

A NOTE ON GENERALIZED BERNOULLI NUMBERS

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In this paper, we consider the zeta function $Z(P, \chi, s)$ associated with a polynomial $P(X) \in \mathbb{R}[X_1, \dots, X_r]$ and $\chi = (\chi_1, \dots, \chi_r)$ with χ_j non-trivial Dirichlet characters, defined by

$$Z(P, \chi, s) = \sum_{n_1=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} \chi_1(n_1) \cdots \chi_r(n_r) P(n_1, \dots, n_r)^{-s},$$

which is absolutely convergent for sufficiently large $\operatorname{Re} s$ under some conditions on $P(X)$. We shall prove that the special value $Z(P, \chi, -m)$ is completely determined by $P^m(X)$ in a simple way. As an immediate application, we give a closed expression for sums of products of any number of generalized Bernoulli numbers.

1. Introduction and Notation.

As usual, \mathbb{N} denotes the set of positive numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and \mathbb{R} denotes the field of real numbers. Let χ be a non-trivial Dirichlet character with conductor N . The L -series attached to χ is defined by

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}, \quad \operatorname{Re} s > 1.$$

It is well known [14] that $L(s, \chi)$ may be continued analytically to the whole complex s -plane. Furthermore, the special values at non-positive integers $s = -m$ ($m = 0, 1, 2, \dots$) can be expressed by the generalized Bernoulli numbers B_χ^n ($n = 0, 1, 2, \dots$) defined by

$$\sum_{a=1}^N \frac{\chi(a) t e^{at}}{e^{Nt} - 1} = \sum_{n=0}^{\infty} \frac{B_\chi^n t^n}{n!}, \quad |t| < \frac{2\pi}{N}.$$

Indeed, $L(-m, \chi) = -\frac{B_\chi^{m+1}}{m+1}$ as given on Page 30 of [14]. The generalized Bernoulli numbers can be expressed in terms of Bernoulli polynomials as

$$B_\chi^n = N^{n-1} \sum_{a=1}^N \chi(a) B_n \left(\frac{a}{N} \right)$$

where the Bernoulli polynomials $B_n(X)$ are defined by

$$\frac{te^{Xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(X) \frac{t^n}{n!}, \quad |t| < 2\pi.$$

Also

$$B_n(X) = \sum_{k=0}^n \binom{n}{k} B_{n-k} X^k$$

where the Bernoulli numbers B_n ($n = 0, 1, 2, \dots$) are defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad |t| < 2\pi.$$

Consequently, we can express the generalized Bernoulli numbers in terms of Bernoulli numbers as follows:

$$B_{\chi}^n = \sum_{a=1}^N \chi(a) \sum_{k=0}^n \binom{n}{k} B_k a^{n-k} N^{k-1}.$$

Let $P(X) = P(X_1, \dots, X_r)$ be a polynomial of r variables with non-negative real coefficients such that $P(n) > 0$ for all $n \in \mathbb{N}^r$ and the series

$$\sum_{n \in \mathbb{N}^r} P(n)^{-s} = \sum_{n_1=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} P(n_1, \dots, n_r)^{-s}$$

is absolutely convergent for $\operatorname{Re} s > \sigma > 0$. χ_1, \dots, χ_r are non-trivial Dirichlet characters with conductors N_1, \dots, N_r , respectively. Consider the zeta function associated with P and $\chi = (\chi_1, \dots, \chi_r)$ defined by

$$Z(P, \chi, s) = \sum_{n_1=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} \chi_1(n_1) \cdots \chi_r(n_r) P(n_1, \dots, n_r)^{-s}, \quad \operatorname{Re} s > \sigma.$$

It is the main purpose of this paper to prove the following result.

Theorem. $Z(P, \chi, s)$ defined above has a meromorphic analytic continuation to the whole complex s -plane. For any integer $m \geq 0$, if

$$P^m(X) = \sum_{|\alpha|=0}^{mp} C_{\alpha} X_1^{\alpha_1} \cdots X_r^{\alpha_r}, \quad p = \deg P,$$

then

$$\begin{aligned} Z(P, \chi, -m) &= \sum_{|\alpha|=0}^{mp} C_{\alpha} L(-\alpha_1, \chi_1) \cdots L(-\alpha_r, \chi_r) \\ &= (-1)^r \sum_{|\alpha|=0}^{mp} C_{\alpha} \prod_{j=1}^r \frac{B_{\chi_j}^{\alpha_j+1}}{\alpha_j + 1}. \end{aligned}$$

Another zeta function $Z(P, \xi, s)$ defined by

$$Z(P, \xi, s) = \sum_{n_1=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} \xi_1^{n_1} \cdots \xi_r^{n_r} P(n_1, \dots, n_r)^{-s}$$

was considered by P. Cassou-Nougès in [2]. Her result for the special values of $Z(P, \xi, s)$ can be restated as

$$Z(P, \xi, -m) = \lim_{t \rightarrow 0^+} \sum_{n_1=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} \xi_1^{n_1} \cdots \xi_r^{n_r} P^m(n) e^{-(n_1 + \cdots + n_r)t}.$$

Here we also have the same formula for the special values of $Z(P, \chi, s)$, i.e.,

$$Z(P, \chi, -m) = \lim_{t \rightarrow 0^+} \sum_{n_1=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} \chi_1(n_1) \cdots \chi_r(n_r) P^m(n) e^{-(n_1 + \cdots + n_r)t}.$$

However $\chi(n) = \prod_{j=1}^r \chi_j(n_j)$ is a multiplicative character while $\xi^n = \prod_{j=1}^r \xi_j^{n_j}$ is an additive character. Hence the treatments are different in some respect. As shown in Section 4, P. Cassou-Nougès' formula for the special values of $Z(P, \xi, s)$ follows from our formula for the special values of $Z(P, \chi, s)$. In addition we have another explicit expression for the special values of $Z(P, \xi, s)$.

