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AN ENDPOINT ESTIMATE FOR SOME MAXIMAL OPERATORS ASSOCIATED TO SUBMANIFOLDS OF LOW CODIMENSION

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We show that the maximal operator

$$\mathcal{M}f(x) = \sup_{j\in\mathbb{Z}} \left| \int_{\mathbb{R}^d} f(x - 2^j y)\, d\mu(y) \right|$$

maps H^1 into $L^{1,\infty}$ under certain assumptions on the decay of $\hat{\mu}$ and the geometry of supp(μ).

1. Introduction and statement of results.

In this paper we consider the lacunary maximal operator $\mathcal{M}$ defined by

$$\mathcal{M}f(x) = \sup_{j\in\mathbb{Z}} \left| \int_{\mathbb{R}^d} f(x - 2^j y)\, d\mu(y) \right|. \tag{1}$$

Here $d \geq 1$ is an integer. When μ is a finite positive Borel measure on $\mathbb{R}^d$, it is proved in [**DR**] that if the Fourier transform of μ satisfies

$$|\hat{\mu}(\xi)| \leq c\,(1 + |\xi|)^{-\alpha} \tag{2}$$

for some $\alpha > 0$, then (1) is bounded on $L^p(\mathbb{R}^d)$ for $1 < p \leq \infty$. Also when $\alpha = \frac{d}{2}$, it is proved in [**O**] that (1) maps $H^1(\mathbb{R}^d)$ into $L^{1,\infty}(\mathbb{R}^d)$. Here H^1 denotes the usual real-variable Hardy space. On the other hand, Theorem 4 in [**C2**] states that if μ is the Lebesque measure σ_{d-1} on the unit sphere $\sum_{d-1}$ in $\mathbb{R}^d$ then (1) maps $H^1(\mathbb{R}^d)$ into $L^{1,\infty}(\mathbb{R}^d)$. The purpose of this paper is to prove a result which includes the results in [**O**] and Theorem 4 in [**C2**] as special cases and which also applies to maximal operators associated to some submanifolds of codimension greater than 1. The method of proof is an adaptation of the argument in [**O**], which is based on the basic approach in [**C2**].

For each bounded subset A of $\mathbb{R}^d$ and $0 < \epsilon < 1$, define $N(A,\epsilon)$ as the smallest number of ϵ-balls needed to cover A, i.e.,

$$N(A,\epsilon) = \min\left\{ m : A \subset \bigcup_{i=1}^{m} B(x_i,\epsilon) \text{ for some } x_i \in \mathbb{R}^d \right\}.$$

Now we state our main result.

Theorem 1. *Suppose μ is a finite positive Borel measure on $\mathbb{R}^d$ with compact support such that for $0 < \epsilon < 1$*

$$N(\mathrm{supp}(\mu), \epsilon) \leq c\,\epsilon^{-n}, \quad |\hat{\mu}(\xi)| \leq c\,(1 + |\xi|)^{-\frac{n}{2}}$$

then (1) maps $H^1(\mathbb{R}^d)$ into $L^{1,\infty}(\mathbb{R}^d)$ when $0 < n \leq d$.

In particular if $n = d$, then we obtain the result of [**O**]. Moreover we have the following.

Corollary 2. *Suppose $M \subset \mathbb{R}^d$ is a C^1 submanifold of dimension n equipped with a finite positive Borel measure μ which has compact support. If the Fourier transform of μ satisfies the decay estimate*

$$|\hat{\mu}(\xi)| \leq c\,(1 + |\xi|)^{-\frac{n}{2}}$$

then (1) maps $H^1(\mathbb{R}^d)$ into $L^{1,\infty}(\mathbb{R}^d)$ when $0 < n \leq d$.

Proof. Let A be a bounded subset of $\mathbb{R}^n$ and $f : A \mapsto \mathbb{R}^d$ be a Lipschitz map. Then it is easy to show that

$$N(f(A), \epsilon) \leq c\,N(A, \epsilon) \leq c\,\epsilon^{-n}. \tag{3}$$

If M is a C^1 submanifold of $\mathbb{R}^d$, then we can view M locally as the graph of a vector-valued C^1 function defined on its tangent plane. Hence by (3) and compactness of $\mathrm{supp}(\mu)$, we have $N(\mathrm{supp}(\mu), \epsilon) \leq c\,\epsilon^{-n}$. By applying Theorem 1, we obtain the conclusion. □

In particular if M is $\sum_{d-1}$ and μ is σ_{d-1}, then we obtain Theorem 4 in [**C2**]. Also, as was treated in [**CDMM**] and [**CM**], if M is a smooth compact convex hypersurface of finite type in $\mathbb{R}^{1+n}$, with Gaussian curvature κ and surface measure μ, then the Fourier transform $\widehat{\kappa^{1/2}\mu}(\xi)$ decays as $|\xi|^{-\frac{n}{2}}$ as $|\xi|$ goes to infinity. Hence Corollary 2 holds for $\kappa^{1/2}\mu$ when $n \geq 1$.

Our proof follows the methods of [**C2**] and [**O**]. What is different from [**O**] is the use of the geometry of $\mathrm{supp}(\mu)$. We use the geometry of $\mathrm{supp}(\mu)$ in proving Lemma 5. The use of geometry of $\mathrm{supp}(\mu)$ allows us to put a weaker decay condition on $\widehat{\mu}$. Littman [**L**] showed that, if $M \subset \mathbb{R}^{1+n}$ is a smooth submanifold of dimension n and has at least l nonzero principal curvatures everywhere on $\mathrm{supp}(\mu)$, where μ is smooth and compactly supported, then

$$|\hat{\mu}(\xi)| \leq c(1 + |\xi|)^{-\frac{l}{2}}.$$

Hence when $l = n \geq 1$, Corollary 2 can be applied.

