

DECOMPOSITION OF DOUBLE COSETS IN \mathfrak{p} -ADIC GROUPS

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Let G be a split group over a locally compact field F with non-trivial discrete valuation. Employing the structure theory of such groups and the theory of Coxeter groups, we obtain a general formula for the decomposition of double cosets $P_1\sigma P_2$ of subgroups $P_1, P_2 \subset G(F)$ containing an Iwahori subgroup into left cosets of P_2 . When P_1 and P_2 are the same hyperspecial subgroup, we use this result to derive a formula of Iwahori for the degrees of elements of the spherical Hecke algebra.

1. Introduction.

Let G be a semisimple algebraic group which is split over a locally compact field F with non-trivial discrete valuation and let I be an Iwahori subgroup of $G(F)$. In [6], Iwahori and Matsumoto show that the double cosets in $I \backslash G(F) / I$ are indexed by the elements of the extended affine Weyl group \tilde{W} of G , and for w in \tilde{W} , they exhibit an explicit set of representatives for the left cosets of I in IwI/I . They also show that the number of single cosets of I in IwI/I is $q^{l(w)}$, where q is the cardinality of the residue field of F and l is the standard combinatorial length function on \tilde{W} .

Let \mathcal{O}_F be the ring of integers of F and let $K \subset G(F)$ be a hyperspecial subgroup, that is, a subgroup isomorphic to $\underline{G}(\mathcal{O}_F)$, where \underline{G} is a smooth group scheme over \mathcal{O}_F with general fiber G . In [5], Iwahori gives a formula for the number of left cosets of K contained in a double coset in $K \backslash G(F) / K$ (i.e., the degree of the characteristic function of this double coset as an element of the spherical Hecke algebra), implicitly making use of the decomposition in [6] and the fact that K contains an Iwahori subgroup. Suppose that π is a uniformizer of F . The double cosets in $K \backslash G(F) / K$ are indexed by the dominant co-characters of a maximal torus of G via the bijection $\lambda \mapsto K\lambda(\pi)K$. Let W_0 be the Weyl group of G and W_0^λ the stabilizer of λ in W_0 . Then there exists a unique set $[W_0/W_0^\lambda] \subset W_0$ of representatives of cosets of W_0^λ of minimal length. Iwahori states that the index

$$[K\lambda(\pi)K : K] = q^{l(\sigma_\lambda)} \sum_{\tau \in [W_0/W_0^\lambda]} q^{l(\tau)},$$

where σ_λ is a certain element of \tilde{W} associated with the co-character λ (see Section 6).

In this paper, we give a summary of the above results and generalize them, finding representatives for the left cosets of P_2 in $P_1\sigma P_2$, where P_1 and P_2 are subgroups of $G(F)$ containing an Iwahori subgroup I . To accomplish this, we make use of the structure theory of groups over fields with discrete valuation (as summarized in Section 3). Because of the close connection between subgroups of $G(F)$ of the above type and subgroups of the affine Weyl group (as given in [6]), the proof of the coset decomposition formula necessitates the use of the theory of Coxeter groups to prove certain results about the additivity of lengths of elements of \tilde{W} (Section 4). In Section 5, we find the coset representatives mentioned above and give a formula for their number when the groups P_1 and P_2 are compact. In addition, we give several examples of this coset decomposition and an explanation of how our general results imply Iwahori's in the case $P_1 = P_2 = K$ (Section 6).

This information on the decomposition of double cosets is useful in computing the action of Hecke operators on spaces of modular forms as defined in [3]. In fact, the results of this paper were used in [8] when G is a compact form of G_2 or PGSp_4 over \mathbb{Q} and $P_1 = P_2 = K$ to compute the action on certain spaces of forms of the spherical Hecke algebra of functions on $G(\mathbb{Q}_p)$ bi-invariant by the subgroup K for several primes p at which G is split. The action of a function in this algebra is given by integrating it against a form f with respect to a Haar measure on $G(\mathbb{Q}_p)$. This integral turns out to be a finite sum. Indeed, when the Hecke operator is the characteristic function of a double coset $K\sigma K$ of K , the integral is simply the sum over the right translations of f by a set of representatives of the left cosets of K inside $K\sigma K$. Since the Hecke algebra is generated by such characteristic functions, the decomposition of double cosets is therefore fundamental to the explicit determination of the actions these algebras.

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2. Notation.

In the following, we will denote by G a connected semisimple algebraic group that is split over a locally compact field F with non-trivial discrete valuation. Let \mathcal{O}_F be the ring of integers of F and let \mathfrak{p} be the prime ideal. We choose a uniformizing parameter π in \mathfrak{p} , and denote by k the residue field $\mathcal{O}_F/\mathfrak{p}$. Let q be the (finite) cardinality of k and let $R \subset \mathcal{O}_F$ be a set of representatives for k containing 0. The group G is the general fiber of a Chevalley group scheme \underline{G} over \mathcal{O}_F whose special fiber is semisimple. We let

$K = \underline{G}(\mathcal{O}_F) \subset \underline{G}(F) = G(F)$ be the set of integral points of G . K is then a hyperspecial maximal compact subgroup of $G(F)$ (cf. [9, 3.8.1, 3.8.2]).

Let $\underline{T} \subset \underline{G}$ be a split maximal torus scheme, and let T be its general fiber. We define N_T to be the normalizer of T in G . Denote by $X^*(T)$ the character module $\text{Hom}(T, \mathbb{G}_m)$ of T and by $X_*(T)$ the co-character module $\text{Hom}(\mathbb{G}_m, T)$ of T . Let $\Phi \subset X^*(T)$ be the set of roots of T , $\Phi^+ \subset \Phi$ a subset of positive roots, and $\Delta \subset \Phi^+$ the corresponding set of simple roots. Also, let $\Phi^\vee \subset X_*(T)$ be the coroots of T and $\alpha \mapsto \alpha^\vee$ the standard bijection between Φ and Φ^\vee . For each $\alpha \in \Phi$ let \underline{U}_α be the one-dimensional unipotent subgroup scheme of \underline{G} corresponding to α . Denote the general fiber of \underline{U}_α by U_α . We choose for each α an isomorphism

$$x_\alpha : \mathbb{G}_a \longrightarrow U_\alpha.$$

When considered as a map $F \longrightarrow \underline{U}_\alpha(F)$, x_α restricts to an isomorphism of \mathcal{O}_F with $\underline{U}_\alpha(\mathcal{O}_F) = U_\alpha(F) \cap K$.

