

## THE HECKE ALGEBRAS OF TYPE B AND D AND SUBFACTORS

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We define a nontrivial homomorphism from the Hecke algebra of type  $B$  onto a reduced algebra of the Hecke algebra of type  $A$  at roots of unity. We use this homomorphism to describe semisimple quotients of the Hecke algebra of type  $B$  at roots of unity. Using these quotients we determine subfactors obtained from the inclusion of Hecke algebra of type  $A$  into Hecke algebras of type  $B$ . We also study intermediate subfactors related to the Hecke algebra of type  $D$ .

### Introduction.

In [W1] Wenzl found examples of subfactors of the hyperfinite  $\text{II}_1$  factor by studying the complex Hecke algebras of type  $A$ , denoted by  $H_n(q)$ . In this paper we construct examples of subfactors obtained from the inclusion of the Hecke algebra of type  $A$  into the Hecke algebra of type  $B$ , denoted by  $H_n(q, Q)$ . To do this we must find the values of the parameters of the Hecke algebra of type  $B$  for which the inductive limit, i.e.,  $H_\infty(q, Q) = \bigcup_{n \geq 0} H_n(q, Q)$ , has  $C^*$  representations. We show that there are  $C^*$  representations when  $q = e^{2\pi i/l}$  and  $Q = -q^k$  for some positive integers  $l$  and  $k$ .

In [O] we defined a surjective homomorphism from the specialized Hecke algebra of type  $B$ ,  $H_n(q, -q^{r_1+m})$ , onto a reduced Hecke algebra of type  $A$ . Here we show that this homomorphism is well-defined and onto when  $q$  is a root of unity. This implies that there exist quotients of the reduced Hecke algebra of type  $A$  which are isomorphic to quotients of  $H_n(q, -q^k)$  at roots of unity. These quotients are  $C^*$  algebras and we use them to construct the  $\text{II}_1$  hyperfinite factor.

Geck and Lambropoulou [GL] have defined a two parameter trace, called Markov trace, on the Hecke algebra of type  $B$ . In [O] we showed that when  $Q = -q^k$ ,  $k$  a positive integer, this trace can be obtained as a pull back of the Markov trace on the Hecke algebra of type  $A$ . Moreover, this trace satisfies the commuting square property needed for the construction of subfactors. We use this trace and the Hecke algebra of type  $B$  to construct the hyperfinite  $\text{II}_1$  factor.

The subfactors obtained from the inclusion of the Hecke algebra of type  $A$  into the Hecke algebra of type  $B$  are equivalent to special cases of subfactors already obtained in [W1] for the Hecke algebras of type  $A$ . We compute the index and higher relative commutants for these subfactors. We found that the index is related to the Schur function of a rectangular Young diagram.

We also obtain intermediate subfactors of index two by studying the inclusion of the Hecke algebra of type  $D$  into the Hecke algebra of type  $B$ . We also consider the inclusion of the Hecke algebra of type  $A$  into the Hecke algebra of type  $D$ . We compute the index for these subfactors.

This paper is organized as follows: In the [first](#) section we introduce notations and definitions which will be used throughout the paper. In the [second](#) section we define the Hecke algebras and recall the homomorphism in [O]. We begin Section [Three](#) by reviewing the representation theory of the Hecke algebra of type  $B$ , [H]. We then proceed to the study of a quotient of this algebra when  $q$  is a root of unity and  $Q = -q^k$  for  $k \in \mathbb{N}$ . We also give the well-defined irreducible representations for this quotient.

In Section [Four](#) we summarize the necessary results about Markov traces and give the weight vector for this trace. We also show that this trace is well-defined on the quotients at roots of unity. In Section [5](#) we show that there are  $C^*$  representations for  $H_\infty(q, -q^k)$ , i.e., representations on a Hilbert space such that the images of the generators  $t, g_1, \dots, g_{n-1}, \dots$  are unitary. In Section [6](#) we give the index and higher relative commutants for the subfactors obtained from the inclusion of the Hecke algebra of type  $A$  into the Hecke algebra of type  $B$ . We conclude this paper with the study of intermediate subfactors involving the Hecke algebra of type  $D$ . In particular, we compute the index for these subfactors.

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## 1. Preliminaries.

We take the ground field to be the complex numbers  $\mathbb{C}$ . For convenience, an algebra  $A$  will be called *semisimple* if it is a direct sum of full matrix rings. Let  $M_n(\mathbb{C})$  denote the algebra of  $n \times n$  matrices over  $\mathbb{C}$ . If  $A \subset B$  are semisimple then  $A = \bigoplus A_i$  and  $B = \bigoplus B_j$  with  $A_i = M_{a_i}(\mathbb{C})$  and  $B_j = M_{b_j}(\mathbb{C})$  for some  $a_i, b_j \in \mathbb{N}$ . Furthermore, any simple  $B_j$  module is also an  $A$ -module. Let  $g_{ij}$  be the number of simple  $A_i$  modules in the decomposition of  $B_j$  into simple  $A$  modules. The matrix  $G = (g_{ij})$  is called the *inclusion matrix* for  $A \subset B$ .

The inclusion of  $A$  in  $B$  can be described by the *Bratteli diagram*. This is a graph with vertices arranged in 2 lines. In one line, the vertices are in one-to-one correspondence with the minimal direct summands  $A_i$  of  $A$ , in the other one with the summands  $B_j$  of  $B$ . Then a vertex corresponding to

$A_i$  is joined with a vertex corresponding to  $B_j$  by  $g_{ij}$  edges. See Figure 1 for the Bratteli diagram of the inclusion of the group algebra of the symmetric group in 3 letters into the group algebra of the symmetric group in 4 letters, i.e.,  $\mathbb{C}S_3 \subset \mathbb{C}S_4$ .

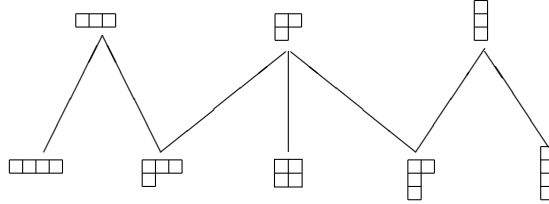


Figure 1.

A *trace* is a linear functional  $\text{tr} : B \rightarrow \mathbb{C}$  such that  $\text{tr}(ab) = \text{tr}(ba)$  for all  $a, b \in B$ . There is only one trace on  $M_n(\mathbb{C})$  up to scalar multiples. Thus, any trace  $\text{tr}$  on  $B = \bigoplus B_j$  is completely determined by a vector  $\vec{t} = (t_j)$ , where  $t_j = \text{tr}(p_j)$  and  $p_j$  is a minimal idempotent in  $B_j$ ,  $\vec{t}$  is called the *weight vector* and the  $t_i$  are called the *weights*. A trace is *nondegenerate* if for any  $b \in B$ , there is a  $b' \in B$  such that  $\text{tr}(bb') \neq 0$ . It is not hard to show that  $\text{tr}$  is nondegenerate if and only if  $t_j \neq 0$  for every  $j$ .

Recall that there is an isomorphism between  $B$  and its dual  $B^*$  defined by  $b \in B \rightarrow \text{tr}(b \cdot) \in B^*$ , where  $\text{tr}(b \cdot)$  denotes the function  $x \rightarrow \text{tr}(bx)$ . Assuming  $\text{tr}$  is nondegenerate on both  $A$  and  $B$ , and using the above isomorphism for  $A$  and  $A^*$ , we obtain for every  $b \in B$  a unique element  $\varepsilon_A(b) \in A$  such that  $\text{tr}(b \cdot)|_A = \text{tr}(\varepsilon_A(b) \cdot)|_A$ . The linear map  $\varepsilon_A : B \rightarrow A$  defined by  $b \rightarrow \varepsilon_A(b)$  is called a trace preserving *conditional expectation* from  $B$  onto  $A$ , the element  $\varepsilon_A(b) \in A$  is uniquely determined by the equation

$$\text{tr}(\varepsilon_A(b)a) = \text{tr}(ba) \quad \text{for all } a \in A.$$

### 1.1. Young Diagrams.

In this section we use notation and terminology from [M].

A *partition* is a finite sequence of nonnegative integers in decreasing order:  $\lambda = [\lambda_1 \geq \lambda_2 \geq \dots]$ . We make no distinction between two sequences that differ only by zeros. The number of parts is called the *length* of  $\lambda$ , and is denoted by  $l(\lambda)$ ;  $|\lambda| = \lambda_1 + \lambda_2 + \dots$  is called the *weight* of  $\lambda$ . If  $|\lambda| = n$  then  $\lambda$  is a partition of  $n$ , denoted  $\lambda \vdash n$ .

It is common to associate partitions with *Young diagrams*. The Young diagrams of  $\lambda$  is an array of  $n$  boxes with  $\lambda_1$  boxes in the first row,  $\lambda_2$  boxes in the second row, and so on. We count rows from top to bottom.

A *standard tableau* is a Young diagram with  $n$  boxes such that the boxes have been filled with numbers from 1 to  $n$ , in such a manner that the numbers

increase along the rows and along the columns. Let  $T_\lambda$  be the set of all standard tableaux of shape  $\lambda$ .

If  $\lambda$  and  $\mu$  are partitions, we shall write  $\mu \subset \lambda$  to mean that the diagram of  $\lambda$  contains the diagram of  $\mu$ , i.e.,  $\lambda_i \geq \mu_i$  for all  $i \geq 1$ .

A *double partition* of size  $n$ ,  $(\alpha, \beta)$ , is an ordered pair of partitions  $\alpha$  and  $\beta$  such that  $|\alpha| + |\beta| = n$ . If  $(\gamma, \rho)$  is another double partition we write  $(\gamma, \rho) \subset (\alpha, \beta)$  if  $\gamma \subset \alpha$  and  $\rho \subset \beta$ .

A pair of standard tableaux is a pair  $\tau_{(\alpha, \beta)} = (t^\alpha, t^\beta)$  of Young diagrams filled with numbers from 1 to  $n$  such that  $t^\alpha$  and  $t^\beta$  are each a standard tableau.

We say that a box in  $(\alpha, \beta)$  has coordinates  $(i, j)$  if it is in the  $i$ -th row and  $j$ -th column of either  $\alpha$  or  $\beta$ . Two boxes in  $(\alpha, \beta)$  can have the same coordinates if they occur in the same box in  $\alpha$  as in  $\beta$ , for instance the most upper-left box in  $\alpha$  and the most upper-left box in  $\beta$ , they both have coordinates  $(1, 1)$ .

