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A PROBLEM OF MCMILLAN ON CONFORMAL MAPPINGS

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We answer one of two questions asked by McMillan in 1970 concerning distortion at the boundary by conformal mappings of the disk.

1. Introduction.

The purpose of this note is to answer a question of J.E McMillan concerning boundary behavior of conformal mappings which was raised in the paper [4]. In that paper, McMillan gave a sufficient geometric condition for a subset of the boundary of a domain to have harmonic measure zero and used it to prove a result which we will describe below. A similar geometric lemma was the key to the original proof of the twist point theorem in [5]. The reader can refer to both of McMillan's papers and to [6] for background on these problems and more generally to [1], [3] and [7] for the ideas used in this paper.

We will use $\omega(z_0, F, \Omega)$ to denote the harmonic measure of the set F in the domain Ω from the point z_0. Let $\mathbb{D}$ denote the unit disk in the complex plane and let $f : \mathbb{D} \to \Omega$ be a conformal map. Let A denote the set of all ideal accessible boundary points $f(e^{i\theta})$ of Ω when f has the nontangential limit $f(e^{i\theta})$ at $e^{i\theta}$. Note that points of A are prime ends of Ω so that a single complex coordinate may represent more than one point of A.

Let $D(a,r)$ denote a disk with center a and radius r. Choose $r_0 < d(f(0), A)$ where d denotes Euclidean distance. For each $a \in A$ and for each $r < r_0$ let $\gamma(a,r) \subset \partial D(a,r)$ be the crosscut of Ω seperating a from $f(0)$ which can be joined to a by a Jordan arc in $\Omega \cap D(a,r)$. Let $L(a,r)$ denote the Euclidean length of $\gamma(a,r)$ and let $U(a,r) = \bigcup_{r'<r} \gamma(a,r')$.

Let

$$A(a,r) = \int_0^r L(a,\rho)\, d\rho$$

denote the Lebesgue measure of $U(a,r)$.

McMillan proved:

Theorem 1.1. *The set of $a \in A$ such that*

$$\limsup_{r\to 0} \frac{A(a,r)}{\pi r^2} < \frac{1}{2}$$

has harmonic measure zero.

Notice that this theorem implies that the set of $a \in A$ such that

$$\limsup_{r\to 0} \frac{L(a,r)}{2\pi r} < \frac{1}{2}$$

has harmonic measure zero.

McMillan also gave an example of a domain for which both

$$\limsup_{r\to 0} \frac{A(a,r)}{\pi r^2} = 1 \quad \omega \quad a.e.$$

and

$$\liminf_{r\to 0} \frac{A(a,r)}{\pi r^2} = 0 \quad \omega \quad a.e.$$

(implying the corresponding limits for $\frac{L(a,r)}{2\pi r}$) and conjectured that

$$E_1 = \left\{ a \in A : \liminf_{r\to 0} \frac{A(a,r)}{\pi r^2} > \frac{1}{2} \right\}$$

and

$$E_2 = \left\{ a \in A : \liminf_{r\to 0} \frac{L(a,r)}{2\pi r} > \frac{1}{2} \right\}$$

must be sets of harmonic measure zero.

Here, we will verify McMillan's conjecture that the set E_2 must always have zero harmonic measure.

2. There are no points of density in $f^{-1}(E_2)$.

With the notations and definitions of the introduction we prove:

Theorem 2.1. *The harmonic measure of the set E_2 is zero.*

Proof. For any positive integers m and k, let

$$E_{m,k} = \left\{ a \in A |\, L(a,r) > \left(\frac{1}{2} + \frac{1}{m}\right) 2\pi r \quad \forall r < \frac{1}{k} \right\}.$$

Since E_2 is the countable union of sets $E_{m,k}$, it suffices to show that each $E_{m,k}$ has harmonic measure zero.

We will require the following lemma (see [7], p. 142) which is a consequence of results of Beurling, [2].