As was indicated in [**C3**], the proof of Littman's theorem goes unchanged to establish the following. Suppose that $M \subset \mathbb{R}^d$ is a smooth manifold of dimension n, and μ is a smooth compactly supported measure on M. For fixed $b \in M$, we can view M locally as a graph of a vector-valued function

$\psi(x)$ defined on its tangent plane. Let $N_b(M)$ be a collection of a unit vector normal to M at b then for each $v \in N_b(M)$ the function $\langle \psi(x), v\rangle$ has a critical point at $x = b$. Suppose that for all $b \in M$ in some neighborhood of $\mathrm{supp}(\mu)$ and for all $v \in N_b(M)$ we have

$$\det \mathrm{D}^2 \langle \psi(x), v\rangle |_{x=b} \neq 0. \tag{4}$$

Then

$$|\hat{\mu}(\xi)| \leq c\,(1 + |\xi|)^{-\frac{n}{2}}. \tag{5}$$

Hence Corollary 2 can be applied in this case also. The condition (4) is controlled by the second-order terms in the Taylor expansion of ψ at b. We give some examples which satisfy (5).

Example 3.

(3.1) For $n = 2m$ and $d = n + 2$, let $x,\, y \in \mathbb{R}^m$ and M be the manifold described by $(x,\, y\,;\, |x|^2 - |y|^2,\, x \cdot y)$, then a smooth measure μ supported in a sufficiently small neighborhood of the origin satisfies (5) when $m \geq 1$. So Corollary 2 holds for this μ when $m \geq 1$.

(3.2) For $n = 4m$ and $d = n+2$, let $x,\, y,\, z,\, u \in \mathbb{R}^m$ and M be the manifold described by $(x,\, y,\, z,\, u\,;\, x{\cdot}z{+}y{\cdot}u,\, x{\cdot}u{-}y{\cdot}z)$, then a smooth measure μ supported in a sufficiently small neighborhood of the origin satisfies (5) when $m \geq 1$. So Corollary 2 holds for this μ when $m \geq 1$.

(3.3) For $n = 4m$ and $d = n+3$, let $x,\, y,\, z,\, u \in \mathbb{R}^m$ and M be the manifold described by $(x,\, y,\, z,\, u\,;\, |x|^2{-}|y|^2{-}|z|^2{+}|u|^2,\, x{\cdot}y{-}z{\cdot}u,\, x{\cdot}z{+}y{\cdot}u)$, then a smooth measure μ supported in a sufficiently small neighborhood of the origin satisfies (5) when $m \geq 1$. So Corollary 2 holds for this μ when $m \geq 1$.

2. Preliminaries.

Notation. If Q is a dyadic cube in $\mathbb{R}^d$ with side-length 2^j, we write $\sigma(Q) = j$. For $\sigma \in \mathbb{Z}$, $\Re_\sigma$ denotes the collection of dyadic cubes $Q \in \mathbb{R}^d$ with $\sigma(Q) = \sigma$. And for $Q \in \Re_\sigma$, Q^* denotes $Q + [-2^\sigma, 2^\sigma]^d$. $|\cdot|$ denotes the Lebesgue measure.

The following Lemma is taken from [**O**] (see Lemma 1).

Lemma 4. *Suppose $\alpha > 0$ is given, and given any finite collection of dyadic cubes $\{Q\}_{Q\in\mathcal{C}}$ in $\mathbb{R}^d$, and corresponding collection of positive numbers $\{\lambda_Q\}_{Q\in\mathcal{C}}$ there exists a finite collection of pairwise disjoint dyadic cubes $\{S\}_{S\in\mathcal{S}}$ such that each $Q \in \mathcal{C}$ is contained for some $S \in \mathcal{S}$ and*

(4.1) $\sum_{Q\subset S} \lambda_Q \leq 3^d \alpha |S|$

(4.2) $\sum_{S\in\mathcal{S}} |S| \leq \frac{1}{\alpha} \sum \lambda_Q$

(4.3) $\left\| \sum_{\substack{\text{Q: not contained}\\ \text{in any S}}} \lambda_Q |Q|^{-1} \chi_Q \right\|_{L^\infty} \leq \alpha.$

Lemma 5 (cf. [**C2**, Lemma 5.1])**.** *Suppose given the following: $0 < n \le d$, a Borel measure μ defined on a compact subset of $\mathbb{R}^d$ with $N(\mathrm{supp}(\mu), \epsilon) \le c\,\epsilon^{-n}$ for $0 < \epsilon < 1$, some $\alpha > 0$, a finite collection $\mathcal{S}$ of pairwise disjoint dyadic cubes $S \subset \mathbb{R}^d$, a finite collection $\mathcal{C}$ of dyadic cubes $Q \subset \mathbb{R}^d$ such that each $Q \in \mathcal{C}$ is contained in some $S = S(Q) \in \mathcal{S}$ and for each $Q \in \mathcal{C}$ a positive number λ_Q is assigned. Then there exist a function $K : \mathcal{C} \mapsto \mathbb{Z}$ and a measurable set E such that*

(5.1) $|E| \le c\left(\frac{1}{\alpha}\sum \lambda_Q + \sum |S|\right)$
(5.2) $\left\{Q + 2^j \mathrm{supp}(\mu)\right\} \subset E$ *if* $j < K(Q)$ *and* $Q \in \mathcal{C}$
(5.3) $\sigma(S(Q)) < K(Q) \quad (Q \in \mathcal{C})$
(5.4) *For each $\tau, \sigma \in \mathbb{Z}$ with $\sigma \le \tau$, and any $q \in \Re_\sigma$*

$$\sum_{Q \subset q,\ K(Q) \le \tau} \lambda_Q \le 2^n \alpha 2^{(d-n)\sigma + n\tau}.$$