## 1.2. Subfactors.

In this section we recall some definitions and basic results for constructing subfactors and for computing their invariants. For details and proof of the following statements see [Jo]. A *von Neumann algebra*  $A$  is a  $*$ -subalgebra of the algebra of bounded operators on a Hilbert space.  $A$  contains 1 and is closed in the weak operator topology. A von Neumann algebra  $A$  whose center is trivial, i.e.,  $Z(A) = \mathbb{C} \cdot 1$ , is called a *factor*. A  $\text{II}_1$  *factor* is an infinite dimensional factor  $A$  which admits a normalized finite trace  $\text{tr} : A \rightarrow \mathbb{C}$  such that (i)  $\text{tr}(1) = 1$ ; (ii)  $\text{tr}(xy) = \text{tr}(yx)$ , for all  $x, y \in A$ ; and (iii)  $\text{tr}(x^*x) \geq 0$ ,  $x \in A$ . This trace is unique.

An algebra is *approximately finite* (AF-algebra) if it is a  $C^*$  algebra that contains an increasing sequence  $(A_n)_{n=1}^\infty$  of finite dimensional  $C^*$ -subalgebras such that  $\bigcup_{n=1}^\infty A_n$  is dense in  $A$ . The *hyperfinite*  $\text{II}_1$  factor is a separable  $\text{II}_1$  factor which is approximately finite.

The trace induces a Hilbert norm on  $A$ . Moreover, we can perform the GNS construction with respect to the trace and obtain a faithful representation of  $A$  on  $L^2(A, \text{tr})$ ; this Hilbert space is obtained as the closure of  $A$  in the norm induced by the trace.  $A$  acts by left multiplication operators on itself and the GNS representation is precisely this representation extended to  $L^2(A, \text{tr})$ . Observe that the identity is the cyclic and separating vector in  $L^2(A, \text{tr})$ . This representation is called the *standard form* of  $A$ .

From now on all factors and subfactors discussed will be  $\text{II}_1$  factors. If  $A$  and  $B$  are a pair of factors, then  $A$  is a *subfactor* of  $B$  if  $A$  is a sub-von Neumann algebra of  $B$ , which is itself a factor and has the same identity as  $B$ , i.e.,  $1_A = 1_B$ . The von Neumann algebra  $A' \cap B$  is called *relative commutant* of  $A$  in  $B$ .

Let  $A \subset B$  be the inclusion of  $\text{II}_1$  factors with  $1_A = 1_B$ . If  $\text{tr}$  is the unique normalized trace on  $B$  then  $\text{tr}|_A$  is the unique normalized trace on  $A$  by uniqueness of the trace. We define the *orthogonal projection*  $e_A : L^2(B, \text{tr}) \rightarrow L^2(A, \text{tr}|_A)$  by

$$e_A(\vec{x}) = \overline{\varepsilon_A(x)}, \quad \vec{x} \in L^2(B, \text{tr}) \text{ and } x \in B,$$

where  $\varepsilon_A$  is the trace preserving conditional expectation. We denote by  $\langle B, e_A \rangle$  the von Neumann algebra generated by  $B$  and  $e_A$  on  $L^2(B, \text{tr})$ , this is called the *basic construction*. In particular, if  $A$  is a factor, then so is  $\langle B, e_A \rangle$ . If it is a finite factor, we define the index  $[B : A]$  of  $A$  in  $B$  to be the number  $1/\text{tr}(e_A)$ , where  $\text{tr}$  denotes the unique normalized trace on  $\langle B, e_A \rangle$ . If  $\langle B, e_A \rangle$  is not finite, the index is defined to be infinite.

In this paper we will study examples of subfactors constructed using the following set-up.

- (i) Let  $(B_n)$  be an ascending sequence of  $C^*$  algebras with  $B_n$  a proper subalgebra of  $B_{n+1}$  for all  $n \in \mathbb{N}$ . Furthermore, let  $\text{tr}$  be a positive finite extremal trace on its inductive limit  $B_\infty = \bigcup_{n \geq 0} B_n$  and  $\pi_{\text{tr}}$  be the GNS construction with respect to  $\text{tr}$ . Then it is well-known that the weak closure  $B$  of  $\pi_{\text{tr}}(B_\infty)$  is isomorphic to  $R$ , the hyperfinite  $\text{II}_1$  factor.
- (ii) Let  $(A_n)$  be an ascending sequence of subalgebras such that  $A_n \subset B_n$  and the weak closure  $A$  of  $\pi_{\text{tr}}(A_\infty)$  is a subfactor.
- (iii) Consider the following square

$$\begin{array}{ccc} B_n & \xleftarrow{\varepsilon_{B_n}} & B_{n+1} \\ \varepsilon_{A_n} \downarrow & & \downarrow \varepsilon_{A_{n+1}} \\ A_n & \subset & A_{n+1} \end{array}$$

where  $\varepsilon_{A_{n+1}}, \varepsilon_{A_n}$  and  $\varepsilon_{B_n}$  are the trace preserving conditional expectations onto  $A_{n+1}, A_n$  and  $B_n$  respectively. We require that this diagram commutes, i.e.,

$$\varepsilon_{A_{n+1}} \varepsilon_{B_n} = \varepsilon_{A_n}, \quad \text{for all } n \in \mathbb{N}.$$

This condition is called the *commuting square property*.

The sequence  $(A_n)$  is *periodic* with period  $k$  if there is an  $n_0 \in \mathbb{N}$  such that for all  $n > n_0$  the inclusion matrix for  $A_{n+k} \subset A_{n+k+1}$  is the same (after relabeling of the central projections) as the one for  $A_n \subset A_{n+1}$ .

We say that  $(A_n) \subset (B_n)$  is periodic if both  $(A_n)$  and  $(B_n)$  are periodic with same period  $k$  and if also the inclusion matrices for  $A_{n+k} \subset B_{n+k}$  and  $A_n \subset B_n$  are the same. If the inclusion matrices for  $A_n \subset B_n, A_n \subset A_{n+1}$

and  $B_n \subset B_{n+1}$  become periodic for  $n \geq n_0$  for some  $n_0$ , then the index  $[B : A]$  of the subfactor  $A$  is the square of the norm of the inclusion matrix for  $A_n \subset B_n$  for all  $n \geq n_0$ .

There are finer invariants for the subfactor  $A \subset B$  than the index. If  $B^{(1)} = \langle B, e_A \rangle$  is obtained by the basic construction, then it is known by [Jo] that  $[B : A] = [B^{(1)} : B]$ . Now iterate the basic construction to obtain a tower  $A \subset B \subset B^{(1)} \subset B^{(2)} \subset \dots$  of  $\text{II}_1$  factors. Let  $C_i = A' \cap B^{(i)}$  be the relative commutant of  $A$  in  $B^{(i)}$ . Then the structure of the algebras  $C_1, C_2, \dots$  is an invariant of subfactors of  $B$ . The  $C_i$ 's are called *higher relative commutants* of  $A \subset B$ .

## 2. Hecke algebras.

### 2.1. Hecke algebras of type $A$ .

In this section we summarize results by Wenzl [W1] about the representation theory of the Hecke algebra of type  $A$ .

The Hecke algebra of type  $A_{n-1}$ ,  $H_n(q)$ , is the free complex algebra with generators  $g_1, g_2, \dots, g_{n-1}$  and parameter  $q \in \mathbb{C}$  with defining relations

$$\begin{aligned} \text{(H1)} \quad & g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \quad \text{for } i = 1, 2, \dots, n-2; \\ \text{(H2)} \quad & g_i g_j = g_j g_i, \quad \text{whenever } |i - j| \geq 2; \\ \text{(H3)} \quad & g_i^2 = (q-1)g_i + q \quad \text{for } i = 1, 2, \dots, n-1. \end{aligned}$$

It is well-known that  $H_n(q) \cong \mathbb{C}S_n$  if  $q$  is not a root of unity, where  $\mathbb{C}S_n$  is the group algebra of the symmetric group,  $S_n$ , (see [Bou, pp. 54-56]). It follows from this that  $H_n(q)$  has dimension  $n!$ . Similarly as for the symmetric group, we can label the irreducible representations of  $H_n(q)$  by Young diagrams.

The full-twist,  $\Delta_f^2$ , is a central element in  $H_n(q)$  and it is defined algebraically by

$$\Delta_f^2 := (g_{f-1} \cdots g_1)^f.$$

The following lemma describes the action of the full-twist on the Hecke algebra of type  $A_{n-1}$ .

**Lemma 2.1.** *Let  $\alpha_\lambda$  be the scalar by which the full-twist acts in the irreducible Hecke algebra representation labeled by  $\lambda$ . Then*

$$(1) \quad \alpha_\lambda = q^{n(n-1) - \sum_{i < j} (\lambda_i + 1)\lambda_j}.$$

For the proof of this lemma see [W2, p. 261].

The Hecke algebras of type  $A$  satisfy the following embedding of algebras  $H_0(q) \subset H_1(q) \subset \dots \subset H_n(q) \subset H_{n+1}(q) \subset \dots$ . The inductive limit is defined by  $H_\infty(q) = \bigcup_{n > 0} H_n(q)$ .

The interesting case for defining subfactors is when the parameter  $q$  is a root of unity,  $q \neq 1$ . In what follows we will describe the semisimple quotients of  $H_n(q)$  which are associated with  $\mathfrak{sl}(r)$ , the special linear algebra, for  $1 < r < l$ .

Let  $r, l \in \mathbb{N}$  and  $l > r$ , then an  $(r, l)$ -*diagram* is a Young diagram  $\mu$  with  $r$  rows such that  $\mu_1 - \mu_r \leq l - r$ . We denote the set of all  $(r, l)$  diagrams of size  $n$  by  $\Lambda_n^{(r,l)}$ . An  $(r, l)$  *tableau* of shape  $\mu \in \Lambda_n^{(r,l)}$  is a standard tableau, such that if we remove the box containing  $n$  the Young subdiagram with  $n - 1$  boxes is an  $(r, l)$ -diagram and an  $(r, l)$  tableau. The set of  $(r, l)$  tableaux of shape  $\mu$  is denoted by  $T_\mu^{(r,l)}$ .