A well-known relation among the Bernoulli numbers is

$$\sum_{k=1}^{n-1} \binom{2n}{2k} B_{2k} B_{2n-2k} = -(2n+1) B_{2n}, \quad \text{for } n \geq 2.$$

This was found by many authors, including Euler (ref. [5], [8]). Dilcher [5] generalized the formula for sums of products of any number of both Bernoulli and Euler numbers. Bernoulli and Euler numbers are special cases of the generalized Bernoulli numbers B_χ^n belonging to a residue class character χ . However it is not easy to get the generalized formula for generalized Bernoulli numbers. At the end of this paper, we give a closed expression for the case as an immediate application of our main theorem.

2. Some Basic Results.

We need some classical results reproduced in [15].

Proposition 1. *Suppose that $\varphi(s) = \sum_{\lambda>0} a_\lambda \lambda^{-s}$ (λ ranges over a sequence of positive real numbers tending $+\infty$) is a Dirichlet series converging for sufficiently large $\text{Re } s$. $f(t) = \sum_{\lambda>0} a_\lambda e^{-\lambda t}$ is the corresponding exponential series. Suppose that at $t = 0$, $f(t)$ has the asymptotic expansion*

$$\sum_{n \geq 0} C_n t^{n/p}$$

where p is a fixed positive number. Then:

- (1) $\varphi(s)$ has a meromorphic continuation to the whole complex plane.
- (2) $\varphi(s)$ has possible simple poles at $s = -n/p$, where n is not a multiple of p , with residue $C_n/\Gamma(-n/p)$, and has no other poles.
- (3) $\varphi(-n) = (-1)^n n! C_{np}$.

Note that the above proposition is different from Proposition 2 of [15]. However, it follows from

$$\begin{aligned} \varphi(s)\Gamma(s) &= \int_0^\infty t^{s-1} f(t) dt, \quad \operatorname{Re} s > \sigma \\ &= \int_0^\delta t^{s-1} \sum_{n=0}^\infty C_n t^{n/p} dt + \int_\delta^\infty t^{s-1} f(t) dt \\ &= \sum_{n=0}^\infty C_n \frac{\delta^{s+\frac{n}{p}}}{s+\frac{n}{p}} + \int_\delta^\infty t^{s-1} f(t) dt, \end{aligned}$$

where δ is a small positive number so that $f(t) = \sum_{n=0}^\infty C_n t^{n/p}$. From the above, we get our assertions.

A function $f(x)$ is called a rapidly decreasing function if it belongs to $C^\infty(\mathbb{R}^n)$ and satisfies

$$\lim_{|x| \rightarrow \infty} |x|^k |D^\alpha f(x)| = 0$$

for any α and any integer $k > 0$ (ref. [10], or page 245 in [11]). The following is a consequence of the Euler-Maclaurin summation formula which is also reproduced in [15].

Proposition 2. *Suppose that f is a rapidly decreasing function on $[0, \infty)$ and at $t = 0$, f has the power series expansion*

$$f(t) = \sum_{r=0}^\infty \frac{f^{(r)}(0)}{r!} t^r.$$

Suppose that $g(t) = \sum_{n=1}^\infty f(nt)$. Then at $t = 0$, $g(t)$ has the asymptotic expansion

$$\frac{C}{t} + \sum_{r=0}^\infty (-1)^r \frac{B_{r+1}}{(r+1)!} f^{(r)}(0) t^r \quad \text{with} \quad C = \int_0^\infty f(t) dt.$$

To find the special value at $s = -m$ of the zeta function

$$Z(P, \chi, s) = \sum_{n_1=1}^\infty \cdots \sum_{n_r=1}^\infty \chi_1(n_1) \cdots \chi_r(n_r) P(n)^{-s},$$

by Proposition 1, it is equivalent to find the coefficient of t^m in the asymptotic expansion at $t = 0$ of the function

$$\sum_{n_1=1}^\infty \cdots \sum_{n_r=1}^\infty \chi_1(n_1) \cdots \chi_r(n_r) \exp\{-P(n)t\}.$$

It is also equivalent to find the constant term in the asymptotic expansion at $t = 0$ of the function

$$g(t) = \sum_{n_1=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} \chi_1(n_1) \cdots \chi_r(n_r) P^m(n) \exp\{-P(n)t\}.$$

For the given polynomial

$$P(X) = \sum_{|\alpha|=0}^p A_\alpha X^\alpha, \quad p = \deg P,$$

we let

$$Q(X, Y) = \sum_{|\alpha|=0}^p A_\alpha X^\alpha Y^{p-|\alpha|}$$

be the corresponding homogeneous polynomial in $r+1$ variables. Obviously, $Q(nt, t) = P(n)t^p$ and so

$$\begin{aligned} g(t^p) &= \sum_{n \in \mathbb{N}^r} \chi_1(n_1) \cdots \chi_r(n_r) P^m(n) \exp\{-P(n)t^p\} \\ &= \sum_{n \in \mathbb{N}^r} \chi_1(n_1) \cdots \chi_r(n_r) P^m(n) \exp\{-Q(nt, t)\} \\ &= \sum_{|\alpha|=0}^{mp} C_\alpha \sum_{n \in \mathbb{N}^r} \chi_1(n_1) \cdots \chi_r(n_r) n^\alpha \exp\{-Q(nt, t)\} \end{aligned}$$

where

$$P^m(X) = \sum_{|\alpha|=0}^{mp} C_\alpha X^\alpha \quad \text{and} \quad n^\alpha = n_1^{\alpha_1} \cdots n_r^{\alpha_r}.$$

In the next section, we shall compute the asymptotic expansion at $t = 0$ of the function

$$f_\beta(t) = \sum_{n \in \mathbb{N}^r} \chi_1(n_1) \cdots \chi_r(n_r) n^\beta \exp\{-Q(nt, t)\}.$$

3. The Proof of the [Theorem](#).

First we shall prove the case $r = 1$. Indeed this special case plays an important role in our proof of the [theorem](#).