Proof. The proof is a stopping-time argument controlled by two parameters τ and σ as in the proof of Lemma 5.1 in [**C2**]. Let $m = \min\{\sigma(Q) : Q \in \mathcal{C}\}$. Select an integer τ_0 such that

$$\tau_0 > \max\{\sigma(Q) : Q \in \mathcal{C}\}, \quad \sum_{Q \in \mathcal{C}} \lambda_Q < \alpha 2^{(d-n)m + n\tau_0}.$$

For each fixed $\tau \in \mathbb{Z}$ with $\tau \le \tau_0$, we define a sequence of functions $\Lambda_{\tau,\sigma} : \Re_\sigma \mapsto \mathbb{R}$ by a descending induction on $\sigma \in \mathbb{Z}$ with $\sigma \le \tau$. And proceed with the same construction by a descending induction on τ. At each step, we divide $\mathcal{C}$ into disjoint subcollections $\mathcal{C}_1$ and $\mathcal{C}_2$ which will increase as we proceed. Let $\mathcal{C}_1, \mathcal{C}_2 \subset \mathcal{C}$ and $\tau \in \mathbb{Z}$ be fixed for the moment, and we define [**Inner Loop**] as

[**Inner Loop**] Define $\Lambda_{\tau,\sigma} : \Re_\sigma \mapsto \mathbb{R}$ with $\sigma \le \tau$. For each $q \in \Re_\sigma$ define

$$\Lambda_{\tau,\sigma}(q) = \sum_{Q \subset q;\ Q \notin \mathcal{C}_1 \cup \mathcal{C}_2} \lambda_Q.$$

First, begin with $\sigma = \tau$. If $\Lambda_{\tau,\sigma}(q) > \alpha 2^{(d-n)\sigma + n\tau}$ then we say that "q is selected at step (τ, σ)" and put into $\mathcal{C}_1$ every Q such that $Q \subset q$ and for such a Q define $K(Q) = 1 + \tau$. Next replace σ by $\sigma - 1$ and repeat the process. Repeat until $\sigma < m$. Actually this part of process terminates once σ is smaller than m. Finally, put into $\mathcal{C}_2$ every $Q \in \mathcal{C} \backslash \mathcal{C}_1$ such that $\sigma(Q) \ge \tau$ and for such a Q define $K(Q) = 1 + \sigma(S(Q))$. Actually every $Q \in \mathcal{C} \backslash \mathcal{C}_1 \cup \mathcal{C}_2$ satisfies $\sigma(Q) \le \tau - 1$.

Perform [**Inner Loop**] with $\mathcal{C}_1 = \mathcal{C}_2 = \emptyset$ and $\tau = \tau_0$. Next replace τ by $\tau - 1$ and repeat [**Inner Loop**]. Repeat until $\tau = m - 1$. After this process, we obtain $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, and clearly all selected q are disjoint, and K

is well-defined. Note that there is the usual stopping-time condition

$$\Lambda_{\tau,\sigma}(q) \leq 2^n \alpha 2^{(d-n)\sigma+n\tau} \tag{6}$$

which holds for all $q \in \Re_\sigma$ when $\sigma \leq \tau \leq \tau_0$. This is because, if $\tau = \tau_0$ then the condition is clear from the initial condition on τ_0. And when $\sigma \leq \tau < \tau_0$, suppose this fails. Then $\Lambda_{\tau+1,\sigma}(q) \geq \Lambda_{\tau,\sigma}(q) > \alpha 2^{(d-n)\sigma+n(\tau+1)}$. This means q is selected at step $(\tau+1, \sigma)$, hence $\Lambda_{\tau,\sigma}(q) = 0$ and we have contradiction.

Next we show (5.4), which says that for each $q \in \Re_\sigma$ with $\sigma \leq \tau$

$$\sum_{Q\subset q;\ K(Q)\leq\tau} \lambda_Q \leq 2^n \alpha 2^{(d-n)\sigma+n\tau}.$$

When $\tau \geq \tau_0$, then the condition is clear from the initial condition of τ_0. When $\tau \leq \tau_0$, then we note the fact that for each $q \in \Re_\sigma$ with $\sigma \leq \tau \leq \tau_0$

$$\Lambda_{\tau,\sigma}(q) = \sum_{Q\subset q;\ Q\notin\mathcal{C}_1\cup\mathcal{C}_2} \lambda_Q \geq \sum_{Q\subset q;\ K(Q)\leq\tau} \lambda_Q. \tag{7}$$

Combining (6) and (7), we have (5.4) when $\sigma \leq \tau \leq \tau_0$. (7) will follow from the definition

$$\Lambda_{\tau,\sigma}(q) = \sum_{Q\subset q;\ Q\notin\mathcal{C}_1\cup\mathcal{C}_2} \lambda_Q$$

and the fact that if $Q \in \mathcal{C}_1 \cup \mathcal{C}_2$ at the beginning of step (τ, σ) then $K(Q) > \tau$. This is because, if $Q \in \mathcal{C}_1$ then $K(Q) \geq 1 + \tau > \tau$, and if $Q \in \mathcal{C}_2$ then $K(Q) = 1 + \sigma(S(Q)) \geq 1 + (1 + \tau) > \tau$. Hence we have (5.4).