We will denote by W_0 the Weyl group $N_T/T = (N_T(F) \cap K)/\underline{T}(\mathcal{O}_F)$ of G and by \tilde{W} the extended affine Weyl group $N_T(F)/\underline{T}(\mathcal{O}_F)$. Then W_0 and \tilde{W} act as groups of affine transformations on the space $X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$. The stabilizer in \tilde{W} of $0 \in X_*(T) \otimes \mathbb{R}$ is W_0 , and there is an isomorphism $\tilde{W} \cong X_*(T) \rtimes W_0$, where $X_*(T)$ is embedded in \tilde{W} as the group of elements acting as translations on $X_*(T) \otimes \mathbb{R}$. We denote by e the identity element of \tilde{W} and by $t(\lambda)$ the element of \tilde{W} corresponding to λ in $X_*(T)$. In this notation, if $w \in W_0$ and $\lambda \in X_*(T)$ then

$$wt(\lambda)w^{-1} = t(w\lambda).$$

Denote by w_α the reflection in W_0 through the vanishing hyperplane in $X_*(T) \otimes \mathbb{R}$ of the root α . The Weyl group W_0 is a Coxeter group with $S_0 = \{w_\alpha \mid \alpha \in \Delta\}$ as its set of involutive generators. Let $\Phi = \Phi_1 \cup \dots \cup \Phi_m$ be the decomposition of Φ into irreducible root systems. (Each Φ_i corresponds to the root system of an almost simple factor of G .) Also, let $\Delta_i = \Delta \cap \Phi_i$, and put $l_i = \#\Delta_i$. Then $l_1 + \dots + l_m = l$, the dimension of T , i.e., the rank of G . Let $\alpha_{0,i}$ be the highest root of Φ_i with respect to the basis of simple roots Δ_i . Then the Coxeter group with set of involutive generators

$$\tilde{S} = S_0 \cup \{t(\alpha_{0,i}^\vee)w_{\alpha_{0,i}} \mid 1 \leq i \leq m\}$$

is isomorphic to the affine Weyl group W_{af} of Φ ([6, Prop. 1.1]). Via this isomorphism, we will view W_{af} as a subgroup of \tilde{W} .

Let I be the Iwahori subgroup of $G(F)$ generated by $\underline{T}(\mathcal{O}_F)$, the subgroups $x_\alpha(\mathcal{O}_F) = \underline{U}_\alpha(\mathcal{O}_F)$ for all α in Φ^+ , and the subgroups $x_\alpha(\mathfrak{p})$ for all α in Φ^- . If we denote by \overline{G} the semisimple algebraic group over k obtained by taking the special fiber of \underline{G} then (as in [9, §3.5]) we have a surjective reduction mod \mathfrak{p} map $K \longrightarrow \overline{G}(k)$, and I is the inverse image in K under this map of the Borel subgroup of $\overline{G}(k)$ corresponding to Φ^+ . The triple

$(G(F), I, N_T(F))$ is a generalized Tits system in the sense of [5], a fact which will be used in Section 3 to study the structure of $G(F)$.

Denote the normalizer of I in $G(F)$ by \tilde{I} and set

$$\Omega = (N_T(F) \cap \tilde{I}) / \underline{T}(\mathcal{O}_F) \subset \tilde{W}.$$

The group Ω is finite abelian and canonically isomorphic to $X_*(T)/\Lambda_r$, where Λ_r is the submodule of $X_*(T)$ generated by Φ^\vee (cf. [5, §2]). Moreover, Ω normalizes W_{af} and there is an isomorphism $\tilde{W} \cong W_{\text{af}} \rtimes \Omega$. If $\rho \in \Omega$ and $w \in W_{\text{af}}$ we will abbreviate the element $\rho w \rho^{-1}$ of W_{af} by w^ρ . If Ψ is a subset of W_{af} , we will write Ψ^ρ for the set $\{w^\rho \mid w \in \Psi\}$.

For w in W_{af} , let $l(w)$ denote the length of w as an element of W_{af} with respect to \tilde{S} . If $w \in W_{\text{af}}$ and $\rho \in \Omega$ then the length $l(w\rho)$ of $w\rho$ is defined to be $l(w)$. If $w' \in \tilde{W}$ then we can write $w' = w_1 \cdots w_d \rho$ for some w_1, \dots, w_d in \tilde{S} and ρ in Ω , and we say (by abuse of notation) that the expression $w = w_1 \cdots w_d \rho$ is *reduced* if $l(w) = d$. (Under this definition, the expression $e = e$ is also to be considered reduced.)

Finally, if $V \subset G(F)$, and $\{V_i\}_{i \in I}$ is a collection of subsets of $G(F)$, then the notation $V = \coprod_{i \in I} V_i$ will signify that $V = \bigcup_{i \in I} V_i$ and that the V_i are pairwise disjoint.

3. Structure Theory of p -adic Groups.

We now state several results concerning the structure of $G(F)$ which stem from the fact that the triple $(G(F), I, N_T(F))$ is a generalized Tits system (as defined in [5]). We also state a result of Iwahori and Matsumoto ([6, Cor. 2.7]) which gives a set of representatives for the left cosets of I contained in a double coset of I .

We start with a summary of the structure theory pertaining to subgroups of $G(F)$ containing I . For any subgroup P of $G(F)$ containing I , we denote by W_P the subgroup $(N_T(F) \cap P) / \underline{T}(\mathcal{O}_F)$ of \tilde{W} . In addition, we let $S_P = W_P \cap \tilde{S}$ and $\Omega_P = W_P \cap \Omega$.

Proposition 3.1. *Let $P, P_1, P_2 \subset G(F)$ be subgroups containing I . Then*

(i)

$$P = IW_P I = \coprod_{w \in \tilde{W}_P} IwI.$$

Moreover, W_P is generated by S_P and S_P is stabilized under conjugation by elements of Ω_P . The map $P \mapsto (S_P, \Omega_P)$ is a bijection from subgroups lying between I and $G(F)$ to pairs (S, Ω') such that $S \subset \tilde{S}$ and Ω' is a subgroup of Ω stabilizing S .

- (ii) If $\Sigma_{P_1, P_2} \subset \tilde{W}$ is a set representatives for the double coset space $W_{P_1} \backslash \tilde{W} / W_{P_2}$ then

$$G(F) = \coprod_{\sigma \in \Sigma_{P_1, P_2}} P_1 \sigma P_2.$$

In particular, if $\gamma, \gamma' \in \tilde{W}$ then $W_{P_1} \gamma W_{P_2} = W_{P_1} \gamma' W_{P_2}$ if and only if $P_1 \gamma P_2 = P_1 \gamma' P_2$.

Note that in Proposition 3.1 (i) if $P = K$ then W_P is the Weyl group W_0 , $S_P = \{w_\alpha \mid \alpha \in \Delta\}$, $\Omega_P = \{e\}$, and the decomposition given is essentially the standard Bruhat decomposition for the group $\overline{G}(k)$. When $P = G(F)$, Proposition 3.1 (i) yields the affine Bruhat decomposition of $G(F)$ with respect to I . Note that if P is a subgroup of $G(F)$ containing I , then P is compact if and only if W_P is finite.

We also give the following summary of the structure of the coset space $I \backslash G(F) / I$ (see [6, Prop. 2.8, Theorem 3.3]).

Proposition 3.2. *Let w, w' be elements of \tilde{W} . Then*

- (i) For all $s \in \tilde{S}$
- a) $IsIwI = IswI$ if $l(sw) > l(w)$,
 - b) $IsIwI = IswI \cup IwI$ if $l(sw) < l(w)$.
- (ii) If $l(ww') = l(w) + l(w')$ then
- (1) $IwIw'I = Iww'I$.

In particular, if $s_1, \dots, s_d \in \tilde{S}$, $\rho \in \Omega$ and $w = s_1 \cdots s_d \rho$ is a reduced expression, then

(2) $Is_1 I \cdots Is_d I \rho I = IwI$.