Lemma 1. *Let P be a polynomial with real coefficients such that $P(n) > 0$ for all $n \in \mathbb{N}$ and Q be the corresponding polynomial defined above. Let*

$h(x, t) = x^\beta \exp\{-Q(xt, t)\}$, N a positive integer and $1 \leq j \leq N$. Furthermore, denote

$$\begin{aligned} f_j(t) &= \sum_{k=0}^{\infty} h(kN + j, t) \\ &= \sum_{k=0}^{\infty} (kN + j)^\beta \exp\{-Q((kN + j)t, t)\}. \end{aligned}$$

Then

$$f_j(t) = h(j, t) + \frac{1}{N} \int_j^\infty h(x, t) dx + \sum_{r=0}^{\infty} \frac{(-1)^r B_{r+1}}{(r+1)!} h^{(r)}(j, t) N^r$$

where $h^{(r)}(x, t)$ is the r -th partial derivative with respect to x .

Proof. It follows from the Euler-Maclaurin summation formula that

$$\begin{aligned} \sum_{k=1}^{\infty} h(kN + j, t) &= \int_0^\infty (Nx + j)^\beta \exp\{-Q((Nx + j)t, t)\} dx \\ &\quad + \sum_{r=0}^{\infty} \frac{(-1)^r B_{r+1}}{(r+1)!} h^{(r)}(j, t) N^r \\ &= \frac{1}{N} \int_j^\infty h(x, t) dx + \sum_{r=0}^{\infty} \frac{(-1)^r B_{r+1}}{(r+1)!} h^{(r)}(j, t) N^r. \end{aligned}$$

□

Proposition 3. Let χ be a non-trivial character with conductor N . Let $\beta \geq 0$ be an integer, and P, Q polynomials as given in the previous lemma. Suppose that

$$f(t) = \sum_{n=1}^{\infty} \chi(n) n^\beta \exp\{-Q(nt, t)\}.$$

Then

$$\begin{aligned} f(t) &= \sum_{j=1}^N \chi(j) h(j, t) - \frac{1}{N} \sum_{j=1}^N \chi(j) \int_0^j h(x, t) dx \\ &\quad + \sum_{j=1}^N \chi(j) \sum_{r=0}^{\infty} \frac{(-1)^r B_{r+1}}{(r+1)!} h^{(r)}(j, t) N^r. \end{aligned}$$

In particular at $t = 0$, $f(t)$ has an asymptotic expansion of the form

$$\sum_{n=0}^{\infty} d_n t^n$$

with the constant term d_0 given by

$$d_0 = -\frac{B_{\chi}^{\beta+1}}{\beta+1} = L(-\beta, \chi).$$

Proof. Note that

$$\begin{aligned} f(t) &= \sum_{j=1}^N \chi(j) \sum_{k=0}^{\infty} (Nk+j)^{\beta} \exp\{-Q((Nk+j)t, t)\} \\ &= \sum_{j=1}^N \chi(j) f_j(t). \end{aligned}$$

So the first assertion follows from Lemma 1 by noting that

$$\int_j^{\infty} h(x, t) dx = \int_0^{\infty} h(x, t) dx - \int_0^j h(x, t) dx$$

and

$$\sum_{j=1}^N \chi(j) \int_j^{\infty} h(x, t) dx = -\sum_{j=1}^N \chi(j) \int_0^j h(x, t) dx$$

since $\sum_{j=1}^N \chi(j) = 0$. Also, from this expression of $f(t)$ we have a power series expansion of the form

$$\sum_{n=0}^{\infty} d_n t^n$$

with

$$\begin{aligned} d_0 &= \sum_{j=1}^N \chi(j) h(j, 0) - \frac{1}{N} \sum_{j=1}^N \chi(j) \int_0^j h(x, 0) dx \\ &\quad + \sum_{j=1}^N \chi(j) \sum_{r=0}^{\infty} \frac{(-1)^r B_{r+1}}{(r+1)!} h^{(r)}(j, 0) N^r \\ &= \sum_{j=1}^N \chi(j) j^{\beta} - \frac{1}{N} \sum_{j=1}^N \chi(j) \frac{j^{\beta+1}}{\beta+1} + \sum_{j=1}^N \chi(j) \sum_{r=0}^{\infty} \frac{(-1)^r B_{r+1}}{(r+1)!} h^{(r)}(j, 0) N^r. \end{aligned}$$

Now it remains to compute $h^{(r)}(j, 0)$. The Leibniz rule for differentiation yields that

$$\begin{aligned} D_x^r h(x, t) &= D_x^r [x^{\beta} \exp\{-Q(xt, t)\}] \\ &= \sum_{u=0}^r \binom{r}{u} D_x^u (x^{\beta}) D_x^{r-u} \exp\{-Q(xt, t)\}. \end{aligned}$$

From the above, we see that

$$D_x^r h(x, t) \Big|_{x=j, t=0} = \begin{cases} \frac{\beta!}{(\beta-r)!} j^{\beta-r}, & \text{if } r \leq \beta; \\ 0, & \text{otherwise.} \end{cases}$$