Next, we construct an exceptional set E. If q is selected at step (τ, σ), then we define $\tau(q) = \tau$ and

$$T(q) = \bigcup_{j\leq\tau(q)+1} \left\{q + 2^j \mathrm{supp}(\mu)\right\}$$

$$E = E_1 \bigcup E_2, \qquad E_1 = \bigcup_{S\in\mathcal{S}} S^*, \qquad E_2 = \bigcup_{q:\text{selected}} T(q).$$

Thus we have

$$|E_1| \leq c \sum |S|$$

and

$$\begin{aligned} T(q) &= \bigcup_{j\leq\tau(q)+1} \left\{q + 2^j \mathrm{supp}(\mu)\right\} \\ &= \bigcup_{j\leq\sigma(q)} \left\{q + 2^j \mathrm{supp}(\mu)\right\} \bigcup_{\sigma(q)<j\leq\tau(q)+1} \left\{q + 2^j \mathrm{supp}(\mu)\right\}. \end{aligned}$$

Because $\mathrm{supp}(\mu)$ is compact, if we regard q^* as a proper expansion of q then $\bigcup_{j<\sigma(q)} \left\{q + 2^j \mathrm{supp}(\mu)\right\} \subset q^*$. And for $j > \sigma(q)$, if x_0 is the center of q, then by using translation invariance and dilation property of Lebesque measure, we have

$$\begin{aligned} \left|\{q + 2^j \mathrm{supp}(\mu)\}\right| &\leq \left|\left\{B(x_0, 2^{\sigma(q)}) + 2^j \mathrm{supp}(\mu)\right\}\right| \\ &= \left|\left\{B(0, 2^{\sigma(q)}) + 2^j \mathrm{supp}(\mu)\right\}\right| \\ &= 2^{dj} \left|\left\{B(0, 2^{\sigma(q)-j}) + \mathrm{supp}(\mu)\right\}\right| \\ &\leq c\, 2^{dj} 2^{d(\sigma(q)-j)} N(\mathrm{supp}(\mu), 2^{\sigma(q)-j}) \\ &\leq c\, 2^{(d-n)\sigma(q)+nj}. \end{aligned}$$

Hence

$$|T(q)| \leq c \left(|q| + \sum_{\sigma(q)<j\leq\tau(q)+1} 2^{(d-n)\sigma(q)+nj} \right) \leq c\, 2^{(d-n)\sigma(q)+n\tau(q)}$$

and we have

$$\begin{aligned} |E_2| &\leq \sum_{q:\text{selected}} |T(q)| \\ &\leq c \sum_{q:\text{selected}} 2^{(d-n)\sigma(q)+n\tau(q)} \\ &\leq \frac{c}{\alpha} \sum_{q:\text{selected}} \Lambda_{\tau,\sigma}(q) \\ &\leq \frac{c}{\alpha} \sum \lambda_Q. \end{aligned}$$

So we obtain (5.1). For (5.2), observe that if $Q \in \mathcal{C}_1$ then Q belongs to some selected q, hence

$$\bigcup_{j<K(Q)} \{Q + 2^j \mathrm{supp}(\mu)\} \subset \bigcup_{j\leq K(Q)=\tau(q)+1} \{q + 2^j \mathrm{supp}(\mu)\} = T(q) \subset E_2$$

and if $Q \in \mathcal{C}_2$ then Q belongs to some $S = S(Q) \in \mathcal{S}$, hence

$$\bigcup_{j<K(Q)=1+\sigma(S(Q))} \{Q + 2^j \mathrm{supp}(\mu)\} \subset S^* \subset E_1$$

if we regard S^* as a proper expansion of S. For (5.3), we replace K by K' and define

$$K(Q) = \max\{K'(Q), 1 + \sigma(S(Q))\}.$$

Then (5.1) and (5.3) are satisfied. We must check (5.2) and (5.4). For (5.2), if $K(Q) = K'(Q)$ then there is no problem. If $K(Q) = 1 + \sigma(S(Q)) > j$

then the argument is the same as above. For (5.4)

$$\sum_{Q\subset q;\ K(Q)\leq\tau}\lambda_Q \leq \sum_{Q\subset q;\ K'(Q)\leq\tau}\lambda_Q$$

and Lemma 5 follows. □

3. Proof of Theorem 1.

Let $f \in H^1(\mathbb{R}^d)$ have the form of a finite sum

$$f = \sum \lambda_Q a_Q$$

where $\lambda_Q > 0$ and a_Q, supported in Q, satifies

$$\|a_Q\|_{L^\infty} \leq \frac{1}{|Q|}, \qquad \int a_Q = 0.$$

As was pointed out in [**C2**], a device of Garnett and Jones involving auxiliary dyadic grids allows us to assume that each Q is dyadic. For $\alpha > 0$, it is enough to show

$$\big|\{x : \mathcal{M}f(x) > 2\alpha\}\big| \leq \frac{c}{\alpha}\sum \lambda_Q. \tag{8}$$

Let $\mathcal{S}$ be as in Lemma 4 and define

$$b = \sum_{S\in\mathcal{S}}\sum_{Q\subset S}\lambda_Q a_Q, \qquad g = f - b.$$

Then $\|g\|_{L^\infty} \leq \alpha$ from (4.3) and so $|\mathcal{M}g| \leq \alpha$ (by assuming μ has mass 1). Thus (8) will follow from

$$|\{x : \mathcal{M}b(x) > \alpha\}| \leq \frac{c}{\alpha}\sum \lambda_Q.$$

Let $\mathcal{S}$ be as above and $\mathcal{C}$ be the collection of Q's appearing in the definition of b. With K and E as in Lemma 5, it is enough to prove

$$\|\mathcal{M}b\|^2_{L^2(\mathbb{R}^d\setminus E)} \leq c\alpha \sum \lambda_Q. \tag{9}$$

Let μ_j be the dilate of μ defined by

$$\langle \phi, \mu_j\rangle = \int_{\mathbb{R}^d} \phi(2^j x)\, d\mu(x)$$

then

$$\mathcal{M}b(x) = \sup_{j\in\mathbb{Z}} |b * \mu_j(x)|\,.$$

If $Q \in \mathcal{C}$, then by (5.3) $a_Q * \mu_j$ is supported in E unless $j \geq K(Q)$. Thus for $x \notin E$, we have

$$\begin{aligned} |\mathcal{M}b(x)|^2 &\leq \sum_j |b * \mu_j(x)|^2 \\ &= \sum_j \left| \left(\sum_{K(Q) \leq j} \lambda_Q a_Q \right) * \mu_j(x) \right|^2 \\ &= \sum_j \left| \sum_{s=0}^{\infty} \left(\sum_{K(Q)=j-s} \lambda_Q a_Q \right) * \mu_j(x) \right|^2 . \end{aligned}$$