In addition to the information resulting from the fact that the triple $(G(F), I, N_T(F))$ is a generalized Tits system, we will also need the following statement (cf. [6, Cor. 2.7]) concerning representatives for the left cosets of I inside the double cosets of I corresponding to the elements of \tilde{S} .

Proposition 3.3. *Suppose $\alpha \in \Delta$ and $i \in \{1, \dots, m\}$, where m is the number of irreducible root systems into which Φ decomposes. Then*

- (i) $Iw_\alpha I = \coprod_{\nu \in R} x_\alpha(\nu) w_\alpha I$
- (ii) $It(\alpha_{0,i}^\vee) w_{\alpha_{0,i}} I = \coprod_{\nu \in R} x_{-\alpha_{0,i}}(\pi\nu) t(\alpha_{0,i}^\vee) w_{\alpha_{0,i}} I$.

We now develop notation which will allow us to give a formula for the representatives of the left cosets of I in an arbitrary double coset in $I \backslash G(F) / I$. This formula will follow easily from the above results. For each s in \tilde{S} , we fix a lifting \bar{s} of s to $N_T(F)$. We define elements $g_s(\nu) \in G(F)$ for all s in \tilde{S} and ν in R by setting

$$g_s(\nu) = \begin{cases} x_\alpha(\nu) \bar{s} & \text{if } s = w_\alpha \text{ for some } \alpha \text{ in } \Delta \\ x_{-\alpha_{0,i}}(\pi\nu) \bar{s} & \text{if } s = t(\alpha_{0,i}^\vee) w_{\alpha_{0,i}} \text{ for some } i \text{ in } \{1, \dots, m\}. \end{cases}$$

In this notation, Proposition 3.3 says that for each $s \in \tilde{S}$

$$IsI = \coprod_{\nu \in R} g_s(\nu)I.$$

For each ρ in Ω we also choose some lifting $\bar{\rho}$ of ρ to $N_T(F)$.

For every w in \tilde{W} we fix an $(l(w) + 1)$ -tuple $e(w) = (s_{w,1}, \dots, s_{w,l(w)}, \rho_w)$ in $\tilde{S}^{l(w)} \times \Omega$ such that $w = s_{w,1} \cdots s_{w,l(w)} \rho_w$. We define $g_w : R^{l(w)} \rightarrow G(F)$ to be the function which assigns to each $(\nu_1, \dots, \nu_{l(w)})$ in $R^{l(w)}$ the element

$$g_{s_{w,1}}(\nu_1) \cdots g_{s_{w,l(w)}}(\nu_{l(w)}) \bar{\rho}_w,$$

using the notation of the previous paragraph. Then we have the following fact concerning the coset space IwI/I .

Corollary 3.4. *Suppose that $w \in \tilde{W}$ and that $w = s_1 \cdots s_d \rho$ is a reduced expression (i.e., $d = l(w)$), where $s_1, \dots, s_d \in \tilde{S}$ and $\rho \in \Omega$. Then the index $[IwI : I]$ is $q^{l(w)}$. In fact,*

$$IwI = \coprod_{\nu_i \in R} g_{s_1}(\nu_1) \cdots g_{s_d}(\nu_d) \bar{\rho}I = \coprod_{\nu \in R^{l(w)}} g_w(\nu)I.$$

Proof. For $U \subset G(F)$ let $\text{char}_U : G(F) \rightarrow \{0, 1\}$ be the characteristic function of U . Since $\text{char}_{Iw'I} \mapsto [Iw'I : I]$ (w' in \tilde{W}) defines a character of the Iwahori Hecke algebra of G with respect to I [6, §3], it follows from Propositions 3.2 and 3.3 that

$$[IwI : I] = [Is_1 \cdots s_d \rho I : I] = [Is_1 I] \cdots [Is_d I : I] = q^{l(w)}$$

(cf. [6, Prop. 3.2]). To complete the proof it suffices to show that the union of the $q^{l(w)}$ cosets given above is all of IwI . This also follows from Propositions 3.2 and 3.3 since

$$\begin{aligned} IwI = Is_1 s_2 \cdots s_d \rho I &= Is_1 Is_2 I \cdots Is_d I \rho I \\ &= \bigcup_{\nu_1 \in R} g_{s_1}(\nu_1) Is_2 I \cdots Is_d I \rho I \\ &= \bigcup_{\nu_1, \dots, \nu_d \in R} g_{s_1}(\nu_1) g_{s_2}(\nu_2) \cdots g_{s_d}(\nu_d) \bar{\rho}I. \end{aligned}$$

□

4. Coxeter Subgroups of the Extended Affine Weyl Group.

Let W be a subgroup of the affine Weyl group W_{af} which is generated by the set $S = W \cap \tilde{S}$. Then W is a Coxeter group. If W' is a subgroup of W generated by $S \cap W'$, then W' is also a Coxeter group, and we will refer to such a subgroup W' as a *special subgroup* of W . Define $[W/W']$ to be the set

$$\{w \in W \mid l(ww') = l(w) + l(w') \text{ for all } w' \in W'\}.$$

The elements of $[W/W']$ are the representatives for W/W' of minimal length [4, §5.12]. We will have need of the following fact concerning $[W/W']$.

Lemma 4.1. *Suppose that W is a Coxeter subgroup of W_{af} with set of generators $S = \tilde{S} \cap W$. Let W' be a special subgroup of W . If τ in $[W/W']$ and s in S satisfy $l(s\tau) < l(\tau)$, then $s\tau$ is in $[W/W']$.*

Proof. Since $l(s\tau) < l(\tau)$ we know that $l(s\tau) = l(\tau) - 1$. For w' in W' we therefore have that

$$l(\tau w') - 1 = l(\tau) + l(w') - 1 = l(s\tau) + l(w') \geq l(s\tau w') \geq l(\tau w') - 1.$$

Thus, $l(s\tau w') = l(s\tau) + l(w')$ and $s\tau$ is in $[W/W']$. □

For the remainder of the section, we fix two special subgroups W_1 and W_2 of W_{af} with $S_1 = W_1 \cap \tilde{S}$ and $S_2 = W_2 \cap \tilde{S}$ as their respective sets of involutive generators. For σ in \tilde{W} define $W_1^{\sigma W_2}$ to be the stabilizer under left multiplication of the coset σW_2 in W_1 , namely, $W_1 \cap \sigma W_2 \sigma^{-1}$. Let $[W_1 \setminus \tilde{W} / W_2] \subset \tilde{W}$ be a set of representatives for $W_1 \setminus \tilde{W} / W_2$ of minimal length, i.e., each σ in $[W_1 \setminus \tilde{W} / W_2]$ is to be an element of shortest length in $W_1 \sigma W_2$. Our first order of business will be to show that $W_1^{\sigma W_2}$ is a special subgroup of W_1 for any σ in $[W_1 \setminus \tilde{W} / W_2]$. Our goal will then be to show that

$$l(\tau \sigma w) = l(\tau) + l(\sigma) + l(w)$$

for all w in W_2 , σ in $[W_1 \setminus \tilde{W} / W_2]$, and τ in $[W_1 / W_1^{\sigma W_2}]$. This fact is a simple generalization of a result of Howlett concerning finite reflection groups (cf. [2, §2.7]). It will prove very important in our analysis of the decomposition of double cosets in Chapter 5.