It follows that

$$\begin{aligned} d_0 &= \sum_{j=1}^N \chi(j) j^\beta - \frac{1}{N} \sum_{j=1}^N \chi(j) \frac{j^{\beta+1}}{\beta+1} \\ &\quad + \sum_{j=1}^N \chi(j) \sum_{r=0}^{\beta} \frac{(-1)^r \beta!}{(r+1)! (\beta-r)!} B_{r+1} N^r j^{\beta-r}. \end{aligned}$$

Note that $B_1 = -\frac{1}{2}$ and $(-1)^r B_{r+1} = -B_{r+1}$ if $r \geq 1$. So

$$\begin{aligned} d_0 &= -\frac{1}{N} \sum_{j=1}^N \chi(j) \frac{j^{\beta+1}}{\beta+1} - \sum_{j=1}^N \chi(j) \sum_{r=0}^{\beta} \frac{\beta!}{(r+1)! (\beta-r)!} B_{r+1} N^r j^{\beta-r} \\ &= -\frac{B_\chi^{\beta+1}}{\beta+1}. \end{aligned}$$

□

Our [theorem](#) is a direct consequence of the following proposition.

Proposition 4. *Let $\chi = (\chi_1, \dots, \chi_r)$, $\beta = (\beta_1, \dots, \beta_r)$, P and Q as given in Section 2. Suppose that*

$$f_\beta(t) = \sum_{n_1=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} \chi_1(n_1) \cdots \chi_r(n_r) n^\beta \exp\{-Q(nt, t)\}.$$

Then $f_\beta(t)$ has an asymptotic expansion of the form

$$\sum_{n=0}^{\infty} d_n t^n$$

with the constant term d_0 given by

$$\begin{aligned} d_0 &= L(-\beta_1, \chi_1) \cdots L(-\beta_r, \chi_r) \\ &= (-1)^r \prod_{j=1}^r \frac{B_{\chi_j}^{\beta_j+1}}{\beta_j+1}. \end{aligned}$$

Proof. We prove the assertion by induction on r . The case $r = 1$ was already proved in the previous proposition. Suppose that $r \geq 2$ and the assertion is true for the case of $r-1$ variables. Consider the case of r variables. Applying

the previous proposition to the first summation of $f_\beta(t)$, where n_1 ranges over all positive integers, we obtain

$$f_\beta(t) = \sum_{j=1}^{N_1} \chi_1(j) j^{\beta_1} h(j, t) - \frac{1}{N_1} \sum_{j=1}^{N_1} \chi_1(j) \int_0^j x^{\beta_1} h(x, t) dx \\ + \sum_{j=1}^{N_1} \chi_1(j) \sum_{r=0}^{\infty} \frac{(-1)^r B_{r+1}}{(r+1)!} \tilde{h}_j^{(r)}(0, t)$$

where

$$h(x, t) = \sum_{n_2=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} \chi_2(n_2) \cdots \chi_r(n_r) n_2^{\beta_2} \cdots n_r^{\beta_r} \\ \cdot \exp\{-Q(xt, n_2t, \dots, n_rt, t)\},$$

and

$$\tilde{h}_j(x, t) = (N_1x + j)^{\beta_1} \sum_{n_2=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} \chi_2(n_2) \cdots \chi_r(n_r) n_2^{\beta_2} \cdots n_r^{\beta_r} \\ \cdot \exp\{-Q((N_1x + j)t, n_2t, \dots, n_rt, t)\} \\ = (N_1x + j)^{\beta_1} h(N_1x + j, t).$$

Note that

$$Q(\alpha t, n_2t, \dots, n_rt, t) = P(\alpha, n_2, \dots, n_r) t^{p'}, \quad p' = \deg P(\alpha, X_2, \dots, X_r)$$

for any fixed number $\alpha > 0$. Applying our induction hypothesis to $h(j, t)$, $h(x, t)$, and $\tilde{h}_j(x, t)$, we get the asymptotic expansion of $f_\beta(t)$, and the constant term d_0 is

$$d_0 = f_\beta(0) \\ = \sum_{j=1}^{N_1} \chi_1(j) j^{\beta_1} h(j, 0) - \frac{1}{N_1} \sum_{j=1}^{N_1} \chi_1(j) \int_0^j x^{\beta_1} h(x, 0) dx \\ + \sum_{j=1}^{N_1} \chi_1(j) \sum_{r=0}^{\infty} \frac{(-1)^r B_{r+1}}{(r+1)!} \tilde{h}_j^{(r)}(0, 0).$$

To compute $\tilde{h}_j^{(r)}(0, 0)$, we use a trick similar to the one in Proposition 3 for computing $h^{(r)}(j, 0)$. The Leibniz rule for differentiation yields that

$$D_x^r \tilde{h}_j(x, t) = \sum_{u=0}^r \binom{r}{u} D_x^u [(N_1x + j)^{\beta_1}] D_x^{r-u} [h(N_1x + j, t)].$$

From the above, we see that

$$D_x^r \tilde{h}_j(x, t) \Big|_{x=0, t=0} = \begin{cases} \frac{\beta_1! N_1^r}{(\beta_1 - r)!} j^{\beta_1 - r} h(j, 0), & \text{if } r \leq \beta_1; \\ 0, & \text{otherwise.} \end{cases}$$

It follows that

$$\begin{aligned} d_0 &= \sum_{j=1}^{N_1} \chi_1(j) j^{\beta_1} h(j, 0) - \frac{1}{N_1} \sum_{j=1}^{N_1} \chi_1(j) \int_0^j x^{\beta_1} h(x, 0) dx \\ &\quad + \sum_{j=1}^{N_1} \chi_1(j) \sum_{r=0}^{\beta_1} \frac{(-1)^r B_{r+1} \beta_1! N_1^r j^{\beta_1 - r}}{(r+1)! (\beta_1 - r)!} h(j, 0). \end{aligned}$$