For each  $\mu \in \Lambda_n^{(r,l)}$  let  $V_\mu$  be the vector space with basis  $\{v_\tau\}$  indexed by elements of  $T_\mu^{(r,l)}$ . The following representations of  $H_n(q)$  at roots of unity were defined in [W1]

$$\pi_\mu^{(r,l)}(g_i)v_\tau = b_d(q)v_\tau + c_d = v_{s_i(\tau)},$$

where  $d = d_{\tau,i} = c(i+1) - c(i) + r(i) - r(i+1)$  with  $c(j)$  and  $r(j)$  the column and row of the box containing  $j$ , respectively. Here  $b_d = \frac{q^d(1-q)}{1-q^d}$ ,  $c_d = \frac{\sqrt{(1-q^{d+1})(1-q^{d-1})}}{1-q^d}$  and  $s_i(\tau)$  is the tableau obtained from  $\tau$  by interchanging the numbers  $i$  and  $i + 1$ . Note that if the  $s_i(\tau)$  is not standard then  $c_d$  is 0.

**Theorem 2.2** (Wenzl [W1, Corollary 2.5]). *Let  $q$  be a primitive  $l$ -th root of unity with  $l \geq 4$ . Then there exists for every  $\mu \in \Lambda_n^{(r,l)}$  a semisimple irreducible representation  $\pi_\mu^{(r,l)}$  of  $H_n(q)$ . Then*

$$\pi_n^{(r,l)} : x \in H_n(q) \rightarrow \bigoplus_{\mu \in \Lambda_n^{(r,l)}} \pi_\mu^{(r,l)}(x)$$

*is semisimple but generally not a faithful representation. Also representations corresponding to different  $(r, l)$  diagrams are nonequivalent.*

Using this theorem Wenzl defines a representation for the inductive limit  $H_\infty(q)$  at roots of unity, [W1]. By definition one has the following inclusion of algebras:

$$H_0(q) \subset H_1(q) \subset \dots \subset H_n(q) \subset H_{n+1}(q) \subset \dots$$

Furthermore, one has by definition the inclusion of the representations  $\pi_n^{(r,l)}(H_n(q)) \subset \pi_{n+1}^{(r,l)}(H_{n+1}(q))$  which is compatible with the inclusion  $H_n(q) \subset H_{n+1}(q)$  for all  $n$ . This is equivalent to saying that the following diagram commutes

$$\begin{array}{ccc} H_n(q) & \longrightarrow & \pi_n^{(r,l)}(H_n(q)) \\ \cap \downarrow & & \cap \downarrow \\ H_{n+1}(q) & \longrightarrow & \pi_{n+1}^{(r,l)}(H_{n+1}(q)) \end{array}$$

The representation of the inductive limit is denoted by  $\pi^{(r,l)}$  with the understanding that  $\pi^{(r,l)}(x) = \pi_n^{(r,l)}(x)$  if  $x \in H_n(q)$ . Furthermore, the representation  $\pi^{(r,l)}$  is a unitary representation, i.e., the image of the generators under this representation are unitary. Wenzl also showed that the ascending sequence of finite dimensional  $C^*$ -algebras  $(A_n)$  is periodic with period  $r$ .

Thus when  $q = e^{2\pi i/l}$  Wenzl obtains from the Hecke algebras,  $H_n(q)$ , an  $AF$  algebra with periodic Bratteli diagram for the sequence  $(\pi^{(r,l)}(H_n(q)))$ .

Let  $t^\mu$  be a Young tableau with  $n$  boxes and  $(t^\mu)'$  be the Young tableau obtained from  $t^\mu$  by removing the box containing  $n$ . The map  $t^\mu \rightarrow (t^\mu)'$  defines a bijection between  $T_\mu$  and  $\bigcup_{\mu' \subset \mu} T_{\mu'}$ , where  $T_\mu$  denotes the set of all standard tableaux of shape  $\mu$ . Therefore, we have the following decomposition of modules

$$(2) \quad V_\mu|_{H_{n-1}(q)} = \bigoplus_{\mu' \subset \mu} V_{\mu'}.$$

## 2.2. Hecke algebra of type $B$ .

In this context, we will mean by the Hecke algebra  $H_n(q, Q)$  of type  $B_n$  the free complex algebra with generators  $t, \tilde{g}_1, \dots, \tilde{g}_{n-1}$  and parameters  $q, Q \in \mathbb{C}$  the generators  $\tilde{g}_i$ 's satisfy (H1)-(H3) as in the definition of the Hecke algebra of type  $A$  and the following relations:

$$(B1) \quad t^2 = (Q - 1)t + Q;$$

$$(B2) \quad t\tilde{g}_1 t\tilde{g}_1 = \tilde{g}_1 t\tilde{g}_1 t;$$

$$(B3) \quad t\tilde{g}_i = \tilde{g}_i t \quad \text{for } i \geq 2.$$

Hoefsmit [H] has written down explicit irreducible representations of  $H_n(q, Q)$  indexed by ordered pairs of Young diagrams. It is clear that there exists an inclusion  $H_n(q) \subset H_n(q, Q)$ .

The Hecke algebras of type  $B$  satisfy the following embedding of algebras  $H_0(q, Q) \subset H_1(q, Q) \subset H_2(q, Q) \subset \dots$ . The inductive limit of the Hecke algebra of type  $B$  is defined by

$$H_\infty(q, Q) := \bigcup_{n \geq 0} H_n(q, Q).$$

Observe that (H3) and (B1) imply that  $t$  and  $\tilde{g}_i$  have at most 2 eigenvalues each, hence also at most 2 projections corresponding to these eigenvalues. There exists an alternate presentation for the Hecke algebra in terms of these projections.

For  $q \neq -1$  and  $Q \neq -1$  let

$$e_t = \frac{(Q - t)}{(Q + 1)}, \quad e_i = \frac{(q - g_i)}{(q + 1)} \quad \text{for } i = 1, \dots, n - 1$$

be the projections corresponding to the eigenvalue  $-1$ . Then  $g_i = q(1 - e_i) - e_i = q - (q + 1)e_i$ . So  $\langle 1, t, g_1, g_2, \dots, g_{n-1} \rangle = \langle 1, e_t, e_1, e_2, \dots, e_{n-1} \rangle$  and the defining relations (H1)-(H3) and (B1)-(B3) of  $H_n(q, Q)$  translate to

- (PH1)  $e_i e_{i+1} e_i - q/(q+1)^2 e_i = e_{i+1} e_i e_{i+1} - q/(q+1)^2 e_{i+1}$   
for  $i = 1, 2, \dots, n-2$ ;
- (PH2)  $e_i e_j = e_j e_i$ , whenever  $|i - j| \geq 2$ ;
- (PH3)  $e_i^2 = e_i$  for  $i = 1, 2, \dots, n-1$ ;
- (PH4)  $e_t^2 = e_t$ ;
- (PH5)  $e_t e_1 e_t e_1 - (Q+q)/(q+1)(Q+1) e_t e_1$   
 $= e_1 e_t e_1 e_t - (Q+q)/(q+1)(Q+1) e_1 e_t$ ;
- (PH6)  $e_t e_i = e_i e_t$  for  $i \geq 2$ .

**2.3. Representations of the Hecke algebra of type  $B$  onto a reduced Hecke algebra of type  $A$ .**

In this section we describe a way to obtain representations of the Hecke algebra of type  $B$  onto a reduced Hecke algebra of type  $A$ . First, we introduce some necessary background.

Let  $p \in H_n(q)$  be an idempotent then the *reduced algebra* with respect to  $p$  is defined by  $pH_n(q)p = \{pap \mid a \in H_n(q)\}$ . In [W1] Wenzl defines a set of minimal idempotents of the Hecke algebra of type  $A$ . These idempotents are indexed by the standard tableaux. So if  $\lambda \vdash f$ , then we denote by  $t^\lambda$  a standard tableau of shape  $\lambda$ . And accordingly  $p_{t^\lambda}$  will denote a minimal idempotent in  $H_f(q)$  corresponding to  $t^\lambda$ .

Fix an integer  $n$ , and let  $m, r_1 \in \mathbb{N}$ . Throughout this section we assume  $n < m$  and  $n < r_1$ . Set  $\lambda = [m^{r_1}]$  and  $\gamma = [m^{r_1}, 1]$ . Then  $p_{t^\lambda} H_{n+f}(q) p_{t^\lambda}$  is the reduced algebra associated to  $p_{t^\lambda}$ . We define a map  $\rho_{f,n}$  from the generators of  $H_n(q, -q^{r_1+m})$  into the reduced Hecke algebra  $p_{t^\lambda} H_{n+f}(q) p_{t^\lambda}$  as follows:

$$\begin{aligned} \rho_{f,n}(1) &= p_{t^\lambda}, & \rho_{f,n}(t) &= -\frac{\alpha_\lambda}{\alpha_\gamma} p_{t^\lambda} \Delta_f^{-2} \Delta_{f+1}^2, & \text{and} \\ \rho_{f,n}(\tilde{g}_i) &= p_{t^\lambda} g_{f+i} & \text{for } i &= 1, \dots, n-1 \end{aligned}$$

where  $\alpha_\lambda$  is the scalar by which the full-twist acts on the irreducible module  $V_\lambda$  of the Hecke algebra of type  $A$ , see Lemma 2.1, and  $\Delta_f^{-2} \Delta_{f+1}^2 = g_f \cdots g_2 g_1^2 g_2 \cdots g_f$ .

**Theorem 2.3.** *With the conditions stated above we have that  $\rho_{f,n}$  extends to a well-defined surjective homomorphism from  $H_n(q, -q^{r_1+m})$  onto  $p_{t^\lambda} H_{n+f}(q) p_{t^\lambda}$ .*

This theorem was proved in [O].

**3. Representations of the Hecke algebra of type  $B$ .**

In this section we give some of the representation theory of  $H_n(q, Q)$ . Then we use the homomorphism of Section 2.3 to find simple representations of  $H_n(q, Q)$  when  $q$  is a root of unity and  $Q = -q^{m+r_1}$ , where  $m, r_1 \in \mathbb{N}$ .

We briefly describe the semi-orthogonal representations of  $H_n(q, Q)$ . Hoefsmit [H] constructed for each double partition  $(\alpha, \beta)$  of  $n$  an irreducible representation  $(\pi_{(\alpha, \beta)}, V_{(\alpha, \beta)})$  of  $H_n(q, Q)$  of degree  $\binom{n}{|\alpha|} f^\alpha f^\beta$  where  $f^\alpha$  is the number of standard tableaux of shape  $\alpha$ .

Let  $T_{(\alpha, \beta)}$  denote the set of pairs of standard tableaux of shape  $(\alpha, \beta)$ . We define the complex vector space  $V_{(\alpha, \beta)}$  with orthonormal basis given by  $\{v_\tau \mid \tau \in T_{(\alpha, \beta)}\}$ . In what follows we describe the action of the generators of  $H_n(q, Q)$  on  $V_{(\alpha, \beta)}$ . The following notations and definitions are needed to define this action.