In order to show that $W_1^{\sigma W_2}$ is a special subgroup of W_1 , we will need the following result, which is a simple generalization of the exchange condition (cf. [1, §2.3A]), one of several equivalent properties that distinguish Coxeter groups from among the more general class of groups generated by finitely many involutions.

Proposition 4.2. *Suppose w in \tilde{W} has reduced expression $w = s_1 \cdots s_d \rho$ for some s_1, \dots, s_d in \tilde{S} and ρ in Ω . Then for all s in \tilde{S} either*

- (i) $l(sw) = l(w) + 1$, or
- (ii) $w = ss_1 \cdots \hat{s}_i \cdots s_d \rho$ for some i in $\{1, \dots, d\}$.

The following is an easy consequence of Proposition 4.2.

Lemma 4.3. *Suppose that w and w' are elements of \tilde{W} such that $l(ww') = l(w) + l(w')$. If s in S satisfies $l(sw) = l(w) + 1$, then either*

- (i) $l(sww') = l(ww') + 1$, or
- (ii) $ww' = sw\hat{w}'$ for some \hat{w}' in \tilde{W} with $l(\hat{w}') < l(w')$.

In fact, in the second case, if $s_1, \dots, s_d \in \tilde{S}$, $\rho \in \Omega$ and $w' = s_1 \cdots s_d \rho'$ is a reduced expression, then

$$\hat{w}' = s_1 \cdots \hat{s}_i \cdots s_d \rho'$$

for some i in $\{1, \dots, d\}$. In particular, if w' is an element of a special subgroup of W_{af} then \hat{w}' is also an element of that subgroup.

Proof. $l(sw w')$ equals either $l(w w') + 1$ or $l(w w') - 1$ so suppose the latter is true. This is clearly impossible if $l(w') = 0$. Moreover, if $l(w) = 0$ then Proposition 4.2 implies that (ii) holds. Thus we may assume that $l(w), l(w') > 0$.

Let $w = t_1 \cdots t_r \rho$ and $w' = s_1 \cdots s_d \rho'$ be reduced expressions for some reflections $t_1, \dots, t_r, s_1, \dots, s_d$ in \tilde{S} and some ρ, ρ' in Ω . Then we have the reduced expression

$$w w' = t_1 \cdots t_r \rho s_1 \cdots s_d \rho' = t_1 \cdots t_r s_1^\rho \cdots s_d^\rho \rho \rho'.$$

Since $s_i^\rho \in \tilde{S}$, it follows from Proposition 4.2, that either

$$w w' = s t_1 \cdots \hat{t}_i \cdots t_r s_1^\rho \cdots s_d^\rho \rho \rho'$$

for some i in $\{1, \dots, r\}$ or

$$w w' = s t_1 \cdots t_r s_1^\rho \cdots \hat{s}_i^\rho l \cdots s_d^\rho \rho \rho'$$

for some i in $\{1, \dots, d\}$. If the former holds then

$$s w w' = t_1 \cdots \hat{t}_i \cdots t_r \rho w'$$

which implies that $l(s w) < l(w)$, a contradiction. Therefore, we must have

$$w w' = s t_1 \cdots t_r s_1^\rho \cdots \hat{s}_i^\rho \cdots s_d^\rho \rho \rho' = s w s_1 \cdots \hat{s}_i \cdots s_d \rho'.$$

Setting $\hat{w}' = s_1 \cdots \hat{s}_i \cdots s_d \rho'$, the first and second statements follow since $l(\hat{w}') \leq l(w') - 1$. The third statement holds since if w' is in a special subgroup of W_{af} , then each of the generators $s_1 \dots s_d$ lies in that subgroup. \square

We are now able to state and prove our first result on the additivity of lengths for certain elements of \tilde{W} , which we will need to show that $W_1^{\sigma W_2}$ is special.

Lemma 4.4. *For all σ in $[W_1 \setminus \tilde{W} / W_2]$, w in W_1 and w' in W_2 , we have $l(w\sigma) = l(w) + l(\sigma)$ and $l(\sigma w') = l(w') + l(\sigma)$.*

Proof. We will prove the first statement; the second statement follows from the first by taking inverses. Suppose $w_0 \in W_1$. The result is trivial if $l(w_0) \leq 1$ as $\sigma \in [W_1 \setminus \tilde{W} / W_2]$. So assume $l(w_0) > 1$ and suppose by induction that $l(w\sigma) = l(w) + l(\sigma)$ for all w in W_1 with $l(w) < l(w_0)$.

We may write w_0 as sw'_0 where $s \in S_1$ and w'_0 in W_1 has length $l(w_0) - 1$. Then, by induction,

$$l(sw_0\sigma) = l(w'_0\sigma) = l(w'_0) + l(\sigma) = l(sw_0) + l(\sigma) = l(w_0) + l(\sigma) - 1.$$

We must therefore show that $l(sw_0\sigma) = l(w_0\sigma) - 1$. Since $s \in S_1$, either this is true or $l(sw_0\sigma) = l(w_0\sigma) + 1$. But if the latter holds then, by applying Lemma 4.3 (with $w = sw_0$ and $w' = \sigma$) we obtain that $sw_0\sigma = w_0\hat{\sigma}$ for some $\hat{\sigma}$ of length less than $l(\sigma)$. This, however, contradicts the fact that $\sigma \in [W_1 \setminus \tilde{W}/W_2]$ is of minimal length in its double coset. \square

Proposition 4.5. *If σ is an element of $[W_1 \setminus \tilde{W}/W_2]$, then $W_1^{\sigma W_2}$ is a special subgroup of W_1 .*

Proof. We must show that if $s_1, \dots, s_d \in S_1$ and $s_1 \cdots s_d \in W_1^{\sigma W_2}$ then $s_1, \dots, s_d \in W_1^{\sigma W_2}$. Fix w in $W_1^{\sigma W_2}$. We may write w as sw' where $s \in S_1$ and $w' \in W_1$ has length $l(w) - 1$. By induction, it suffices to show that w' and hence s are in $W_1^{\sigma W_2}$.

Since $w = sw' \in W_1^{\sigma W_2}$, $w'\sigma W_2 = s\sigma W_2$, so that $w'\sigma = s\sigma w_0$ for some w_0 in W_2 . Therefore, in order to show that w' is in the stabilizer $W_1^{\sigma W_2}$ of σW_2 in W_1 , it suffices to prove that $s\sigma w_0 = \sigma \hat{w}_0$ for some \hat{w}_0 in W_2 .

Now $l(s\sigma w_0)$ is either equal to $l(\sigma w_0) - 1$ or $l(\sigma w_0) + 1$. Suppose the former is true. Then, by Lemma 4.3 (with $w = \sigma$ and $w' = w_0$ in W_2) we have that $\sigma w_0 = \sigma \hat{w}_0$ for some \hat{w}_0 in W_2 so that $s\sigma w_0 = \sigma \hat{w}_0$. Thus it suffices to rule out the case $l(s\sigma w_0) = l(\sigma w_0) + 1$.