So for $x \notin E$, by Minkowski's inequality

$$|\mathcal{M}b(x)| \leq \sum_{s=0}^{\infty} \left[\sum_j \left| \left(\sum_{K(Q)=j-s} \lambda_Q a_Q \right) * \mu_j(x) \right|^2 \right]^{\frac{1}{2}} .$$

Now (9) will follow from

$$\left\| \left[\sum_j \left| \left(\sum_{K(Q)=j-s} \lambda_Q a_Q \right) * \mu_j \right|^2 \right]^{\frac{1}{2}} \right\|_{L^2}^2 \leq c(s+3)\alpha 2^{-\epsilon s} \sum \lambda_Q$$

where $\epsilon = \min(1, n)$. And so from

$$\left\| \left(\sum_{K(Q)=j-s} \lambda_Q a_Q \right) * \mu_j \right\|_{L^2}^2 \leq c\alpha(s+3) 2^{-\epsilon s} \sum_{K(Q)=j-s} \lambda_Q. \tag{10}$$

By scaling we may take $j = 0$. And (10) will follow from

$$\left\| \left(\sum_{K(Q)=-s} \lambda_Q a_Q \right) * \mu \right\|_{L^2}^2 \leq c\alpha(s+3) 2^{-\epsilon s} \sum_{K(Q)=-s} \lambda_Q. \tag{11}$$

Next as in Lemma 3 in [**O**], for each positive integer N, we define a sequence of functions h_N and L_N. First we define h_N by

$$\hat{h}_N(\xi) = \frac{\chi_{|\xi| \leq N}(\xi)}{(1+|\xi|)^n}.$$

Choose a radial function $\rho \in C_c^{\infty}(\mathbb{R}^d)$ such that

$$\int \rho = 1, \quad \operatorname{supp}(\rho) \subset [-1,1]^d, \quad \hat{\rho} \geq 0.$$

Now let $L_N = \rho h_N$ and

$$\hat{L}(\xi) = \lim_{N\to\infty} \hat{L}_N(\xi) = \int \frac{\hat{\rho}(y)dy}{(1+|\xi - y|)^n}.$$

Lemma 6. *We have the following:*
(6.1) $\operatorname{supp}(L_N) \subset [-1,1]^d$
(6.2) $\hat{L}_N(\xi) \geq \frac{c}{(1+|\xi|)^n}$ *if* $|\xi| \leq N-1$
(6.3) *For each β, we have*

$$\left|\partial_\xi^\beta \hat{L}(\xi)\right| \leq \frac{A_\beta}{(1+|\xi|)^{n+|\beta|}}.$$

Proof. It is easy to check (6.1), (6.2). For (6.3), first we assume $d \geq 2$, then we have

$$\begin{aligned} \left|\partial_\xi^\beta \hat{L}(\xi)\right| &= \left|\int \hat{\rho}(y)\partial_\xi^\beta \frac{1}{(1+|\xi-y|)^n}\, dy\right| \\ &\leq c\int \frac{|\hat{\rho}(y)|dy}{(1+|\xi-y|)^{n+|\beta|}} \leq \frac{c}{(1+|\xi|)^{n+|\beta|}}. \end{aligned}$$

When $d = 1$, we use

$$\hat{L}(\xi) = \int_\xi^\infty \frac{\hat{\rho}(y)dy}{(1+y-\xi)^n} + \int_{-\infty}^\xi \frac{\hat{\rho}(y)dy}{(1+\xi-y)^n},$$

and do similarly as before. □

Next, let ϕ_N be the inverse Fourier transform of $(\hat{L}_N)^{\frac{1}{2}}$, then $L_N = \phi_N * \widetilde{\phi_N}$. And we have

$$|\hat{\phi}_N(\xi)|^2 \geq \frac{c}{(1+|\xi|)^n} \qquad \text{when } |\xi| \leq N-1.$$

Therefore, returning to (11) we have

$$\begin{aligned} \left\|\left(\sum_{K(Q)=-s} \lambda_Q a_Q\right) * \mu\right\|_{L^2}^2 &= c\int \left|\left(\sum_{K(Q)=-s} \lambda_Q \hat{a}_Q\right)(\xi)\right|^2 |\hat{\mu}(\xi)|^2\, d\xi \\ &\leq c\int \left|\left(\sum_{K(Q)=-s} \lambda_Q \hat{a}_Q\right)(\xi)\right|^2 \liminf_{N\to\infty} \left|\hat{\phi}_N(\xi)\right|^2 d\xi \\ &\leq c\liminf_{N\to\infty} \int \left|\left(\sum_{K(Q)=-s} \lambda_Q \hat{a}_Q\right)(\xi)\right|^2 \left|\hat{\phi}_N(\xi)\right|^2 d\xi \\ &\leq c\liminf_{N\to\infty} \left\|\left(\sum_{K(Q)=-s} \lambda_Q a_Q\right) * \phi_N\right\|_{L^2}^2. \end{aligned}$$

So (11) will follow from

$$\liminf_{N\to\infty}\left\|\left(\sum_{K(Q)=-s}\lambda_Q a_Q\right)*\phi_N\right\|_{L^2}^2 \leq c\,\alpha(s+3)2^{-\epsilon s}\sum_{K(Q)=-s}\lambda_Q. \tag{12}$$