Let  $(\alpha, \beta)$  be a pair of Young diagrams and  $\tau = (t^\alpha, t^\beta)$  be a pair of standard tableaux of shape  $(\alpha, \beta)$ . Define the *content* of a box  $b$  as follows:

$$ct(b) = \begin{cases} Qq^{j-i} & \text{if } b \text{ is in position } (i, j) \text{ in } t^\alpha \\ -q^{j-i} & \text{if } b \text{ is in position } (i, j) \text{ in } t^\beta. \end{cases}$$

Now define for each  $1 \leq i \leq n-1$

$$(\tilde{g}_i)_\tau = \frac{q-1}{1 - \frac{ct(\tau(i))}{ct(\tau(i+1))}}$$

where  $\tau(i)$  denotes the coordinates of the box containing the number  $i$ . Notice that  $(\tilde{g}_i)_\tau$  depends only on the position of  $i$  and  $i+1$ . We are now ready to define the action of the generators on  $V_{(\alpha, \beta)}$ .

$$(3) \quad \begin{aligned} tv_\tau &= ct(\tau(1))v_\tau \\ \tilde{g}_i v_\tau &= (\tilde{g}_i)_\tau v_\tau + (q - (\tilde{g}_i)_\tau)v_{s_i(\tau)}, \quad \text{for } i = 1, \dots, n-1 \end{aligned}$$

where  $s_i(\tau)$  is the standard tableau obtained from  $\tau$  by switching  $i$  and  $i+1$  in  $\tau$ . If  $i$  and  $i+1$  do not occur in the same row or column of  $t^\alpha$  or  $t^\beta$ , then  $s_i(t^\alpha, t^\beta)$  is again a pair of standard tableaux. Let  $V$  be the span of  $\{v_\tau, v_{s_i(\tau)}\}$ . Obviously,  $V$  is  $\tilde{g}_i$ -invariant. The action of  $\tilde{g}_i|_V$  is given by the following  $2 \times 2$  matrix

$$(4) \quad \begin{pmatrix} (\tilde{g}_i)_\tau & (q - (\tilde{g}_i)_\tau) \\ (q - (\tilde{g}_i)_{s_i(\tau)}) & (\tilde{g}_i)_{s_i(\tau)} \end{pmatrix}.$$

Finally, we have that if  $i$  and  $i+1$  occur in the same row then  $\tilde{g}_i v_\tau = qv_\tau$ ; and if  $i$  and  $i+1$  occur in the same column then  $\tilde{g}_i v_\tau = -v_\tau$ .

**Theorem 3.1** (Hoefsmit [H, Thm. 2.2.7]). *The modules  $V_{(\alpha, \beta)}$ , where  $(\alpha, \beta)$  runs over all double partitions of  $n$ , form a complete set of non-isomorphic irreducible modules of  $H_n(q, Q)$ .*

**Remark.** Let  $q \neq -1$  and  $Q \neq -1$ . One can easily obtain representations for the spectral projections defined in Section 2.2. Recall the equations  $e_t = \frac{Q-t}{Q+1}$  and  $e_i = \frac{q-\tilde{g}_i}{q+1}$  for  $i = 1, \dots, n-1$ . The matrix representation of these projections is obtained via the substitution  $(\tilde{g}_i)_\tau = q - (q+1)(e_i)_\tau$ .

Let the numbers  $i$  and  $j$  be contained in  $(r_i, c_i)$  and  $(r_j, c_j)$  respectively, then define

$$d_{\tau,i,j} = c_i - c_j + r_j - r_i$$

we will refer to  $d_{\tau,i,j}$  as the *axial distance* from  $j$  to  $i$  in  $\tau$ . Note that  $d_{\tau,j,i} = -d_{\tau,i,j}$ . Let  $d = d_{\tau,i,i+1}$ . We have two possibilities for the denominator of  $(\tilde{g}_i)_\tau$

$$(5) \text{ denominator}((\tilde{g}_i)_\tau) = \begin{cases} 1 - q^d & \text{if } i \text{ and } i + 1 \text{ are both in } t^\alpha \text{ or } t^\beta \\ 1 + Qq^d & \text{otherwise.} \end{cases}$$

Observe that  $\pi_{(\alpha,\beta)}(\tilde{g}_i)$  is undefined if and only if  $(\tilde{g}_i)_\tau$  is undefined. This implies that if  $Q \neq -q^k$  for  $k \in \mathbb{Z}$  and if  $q$  is not an  $l$ -th root of unity for  $1 \leq l \leq n - 1$  then  $\pi_{(\alpha,\beta)}(\tilde{g}_i)$  is well-defined in  $V_{(\alpha,\beta)}$  for all pairs  $(\alpha, \beta)$  and  $i = 1, 2, \dots, n - 1$ . Notice that  $(\tilde{g}_i)_\tau$  is also undefined when  $d_{\tau,i,i+1} = 0$  and both  $i$  and  $i + 1$  are in  $t^\alpha$  or  $t^\beta$ , but this never happens if  $\tau$  is a pair of standard tableaux, see [W1, Lemma 2.11].

Observe that the map  $\tau \rightarrow \tau'$  (where  $\tau'$  is obtained from  $\tau$  by removing the box containing  $n$ ) defines a bijection between  $T_{(\alpha,\beta)}$  and  $\bigcup_{(\alpha,\beta)' \subset (\alpha,\beta)} T_{(\alpha,\beta)'}$ . So, in particular, we have

$$(6) \quad V_{(\alpha,\beta)} \Big|_{H_{n-1}(q,Q)} \cong \bigoplus_{(\alpha,\beta)' \subset (\alpha,\beta)} V_{(\alpha,\beta)'}$$

where  $(\alpha, \beta)'$  is a pair of Young tableaux obtained by removing one box from either  $\alpha$  or  $\beta$ . From the definition of  $\pi_{(\alpha,\beta)}$  and  $\pi_{(\alpha,\beta)'}$  we see that this equation yields the decomposition of  $V_{(\alpha,\beta)}$  as an  $H_{n-1}(q, Q)$ -module

$$(7) \quad \pi_{(\alpha,\beta)} \Big|_{H_{n-1}(q,Q)} \cong \bigoplus_{(\alpha,\beta)' \subset (\alpha,\beta)} \pi_{(\alpha,\beta)'}$$

### 3.1. The Hecke algebra of type $B$ at roots of unity.

In the previous section we observed that the irreducible representations of  $H_n(q, Q)$  depend on rational functions with denominator  $(Qq^d + 1)$  or  $(1 - q^d)$ ,  $d \in \mathbb{Z}$ . Thus some of the representations will be undefined when  $Q = -q^k$  for some  $k \in \mathbb{Z}$  or when  $q$  is a root of unity. It is the objective of this section to describe the simple decomposition of quotients of the Hecke algebra of type  $B$  when  $Q = -q^k$  and  $q$  is an  $l$ -th root of unity.

In Section 2.3 we defined for  $r_1, m \in \mathbb{N}$  such that  $r_1 > n$  and  $m > n$  an onto homomorphism from the specialized Hecke algebra of type  $B$ ,  $H_n(q, -q^{r_1+m})$ , onto a reduced Hecke algebra of type  $A$ ,  $p_{t^\lambda} H_{n+f}(q) p_{t^\lambda}$ , where  $p_{t^\lambda}$  is an idempotent indexed by  $t^\lambda$ , a standard tableau corresponding to  $\lambda = [m^{r_1}]$ , i.e.,

$$\rho_{f,n} : H_n(q, -q^{r_1+m}) \longrightarrow p_{t^\lambda} H_{n+f}(q) p_{t^\lambda}.$$

In what follows we show that there is a well-defined surjective homomorphism when  $q$  is a root of unity and  $Q = -q^{m+r_1}$  if we map onto a well-defined quotient of  $p_{t^\lambda} H_{n+f}(q) p_{t^\lambda}$ .

By Theorem 2.2 when  $q$  is an  $l$ -th root of unity then  $\pi_n^{(r,l)}(H_n(q))$  is a well-defined quotient of the Hecke algebra of type  $A$  which is semisimple. The simple components are indexed by  $(r, l)$ -diagrams. We will denote this quotient by  $H_n^{(r,l)}(q)$ .

In [W1] Wenzl showed that there exist well-defined minimal idempotents of  $H_n^{(r,l)}(q)$  for every  $(r, l)$  tableau. We denote these idempotents by  $p_{t^\lambda}^{(l)}$ . In particular, we have the following well-defined reduced algebra  $p_{t^\lambda}^{(l)} H_n^{(r,l)}(q) p_{t^\lambda}^{(l)}$ . Throughout the sequel we will only be interested in the case when  $\lambda = [m^{r_1}]$ . Notice that  $\lambda$  is an  $(r, l)$ -diagram if  $m \leq l - r$ . Now we choose a Young tableau  $t^\lambda \in T_\lambda^{(r,l)}$  such that  $p_{t^\lambda}^{(l)}$  is well-defined. Define a map from the generators of  $H_n(q, -q^{r_1+m})$  into the reduced algebra  $p_{t^\lambda}^{(l)} H_{n+f}^{(r,l)}(q) p_{t^\lambda}^{(l)}$  as follows:

$$\begin{aligned} \tilde{\rho}_{f,n}(1) &= p_{t^\lambda}^{(l)}, \quad \tilde{\rho}_{f,n}(t) = -\frac{\alpha_\lambda}{\alpha_\gamma} p_{t^\lambda}^{(l)} \Delta_f^{-2} \Delta_{f+1}^2 \quad \text{and} \\ \tilde{\rho}_{f,n}(\tilde{g}_i) &= p_{t^\lambda}^{(l)} g_{f+i} \quad \text{for } i = 1, \dots, n-1. \end{aligned}$$

**Theorem 3.2.** *Let  $m, r_1, r_2, l \in \mathbb{N}$ ,  $l \geq 4$  and  $r = r_1 + r_2 < l$ . Assume  $q$  is a primitive  $l$ -th root of unity and  $Q = -q^{m+r_1}$  with  $r_1 < m + r_1 \leq l - r_2$ . Then  $\tilde{\rho}_{f,n}$  as defined above is a nontrivial onto homomorphism.*

*Proof.*  $\tilde{\rho}_{f,n}$  is well-defined at roots of unity. Thus the proof that  $\tilde{\rho}_{f,n}$  is a homomorphism is the same as in [O].