Since the constant term in the asymptotic expansion of $h(j, t)$ or $h(x, t)$ is

$$(-1)^{r-1} \prod_{j=2}^r \frac{B_{\chi_j}^{\beta_j + 1}}{\beta_j + 1},$$

we have

$$\begin{aligned} d_0 &= (-1)^{r-1} \prod_{j=2}^r \frac{B_{\chi_j}^{\beta_j + 1}}{\beta_j + 1} \left[\sum_{j=1}^{N_1} \chi_1(j) j^{\beta_1} - \frac{1}{N_1} \sum_{j=1}^{N_1} \chi_1(j) \frac{j^{\beta_1 + 1}}{\beta_1 + 1} \right. \\ &\quad \left. + \sum_{j=1}^{N_1} \chi_1(j) \sum_{r=0}^{\beta_1} \frac{(-1)^r \beta_1!}{(r+1)! (\beta_1 - r)!} B_{r+1} N^r j^{\beta_1 - r} \right] \\ &= (-1)^{r-1} \prod_{j=2}^r \frac{B_{\chi_j}^{\beta_j + 1}}{\beta_j + 1} \cdot \left(-\frac{B_{\chi_1}^{\beta_1 + 1}}{\beta_1 + 1} \right) \\ &= (-1)^r \prod_{j=1}^r \frac{B_{\chi_j}^{\beta_j + 1}}{\beta_j + 1}. \end{aligned}$$

This proves our assertions. \square

Corollary. *Suppose that*

$$F(t) = \sum_{n \in \mathbb{N}^r} \chi_1(n_1) \cdots \chi_r(n_r) P^m(n) e^{-(n_1 + \cdots + n_r)t}, \quad t > 0$$

then

$$Z(P, \chi, -m) = \lim_{t \rightarrow 0^+} F(t).$$

Proof. From the notation in our main [theorem](#) it follows that

$$F(t) = \sum_{|\beta|=0}^{mp} C_\beta F_\beta(t)$$

with

$$\begin{aligned} F_\beta(t) &= \sum_{n \in \mathbb{N}^r} \chi_1(n_1) \cdots \chi_r(n_r) n^\beta e^{-(n_1 + \cdots + n_r)t} \\ &= \prod_{j=1}^r \left[\sum_{n=1}^{\infty} \chi_j(n) n^{\beta_j} e^{-nt} \right]. \end{aligned}$$

From

$$\begin{aligned} \sum_{n=1}^{\infty} \chi_j(n) e^{-nt} &= \sum_{k=0}^{\infty} \sum_{a=1}^{N_j} \chi_j(a) e^{-(a+kN_j)t} \\ &= \sum_{a=1}^{N_j} \frac{\chi_j(a) e^{-at}}{1 - e^{-N_j t}} \\ &= \sum_{n=1}^{\infty} \frac{-B_{\chi_j}^n(-t)^{n-1}}{n!} \end{aligned}$$

and differentiating term-by-term β_j times with respect to t , we get

$$\sum_{n=1}^{\infty} \chi_j(n) n^{\beta_j} e^{-nt} = \sum_{n=\beta_j+1}^{\infty} \frac{-B_{\chi_j}^n(-t)^{n-\beta_j-1}}{n \cdot (n - \beta_j - 1)!}.$$

Consequently we have

$$\lim_{t \rightarrow 0^+} F_\beta(t) = \prod_{j=1}^r \left(-\frac{B_{\chi_j}^{\beta_j+1}}{\beta_j + 1} \right)$$

and hence our assertion follows. \square

4. A Consequence.

Let $P(X) \in \mathbb{R}[X_1, \dots, X_r]$ be a polynomial as given before and $\xi = (\xi_1, \dots, \xi_r) \in \mathbb{C}^r$ such that $|\xi_j| = 1$ and $\xi_j \neq 1$ for all j . In 1982, P. Cassou-Noguès considered the zeta function

$$\begin{aligned} Z(P, \xi, s) &= \sum_{n \in \mathbb{N}^r} \xi^n P(n)^{-s} \\ &= \sum_{n_1=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} \xi_1^{n_1} \cdots \xi_r^{n_r} P(n)^{-s}, \quad \operatorname{Re} s > \sigma, \end{aligned}$$

and she proved that

$$Z(P, \xi, -m) = R(P^m)(\xi)$$

where

$$R(P^m)(T) = \sum_{n \in \mathbb{N}^r} P^m(n) T^n$$

which is a power series and can be realized as a rational function in T .

Here we change the dummy variable n and reformulate the above result so that we can use our [theorem](#) to give a new proof.

Theorem (P. Cassou-Noguès). *Suppose that*

$$Z(P, \xi, s) = \sum_{n_1=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} \xi_1^{n_1} \cdots \xi_r^{n_r} P(n_1, \dots, n_r)^{-s}, \quad \text{Re } s > \sigma.$$

Then $Z(P, \xi, s)$ has a meromorphic continuation to the whole complex s -plane and for any integer $m \geq 0$,

$$Z(P, \xi, -m) = \lim_{t \rightarrow 0^+} \sum_{n \in \mathbb{N}^r} \xi^n P^m(n) e^{-(n_1 + \cdots + n_r)t}.$$

Proof. Recall that in the proof of our result, we use only the following two properties of the Dirichlet characters χ_1, \dots, χ_r .

- (1) χ_j is a periodic function, $\chi_j(n + N_j) = \chi_j(n)$ for all $n \in \mathbb{N}$.
- (2) χ_j is non-trivial and $\sum_{a=1}^{N_j} \chi_j(a) = 0$.