Lemma 2.1. *Let f map $\mathbb{D}$ conformally into $\mathbb{C}$ and let $0 < \delta < 1$. If $z \in \mathbb{D}$ and I is an arc of $\mathbb{T}$ with $\omega(z, I) \geq \alpha > 0$ then there exists a Borel set $B \subset I$ with $\omega(z, B) > (1 - \delta)\omega(z, I)$ such that*

$$|f(\xi) - f(z)| \leq \Lambda(f(S)) < K(\delta, \alpha)d_f(z) \qquad \text{for } \xi \in B$$

where Λ denotes linear measure, $d_f(z)$ is the euclidean distance from $f(z)$ to the boundary of $f(\mathbb{D})$, S is the non-euclidean segment from z to ξ and where $K(\delta, \alpha)$ depends only on δ and α.

The basic idea of the proof of Theorem 2.1 is that since points of $E_{m,k}$ are separated from $f(0)$ by circular arcs of wide angle and large radius, if $f^{-1}(E_{m,k})$ has a point of density then Lemma 2.1 will provide enough wide angled circular arcs of a fixed radius to wrap around on themselves and disconnect the domain Ω.

Suppose then that $\eta \in \mathbb{T}$ is a point of density of $f^{-1}(E_{m,k})$ and let I denote an arc of $\mathbb{T}$ centered at η.

Given $\delta_1 > 0$ we can choose I such that

$$\frac{|f^{-1}(E_{m,k}) \cap I|}{|I|} > (1 - \delta_1). \tag{1}$$

Given $\delta_2 > 0$ we can find $0 < r(I, \delta_2) < 1$ such that

$$\omega((1 - r(I, \delta_2))\eta, I, \mathbb{D}) = 1 - \delta_2$$

and this determines the point $z_I = (1 - r(I, \delta_2))\eta$.

If we are given $\delta_3 > 0$ then if δ_1 is sufficiently small, (1) implies that

$$\omega(f^{-1}(E_{m,k}), z_I, \mathbb{D}) > (1 - \delta_3).$$

By Lemma 2.1, if we are given $\delta_4 > 0$ then there is a Borel set $B \subset I$ such that

$$\omega(z_I, B, \mathbb{D}) > (1 - \delta_4)(1 - \delta_2)$$

and such that

$$|f(\xi) - f(z_I)| < K(\delta_4, (1 - \delta_2))d_f(z_I) \quad \forall \xi \in B. \tag{2}$$

It follows that

$$\omega(f^{-1}(E_{m,k}) \cap B, z_I, \mathbb{D}) > 1 - (\delta_2 + \delta_3 + \delta_4 - \delta_2\delta_4) \tag{3}$$

and that (2) holds for all $\xi \in f^{-1}(E_{m,k}) \cap B$. Notice that the constant K only depends on δ_2 and δ_4.

Since $f(\eta) \in A$ we can choose I so that $Kd_f(z_I) << \frac{1}{k}$ where k is the integer in the definition of $E_{m,k}$. The finite number of steps required to get a contradiction in the construction to follow will only depend on the number m in the definition of $E_{m,k}$. By choosing a sufficiently small arc I, we can arrange that in each step of our construction, the positive number

$$\delta \equiv \delta_2 + \delta_3 + \delta_4 - \delta_2\delta_4$$

is small enough so that the construction can proceed to the next step. We assume that these conditions hold on the size of the interval I.

Let $w_0 = f(0)$, $w_1 = f(z_I)$, $d_0 = d_f(z_I)$ and let x_1 be a point of $\partial\Omega$ such that $|x_1 - w_1| = d_0$. Let the letters $c_1, c_2, \dots$ denote positive constants which will be assumed to be sufficiently small in each step of the construction but will ultimately depend only on the number m in the definition of the set $E_{m,k}$ and not on f, Ω, or δ. Let $C_1, C_2, \dots$ denote other constants which may be purely numerical or which may depend only on the number m.

First let $0 < c_0 \ll 1$ and $c_1 \ll \frac{\pi}{m} c_0$. We will see that these choices allow for rotation by a fixed positive angle of certain separating circular arcs in consecutive steps of the construction to follow. The arc of $\partial D(x_1, c_1 d_0)$ which intersects the interior of $D(w_1, d_0)$ extends to a crosscut of Ω and determines a unique subdomain $U_1 \subset \Omega$ not containing w_1. We proceed to find a point close to x_1 which is contained in $E_{m,k}$. By Harnack's inequality,

$$\omega(w_1, \partial U_1 \cap \partial\Omega \cap D(x_1, c_1 d_0), \Omega) \geq C_1 \omega(w_1', \partial U_1 \cap \partial\Omega \cap D(x_1, c_1 d_0), \Omega)$$

where w_1' is the point on the line between w_1 and x_1 such that $|x_1 - w_1'| = \frac{c_1 d_0}{2}$. By the comparison principle for harmonic measure and the Beurling projection theorem, ([1], p. 43),

$$\omega(w_1', \partial U_1 \cap \partial\Omega \cap D(x_1, c_1 d_0), \Omega) \geq C_2 > 0.$$