To show that  $\tilde{\rho}_{f,n}$  is onto, it suffices to show that every irreducible representation of the reduced algebra is an irreducible representation of  $H_n(q, -q^{r_1+m})$ . The proof is by induction on  $n$ . For  $n = 1$ , we have  $\tilde{\rho}_{f,1} : H_1(q, -q^{r_1+m}) \rightarrow p_{t^\lambda}^{(l)} H_{f+1}^{(r,l)}(q) p_{t^\lambda}^{(l)}$ . Since  $\lambda \vdash f$  is a rectangular diagram there are only two Young diagrams with  $f + 1$  boxes which contain  $\lambda$ , i.e.,  $[m + 1, m^{r_1-1}]$  and  $[m^{r_1}, 1]$ . Note that  $V_{[m+1, m^{r_1-1}]}^{(l)}$  is well-defined as long as  $m + 1 \leq l - r$  and  $V_{[m^{r_1}, 1]}^{(l)}$  is well-defined as long as  $r_2 > 0$ . The action of  $\tilde{\rho}_{f,1}(t)$  on the representation indexed by  $[m + 1, m^{r_1-1}]$  (resp.  $[m^{r_1}, 1]$ ) is  $-q^{r_1+m}$  (resp.  $-1$ ). And both representations are 1 dimensional.

The algebra  $H_1(q, -q^{r_1+m})$  has two irreducible representations indexed by  $([1], \emptyset)$  and  $(\emptyset, [1])$ . Both representations are 1 dimensional and  $t \in H_1(q, -q^{r_1+m})$  acts by a scalar on these representations. The action of  $t$  on  $V_{([1], \emptyset)}$  (resp.  $V_{(\emptyset, [1])}$ ) is  $-q^{r_1+m}$  (resp.  $-1$ ). Notice that  $l \neq r_1 + m$ , since we assumed that  $m \leq l - r_1 - r_2$ . Therefore, these representations are irreducible and nonequivalent. This shows that  $\pi_{([1], \emptyset)} \cong \pi_{[m+1, m^{r_1-1}]}$  and

$\pi_{(\emptyset, [1])} \cong \pi_{[m^{r_1, 1}]}$  whenever the representations  $\pi_{[m+1, m^{r_1-1}]}$  and  $\pi_{[m^{r_1, 1}]}$  in the reduced algebra are well-defined.

In what follows if  $\lambda \subset \nu$ , then  $\nu/\lambda$  will be identified with the pair  $(\alpha, \beta)$  of Young diagrams which remain after removing  $\lambda$ , see Figure 2.

Assume that for  $n > 1$  we have  $\tilde{\rho}_{f, n}$  is onto. If  $\nu \vdash n+f$  is an  $(r, l)$ -diagram containing  $\lambda$ , then  $V_{\nu/\lambda}^{(l)}$  is an irreducible module of  $H_n(q, -q^{r_1+m})$ . Now let  $\mu \vdash n+f+1$  be an  $(r, l)$ -diagram which contains  $\lambda$ , then  $V_{\mu/\lambda}^{(l)}|_{H_n(q, -q^{r_1+m})} \cong \bigoplus_{\lambda \subset \mu' \subset \mu} V_{\mu'/\lambda}$ , as in Equation (6). Clearly  $V_{\mu/\lambda}^{(l)}$  is a representation of  $H_{n+1}(q, -q^{r_1+m})$ .

The irreducibility can be shown exactly as in [W1, Theorem 2.2 and Corollary 2.5]. The fact that representations belonging to different Young diagrams are inequivalent is also shown as in [W1, Theorem 2.2 and Lemma 2.11].  $\square$

This theorem constructs a semisimple quotient of  $H_n(q, -q^{r_1+m})$ , which we denote by  $H_n^{(r, l)}(q, -q^{r_1+m})$ .

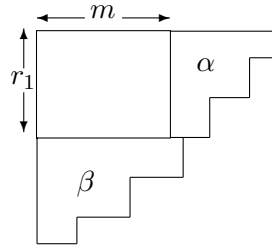


Figure 2.

**Observation.** There is a 1-1 correspondence between pairs of Young diagrams  $(\alpha, \beta)$  satisfying the condition  $\alpha_{r_1} - \beta_1 \geq -m$  with  $l(\alpha) \leq r_1$  and Young diagrams containing a rectangular diagram  $[m^{r_1}]$ , see Figure 2.

Now we define a subset  $\Gamma_n(l, m, r)$  of the set of double partitions. We will show that the quotient  $H_n^{(r, l)}(q, -q^{r_1+m})$  which is isomorphic to the image of  $\tilde{\rho}_{f, n}$  is indexed by the ordered pairs of Young diagrams which we now define.

**Definition.** Let  $m, l, r \in \mathbb{N}$  with  $r \leq l - 1$ . A pair of Young diagrams  $(\alpha, \beta)$  such that  $l(\alpha) \leq r_1$  and  $l(\beta) \leq r_2$  is called a  $(m, l, r)$ -diagram if

- (1)  $\alpha_1 - \beta_{r_2} \leq l - r - m$  and
- (2)  $\alpha_{r_1} - \beta_1 \geq -m$ .

Let  $\Gamma_n(l, m, r)$  denote the set of all  $(m, l, r)$ -diagrams with  $n$  boxes.

We have the following corollary of Theorem 3.2.

**Corollary 3.3.** (i) Let  $\mu = [m + \alpha_1, \dots, m + \alpha_{r_1}, \beta_1, \dots, \beta_{r_2}]$  be an  $(r, l)$  diagram. Then there exists a 1-1 correspondence between  $\mu \in \Lambda_{n+f}^{(r,l)}$  and  $(\alpha, \beta) \in \Gamma(l, m, r)$ .

(ii) If the representation indexed by  $(\alpha, \beta)$  is well-defined, then the bijection in (i) is compatible with the homomorphism,  $\tilde{\rho}_{f,n}$ .

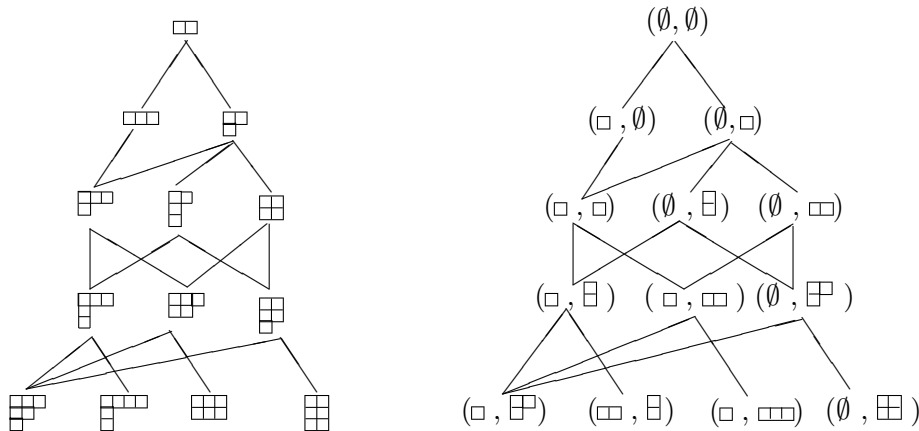
*Proof.* (i) Recall that  $\mu \in \Lambda_{n+f}^{(r,l)}$  implies that  $\mu_1 - \mu_r \leq l - r$ , where  $l(\mu) \leq r = r_1 + r_2$ . By substituting  $\mu_1 = \alpha_1 + m$  and  $\mu_r = \beta_{r_2}$  one gets  $\alpha_1 - \beta_{r_2} + r_2 \leq l - r_1 + m$  which is condition (1) in the definition of the elements in  $\Gamma(l, m, r)$ . The other condition is easily seen by the definition of a Young diagram.  $\mu_{r_1} \leq \mu_{r_1+1}$  implies condition (2)  $\alpha_{r_1} - \beta_1 > -m$ . Clearly, having  $(\alpha, \beta) \in \Gamma(l, m, r)$  one can construct  $\mu$  by adjoining the box  $[m^{r_1}]$ .

(ii) By (i) we have two indexing sets for the irreducible representations of  $H_n^{(r,l)}(q, -q^{r_1+m})$ . If  $(\pi_\mu^{(r,l)}, V_\mu^{(r,l)})$  is a well-defined irreducible representation then we can also index it with a pair  $(\alpha, \beta) \in \Gamma(l, m, r)$ . Furthermore, if we restrict  $V_\mu^{(r,l)}$  to  $H_{n-1}^{(r,l)}(q, -q^{r_1+m})$  we obtain the decomposition

$$V_\mu^{(r,l)} \Big|_{H_{n-1}^{(r,l)}(q, -q^{r_1+m})} = \bigoplus_{\mu' \subset \mu} V_{\mu'}^{(r,l)}$$

where  $\mu' \in \Lambda_{n-1}^{(r,l)}$  and  $\mu' \supset \lambda$  by Theorem 3.2. Note that  $\mu'$  can be associated with a pair  $(\alpha, \beta)' \in \Gamma_{n-1}(l, m, r)$  and  $V_{(\alpha, \beta)'}^{(r,l)}$  can be associated with  $V_{\mu'}^{(r,l)}$  whenever the representations are well-defined. Therefore, the bijection in (i) is compatible with the homomorphism  $\hat{\rho}_{f,n}$ .  $\square$

In Figure 3 we show the Bratteli diagrams for the example,  $l = 5, m = 2, r_1 = 1$  and  $r_2 = 2$ . In this case  $\lambda = [2]$ .



**Figure 3.** Bratteli Diagrams for  $p_{t[2]}^{(5)} H_n^{(3,5)}(q) p_{t[2]}^{(5)}$  and  $H_n^{(3,5)}(q, -q^3)$ .

Let us fix  $m, l, r \in \mathbb{N}$  with  $l \geq 4$  and let  $q = e^{2\pi i/l}$ . Set

$$(8) \quad B_n = \bigoplus_{(\alpha, \beta) \in \Gamma_n(l, m, r)} \pi_{(\alpha, \beta)}^{(l)}(H_n(q, -q^{r_1+m}))$$

where  $\pi_{(\alpha, \beta)}^{(l)}$  denotes the representation indexed by  $(\alpha, \beta)$  when  $q$  is an  $l$ -th root of unity obtained through the homomorphism  $\hat{\rho}_{f,n}$ . By definition of the  $\pi_{(\alpha, \beta)}^{(l)}$ 's, the restriction of this representation to  $H_{n-1}(q, -q^{r_1+m})$  is isomorphic to  $B_{n-1}$ . With this identifications we can define the representation

$$\pi^{(l)} : H_\infty(q, -q^{r_1+m}) \longrightarrow B_\infty$$

of the corresponding inductive limits by

$$(9) \quad \pi^{(l)}(x) = \bigoplus_{(\alpha, \beta) \in \Gamma_n(m, l, r)} \pi_{(\alpha, \beta)}^{(l)}(x)$$

for all  $x \in H_n(q, -q^{r_1+m})$ .