If this holds then

$$(3) \quad l(w'\sigma) = l(s\sigma w_0) = l(\sigma w_0) + 1.$$

But by Lemma 4.4, $l(w'\sigma) = l(w') + l(\sigma)$ and $l(\sigma w_0) = l(\sigma) + l(w_0)$. Thus, by (3), $l(w') = l(w_0) + 1$ and hence $l(w) = l(w_0) + 2$. On the other hand, $w\sigma = sw'\sigma = s\sigma w_0$ so, by Lemma 4.4 again, $l(w) = l(w_0)$. This contradiction implies that $l(s\sigma w_0)$ cannot equal $l(\sigma w_0) + 1$ and the proof is complete. \square

We now state and prove the main result on length additivity.

Theorem 4.6. *Suppose $\sigma \in [W_1 \setminus \tilde{W}/W_2]$, $\tau \in [W_1/W_1^{\sigma W_2}]$ and $w \in W_2$. Then*

$$l(\tau\sigma w) = l(\tau) + l(\sigma) + l(w).$$

Proof. We will prove the theorem by induction on $l(\tau)$. The theorem is true for $\tau = e$ by Lemma 4.4. So suppose $l(\tau) > 0$ and that the statement is true for all τ' in $[W_1/W_1^{\sigma W_2}]$ with $l(\tau') < l(\tau)$. Let s in S_1 be such that $l(s\tau) = l(\tau) - 1$. Note that the element $s\tau$ is in $[W_1/W_1^{\sigma W_2}]$ by Lemma 4.1. By Lemma 4.3 applied to $(s\tau)(\sigma w)$, we obtain that either

- (i) $l(s(s\tau)(\sigma w)) = l((s\tau)(\sigma w)) + 1$, or
- (ii) $s\tau\sigma w = \tau\gamma$ for some γ in \tilde{W} with $l(\gamma) < l(\sigma w)$.

In the first case, we obtain that

$$\begin{aligned}
l(\tau\sigma w) &= l(s(s\tau)(\sigma w)) \\
&= l((s\tau)(\sigma w)) + 1 && \text{by (i)} \\
&= l((s\tau)\sigma w) + 1 \\
&= l(s\tau) + l(\sigma) + l(w) + 1 && \text{by induction} \\
&= l(\tau) + l(\sigma) + l(w)
\end{aligned}$$

so it remains to rule out the second case.

We can assume that $l(\sigma w) > 0$ since the statement of the theorem is trivially true if $l(\sigma w) = l(\sigma) + l(w) = 0$. Since $l(\sigma w) = l(\sigma) + l(w)$, we have a reduced expression

$$\sigma w = \rho s_1 \cdots s_t$$

where $s_1, \dots, s_t \in \tilde{S}$, $\rho \in \Omega$, and for some r in $\{1, \dots, t\}$, ρ together with the first r involutions in the product yield a reduced expression for σ while the next $t - r$ involutions yield a reduced expression for w . According to Lemma 4.3, the element γ in case (ii) is obtained from σw by deleting one of the involutions s_i in the above reduced word. If $i \leq r$ then $\gamma = \hat{\sigma} w$ for some $\hat{\sigma}$ with $l(\hat{\sigma}) < l(\sigma)$ so $s\tau\sigma w = \tau\hat{\sigma}w$. Then $W_1\sigma W_2 = W_1\hat{\sigma}W_2$, a contradiction as $\sigma \in [W_1 \setminus \tilde{W}/W_2]$ is an element of minimal length in $W_1\sigma W_2$. On the other hand, if $i > r$ then $\gamma = \sigma\hat{w}$ for some \hat{w} in W_2 with $l(\hat{w}) < l(w)$ so $s\tau\sigma w = \tau\sigma\hat{w}$. But then $\tau\sigma W_2 = s\tau\sigma W_2$ so $\tau \equiv s\tau \pmod{W_1^{\sigma W_2}}$, a contradiction since τ is the shortest element in $\tau W_1^{\sigma W_2}$ as it lies in $[W_1/W_1^{\sigma W_2}]$. \square

Corollary 4.7. *If $\sigma \in [W_1 \setminus \tilde{W}/W_2]$ then σ is the unique element of minimal length in $W_1\sigma W_2$.*

Proof. Let w and w' be elements of W_1 and W_2 respectively such that $l(w\sigma w') = l(\sigma)$. Write $w = \tau\gamma$, where $\tau \in [W_1/W_1^{\sigma W_2}]$ and $\gamma \in W_1^{\sigma W_2}$. Then

$$w\sigma w' = \tau\gamma\sigma w' = \tau\sigma w''$$

for some w'' in W_2 since $\gamma\sigma W_2 = \sigma W_2$. But then

$$l(\sigma) = l(w\sigma w') = l(\tau\sigma w'') = l(\tau) + l(\sigma) + l(w''),$$

which implies that $\tau, w'' = e$ so that $w\sigma w' = \sigma$. \square

5. Double Coset Decomposition.

Throughout this section we will use the notation developed in Sections 3 and 4. Fix two subgroups P_1 and P_2 of $G(F)$ containing the Iwahori subgroup I . The goal of this section is to find representatives for the left cosets of P_2 in a double coset $P_1\sigma P_2$. We also give a formula for the number of left cosets in a double coset when P_1 and P_2 are compact. Using the notation

of Section 3, let $W_i = W_{P_i}$, the set of elements of \tilde{W} with representatives in P_i ($i = 1, 2$). We suppose for now that both W_1 and W_2 are contained in W_{af} . We will deal with general subgroups containing I later on in the section. Let $S_i = S_{P_i}$ be the canonical set of involutive generators of W_i .

By Proposition 3.1,

$$G(F) = \coprod_{\sigma \in [W_1 \backslash \tilde{W} / W_2]} P_1 \sigma P_2.$$

Let us therefore fix σ in $[W_1 \backslash \tilde{W} / W_2]$ and consider the coset $P_1 \sigma P_2$. We first decompose $P_1 \sigma P_2$ into a disjoint union of double cosets in $I \backslash G(F) / P_2$.

Lemma 5.1. *The double coset $P_1 \sigma P_2$ is the disjoint union of the cosets $I \tau \sigma P_2$ as τ ranges over $[W_1 / W_1^{\sigma W_2}]$.*

Proof. By Proposition 3.1 (i), we have that

$$P_1 = \coprod_{w \in W_1} I w I.$$

It follows that

$$P_1 \sigma P_2 = \bigcup_{w \in W_1} I w I \sigma P_2.$$

We claim that this last expression is equal to $\bigcup_{w \in W_1} I w \sigma P_2$.

By Equation (2) in Proposition 3.2, if we write w' in W_1 as a reduced expression $w' = s_1 \cdots s_d$ where $s_1, \dots, s_d \in S_1$, we have that

$$I w' I \sigma I = I s_1 \cdots s_d I \sigma I = I s_1 I \cdots I s_d I \sigma I.$$

But by Proposition 3.2 (i) applied repeatedly

$$I w' \sigma I = I s_1 \cdots s_d \sigma I \subset I s_1 I \cdots I s_d I \sigma I = I w' I \sigma I$$

and

$$I w' I \sigma I = I s_1 I \cdots I s_d I \sigma I \subset \bigcup_{w \in W_1} I w \sigma I.$$

Therefore

$$\bigcup_{w \in W_1} I w \sigma I = \bigcup_{w \in W_1} I w I \sigma I$$

and the claim follows.