So by Lemma 2.1 and Equation (3), if δ is sufficiently small, $(\delta << C_1 C_2)$, there is a constant C_3 such that

$$\omega(w_1, \partial U_1 \cap \partial\Omega \cap D(x_1, c_1 d_0) \cap E_{m,k}, \Omega) \geq C_3 > 0.$$

Choose a point $x_1^* \in \partial U_1 \cap \partial\Omega \cap D(x_1, c_1 d_0) \cap E_{m,k}$. If c_0 is sufficiently small then the arc of $\{z \in \mathbb{C} : |x_1^* - z| = c_0 d_0\}$ which intersects $D(w_1, d_0)$ has an angle greater than $\pi(1 - \frac{1}{2m})$. This arc must therefore be part of the crosscut whose length is $L(x_1^*, c_0 d_0) > \pi(1 + \frac{1}{m})$. Denote by $\overline{ab}$ the segment with endpoints $a \in \mathbb{C}$ and $b \in \mathbb{C}$. Let w_1^* be the point on $\overline{x_1^* w_1}$ with $|x_1 - w_1^*| = c_0 d_0$ and consider the annulus

$$R_1 = \{z \in \mathbb{C} : (1 - c_2)|x_1^* - w_1^*| < |x_1^* - z| < (1 + c_2)|x_1^* - w_1^*|\}$$

where $c_2 \ll \frac{\pi}{m} c_0$. Let S_1 be the component of $R_1 \cap \Omega$ which intersects $D(w_1, d_0)$ and let x_2 be a point of $\partial\Omega \cap S_1$ such that $\overline{x_1^* x_2}$ has minimal angle clockwise from $\overline{x_1^* w_1^*}$.

Let S_1^* denote the sector of R_1 clockwise between $\overline{x_1^* w_1^*}$ and $\overline{x_1^* x_2}$. The circular arc $\partial D(x_2, c_2 d_0) \cap S_1^*$ is part of a crosscut of Ω which determines a unique subdomain U_2 of Ω not containing w_1^*. By an argument similar to the previous one using Harnack's inequality, the comparison principle for harmonic measure and the Beurling projection theorem but now in the annular sector S_1, it follows that

$$\omega(w_1, \partial\Omega \cap \partial U_2 \cap D(x_2, c_2 d_0), \Omega) > C_4 > 0.$$

We remark that C_4 depends on c_0, c_1, c_2 and therefore only on m and that the remaining constants C_j may have similar dependence on m.

A simple geometric argument shows that there is a point x_2^* in $D(x_2, c_2d_0)$ $\cap E_{m,k}$ and a constant $c_3 > 0$ determined by the diameter of the $E_{m,k} \cap D(x_2, c_2d_0)$ such that the set of distances

$$\{|x_2^* - w| : w \in D(x_1^*, c_1d_0) \cap \partial\Omega\}$$

contains an interval J_1 of length greater than c_3d_0.

Let $R_2 = \{w \in \mathbb{C} : |w - x_2^*| \in J_1\}$ and let S_2 be the component of $R_2 \cap \Omega$ which intersects S_1. Each of the circular arcs of S_2 centered at x_2^* is a crosscut of Ω. If there is such a crosscut $L_1 \subset S_2 \cap \Omega$ which does not separate x_2^* from w_0 then we repeat the above construction of S_2 but in the counterclockwise direction from $\overline{x_1^*w_1^*}$. Then any circular arc $L_2 \subset S_2 \cap \Omega$ centered at x_2^* which intersects S_1, separates x_2^* from w_0. For otherwise, w_0 is contained in both subdomains of Ω determined by the concave sides of L_1 and L_2. Since w_0 lies on the convex side of any circular arc which defines $L(a, r)$ for some $a \in A$ and $r > 0$ and therefore of any arc of S_1, this is impossible. If one choice of S_2, clockwise or counterclockwise from $\overline{x_1^*w_1^*}$, fails to separate x_2^* from w_0 we choose the other. Otherwise, the construction can continue, as described below, in both directions until the non-separating case occurs and after that point, a topological argument similar to the above allows the construction to continue in the remaining direction.

