{\beta}\in\widetilde{\Delta}$. Then $w_{\widetilde{\beta}}(\widetilde{\alpha})=\widetilde{\alpha}$ and $w_{\widetilde{\alpha}}(\widetilde{\beta})=\widetilde{\beta}$. It follows that (6) holds again as

$$[w_{\widetilde{\alpha}},w_{\widetilde{\beta}},w_{\widetilde{\alpha}},w_{\widetilde{\beta}},]=a_{\widetilde{\alpha}}a_{w_{\widetilde{\alpha}}\widetilde{\beta}}a_{w_{\widetilde{\alpha}}w_{\widetilde{\beta}}\widetilde{\alpha}}a_{w_{\widetilde{\alpha}}w_{\widetilde{\beta}}w_{\widetilde{\alpha}}\widetilde{\beta}}=a_{\widetilde{\alpha}}a_{\widetilde{\beta}}a_{-\widetilde{\alpha}}a_{-\widetilde{\beta}}=e.$$

□

Let $t_1, \dots, t_n \in \nu(S)$. Since $[t_1, \dots, t_n]$ depends only on the product $t_1 \cdots t_n$, $[\ \ ]$ gives a function $\nu(W_K) \to W_{\mathrm{aff}}$, which we will also denote by $[\ \ ]$. Explicitly, for $w \in \nu(W_K)$, $[w] = [t_1, \dots, t_n]$ for any $t_1, \dots, t_n \in \nu(S)$ with $w = t_1 \cdots t_n$. Note that $[\ \ ]$ is a 1-cocycle from $\nu(W_K)$ to the group of translations in W_{aff}.

Proposition 4.3. *The space V of functions $W \to \mathbb{C}$ satisfying (3) and (4) has dimension $|W/\nu(W_K)|$.*

Proof. Let R be a set of representatives for the left cosets of $\nu(W_K)$ in W. For each $\sigma \in R$, define the function $c_\sigma : W \to \mathbb{C}$ by setting

$$c_\sigma(w) = \begin{cases} \chi\delta^{1/2}([w']) & \text{if } w = \sigma w' \in \sigma\nu(W_K) \\ 0 & \text{if } w \notin \sigma\nu(W_K). \end{cases}$$

The c_σ are clearly linearly independent and are $|W/\nu(W_K)|$ in number. It suffices then to show that the c_σ are in V.

Fix $\sigma \in R$. Let α be an element of Δ such that $w_\alpha \in S$. If $w \notin \sigma\nu(W_K)$ then $ww_\alpha \notin \sigma\nu(W_K)$ so

$$c_\sigma(w) = 0 = c_\sigma(ww_\alpha).$$

If $w = \sigma w' \in \sigma\nu(W_K)$ then

$$c_\sigma(ww_\alpha) = c_\sigma(\sigma w' w_\alpha) = \chi\delta^{1/2}([w'w_\alpha]) = \chi\delta^{1/2}([w']) = c_\sigma(w).$$

Thus (3) holds for c_σ.

Now let $\widetilde{\alpha}$ be an element of $\widetilde{\Delta}$ such that $w_{\widetilde{\alpha}-1} \in S$. As before, if $w \notin \sigma\nu(W_K)$ then

$$c_\sigma(w) = 0 = \chi\delta^{1/2}(a_{w\widetilde{\alpha}})c_\sigma(ww_{\widetilde{\alpha}}).$$

And if $w = \sigma w' \in \sigma\nu(W_K)$ then

$$\begin{aligned} c_\sigma(ww_{\widetilde{\alpha}}) &= c_\sigma(\sigma w' w_{\widetilde{\alpha}}) \\ &= \chi\delta^{1/2}([w'w_{\widetilde{\alpha}}]) \\ &= \chi\delta^{1/2}([w']a_{w'\widetilde{\alpha}}) \\ &= \chi\delta^{1/2}([w'])\chi\delta^{1/2}(a_{w'\widetilde{\alpha}}) \\ &= \chi\delta^{1/2}(a_{w'\widetilde{\alpha}})c_\sigma(w). \end{aligned}$$

Thus c_σ satisfies (4) and lies in V. □

It follows that $\dim I(\chi)^K = \dim V = |W/\nu(W_K)|$.

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