We must now determine which of the terms in the above union are the same. To this end, we apply Proposition 3.1 (ii) to the subgroups I and P_2 of $G(F)$. Since $W_I = \langle e \rangle$ and $W_{P_2} = W_2$, it follows that for any w, w' in W_1 , $I w \sigma P_2 = I w' \sigma P_2$ if and only if $w \sigma \equiv w' \sigma \pmod{W_2}$, i.e., if and only if $w \equiv w' \pmod{W_1^{\sigma W_2}}$. Therefore, to obtain a disjoint union of cosets $I w \sigma P_2$ we take the union over w in the set of representatives $[W_1 / W_1^{\sigma W_2}]$. \square

We now decompose $P_1\sigma P_2$ into left cosets of P_2 by expressing each double coset $I\tau\sigma P_2$ as a union of such left cosets. Recall that R is a set of representatives in \mathcal{O}_F for the residue field k which contains 0.

Theorem 5.2. *Suppose that P_1 and P_2 are subgroups of $G(F)$ containing I . If $W_{P_1}, W_{P_2} \subset W_{af}$ and $\sigma \in W_{P_1} \backslash \tilde{W} / W_{P_2}$, then the double coset $P_1\sigma P_2$ is equal to the disjoint union*

$$P_1\sigma P_2 = \coprod_{\tau \in [W_{P_1} / W_{P_1}^{\sigma W_{P_2}}]} \coprod_{\nu \in R^{l(\tau\sigma)}} g_{\tau\sigma}(\nu)P_2.$$

Proof. By Corollary 3.4, we have the decomposition

$$(4) \quad \begin{aligned} I\tau\sigma P_2 = I\tau\sigma I P_2 &= \left(\coprod_{\nu \in R^{l(\tau\sigma)}} g_{\tau\sigma}(\nu)I \right) P_2 \\ &= \bigcup_{\nu \in R^{l(\tau\sigma)}} g_{\tau\sigma}(\nu)P_2. \end{aligned}$$

Because of Lemma 5.1, the theorem will follow if we show that the cosets in the union (4) are distinct. So suppose that

$$g_{\tau\sigma}(\nu)P_2 = g_{\tau\sigma}(\nu')P_2$$

for some $\tau \in [W_1 / W_1^{\sigma W_2}]$ and $\nu, \nu' \in R^{l(\tau\sigma)}$. We will show that $\nu = \nu'$. The main idea of the argument is to transfer the problem from P_2 -cosets in $G(F)$ to W_2 -cosets in \tilde{W} and then to bring to bear our results on Coxeter groups from Section 4.

First we note that by Proposition 3.1 (i) and Corollary 3.4,

$$\begin{aligned} g_{\tau\sigma}(\nu)P_2 &= \coprod_{w \in W_2} g_{\tau\sigma}(\nu)IwI \\ &= \coprod_{w \in W_2} \coprod_{\nu'' \in R^{l(w)}} g_{\tau\sigma}(\nu)g_w(\nu'')I \end{aligned}$$

and similarly

$$g_{\tau\sigma}(\nu')P_2 = \coprod_{w \in W_2} \coprod_{\nu'' \in R^{l(w)}} g_{\tau\sigma}(\nu')g_w(\nu'')I.$$

Since these two P -cosets are equal, there must exist some w in W_2 and ν'' in $R^{l(w)}$ such that $g_{\tau\sigma}(\nu')g_w(\nu'')I$ equals $g_{\tau\sigma}(\nu)g_e(0)I = g_{\tau\sigma}(\nu)I$. We will show that this equality can only hold if $w = e$. Then we will have that $g_{\tau\sigma}(\nu)I = g_{\tau\sigma}(\nu')I$, which immediately implies that $\nu = \nu'$ by Corollary 3.4.

So suppose that $g_{\tau\sigma}(\nu)I = g_{\tau\sigma}(\nu')g_w(\nu'')I$, where $w \in W_2$ and $\nu'' \in R^{l(w)}$. By the definition of $g_{\tau\sigma}(\nu)$ and Proposition 3.2 (ii), we have that

$$(5) \quad g_{\tau\sigma}(\nu)I \subset I\tau\sigma I.$$

Similarly, for each ν'' in $R^{l(w)}$,

$$g_{\tau\sigma}(\nu')g_w(\nu'')I \subset I\tau\sigma IwI.$$

We are now able to use Section 4 since $\sigma \in [W_1 \setminus \tilde{W}/W_2]$, $\tau \in [W_1/W_1^{\sigma W_2}]$ and $w \in W_2$. By Theorem 4.6, we conclude that $l(\tau\sigma w) = l(\tau\sigma) + l(w)$. This implies via Equation (1) in Proposition 3.2 that $I\tau\sigma IwI = I\tau\sigma wI$. Hence,

$$(6) \quad g_{\tau\sigma}(\nu')g_w(\nu'')I \subset I\tau\sigma wI.$$

Since the double cosets in (5) and (6) both contain the left I -coset $g_{\tau\sigma}(\nu)I = g_{\tau\sigma}(\nu')g_w(\nu'')I$, we conclude that they must be equal. But $I\tau\sigma I = I\tau\sigma wI$ implies $w = e$ since $I \setminus G(F)/I$ is represented by \tilde{W} (Proposition 3.1). \square

Remark 5.3. If we take as a representative of $P_1\sigma P_2$ an element σ' of σW_2 not equal to σ then

$$P_1\sigma P_2 = \coprod_{\tau \in [W_1/W_1^{\sigma' W_2}]} \bigcup_{\nu \in R^{l(\tau\sigma')}} g_{\tau\sigma'}(\nu)P_2,$$

but this union is no longer disjoint. For the number of cosets in the preceding union is larger than than the number of cosets given in Theorem 5.2 as $l(\tau\sigma') = l(\tau) + l(\sigma') > l(\tau) + l(\sigma) = l(\tau\sigma)$ for any $\tau \in [W_1/W_1^{\sigma' W_2}]$ by Theorem 4.6.

We now give a decomposition of double cosets into left cosets for arbitrary subgroups of $G(F)$ containing I . Adjusting our notation slightly, we let P'_1 and P'_2 be two such subgroups. Set $W'_i = W_{P'_i}$ and $W_i = W'_i \cap W_{\text{af}}$. As before let $S_i = W_i \cap \tilde{S}$. Recall that S_i is stabilized by $\Omega_{P'_i}$ under conjugation (Proposition 3.1). For $i = 1, 2$, let $P_i \subset P'_i$ be the subgroup IW_iI . Then

$$P'_i = \coprod_{\rho \in \Omega_{P'_i}} P_i\rho = \coprod_{\rho \in \Omega_{P'_i}} \rho P_i$$

by Proposition 3.1.

Let $[W'_1 \setminus \tilde{W}/W'_2] = [W_{P'_1} \setminus \tilde{W}/W_{P'_2}]$ be a set of representatives of smallest possible length for the double cosets in $W'_1 \setminus \tilde{W}/W'_2$. (Note that this set of representatives is no longer unique in contrast to $[W_1 \setminus \tilde{W}/W_2]$.)