We have now arranged that each of the circular arcs of S_2 centered at x_2^* separates x_2^* from w_0 and can be joined to x_2^* by a Jordan arc lying inside S_1. Therefore, since $x_2^* \in E_{m,k}$, each circular arc of S_2 has an angular measure greater than $(1 + \frac{2}{m})\pi$. Let w_2 be a point of $S_2 \cap S_1$ and let x_3 be a point of $\partial\Omega \cap \overline{S_2}$ which minimizes the clockwise angle from $\overline{x_2^*w_2}$ to $\overline{x_2^*x_3}$. Let S_2^* denote the sector of R_2 clockwise between $\overline{x_2^*w_2}$ and $\overline{x_2^*x_3}$. As before the circular arc $\partial D(x_3, c_3d_0) \cap S_2^*$ extends to a crosscut of Ω which determines a unique subdomain of $U_3 \subset \Omega$ not containing w_1. The same harmonic measure argument as before but now done in the union of annular corridors $S_1 \cup S_2$ shows that

$$\omega(w_1, \partial\Omega \cap \partial U_3 \cap D(x_3, c_3d_0), \Omega) > C_6 > 0.$$

If $\delta > 0$ is sufficiently small, then as before, Lemma 2.1 and (3) imply that

$$\omega(w_1, \partial\Omega \cap \partial U_3 \cap D(x_3, c_3d_0) \cap E_{m,k}, \Omega) > C_7 > 0$$

and we find $x_3^* \in \partial\Omega \cap \partial U_3 \cap D(x_3, c_3d_0) \cap E_{m,k}$ such that the set of distances

$$\{|x_3^* - w| : w \in D(x_2^*, c_3d_0) \cap \partial\Omega\}$$

contain an interval J_3 of length greater than c_4d_0, where c_4 depends only on the previous c_i and on m. Note that since the constants satisfy $c_i << c_0\frac{\pi}{m}$, there is a numerical constant $c > 0$ such that the clockwise angle from $\overline{x_1^*x_2^*}$

to $\overline{x_2^* x_3^*}$ is at least $(1+\frac{c}{m})\pi$. The construction continues in this way so that having found annular corridors $S_1, \dots S_j$ with centers $x_1^*, x_2^*, \dots x_j^*$ we find $x_{j+1}^* \in E_{m,k}$ so that there is an interval of distances J_j between x_{j+1}^* and the part of $\partial\Omega$ in a disk of radius $c_\ell d_0$ centered at x_j^*. The intersection of the annulus centered at x_{j+1}^* determined by J_j with Ω contains a component S_{j+1} which intersects S_j. Concentric circular arcs of this annular piece separate x_{j+1}^* from w_0 (or else the construction continues in the other direction) and each such circular arc can be joined to x_{j+1}^* through the annular corridor S_j by a Jordan arc contained in the circle. Therefore, each such arc has an angle greater than $(1+\frac{2}{m})\pi$. Let w_{j+1} be a point of $S_{j+1} \cap S_j$ and find x_{j+2} which minimizes the clockwise angle between $\overline{x_{j+1}^* w_{j+1}}$ and $\overline{x_{j+1}^* x_{j+2}}$. The construction can continue if $\delta > 0$ is sufficiently small since the harmonic measure of the end of S_{j+1} near x_{j+2} from w_1 in $S_1 \cup S_2 \cup \dots S_{j+1} \cup D(w_1, d_0)$ is greater than some positive numerical constant.

But it is clear from the construction that the union of annular corridors $S_1 \cup \dots \cup S_j$ must wrap around on itself after a finite number of steps which only depends on m. The union of annular corridors thus formed, being a subset of Ω, would contain a closed curve in Ω whose interior component contains the points $x_i^* \in \partial\Omega$. Since Ω is simply connected, this contradiction shows that $f^{-1}(E_{m,k})$ does not contain a point of density and therefore must have measure zero. Therefore $E_{m,k}$ has harmonic measure zero in Ω. □

Note. The authors have now answered the question left open here. The result will appear in a forthcoming paper.

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