Because $\mathrm{supp}(L_N)\subset[-1,1]^d$, and for each $Q,Q'\in\mathcal{C}$ such that $K(Q)=K(Q')=-s$, we have $\sigma(Q),\sigma(Q')\leq K(Q)=K(Q')=-s$, hence $\left|\left\langle a_{Q'}*L_N,a_Q\right\rangle\right|=0$ when $\mathrm{dist}(Q,Q')>4$. So we have

$$\begin{aligned}
&\liminf_{N\to\infty}\left\|\left(\sum_{K(Q)=-s}\lambda_Q a_Q\right)*\phi_N\right\|_{L^2}^2\\
&\leq 2\liminf_{N\to\infty}\sum_{\substack{Q,Q';\ \sigma(Q')\geq\sigma(Q)\\ \mathrm{dist}(Q,Q')\leq 4}}\lambda_Q\lambda_{Q'}\left|\left\langle a_{Q'}*L_N,a_Q\right\rangle\right|\\
&\leq 2\liminf_{N\to\infty}\sum_{\substack{Q,Q';\ \sigma(Q')\geq\sigma(Q)\\ \mathrm{dist}(Q,Q')\leq 4}}\lambda_Q\lambda_{Q'}\left|\left\langle \hat{a}_{Q'}\hat{L}_N,\hat{a}_Q\right\rangle\right|\\
&\leq 2\sum_{Q'}\sum_{\substack{Q\subset Q'^*\\ \mathrm{dist}(Q,Q')\leq 4}}\lambda_Q\lambda_{Q'}\left|\left\langle \hat{a}_{Q'}\hat{L},\hat{a}_Q\right\rangle\right|\\
&\quad+\ 2\sum_{Q'}\sum_{\substack{Q\bigcap Q'^*=\emptyset\\ \mathrm{dist}(Q,Q')\leq 4}}\lambda_Q\lambda_{Q'}\left|\left\langle \hat{a}_{Q'}\hat{L},\hat{a}_Q\right\rangle\right|\\
&=\mathrm{I}+\mathrm{II}.
\end{aligned}$$

Lemma 7. *We have the following:*

(7.1) $\left|\left\langle \hat{a}_{Q'}\hat{L},\hat{a}_Q\right\rangle\right|\leq c2^{-(d-n)\sigma(Q')}$

(7.2) $\left|\left\langle \hat{a}_{Q'}\hat{L},\hat{a}_Q\right\rangle\right|\leq c\frac{2^{\sigma(Q)}}{\left(\mathrm{dist}(Q,Q')\right)^{d-n+1}}$ when $Q\bigcap Q'^*=\emptyset$.

Proof. For (7.1), we consider as two cases; $d=n$ and $d>n$. When $d=n$, we use the easy estimates.

$$|\hat{a}_Q(\xi)|\leq c\min(1,|\xi|2^{\sigma(Q)}),\qquad ||\hat{a}_Q||_{L^2}^2\leq c2^{-d\sigma(Q)}.$$

Hence we have

$$\begin{aligned}\left|\left\langle \hat{a}_{Q'}\hat{L}, \hat{a}_Q\right\rangle\right| &\le \left\|\hat{a}_Q\right\|_{L^\infty}\int \frac{|\hat{a}_{Q'}(\xi)|}{(1+|\xi|)^d}d\xi \\ &\le c\left(\int_{|\xi|<2^{-\sigma(Q')}} \frac{|\xi|2^{\sigma(Q')}}{(1+|\xi|)^d}d\xi \right.\\ &\qquad \left. + \left\|\hat{a}_{Q'}\right\|_{L^2}\left[\int_{|\xi|\ge 2^{-\sigma(Q')}}(1+|\xi|)^{-2d}d\xi\right]^{1/2}\right) \\ &\le c.\end{aligned}$$

When $d > n$, choose $\eta \in C_c^\infty(\mathbb{R}^d)$ such that $\eta(\xi) = 1$ for $|\xi| \le 1$, and $\eta(\xi) = 0$ for $|\xi| \ge 2$. Define another function δ by $\delta(\xi) = \eta(\xi) - \eta(2\xi)$. Then we have

$$1 = \eta(\xi) + \sum_{j=1}^{\infty}\delta(2^{-j}\xi), \quad \text{for all } \xi,$$

and

$$\hat{L}(\xi) = \eta(\xi)\hat{L}(\xi) + \sum_{j=1}^{\infty}\hat{L}(\xi)\delta(2^{-j}\xi) = m_0(\xi) + \sum_{j=1}^{\infty} m_j(\xi).$$

We set

$$K_j(x) = \int e^{2\pi i x\cdot\xi}\, m_j(\xi)d\xi.$$

Observe that

$$\left|(-2\pi i x)^\gamma \partial_x^\beta K_j(x)\right| = \left|\int \partial_\xi^\gamma\left[(2\pi i\xi)^\beta\, m_j(\xi)\right]\, e^{2\pi i x\cdot\xi}d\xi\right|.$$

By (6.3) and support condition of the integrand, we can show

$$\left|x^\gamma\partial_x^\beta K_j(x)\right| \le A_{\gamma,\beta}2^{j(d-n+|\beta|-|\gamma|)}.$$

Hence, for each positive integer M, we have

$$\left|\partial_x^\beta K_j(x)\right| \le A_{M,\beta}|x|^{-M}2^{j(d-n+|\beta|-M)}, \tag{13}$$

and so

$$\sum_{j=0}^{\infty}\left|\partial_x^\beta K_j(x)\right| = \sum_{2^j\le|x|^{-1}} + \sum_{2^j>|x|^{-1}}.$$

First with $M = 0$, we have

$$\begin{aligned}\sum_{2^j\le|x|^{-1}}\left|\partial_x^\beta K_j(x)\right| &\le A_\beta\sum_{2^j\le|x|^{-1}}2^{j(d-n+|\beta|)} \\ &\le A'_\beta|x|^{-d+n-|\beta|}.\end{aligned}$$

Second with $M > d - n + |\beta|$, we have

$$\begin{aligned}\sum_{2^j > |x|^{-1}} \left|\partial_x^\beta K_j(x)\right| &\leq A_\beta \sum_{2^j > |x|^{-1}} |x|^{-M} 2^{j(d-n+|\beta|-M)} \\ &\leq A'_\beta |x|^{-d+n-|\beta|}.\end{aligned}$$