Lemma 5.4. *The set $[W_1 \setminus \tilde{W}/W_2]$ is equal to the set of all products of the form $\rho_1\sigma\rho_2$ as ρ_i ranges over $\Omega_{P'_i}$ ($i = 1, 2$) and σ ranges over $[W'_1 \setminus \tilde{W}/W'_2]$. In particular, if $\rho_i \in \Omega_{P'_i}$ ($i = 1, 2$) and $\sigma \in [W_1 \setminus \tilde{W}/W_2]$, then $\rho_1\sigma\rho_2$ is also in $[W_1 \setminus \tilde{W}/W_2]$.*

Proof. The elements $\rho_1\sigma\rho_2$ clearly exhaust $W_1\backslash\tilde{W}/W_2$ since

$$\tilde{W} = \coprod_{\sigma \in [W_1' \backslash \tilde{W} / W_2']} W_1' \sigma W_2' = \coprod_{\sigma \in [W_1' \backslash \tilde{W} / W_2']} \bigcup_{\rho_1 \in \Omega_{P_1}'} \bigcup_{\rho_2 \in \Omega_{P_2}'} W_1 \rho_1 \sigma \rho_2 W_2.$$

Moreover, $\rho_1\sigma\rho_2$ is of minimal length in $W_1\rho_1\sigma\rho_2W_2$ since σ is of minimal length in $W_1'\sigma W_2'$. The second statement follows trivially from the first. \square

Let us now fix an element σ in $[W_1' \backslash \tilde{W} / W_2']$ and consider the double coset $P_1'\sigma P_2'$. Denote by $\Omega_{P_1'}^\sigma$ the stabilizer $\{\rho \in \Omega_{P_1'} \mid \sigma^\rho = \sigma\}$ of σ in Ω_{P_1}' and let $\Omega_{P_1', P_2'}^\sigma = \Omega_{P_1'}^\sigma \cap \Omega_{P_2}'$. Also, let $J_{P_1', P_2'}^\sigma$ be a set of representatives for $\Omega_{P_1}' / \Omega_{P_1', P_2'}^\sigma$.

Lemma 5.5. *If $\rho \in \Omega_{P_1}'$, then*

$$\rho[W_1/W_1^{\sigma W_2}] \rho^{-1} = [W_1/W_1^{\sigma^\rho W_2^\rho}].$$

(Recall that W_2^ρ denotes the set $\{w^\rho \mid w \in W_2\}$.) In particular, if $\rho \in \Omega_{P_1', P_2'}^\sigma$, then $\rho[W_1/W_1^{\sigma W_2}] \rho^{-1} = [W_1/W_1^{\sigma W_2}]$.

Proof. The first statement is true since Ω_{P_1}' stabilizes W_1 and since

$$\rho W_1^{\sigma W_2} \rho^{-1} = \rho(W_1 \cap \sigma W_2 \sigma^{-1}) \rho^{-1} = W_1 \cap \sigma^\rho W_2^\rho (\sigma^\rho)^{-1} = W_1^{\sigma^\rho W_2^\rho}.$$

The second statement follows from the first and the fact that if $\rho \in \Omega_{P_1', P_2'}^\sigma$ then ρ stabilizes σ and W_2 . \square

Following our procedure in the beginning of the section, we first decompose $P_1'\sigma P_2'$ into a union of cosets in $I \backslash G(F) / P_2'$.

Lemma 5.6. *The double coset $P_1'\sigma P_2'$ is the disjoint union*

$$\coprod_{\gamma \in J_{P_1', P_2'}^\sigma} \coprod_{\tau \in [W_1/W_1^{\sigma W_2}]} I \gamma \tau \sigma P_2'.$$

Proof. Since $\sigma \in [W_1 \backslash \tilde{W} / W_2]$ and $W_{P_i} = W_i \in W_{\text{af}}$ ($i = 1, 2$), we have that

$$P_1 \sigma P_2 = \coprod_{\tau \in [W_1/W_1^{\sigma W_2}]} I \tau \sigma P_2,$$

by Lemma 5.1. Therefore,

$$\begin{aligned} (7) \quad P_1' \sigma P_2' &= \bigcup_{\rho_i \in \Omega_{P_i}'} \rho_i P_1 \sigma P_2 \rho_i \\ &= \bigcup_{\rho_i \in \Omega_{P_i}'} \bigcup_{\tau \in [W_1/W_1^{\sigma W_2}]} \rho_i I \tau \sigma P_2 \rho_i \end{aligned}$$

$$\begin{aligned}
 &= \bigcup_{\rho_i \in \Omega_{P'_i}} \bigcup_{\tau \in [W_1/W_1^{\sigma W_2}]} I\rho_1\tau\sigma\rho_2P_2 \\
 &= \bigcup_{\rho_i \in \Omega_{P'_i}} \bigcup_{\tau \in [W_1/W_1^{\sigma W_2}]} I\tau^{\rho_1}\sigma^{\rho_1}\rho_1\rho_2P_2 \\
 &= \bigcup_{\rho_1 \in \Omega_{P'_1}} \bigcup_{\tau \in [W_1/W_1^{\sigma W_2}]} I\tau^{\rho_1}\sigma^{\rho_1}\rho_1P'_2.
 \end{aligned}$$

Using the definitions of $\Omega_{P'_1}$ and $J_{P'_1, P'_2}^\sigma$, we can eliminate some of the repetition of cosets in the union (7). For (7) is equal to

$$\begin{aligned}
 (8) \quad &\bigcup_{\rho_1 \in \Omega_{P'_1}} \bigcup_{\tau \in [W_1/W_1^{\sigma W_2}]} I\tau^{\rho_1}\sigma^{\rho_1}\rho_1P'_2 \\
 &= \bigcup_{\gamma \in J_{P'_1, P'_2}^\sigma} \bigcup_{\rho \in \Omega_{P'_1, P'_2}^\sigma} \bigcup_{\tau \in [W_1/W_1^{\sigma W_2}]} I\tau^{\gamma\rho}\sigma^{\gamma\rho}\gamma\rho P'_2 \\
 (9) \quad &= \bigcup_{\gamma \in J_{P'_1, P'_2}^\sigma} \bigcup_{\rho \in \Omega_{P'_1, P'_2}^\sigma} \bigcup_{\tau' \in \gamma\rho[W_1/W_1^{\sigma W_2}](\gamma\rho)^{-1}} I\tau'\sigma^{\gamma\rho}\gamma P'_2.
 \end{aligned}$$

By Lemma 5.5 and the fact that $\Omega_{P'_1, P'_2}^\sigma$ stabilizes σ and W_2 , this last expression is equal to

$$(10) \quad \bigcup_{\gamma \in J_{P'_1, P'_2}^\sigma} \bigcup_{\tau' \in [W_1/W_1^{\sigma^\gamma W_2^\gamma}]} I\tau'\sigma^\gamma\gamma P'_2.$$