If  $q = e^{2\pi i/l}$  then Wenzl [W1] showed that the inclusion diagrams for the Hecke algebras of type  $A$  eventually become periodic with period  $r$  (the maximum number of rows allowed).

**Lemma 3.4.** (a) *If the inclusion diagram for  $\dots \subset H_{n-1}(q) \subset H_n(q) \subset \dots$  has period  $r$ , then the inclusion diagram for  $\dots \subset p_{t^\lambda} H_{n-1}(q) p_{t^\lambda} \subset p_{t^\lambda} H_n(q) p_{t^\lambda} \subset \dots$  has period  $r$ .*

(b) *The inclusion diagram for  $\dots \subset H_{n-1}^{(r,l)}(q, -q^{r_1+m}) \subset H_n^{(r,l)}(q, -q^{r_1+m}) \subset \dots$  has period  $r$  whenever  $\dots \subset p_{t^\lambda} H_{n-1}^{(r,l)}(q) p_{t^\lambda} \subset p_{t^\lambda} H_n^{(r,l)}(q) p_{t^\lambda} \subset \dots$  has period  $r$ .*

*Proof.* The proof of (a) follows immediately from the definition of reduced algebra.

(b) We have shown above that the quotient  $H_n^{(r,l)}(q, -q^{r_1+m})$  of the Hecke algebra of type  $B$  is isomorphic to the reduced algebra  $p_{t^\lambda}^{(l)} H_{n+f}^{(r,l)}(q) p_{t^\lambda}^{(l)}$ . Thus periodicity follows from this isomorphism.  $\square$

#### 4. Markov Traces.

In this section we define special traces which will help us define  $\text{II}_1$  factors. These traces satisfy the commuting square property, which is needed for the construction of subfactors. The existence of these traces on the Hecke algebra of type  $B$  has been proven by Geck and Lambropoulou in [GL].

**Definition.** A trace,  $\text{tr}$ , on  $H_\infty(q, Q)$  is called a *Markov trace* if there is a  $z \in \mathbb{C}(q, Q)$  such that  $\text{tr}(xg_n) = z\text{tr}(x)$  for all  $n \in \mathbb{N}$  and  $x \in H_n(q, Q)$ .

All generators  $g_i, i = 1, 2, \dots$  are conjugate in  $H_\infty(q, Q)$ . Thus, any trace function on  $H_\infty(q, Q)$  must have the same value on these elements.

In particular, this implies that the parameter  $z$  is independent of  $n$  in the definition of Markov trace.

Geck and Pfeiffer [GP] showed that  $\text{tr}$  is uniquely determined on elements of minimal length of the form  $d_1 \cdots d_n$  where  $d_i = g_{i-1}$  or  $d_i = t'_{i-1} = g_{i-1} \cdots g_1 t g_1^{-1} \cdots g_{i-1}^{-1}$ . So if  $\text{tr}$  is a Markov trace then

$$\text{tr}(d_1 \cdots d_n) = z^a \text{tr}(t'_0 \cdots t'_{b-1})$$

where  $a$  is the number of  $d_i$ 's which equal  $g_{i-1}$  and  $b$  is the number which equal  $t'_{i-1}$ .

Geck and Lambropoulou have shown that given  $z, y_1, y_2, \dots \in \mathbb{C}(q, Q)$  then there is a unique Markov trace for  $H_\infty(q, Q)$  such that  $\text{tr}(t'_0 t'_1 \cdots t'_{k-1}) = y_k$  for all  $k \geq 1$ . The case in which we are interested is described in the following proposition which is proved in [GL].

**Proposition 4.1.** *Let  $z, y \in \mathbb{C}(q, Q)$  be a Markov trace with parameter  $z$  such that  $\text{tr}(t'_0 t'_1 \cdots t'_{k-1}) = y^k$  for all  $k \geq 1$ , then*

$$\text{tr}(h t'_{n,0}) = y \text{tr}(h) \quad \text{for all } n \geq 0 \text{ and } h \in H_n(q, Q),$$

where  $t'_{n,0} = g_n \cdots g_1 t g_1^{-1} \cdots g_n^{-1}$  or  $g_n^{-1} \cdots g_1^{-1} t g_1 \cdots g_n$ .

In [O] we computed the weight vector for the trace described in Proposition 4.1. The components of this vector are indexed by double partitions and are given by the following formula:

$$(10) \quad W_{(\alpha, \beta)}(q, Q) = q^{n(\alpha) + n(\beta)} \left( \frac{1-q}{1-q^r} \right)^{|\alpha| + |\beta|} \prod_{1 \leq i < j \leq r_1} \frac{1 - q^{\alpha_i - \alpha_j + j - i}}{1 - q^{j-i}} \\ \cdot \prod_{1 \leq i < j \leq r_2} \frac{1 - q^{\beta_i - \beta_j + j - i}}{1 - q^{j-i}} \prod_{i=1}^{r_1} \prod_{j=1}^{r_2} \frac{Q q^{\alpha_i - i} + q^{\beta_j - j}}{Q q^{-i} + q^{-j}}$$

where  $n(\alpha) = \sum_{i>1} (i-1)\alpha_i$  and  $r = r_1 + r_2$ .

The weights for the Markov trace on the Hecke algebra of type  $A$  can be found in [W1]. They are given by the following formula:

$$(11) \quad s_{\alpha, r}(q) = q^{n(\alpha)} \left( \frac{1-q}{1-q^r} \right)^{|\alpha|} \prod_{1 \leq i < j \leq r} \frac{1 - q^{\alpha_i - \alpha_j + j - i}}{1 - q^{j-i}}.$$

In [O] we also observed that when  $Q = -q^{r_1+m}$  we obtain

$$W_{(\alpha, \beta)}(q, -q^{r_1+m}) = \frac{s_{\mu, r}(q)}{s_{[m^{r_1}], r}(q)}$$

where  $\mu = [m + \alpha_1, \dots, m + \alpha_{r_2}, \beta_1, \dots, \beta_{r_1}]$ .

In [W1] Lemma 3.5, Wenzl showed that if  $l(\mu) > r$  then  $s_{\mu, r}(q) = 0$ . Also he showed that  $s_{\mu, r}(q)$  is well-defined when  $q$  is a primitive  $l$ -th root of unity with  $l > 1$  if  $\mu_1 - \mu_r \leq l - r + 1$ , and  $s_{\mu, r}(q) = 0$  if and only if

$\mu_1 - \mu_r = l - k + 1$ . In particular,  $s_{\mu,r}(q) > 0$  for all  $(r, l)$  diagrams. Thus, he obtained a well-defined, positive trace on  $H_n^{(r,l)}(q)$ . Furthermore, he showed that the weight vector for the restriction of  $\text{tr}$  to  $H_n^{(r,l)}(q)$  is given by the vector  $(s_{\mu,r}(q))_{\mu \in \Lambda_n^{(r,l)}}$ .

**Proposition 4.2.** *The Markov trace defined by the weights in Equation (10) factors over the quotients of the Hecke algebra of type B,  $H_{n+f}^{(r,l)}(q, -q^{r_1+m})$ .*

*Proof.* This proposition is a direct consequence of the results cited before the statement of this proposition. We know that the quotient  $H_n^{(r,l)}(q, -q^{r_1+m})$  is isomorphic to a quotient of the reduced algebra  $p_{t^\lambda}^{(l)} H_{n+f}^{(r,l)}(q) p_{t^\lambda}^{(l)}$  which has weight vectors given by  $\frac{s_{\mu,r}(q)}{s_{[m^{r_1}],r}(q)}$ . Since  $[m^{r_1}]$  is an  $(r, l)$  diagram we have that  $s_{[m^{r_1}],r}(q) \neq 0$ . So the weights are well-defined. Furthermore, they will be zero exactly when  $s_{\mu,r}(q) = 0$ . □

### 5. $C^*$ -Representations of $H_n(q, Q)$ .

In order to use the results in [W1], we need to find nontrivial  $C^*$  representations of the inductive limit of the Hecke algebra of type B,  $H_\infty(q, Q)$ . That is, we need to find the values of the parameters  $q$  and  $Q$  for which the generators  $e_i$  are self-adjoint. From now on  $V_{(\alpha,\beta)}$  is assumed to be a complex Hilbert space with orthonormal basis  $\{v_\tau \mid \tau \in T_{(\alpha,\beta)}\}$ .

In what follows we show that there are  $C^*$  representations of  $H_\infty(q, Q)$  when  $Q = -q^k$  and  $q$  is an  $l$ -th root of unity.

**Definition.** A representation  $\rho$  of  $H_n(q, Q)$  or  $H_\infty(q, Q)$  on a Hilbert space is called a  $C^*$  representation if  $\rho(e_t)$  and  $\rho(e_i)$  for  $i = 1, 2, \dots, n - 1$  or for all  $i \in \mathbb{N}$  are self-adjoint projections.

Wenzl in [W1] showed that there are nontrivial  $C^*$  representations of  $H_\infty(q)$ , if  $q$  is real and positive or if  $q = e^{2\pi i/l}$ , where  $l$  is a positive integer greater than or equal to 4. Since  $H_\infty(q) \subset H_\infty(q, Q)$  it follows that to obtain  $C^*$  representations of  $H_\infty(q, Q)$  it is necessary for  $q$  to be real and positive or an  $l$ -th root of unity. Unfortunately, this is not sufficient, we also need a condition for  $Q$ .

**Proposition 5.1.** *If  $q$  and  $Q$  are both real and positive there are faithful  $C^*$  representations of  $H_n(q, Q)$  for all  $n \in \mathbb{N}$ . If  $q = e^{\pm 2\pi i/l}$  and  $Q = -q^{r_1+m}$  for  $l, m, r_1, r_2 \in \mathbb{N}$  with  $l \geq 4$ , and  $r_1 \leq m + r_1 \leq l - r_2$  then  $\pi_{(\alpha,\beta)}^{(l)}$  is a  $C^*$  representation for all  $(m, l, r)$ -diagrams  $(\alpha, \beta)$ .*

*Proof.* Theorem 3.2 shows that there exists an onto homomorphism from  $H_n^{(r,l)}(q, -q^{r_1+m})$  onto  $p_{t^\lambda}^{(l)} H_{n+f}^{(r,l)}(q) p_{t^\lambda}^{(l)}$ . Thus, it will suffice to show that the reduced algebra has  $C^*$  representations. In [W1] Wenzl showed that the

quotient  $H_{n+f}^{(r,l)}(q)$  is a  $C^*$  algebra when  $q$  is a root of unity. Furthermore, this quotient is semisimple and the irreducible modules are indexed by  $(r, l)$ -diagrams.