Thus, in particular, it works for the case $\chi_j(n) = e^{2\pi i n / N_j}$ or in general $\chi_j(n) = e^{2\pi i n \eta_j}$, η_j is a positive rational number such that $0 < \eta_j < 1$.

Now we suppose that $\xi = (\xi_1, \dots, \xi_r) = (e^{2\pi i q_1}, \dots, e^{2\pi i q_r})$ with $0 < q_j < 1$

1. Let $\eta_k = (\eta_1^{(k)}, \dots, \eta_r^{(k)})$ be a sequence of r -tuples of rational numbers such that

- (1) $0 < \eta_j^{(k)} < 1$ for all $1 \leq j \leq r$, $k \geq 1$,
- (2) $\lim_{k \rightarrow \infty} \eta_k = \xi$.

Consider the sequence of zeta functions $\{Z_k\}$ defined by

$$Z_k(P, \eta_k, s) = \sum_{n \in \mathbb{N}^r} \eta_k^n P(n)^{-s}, \quad \text{Re } s > \sigma.$$

On the half-plane $\text{Re } s > \sigma$, we have

$$\lim_{k \rightarrow \infty} Z_k(P, \eta_k, s) = Z(P, \xi, s).$$

Also all the zeta function $Z_k(P, \eta_k, s)$ and $Z(P, \xi, s)$ have analytic continuation to the whole complex s -plane. So that

$$\lim_{k \rightarrow \infty} Z_k(P, \eta_k, -m) = Z(P, \xi, -m).$$

By our result

$$Z_k(P, \eta_k, -m) = \lim_{t \rightarrow 0^+} \sum_{n \in \mathbb{N}^r} \eta_k^n P^m(n) e^{-(n_1 + \cdots + n_r)t},$$

it follows that

$$Z(P, \xi, -m) = \lim_{t \rightarrow 0^+} \sum_{n \in \mathbb{N}^r} \xi^n P^m(n) e^{-(n_1 + \cdots + n_r)t}.$$

□

The special values $Z(P, \xi, -m)$ can be expressed in terms of special values of the L -series

$$L_q(s) = \sum_{n=1}^{\infty} e^{2\pi i n q} n^{-s}, \quad \operatorname{Re} s > 1, \quad 0 < q < 1.$$

From

$$\begin{aligned} L_q(s)\Gamma(s) &= \sum_{n=1}^{\infty} e^{2\pi i n q} \int_0^{\infty} t^{s-1} e^{-nt} dt \\ &= \int_0^{\infty} \frac{e^{2\pi i q t s-1}}{e^t - e^{2\pi i q}} dt, \quad \operatorname{Re} s > 1, \end{aligned}$$

we conclude that

$$L_q(-m) = (-1)^m m! \times \left\{ \begin{array}{l} \text{the coefficient of } t^m \text{ in the power series} \\ \text{expansion at } t = 0 \text{ of } \frac{e^{2\pi i q}}{e^t - e^{2\pi i q}} \end{array} \right\}.$$

In other words,

$$\sum_{n=1}^{\infty} e^{2\pi i n q} e^{-nt} = \frac{e^{2\pi i q}}{e^t - e^{2\pi i q}} = \sum_{m=0}^{\infty} \frac{(-1)^m L_q(-m) t^m}{m!}, \quad |t| < 2\pi q.$$

Differentiating the above equality β times with respect to t , we obtain

$$\sum_{n=1}^{\infty} e^{2\pi i n q} n^{\beta} e^{-nt} = \sum_{m=\beta}^{\infty} \frac{L_q(-m) (-t)^{m-\beta}}{(m-\beta)!}.$$

Proposition 5. *Suppose that*

$$P^m(X) = \sum_{|\alpha|=0}^{mp} C_{\alpha} X^{\alpha}.$$

Then

$$Z(P, \xi, -m) = \sum_{|\alpha|=0}^{mp} C_{\alpha} L_{q_1}(-\alpha_1) \cdots L_{q_r}(-\alpha_r).$$

Proof. Note that

$$\begin{aligned} \sum_{n \in \mathbb{N}^r} \xi^n P^m(n) e^{-(n_1 + \cdots + n_r)t} &= \sum_{|\alpha|=0}^{mp} C_{\alpha} \sum_{n \in \mathbb{N}^r} \xi^n n^{\alpha} e^{-(n_1 + \cdots + n_r)t} \\ &= \sum_{|\alpha|=0}^{mp} C_{\alpha} \prod_{j=1}^r \left\{ \sum_{n=1}^{\infty} e^{2\pi i n q_j} n^{\alpha_j} e^{-nt} \right\}. \end{aligned}$$

From

$$\lim_{t \rightarrow 0^+} \sum_{n=1}^{\infty} e^{2\pi i n q_j} n^{\alpha_j} e^{-nt} = L_{q_j}(-\alpha_j),$$

we get our assertion by the previous theorem. \square

Now we give expressions for $L_q(-m)$. From the power series expansion

$$\frac{e^{2\pi i q}}{e^t - e^{2\pi i q}} = \sum_{n=0}^{\infty} \frac{(-t)^n \varepsilon_n(e^{2\pi i q})}{n!(1 - e^{2\pi i q})^{n+1}}, \quad |t| < 2\pi q,$$

where $\varepsilon_n(p) = \sum_{k=1}^n A_{n,k} p^k$ is the Eulerian polynomials, the coefficients $A_{n,k}$ are the Eulerian numbers which are the numbers of permutations of the chain $\{1 < 2 < \cdots < n\}$ with precisely $k - 1$ descents (see, e.g., [4]), we have

$$L_q(-m) = \frac{\varepsilon_m(e^{2\pi i q})}{(1 - e^{2\pi i q})^{m+1}}.$$

Meanwhile, we have the following

Proposition 6. *Suppose that*

$$P^m(X) = \sum_{|\alpha|=0}^{mp} C_{\alpha} X^{\alpha},$$

then

$$Z(P, \xi, -m) = \sum_{|\alpha|=0}^{mp} C_{\alpha} \prod_{j=1}^r \frac{\varepsilon_{\alpha_j}(\xi_j)}{(1 - \xi_j)^{\alpha_j+1}}.$$