Hence we have

$$\sum_{j=0}^{\infty} \left|\partial_x^\beta K_j(x)\right| \leq A'_\beta |x|^{-d+n-|\beta|}. \tag{14}$$

Returning to (7.1), by Lebesgue Dominated Convergence Theorem, we have

$$\begin{aligned}\left|\left\langle \hat{a}_Q \hat{L}, \hat{a}_{Q'}\right\rangle\right| &= \left|\sum_{j=0}^{\infty} \left\langle \hat{a}_{Q'} m_j, \hat{a}_Q\right\rangle\right| \\ &= \left|\sum_{j=0}^{\infty} \left\langle a_{Q'} * K_j, a_Q\right\rangle\right| \\ &\leq \left\langle |a_{Q'}| * \sum_{j=0}^{\infty} |K_j|, |a_Q| \right\rangle \\ &\leq \left\|a_Q\right\|_{L^1} \sup_{x\in Q} |a_{Q'}| * \left(\sum_{j=0}^{\infty} \left|K_j(x)\right|\right) \\ &\leq c \left\|a_{Q'}\right\|_{L^\infty} \sup_{x\in Q} \int_{Q'} \sum_{j=0}^{\infty} |K_j(x-y)| dy,\end{aligned}$$

and by (14), we have

$$\sup_{x\in Q} \int_{Q'} \sum_{j=0}^{\infty} |K_j(x-y)| dy \leq c \sup_{x\in Q} \int_{Q'} |x-y|^{-d+n}\, dy \leq c\, 2^{n\sigma(Q')}$$

when $d > n$. Hence when $d > n$, we have

$$\left|\left\langle \hat{a}_{Q'} \hat{L}, \hat{a}_Q\right\rangle\right| \leq c\, 2^{-(d-n)\sigma(Q')},$$

and obtain (7.1). For (7.2), let $\widetilde{x}$ be the center of Q, then

$$\begin{aligned}
&\left|\left\langle \hat{a}_{Q'}\hat{L}, \hat{a}_Q \right\rangle\right| \\
&= \left|\sum_{j=0}^{\infty} \left\langle a_{Q'} * K_j, a_Q \right\rangle\right| \\
&= \left|\sum_{j=0}^{\infty} \iint a_{Q'}(y)\big(K_j(x-y) - K_j(\widetilde{x}-y)\big)a_Q(x)\, dxdy\right| \\
&\le \iint |a_{Q'}(y)|\, |a_Q(x)| \sum_{j=0}^{\infty} \big|\big(K_j(x-y) - K_j(\widetilde{x}-y)\big)\big|\, dxdy \\
&\le \iint |a_{Q'}(y)|\, |a_Q(x)| \sum_{j=0}^{\infty} |x-\widetilde{x}||\bigtriangledown K_j(\widetilde{x}_j - y)|\, dxdy,
\end{aligned}$$

where $\widetilde{x}_j$ lies in the line connecting $\widetilde{x}$ and x. By (13), for each positive integer M, we have

$$\begin{aligned}
|\bigtriangledown K_j(\widetilde{x}_j - y)| &\le A_M|\widetilde{x}_j - y|^{-M} 2^{j(d-n+1-M)} \\
&\le A'_M \operatorname{dist}(Q,Q')^{-M} 2^{j(d-n+1-M)},
\end{aligned}$$

when $Q\bigcap Q'^* = \emptyset$. Hence, by the same method as in (14), we have

$$\sum_{j=0}^{\infty} |\bigtriangledown K_j(\widetilde{x}_j - y)| \le c\ \left(\operatorname{dist}(Q,Q')\right)^{-d+n-1} \text{ when } Q\bigcap Q'^* = \emptyset.$$

And so we have

$$\begin{aligned}
\left|\left\langle \hat{a}_{Q'}\hat{L}, \hat{a}_Q \right\rangle\right| &\le c\frac{2^{\sigma(Q)}}{\left(\operatorname{dist}(Q,Q')\right)^{d-n+1}} \iint |a_{Q'}(y)|\, |a_Q(x)| dxdy \\
&\le c\frac{2^{\sigma(Q)}}{\left(\operatorname{dist}(Q,Q')\right)^{d-n+1}}
\end{aligned}$$

when $Q\bigcap Q'^* = \emptyset$. □

- Estimation of part I:

By (5.4) we have $\sum_{Q\subset Q'^*}\lambda_Q \le c\alpha 2^{(d-n)\sigma(Q')-ns}$ and use (7.1). So we have

$$\begin{aligned} I &\le c\sum_{Q'}\sum_{Q\subset Q'^*}\lambda_Q\lambda_{Q'}2^{-(d-n)\sigma(Q')} \\ &\le c\left(\sum_{Q'}\lambda_{Q'}2^{-(d-n)\sigma(Q')}\right)\left(\alpha 2^{(d-n)\sigma(Q')-ns}\right) \\ &\le c2^{-ns}\alpha\sum_{K(Q)=-s}\lambda_Q. \end{aligned}$$

- Estimation of part II:

If $Q\cap Q'^*=\emptyset$, then by (7.2) and $\sigma(Q)\le\sigma(Q')$, we have

$$\begin{aligned} II &\le c\sum_{Q'}\sum_{\substack{Q\cap Q'^*=\emptyset \\ \operatorname{dist}(Q,Q')\le 4}}\lambda_Q\lambda_{Q'}\frac{2^{\sigma(Q')}}{\operatorname{dist}(Q,Q')^{(d-n)+1}} \\ &\le c\left(\sum_{Q'}2^{\sigma(Q')}\lambda_{Q'}\right)\left(\sum_{Q\cap Q'^*=\emptyset}\frac{\lambda_Q}{\operatorname{dist}(Q,Q')^{(d-n)+1}}\right) \\ &\le c\left(\sum_{Q'}2^{\sigma(Q')}\lambda_{Q'}\right)\left(\sum_{\substack{Q;\ \operatorname{dist}(Q,Q')\sim 2^{m+\sigma(Q')} \\ m+\sigma(Q')\le -s+2}} + \sum_{\substack{Q;\ \operatorname{dist}(Q,Q')\sim 2^{m+\sigma(Q')} \\ -s+3\le m+\sigma(Q')\le 2}}\right) \\ &\le c\left(\sum_{Q'}2^{\sigma(Q')}\lambda_{Q'}\right)(II_1+II_2). \end{aligned}$$