Since  $p_{t^\lambda}^{(l)}$  is an orthogonal projection in  $H_{n+f}^{(r,l)}(q)$  and it is well-defined when  $q$  is a root of unity then it is self-adjoint. Then it follows that  $p_{t^\lambda}^{(l)} H_{n+f}^{(r,l)}(q) p_{t^\lambda}^{(l)}$  has a  $C^*$  representation and the irreducible modules are indexed by  $(r, l)$ -diagrams which contain the diagram  $\lambda$ . Thus, by Corollary 3.3 the irreducibles are indexed by  $(m, l, r)$ -diagrams.  $\square$

## 6. Subfactors, Index and Commutants.

In the previous section we showed that there are quotients of the Hecke algebra of type  $B$ ,  $H_n(q, -q^{r_1+m})$ , which are  $C^*$  algebras. We also showed that these quotients are isomorphic to  $p_{t^\lambda}^{(l)} H_{n+f}^{(r,l)}(q) p_{t^\lambda}^{(l)}$ , when  $q$  is an  $l$ -th root of unity. We are now in position to construct the subfactors which arise from the inclusion of the Hecke algebra of type  $A$  into the Hecke algebra of type  $B$ . We will give the index and relative commutants for these subfactors.

In order to construct the subfactors we will use the following two sequences of algebras. Let  $l, m, r_1, r_2 \in \mathbb{N}$  with  $l > r$  and set  $r = r_1 + r_2$ :

- (i) Let  $B_n = p_{t^\lambda}^{(l)} H_{n+f}^{(r,l)}(q) p_{t^\lambda}^{(l)}$  be the finite dimensional  $C^*$  algebras in the previous section. Then we have the sequence given by the proper inclusion of  $B_n \subset B_{n+1}$  for all  $n \in \mathbb{N}$ .
- (ii) Let  $A_n = p_{t^\lambda}^{(l)} H_{f,n+f}^{(r,l)}(q) p_{t^\lambda}^{(l)}$ , where  $H_{f,n+f}^{(r,l)}(q)$  is the finite dimensional  $C^*$  algebra generated by  $g_{f+1}, \dots, g_{f+n}$  in  $H_{n+f}^{(r,l)}(q)$ . Furthermore, we have that  $A_n \subset B_n$ .

Thus, we have two sequences  $(A_n)$  and  $(B_n)$  of  $C^*$  algebras such that  $A_n \subset B_n$ .  $p_{t^\lambda}^{(l)}$  is the identity in  $A_n$  and  $B_n$ .

From the work of Jimbo [Ji] and Drinfel'd [D] we know that if  $q$  is not a root of unity and  $V$  is the fundamental representation of  $U_q(\mathfrak{sl}(r))$  (quantum group of  $\mathfrak{sl}(r)$ ). Then there is an isomorphism

$$\phi : H_n^{(r)}(q) \rightarrow \text{End}_{U_q(\mathfrak{sl}(r))}(V^{\otimes n})$$

where  $H_n^{(r)}(q)$  is the quotient of  $H_n(q)$  with irreducible representations indexed by Young diagrams with at most  $r$  rows.

By the onto homomorphism  $\rho_{f,n} : H_n(q, -q^{r_1+m}) \rightarrow p_{t^\lambda} H_{n+f}(q) p_{t^\lambda}$ ; and the fact that the image of  $\phi(p_{t^\lambda})$  is  $V_{[m^{r_1}]}$ , We have the following representation

$$H_n^{(r)}(q, -q^{r_1+m}) \rightarrow \text{End}_{U_q(\mathfrak{sl}(r))}(V_{[m^{r_1}]} \otimes V^{\otimes n})$$

where  $V_{[m^{r_1}]}$  is the  $U_q(\mathfrak{sl}(r))$ -module of highest weight  $[m^{r_1}]$ . This isomorphism is given by composing the surjective homomorphism  $\rho_{n,f}$  and the isomorphism  $\phi$ .

The Markov trace defined in Section 4,  $\text{tr}$ , on the Hecke algebras defines a Markov trace on the centralizer algebras  $\text{End}_{U_q(\mathfrak{sl}(r))}(V_{[m^{r_1}]} \otimes V^{\otimes n})$ . We will denote this Markov trace by  $\text{tr}$  again.

We now let  $q$  be an  $l$ -th root of unity. We outline some of the results and definitions about  $U_q(\mathfrak{sl}(r))$ -modules as necessary for our purpose, see [W3] and [A] for the details. A *tilting module* of  $U_q(\mathfrak{sl}(r))$  is a direct summand of a tensor power of the fundamental module  $V$ , or if it is a direct sum of such modules.

Tilting modules satisfy the following properties:

- (1) Tensor products of tilting modules are tilting modules.
- (2) Any tilting module is isomorphic to a direct sum of indecomposable tilting modules.

Each indecomposable tilting module has a  $q$ -dimension. If  $q$  is a root of unity, this  $q$ -dimension can be zero. An indecomposable tilting module with 0  $q$ -dimension will be called *negligible*. The indecomposable negligible modules generate a tensor ideal, which we will denote by  $\mathcal{N}eg(T)$ .

Thus, let  $W_1$  and  $W_2$  be two tilting modules, we now define a tensor product  $\bar{\otimes}$  as follows:

$$W_1 \bar{\otimes} W_2 = (W_1 \otimes W_2) / \mathcal{N}eg(W_1 \otimes W_2)$$

where  $\mathcal{N}eg(W_1 \otimes W_2)$  is defined as follows: First decompose  $W_1 \otimes W_2$  into indecomposable tilting modules then throw away the negligible ones. Using this tensor product we have the following representation of the Hecke algebras at roots of unity. Let  $V$  be the fundamental module of  $U_q(\mathfrak{sl}(r))$

$$H_n^{(r,l)}(q) \rightarrow \text{End}_{U_q(\mathfrak{sl}(r))}(V^{\bar{\otimes} n})$$

and also

$$H_n^{(r,l)}(q, -q^{r_1+m}) \rightarrow \text{End}_{U_q(\mathfrak{sl}(r))} \left( V_{[m^{r_1}]} \bar{\otimes} V^{\bar{\otimes} n} \right).$$

Thus we have that

$$A_n \cong \text{End}_{U_q(\mathfrak{sl}(r))}(1 \bar{\otimes} V^{\bar{\otimes} n})$$

and

$$B_n \cong \text{End}_{U_q(\mathfrak{sl}(r))}(V_{[m^{r_1}]} \bar{\otimes} V^{\bar{\otimes} n})$$

since  $\phi$  is surjective and injective when one restricts to Young diagrams with at most  $r$  rows.

The Markov trace on the Hecke algebras factors over the quotient  $H_n^{(r,l)}(q, -q^{r_1+m})$ , thus the Markov trace on the centralizer algebras also factors over the corresponding quotients.

By Proposition 4.3 in [W3] if we take the above identifications for  $A_n$  and  $B_n$  then we have that the sequence of algebras  $(A_n)$  and  $(B_n)$  satisfy the commuting square property. And the sequences  $(A_n)$  and  $(B_n)$  are periodic.

It is well-known that under the periodicity assumption there exists at most one normalized trace on  $B_\infty = \bigcup_n B_n$ , which must be a factor trace, that is, the weak closure of  $\pi_{\text{tr}}(\bigcup_n B_n)$  must be a factor. Similarly, for  $A_\infty = \bigcup_n A_n$ . Therefore, one obtains a pair of hyperfinite  $\text{II}_1$  factors

$$A = \pi^{(r,l)}(A_\infty)'' \subset B = \pi^{(r,l)}(B_\infty)''.$$

In [W1] Wenzl showed that if a factor is generated by a ladder of commuting squares and if the Bratteli diagrams are periodic, then the index is given as a quotient of the weight vectors of the unique normalized trace,  $\text{tr}$ , i.e., if  $\vec{s}_n$  is the weight vector on  $A_n$  and  $\vec{t}_n$  is the weight vector on  $B_n$  then the index is given by the following formula whenever  $n$  is big enough:

$$(12) \quad [B : A] = \frac{\|\vec{s}_n\|^2}{\|\vec{t}_n\|^2}.$$

**Proposition 6.1.** *Let  $r_1, r_2 \in \mathbb{N}$  and set  $r = r_1 + r_2$ . For each pair  $m, l \in \mathbb{N}$  such that  $m \leq l - r$ ,  $Q = -q^{r_1+m}$  and  $q = e^{2\pi i/l}$ , there is a subfactor of the hyperfinite  $\text{II}_1$  factor obtained from the inclusion  $A \subset B$  with index given by the following formula:*

$$\prod_{i=1}^{r_1} \prod_{j=1}^{r_2} \frac{\sin^2((r_1 + m + j - i)\pi/l)}{\sin^2((r_1 + j - i)\pi/l)}.$$

*Proof.* In [O] we showed that the weights of the Markov trace on  $H_n(q, -q^{r_1+m})$  are given by

$$\frac{s_{\mu,r}(q)}{s_{[m^{r_1}],r}(q)}$$

where  $\mu$  is an  $(r, l)$ -diagram with  $n + f$  boxes containing  $[m^{r_1}]$ . Thus the norm of the weight vector is

$$\|\vec{t}_n\| = \sum_{[m^{r_1}] \subset \mu \vdash n+f} \frac{(s_{\mu,r}(q))^2}{(s_{[m^{r_1}],r}(q))^2}.$$

Now, note that  $H_{f,n+f}(q)$  commutes with  $p_{t^\lambda}^{(l)}$ , thus  $p_{t^\lambda}^{(l)} H_{f,n+f}(q) p_{t^\lambda}^{(l)} = p_{t^\lambda}^{(l)} H_{f,n+f}(q)$ , thus the weight vector is given by

$$\|\vec{s}_n\| = \sum_{[m^{r_1}] \subset \nu \vdash n+f} (s_{\nu,r}(q))^2.$$

Therefore, we have by Wenzl’s index formula in [W1] that the index for this subfactors is

$$\begin{aligned}
 [B : A] &= (s_{[mr_1],r}(q))^2 \\
 &= \prod_{i=1}^{r_1} \prod_{j=1}^{r_2} \frac{\sin^2((r_1 + m + j - i)\pi/l)}{\sin^2((r_1 + j - i)\pi/l)}
 \end{aligned}$$

recall that  $s_{[mr_1],r}(q)$  is the Schur function defined in Section 4. The last equality is obtained by the substitution  $q = e^{2\pi i/l}$  into this Schur function. □

**Remark.** This proposition can also be proved in much more generality using the machinery introduced before the statement of the proposition, see [W3].