5. Sums of Products of Generalized Bernoulli Numbers.

A well-known relation among the Bernoulli numbers is

$$\sum_{k=1}^{n-1} \binom{2n}{2k} B_{2k} B_{2n-2k} = -(2n+1)B_{2n}, \quad \text{for } n \geq 2.$$

This was found by many authors, including Euler (ref. [5], [8]). Dilcher remarked in [5] that it may be of interest to find formulas of the above type for sums of products of generalized Bernoulli numbers. In the following Proposition 7, we give a closed expression for sums of products of generalized Bernoulli numbers.

Proposition 7. *Let r be a positive integer and χ_i be a non-trivial Dirichlet character with conductor N_i , for $i = 1, 2, \dots, r$. Then for any positive*

integer m ,

$$\begin{aligned} & \sum_{\substack{p_1+\dots+p_r=m \\ p_1, \dots, p_r \geq 0}}^m \binom{m}{p_1, \dots, p_r} \frac{B_{\chi_1}^{p_1+1}}{N_1^{p_1+1}(p_1+1)} \cdots \frac{B_{\chi_r}^{p_r+1}}{N_r^{p_r+1}(p_r+1)} \\ &= \sum_{a_1=1}^{N_1} \cdots \sum_{a_r=1}^{N_r} \chi_1(a_1) \cdots \chi_r(a_r) \frac{(-1)^{r-1}}{(r-1)!} \\ & \quad \cdot \sum_{j=0}^{r-1} (-1)^j \left\{ \sum_{k=0}^j \binom{r-1-j+k}{k} s(r, r-j+k) \delta^k \right\} \frac{B_{m+r-j}(\delta)}{m+r-j}, \end{aligned}$$

where $\delta = \frac{a_1}{N_1} + \cdots + \frac{a_r}{N_r}$ and $s(n, k)$ is the Stirling number of the first kind.

Proof. Consider the zeta function

$$Z_r(s) = \sum_{n_1=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} \chi_1(n_1) \cdots \chi_r(n_r) \left(\sum_{j=1}^r \left(\prod_{\substack{i=1 \\ i \neq j}}^r N_i \right) n_j \right)^{-s}.$$

Substitute $n_i = a_i + N_i m_i$ where $a_i = 1, \dots, N_i$ and $m_i \geq 0$ for $i = 1, \dots, r$. Thus $Z_r(s)$ becomes

$$\sum_{a_1=1}^{N_1} \cdots \sum_{a_r=1}^{N_r} \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} \left(\prod_{i=1}^r \chi_i(a_i + m_i N_i) N_i^{-s} \right) \left[\sum_{j=1}^r \left(m_j + \frac{a_j}{N_j} \right) \right]^{-s}.$$

Now we let

$$Z_B(s) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} \left(\prod_{i=1}^r N_i \right)^{-s} \left[\sum_{j=1}^r \left(m_j + \frac{a_j}{N_j} \right) \right]^{-s}.$$

Then we can represent the zeta function $Z_r(s)$ as

$$Z_r(s) = \sum_{a_1=1}^{N_1} \cdots \sum_{a_r=1}^{N_r} \left(\prod_{i=1}^r \chi_i(a_i) \right) Z_B(s).$$

From [8] we know that this zeta function $Z_B(s)$ has an analytic continuation to the whole complex plane, and the special values at non-positive integers $s = -m$ are given by

$$Z_B(-m) = \left(\prod_{i=1}^r N_i^m \right) \sum_{\substack{p_1+\dots+p_r=m+r \\ p_1, \dots, p_r \geq 0}} \frac{m!}{p_1! \cdots p_r!} \prod_{j=1}^r B_{p_j} \left(\frac{a_j}{N_j} \right).$$

Using the following identity ([5], Theorem 3)

$$\sum_{\substack{j_1+\dots+j_r=n \\ j_1, \dots, j_r \geq 0}} \binom{n}{j_1, \dots, j_r} B_{j_1}(x_1) \cdots B_{j_r}(x_r) =$$

$$(-1)^{r-1} r \binom{n}{r} \sum_{j=0}^{r-1} (-1)^j \left\{ \sum_{k=0}^j \binom{r-j-1+k}{k} s(r, r-j+k) y^k \right\} \frac{B_{n-j}(y)}{n-j},$$

where $y = x_1 + \cdots + x_r$ and $s(n, k)$ are Stirling numbers of the first kind, and we can rewrite $Z_B(-m)$ as

$$\frac{(\prod_{i=1}^r N_i^m) (-1)^{r-1}}{(r-1)!} \sum_{j=0}^{r-1} (-1)^j$$

$$\cdot \left\{ \sum_{k=0}^j \binom{r-1-j+k}{k} s(r, r-j+k) \delta^k \right\} \frac{B_{m+r-j}(\delta)}{m+r-j},$$

where $\delta = \frac{a_1}{N_1} + \cdots + \frac{a_r}{N_r}$. Now applying our [theorem](#), the special values at non-positive integers $s = -m$ of the zeta function $Z_r(s)$ are

$$Z_r(-m) = \sum_{\substack{p_1+\dots+p_r=m \\ p_1, \dots, p_r \geq 0}} \binom{m}{p_1, \dots, p_r} \left(\prod_{i=1}^r \frac{N_i^{m-p_i} B_{\chi_i}^{p_i+1}}{p_i+1} \right).$$

On the other hand, using the equality

$$Z_r(-m) = \sum_{a_1=1}^{N_1} \cdots \sum_{a_r=1}^{N_r} \prod_{i=1}^r \chi_i(a_i) Z_B(-m)$$

and the above values of $Z_r(-m)$ and $Z_B(-m)$, we get our assertion. \square

Remark. As special cases we state formulas for sums of products of two, respectively three, generalized Bernoulli numbers.