For each positive integer m, consider the contribution of all λ_Q over all Q disjoint from Q'^* with $\sigma(Q)\le\sigma(Q')$. So we have $\operatorname{dist}(Q,Q')\sim 2^{m+\sigma(Q')}$. All such Q are contained in the union of a fixed number of elements of $\Re_{m+\sigma(Q')}$. Hence when $m+\sigma(Q')\le -s+2$, we can use (5.4) to obtain

$$\begin{aligned} II_1 &= \sum_{\substack{Q;\ \operatorname{dist}(Q,Q')\sim 2^{m+\sigma(Q')} \\ m+\sigma(Q')\le -s+2}}\frac{\lambda_Q}{\operatorname{dist}(Q,Q')^{(d-n)+1}} \\ &\le c\sum_{m\ge 0}\alpha 2^{-(d-n+1)(m+\sigma(Q'))}2^{(d-n)(m+\sigma(Q'))-ns} \\ &\le c\alpha 2^{-\sigma(Q')}2^{-ns}. \end{aligned}$$

Next, consider all Q with $\text{dist}(Q,Q') \sim 2^{m+\sigma(Q')}$ and $m+\sigma(Q') \geq -s+3$. Recall that each $Q \in C$ is contained in $S(Q)$ for some $S(Q) \in \mathcal{S}$. Since $K(Q) = -s$ and $K(Q) > \sigma(S(Q))$, we obtain $\text{dist}(S(Q),Q') \geq 2^{-s}$. Also, by (4.1), we have $\sum_{Q\subset S} \lambda_Q \leq c\alpha|S|$ for every $S \in \mathcal{S}$, hence we obtain

$$\begin{aligned} II_2 &= \sum_{\substack{Q;\, \text{dist}(Q,Q')\sim 2^{m+\sigma(Q')} \\ -s+3\leq m+\sigma(Q')\leq 2}} \frac{\lambda_Q}{\text{dist}(Q,Q')^{(d-n)+1}} \\ &\leq c\sum \frac{\lambda_Q}{\text{dist}(S(Q),Q')^{(d-n)+1}} \\ &\leq c\alpha \sum \frac{|S|}{\text{dist}(S,Q')^{(d-n)+1}} \\ &\leq c\alpha \int_{2^{-s}\leq|y|\leq 4} |y|^{-(d-n+1)} dy \\ &\leq c\alpha(s2^{(1-n)s}+1). \end{aligned}$$

Finally, since $\sigma(Q') < K(Q') = -s$, we obtain

$$\begin{aligned} II &\leq c\sum_{Q'} 2^{\sigma(Q')}\Big(\alpha 2^{-\sigma(Q')}2^{-ns} + \alpha(s2^{(1-n)s}+1)\Big)\lambda_{Q'} \\ &\leq c(s+3)\alpha 2^{-\epsilon s} \sum_{K(Q)=-s} \lambda_Q \end{aligned}$$

where $\epsilon = \min(n,1)$. This completes the proof of (12) and Theorem 1.

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References

[Ca] A.P. Calderón, *An atomic decomposition of distribution in H^p space*, Adv. in Math., **25** (1977), 216-225, MR 56 #6376, Zbl 379.46050.

[C1] M. Christ, *Hilbert transforms along curves,* I: *Nilpotent groups*, Ann. of Math., **122** (1985), 575-596, MR 87f:42039a, Zbl 593.43011.

[C2] ______, *Weak type* $(1,1)$ *bounds for rough operators*, Ann. of Math., **128** (1988), 19-42, MR 89m:42013, Zbl 666.47027.

[C3] ______, *Restriction of Fourier transform to submanifolds of low codimension*, Ph.D. Thesis, University of Chicago, 1982.

[CDMM] M. Cowling, S. Disney, G. Mauceri and D. Müller, *Damping oscillatory integrals*, Invent. Math., **101** (1990), 237-260, MR 91d:58242, Zbl 733.42011.

[CM] M. Cowling and G. Mauceri, *Oscillatory integrals and Fourier transforms of surface carried measures*, Trans. Amer. Math. Soc., **304** (1987), 53-68, MR 89e:42013, Zbl 647.42009.

[DR] J. Duoandikoetxea and J. L. Rubio de Francia, *Maximal and singular integral operators via Fourier transform estimates*, Invent. Math., **84** (1986), 541-561, MR 87f:42046, Zbl 568.42012.

[F] H. Federer, *Geometric Measure Theory*, Springer-Verlag, 1969, MR 41 #1976, Zbl 176.00801.

[Ka] J.P. Kahane, *Some Random Series of Functions*, Cambridge University Press, 1986, MR 87m:60119.

[Kö] T. Körner, *On a theorem of Ivasev-Musatov* III, Proc. London Math. Soc., **53**(**1**) (1986), 143-192.

[L] W. Littman, *Fourier transforms of surface carried measures*, Bull. Amer. Math. Soc., **69** (1963), 776-770, MR 27 #5086, Zbl 143.34701.

[O] D.M. Oberlin, *An endpoint estimate for some maximal operators*, Revista Mat. Iberoamerica, **12**(**3**) (1996), 641-652, MR 97m:42016, Zbl 871.42015.

[SW] E.M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, 1971, MR 46 #4102, Zbl 232.42007.

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