It is well-known that the index is not a complete invariant of  $\text{II}_1$  factors. A finer invariant is given by the higher relative commutants. Consider the following tower of  $\text{II}_1$  factors associated to  $A \subset B$

$$A \subset B \subset B^{(1)} = \langle B, e_1 \rangle \subset B^{(2)} = \langle B^{(1)}, e_2 \rangle \subset B^{(3)} = \langle B^{(2)}, e_3 \rangle \dots$$

where  $B^{(i)} = \langle B, e_i \rangle$  is obtained by the basic construction applied to  $B_{i-2} \subset B_{i-1}$ . Since  $[B^{(i)} : A] = [B : A]^{i+1} < \infty$ ,  $[B^{(i)} : B] = [B : A]^i < \infty$ , the higher relative commutants  $A' \cap B^{(i)}$  are all finite dimensional algebras.

The subfactors we have obtained are special cases of the subfactors obtained in [W3] by Wenzl. The higher relative commutants for the subfactors obtained in this paper are given by the following proposition.

**Proposition 6.2.** *Let  $m, r_1, r_2 \in \mathbb{N}$ . Let  $A \subset B$  be the pair of factors constructed above with index as described in the previous proposition, then the higher relative commutants are given by*

$$(13) \quad A' \cap B^{(i)} = \text{End}_{U_q(\mathfrak{sl}(r))}(\cdots V_{[mr_1]} \bar{\otimes} V_{[mr_2]} \bar{\otimes} V_{[mr_1]}) \quad (i + 1 \text{ factors})$$

where  $U_q(\mathfrak{sl}(r))$  is the quantum group of  $\mathfrak{sl}(r)$  and  $V_{[mr_2]} \cong (V_{[mr_1]})^*$ .

*Proof.* The proof of this proposition follows from the proof of Theorem 4.4 in [W3]. We will outline the proof for the reader’s convenience. To compute the higher relative commutants we need to compute the  $i$ -th extension  $B^{(i)}$  via Jones’ basic construction. The computation of  $B^{(i)}$  is done by induction on  $i$ . Let  $B_n^{(1)} = \text{End}_{U_q(\mathfrak{sl}(r))}(V_{[mr_2]} \bar{\otimes} V_{[mr_1]} \bar{\otimes} V^{\bar{\otimes} n})$ . We define the embedding  $B_n^{(1)} \rightarrow B_{n+1}^{(1)}$  via  $B_n^{(1)} \cong B_n^{(1)} \bar{\otimes} Id_V \subset B_{n+1}^{(1)}$ . Let  $B_\infty^{(1)}$  be the inductive limit of the  $B_n^{(1)}$ ’s. Then define  $\tilde{B}^{(1)}$  as the weak closure of the GNS construction with respect to the normalized Markov trace,  $\text{tr}$ , i.e.,  $\tilde{B}^{(1)} = \overline{\pi_{\text{tr}}(B_\infty^{(1)})}^W$ . Then one can check that there is an isomorphism  $\tilde{B}^{(1)} \rightarrow B^{(1)}$ , where  $B^{(1)}$  is



of type  $D$  is by embedding them into the Hecke algebras of type  $B$ , and then use known results for type  $B$ . Hoefsmit [H] observed that in order to obtain an embedding of  $H_n^D(q)$  into  $H_n^B(q, Q)$  we have to set the parameter  $Q$  equal to 1, thus from now on we assume that  $Q = 1$ . Notice that in this case we have  $t^2 = 1$ . The Hecke algebra of type  $D$  is generated by  $u = t\hat{g}_1t, \hat{g}_1, \dots, \hat{g}_{n-1}$  satisfying the following relations:

- (D1)  $\hat{g}_i\hat{g}_{i+1}\hat{g}_i = \hat{g}_{i+1}\hat{g}_i\hat{g}_{i+1}$  for  $i = 1, \dots, n - 2$ ;
- (D2)  $\hat{g}_i\hat{g}_j = \hat{g}_j\hat{g}_i$  whenever  $|i - j| \geq 2$ ;
- (D3)  $\hat{g}_i^2 = (q - 1)\hat{g}_i + q$  for all  $i$ ;
- (D4)  $\hat{g}_i u = u\hat{g}_i$  for all  $i$ ;
- (D5)  $u^2 = (q - 1)u + q$ .

We have  $H_n^D \subset H_n^B$  for all  $n$ ; then  $H_\infty^D = \bigcup_{n \geq 1} H_n^D \subset H_\infty^B$ . Geck [G] has shown that the restriction of a Markov trace on  $H_\infty^B$  is a Markov trace on  $H_\infty^D$  and both have the same parameter. Furthermore, he shows that every Markov trace on  $H_\infty^D$  can be obtained in this way.

From Hoefsmit [H] we know that the simple components for  $H_n^D$  are indexed by double partitions  $(\alpha, \beta)$ . If  $\alpha \neq \beta$  we have that the  $H_n^B$ -modules  $V_{(\alpha, \beta)}$  and  $V_{(\beta, \alpha)}$  are simple, equivalent  $H_n^D$ -modules. And if  $\alpha = \beta$  we have that the  $H_n^B$ -module  $V_{(\alpha, \alpha)}$  decomposes into two simple nonequivalent  $H_n^D$ -modules, i.e.,  $V_{(\alpha, \alpha)_i}$  with  $i = 1, 2$ . Using Bratteli diagrams we have the following relations for simple modules of  $H_n^B$  and  $H_n^D$

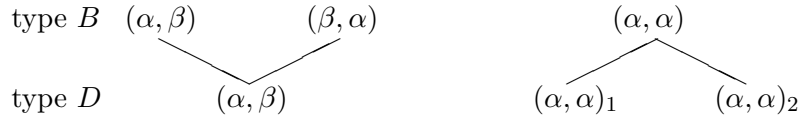


Figure 4.

We now state the proposition about the weights of the Markov trace for the Hecke algebra of type  $D$ . This proposition was proved in [O].

**Proposition 7.1.** *Let  $r_1, r_2 \in \mathbb{N}$  and set  $r = r_1 + r_2$ . Then the weight formula for the Markov trace on the Hecke algebra of type  $D$  with parameters  $z = q^r \frac{(1-q)}{(1-q^r)}$  and  $y = \frac{(Qq^{r^2}+1)(1-q^{r1})}{(1-q^r)} - 1$  is given as follows:*

$$W_{(\alpha, \beta)}^D(q) = W_{(\alpha, \beta)}(q, 1) + W_{(\beta, \alpha)}(q, 1), \quad \text{if } \alpha \neq \beta$$

and

$$W_{(\alpha, \alpha)_i}^D(q) = W_{(\alpha, \alpha)}(q, 1), \quad \text{for } i = 1, 2 \quad \text{if } \alpha = \beta$$

where  $W_{(\alpha, \beta)}(q, 1)$  denote the weight in Section 4 Equation (10) evaluated at  $Q = 1$  of the Hecke algebra of type  $B$ .

Denote the Hecke algebra of type  $A$  by  $H_n^A$ . Now let  $r_1, m \in \mathbb{N}$  and assume that  $q$  is a primitive  $2(r_1 + m)$ -root of unity. This implies that  $Q = -q^{r_1+m} = 1$ . Observe that we have the following inclusion of algebras  $H_n^A \subset H_n^D \subset H_n^B$ . In the previous section we described subfactors obtained from the inclusion  $H_n^A(q) \subset H_n^B(q, -q^{r_1+m})$ . In what follows we would like to consider the subfactors obtained from the inclusions  $H_n^D \subset H_n^B$  and  $H_n^A \subset H_n^D$ .

In Section 5 we showed that there exist  $C^*$ -representations for  $H_n^B(q, -q^{r_1+m})$  with  $r_1 + m < l - r_2$  which holds true for  $l = 2(r_1 + m)$  and  $r_2 < r_1 + m$ . Therefore, we have the following inclusion of hyperfinite  $\text{II}_1$  factors

$$D = \pi^{(l)}(H_\infty^D)'' \subset B = \pi^{(l)}(H_\infty^B)''.$$

**Proposition 7.2.** *Let  $r_1, r_2 \in \mathbb{N}$ . The index for the inclusion of  $D \subset B$  is given as follows:*

$$[B : D] = \begin{cases} 1 & \text{if } r_1 \neq r_2 \\ 2 & \text{if } r_1 = r_2. \end{cases}$$

*Proof.* Choose  $n \gg r_1 + r_2$  and assume  $r_1 \neq r_2$ . Without loss of generality we may assume  $r_1 < r_2$ , then for sufficiently large  $n$  we have pairs of Young diagrams such that  $l(\alpha) = r_1$  and  $l(\beta) = r_2$ . In this case  $W_{(\beta, \alpha)}(q, 1) = 0$ . This implies that the weight vector for type  $B$  is equal to the weight vector of type  $D$ . Thus by Wenzl's index formula  $[B : D] = 1$ .

If  $r_1 = r_2$ , then from Equation (10) in Section 4 we see that  $W_{(\alpha, \beta)}^B(q, 1) = W_{(\beta, \alpha)}^B(q, 1)$  for any pair  $(\alpha, \beta)$ . Thus by the previous proposition we have that  $W_{(\alpha, \beta)}^D(q) = 2W_{(\alpha, \beta)}(q, 1)$ . This implies that

$$[B : D] = \frac{\sum_{\alpha \neq \beta} (2W_{(\alpha, \beta)}^B(q, 1))^2 + \sum_{\alpha = \beta} W_{(\alpha, \alpha)_1}^D(q)^2 + W_{(\alpha, \alpha)_2}^D(q)^2}{\sum_{\alpha \neq \beta} (W_{(\alpha, \beta)}^B(q, 1))^2 + \sum_{\alpha = \beta} W_{(\alpha, \alpha)}^B(q)^2} = 2$$

since  $W_{(\alpha, \alpha)_i}^D(q) = W_{(\alpha, \alpha)}^B(q, 1)$ .  $\square$

**Corollary 7.3.** *Let  $r_1 = r_2 \in \mathbb{N}$ . The index for the inclusion of  $A \subset D$  is given as follows:*

$$[D : A] = [B : A]/2.$$

*Proof.* By Proposition 2.18 in [Jo] we have that if we have an inclusion of three  $\text{II}_1$  factors,  $A \subset D \subset B$  then  $[B : A] = [D : A][B : D]$ . By the previous proposition we have our result.  $\square$

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