- (1) Let χ_1, χ_2 be non-trivial Dirichlet characters with conductors N_1, N_2 , respectively. Then for any positive integer m ,

$$\sum_{k=0}^m \binom{m}{k} \frac{B_{\chi_1}^{k+1}}{N_1^k (k+1)} \frac{B_{\chi_2}^{m-k+1}}{N_2^{m-k} (m-k+1)}$$

$$= \sum_{a_1=1}^{N_1} \sum_{a_2=1}^{N_2} \chi_1(a_1) \chi_2(a_2)$$

$$\cdot \left[\frac{\frac{a_1}{N_1} + \frac{a_2}{N_2} - 1}{m+1} B_{m+1} \left(\frac{a_1}{N_1} + \frac{a_2}{N_2} \right) - \frac{B_{m+2} \left(\frac{a_1}{N_1} + \frac{a_2}{N_2} \right)}{m+2} \right].$$

- (2) Let χ_1, χ_2, χ_3 be non-trivial Dirichlet characters with conductors N_1, N_2, N_3 , respectively. Then for any positive integer m , we have

$$\begin{aligned} & \sum_{\substack{p+q+r=m \\ p,q,r \geq 0}} \binom{m}{p, q, r} \frac{B_{\chi_1}^{p+1}}{N_1^p(p+1)} \frac{B_{\chi_2}^{q+1}}{N_2^q(q+1)} \frac{B_{\chi_3}^{r+1}}{N_3^r(r+1)} \\ &= \frac{1}{2} \sum_{a_1=1}^{N_1} \sum_{a_2=1}^{N_2} \sum_{a_3=1}^{N_3} \chi_1(a_1) \chi_2(a_2) \chi_3(a_3) \\ & \quad \cdot \left[(\delta^2 - 3\delta + 2) \frac{B_{m+1}(\delta)}{m+1} + (3 - 2\delta) \frac{B_{m+2}(\delta)}{m+2} + \frac{B_{m+3}(\delta)}{m+3} \right], \end{aligned}$$

$$\text{where } \delta = \frac{a_1}{N_1} + \frac{a_2}{N_2} + \frac{a_3}{N_3}.$$

As a final example we consider the Euler numbers E_n , $0 \leq n < \infty$. We have $E_{2n+1} = 0$, $n \geq 0$, while E_{2n} , $n \geq 0$, is defined by

$$\sec x = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n}, \quad |x| < \frac{\pi}{2}.$$

The Euler numbers are special cases of the generalized Bernoulli numbers B_χ^n belonging to a residue class character χ . In fact we have

$$E_n = -\frac{2B_\eta^{n+1}}{n+1}, \quad n \geq 0,$$

where η is the primitive character with conductor 4. If we let $r = 2$ and the characters χ_1 and χ_2 in Proposition 7 be the same character η (the primitive character with conductor 4), then we get an identity which is a special case of Eq. (4.9) in [5].

Proposition 8. *For a non-negative integer n , we have the following identity:*

$$\sum_{k=0}^n \binom{2n}{2k} E_{2k} E_{2n-2k} = (2^{2n+2} - 1) \frac{2^{2n+2} B_{2n+2}}{2n+2}.$$

Proof. Let $r = 2$ and χ_1, χ_2 as indicated above, i.e., $N_1 = N_2 = 4$. Then

$$\begin{aligned} & 4^{-m} \sum_{k=0}^m \binom{m}{k} \frac{B_\chi^{k+1} B_\chi^{m-k+1}}{(k+1)(m-k+1)} \\ &= \sum_{a=1}^4 \sum_{b=1}^4 \chi(ab) \left[\frac{\frac{a+b}{4} - 1}{m+1} B_{m+1} \left(\frac{a+b}{4} \right) - \frac{B_{m+2}(\frac{a+b}{4})}{m+2} \right] \\ &= -\frac{B_{m+1}(\frac{1}{2})}{2(m+1)} - \frac{B_{m+2}(\frac{1}{2})}{m+2} + \frac{2B_{m+2}(1)}{m+2} + \frac{B_{m+1}(\frac{3}{2})}{2(m+1)} - \frac{B_{m+2}(\frac{3}{2})}{m+2}. \end{aligned}$$

The left-hand side of the above identity is exactly $4^{-m-1} \sum_{k=0}^m \binom{m}{k} E_k E_{m-k}$. Using some basic properties of the Bernoulli polynomials:

$$\begin{aligned} B_n \left(\frac{3}{2} \right) &= 2^{1-n} \cdot n + B_n \left(\frac{1}{2} \right) , \\ B_n \left(\frac{1}{2} \right) &= (2^{1-n} - 1) B_n , \\ B_n(1-x) &= (-1)^n B_n(x), \end{aligned}$$

the right-hand side of the above identity becomes

$$2 \cdot [1 + (-1)^{m+2} - 2^{-m-1}] \frac{B_{m+2}}{m+2}.$$

The result follows by setting $m = 2n$. □

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