As in the proof of Lemma 5.1, we now determine whether the terms in the union (10) are distinct. Let $\gamma, \gamma' \in J_{P'_1, P'_2}^\sigma$, and let $\tau \in [W_1/W_1^{\sigma^\gamma W_2^\gamma}]$, $\tau' \in [W_1/W_1^{\sigma^{\gamma'} W_2^{\gamma'}}]$. We will show that the terms $I\tau\sigma^\gamma\gamma P'_2$ and $I\tau'\sigma^{\gamma'}\gamma' P'_2$ in (10) are equal only if $\gamma = \gamma'$ and $\tau = \tau'$. By Proposition 3.1 applied to I and P'_2 , we have that the two double cosets are equal if and only if

$$(11) \quad \tau\sigma^\gamma\gamma W_2' = \tau'\sigma^{\gamma'}\gamma' W_2'.$$

Since $W_2' = \coprod_{\gamma \in \Omega_{P'_2}} \gamma W_2$, this is equivalent to the condition that

$$\tau\sigma^\gamma\gamma\rho W_2 = \tau'\sigma^{\gamma'}\gamma' W_2$$

for some $\rho \in \Omega_{P'_2}$. But $\tau, \tau' \in W_1$ so this means that

$$(12) \quad W_1\sigma^\gamma\gamma\rho W_2 = W_1\sigma^{\gamma'}\gamma' W_2.$$

Now both $\sigma^\gamma\gamma\rho = \gamma\sigma\rho$ and $\sigma^{\gamma'}\gamma' = \gamma'\sigma$ are in $[W_1 \setminus \tilde{W}/W_2]$ by Lemma 5.4 since σ is an element of $[W_1 \setminus \tilde{W}/W_2]$. As a consequence of the uniqueness of the coset representatives of shortest length of $W_1 \setminus \tilde{W}/W_2$ (Corollary 4.7), Equation (12) can hold only if $\sigma^\gamma\gamma\rho = \sigma^{\gamma'}\gamma'$. Since $\tilde{W} = W_{\text{af}} \rtimes \Omega$, it follows

easily from this that $\gamma\rho = \gamma'$ and hence that $\sigma^\gamma = \sigma^{\gamma'}$. Thus $\gamma^{-1}\gamma' = \rho \in \Omega_{P'_2}$ and $\gamma^{-1}\gamma' \in \Omega_{P'_1}^\sigma$ so that $\gamma \equiv \gamma' \pmod{\Omega_{P'_1, P'_2}^\sigma}$. Since $\gamma, \gamma' \in J_{P'_1, P'_2}^\sigma$ we must have $\gamma = \gamma'$ and $\rho = e$.

Since $\gamma = \gamma'$, it follows from (11) that $\tau\gamma\sigma W'_2\gamma^{-1} = \tau'\gamma\sigma W'_2\gamma^{-1}$. Thus τ and τ' are elements of $[W_1/W_1^{\sigma^\gamma W_2^\gamma}]$ which lie in the same left coset of $W_1^{\sigma^\gamma W_2^\gamma}$. This forces $\tau = \tau'$. Therefore, we see that the union in (10) is disjoint; i.e.,

$$P'_1\sigma P'_2 = \coprod_{\gamma \in J_{P'_1, P'_2}^\sigma} \coprod_{\tau \in \gamma[W_1/W_1^{\sigma W_2}]\gamma'} I\tau\sigma^\gamma\gamma P'_2 = \coprod_{\gamma \in J_{P'_1, P'_2}^\sigma} \coprod_{\tau \in [W_1/W_1^{\sigma W_2}]} I\gamma\tau\sigma P'_2.$$

□

Proceeding as in the proof of Theorem 5.2, we now use this decomposition of $P'_1\sigma P'_2$ to conclude, in analogy to (4), that

$$P'_1\sigma P'_2 = \coprod_{\gamma \in J_{P'_1, P'_2}^\sigma} \coprod_{\tau \in [W_1/W_1^{\sigma W_2}]} \bigcup_{\nu \in R^{l(\tau\sigma)}} g_{\gamma\tau\sigma}(\nu)P'_2.$$

(Note that $l(\gamma\tau\sigma) = l(\tau\sigma)$ since $\gamma \in \Omega$.) In order to prove that these P'_2 -cosets are distinct, we note that the argument in Theorem 5.2 will still work if P_2 is replaced by P'_2 and W_2 by W'_2 provided that $l(\gamma\tau\sigma w') = l(\gamma\tau\sigma) + l(w')$ for all γ in $J_{P'_1, P'_2}^\sigma$, τ in $[W_1/W_1^{\sigma W_2}]$ and w' in W'_2 . This condition is easily proved to hold—write w' as $w''\rho$ where $w'' \in W_2$ and $\rho \in \Omega_{P'_2}$. Then, since $\sigma \in [W_1 \setminus \tilde{W}/W_2]$ and $\tau \in [W_1/W_1^{\sigma W_2}]$, we have by Theorem 4.6 that

$$\begin{aligned} l(\gamma\tau\sigma w') &= l(\gamma\tau\sigma w''\rho) \\ &= l(\tau\sigma w'') \\ &= l(\tau\sigma) + l(w'') \\ &= l(\gamma\tau\sigma) + l(w''\rho) \\ &= l(\gamma\tau\sigma) + l(w'). \end{aligned}$$

Thus the P'_2 -cosets appearing in the union are distinct and we have proved the following theorem.

Theorem 5.7. *Let P'_1, P'_2 be subgroups of $G(F)$ containing the Iwahori subgroup I . Let σ be an element of $[W_{P'_1} \setminus \tilde{W}/W_{P'_2}]$. Then*

$$P'_1\sigma P'_2 = \coprod_{\gamma \in J_{P'_1, P'_2}^\sigma} \coprod_{\tau \in [W_{P'_1}/W_{P'_1}^{\sigma W_{P'_2}}]} \coprod_{\nu \in R^{l(\tau\sigma)}} g_{\gamma\tau\sigma}(\nu)P'_2.$$

The number of terms in the disjoint union in Theorem 5.7 is calculated in the following corollary when W_1 and W_2 are finite (i.e., P_1 and P_2 are compact).

Corollary 5.8. *Let σ be an element of $[W'_1 \backslash \tilde{W} / W'_2]$. Suppose that W_1 is finite. Then the number of left cosets of P'_2 in $P'_1 \sigma P'_2$ is*

$$\#(J_{P'_1, P'_2}^\sigma) \cdot q^{l(\sigma)} \cdot \sum_{\tau \in [W_1 / W_1^{\sigma W_2}]} q^{l(\tau)} = [\Omega_{P'_1} : \Omega_{P'_1, P'_2}^\sigma] \sum_{\gamma \in W_1 \sigma W_2} q^{l(\gamma)} \Big/ \sum_{w \in W_2} q^{l(w)}.$$

Proof. That the number of cosets is equal to the first expression is immediate from the theorem and the definition of $g_{\gamma\tau\sigma}(\nu)$. To prove that the two expressions are equal, first note that $\#(J_{P'_1, P'_2}^\sigma) = [\Omega_{P'_1} : \Omega_{P'_1, P'_2}^\sigma]$. Also, observe that $\{\tau\sigma \mid \tau \in [W_1 / W_1^{\sigma W_2}]\}$ is a set of coset representatives in $W_1 \sigma W_2$ for $W_1 \sigma W_2 / W_2$ so

$$\sum_{\gamma \in W_1 \sigma W_2} q^{l(\gamma)} = \sum_{\tau \in [W_1 / W_1^{\sigma W_2}]} \sum_{w \in W_2} q^{l(\tau\sigma w)}.$$

On the other hand, since $l(\tau\sigma w) = l(\tau) + l(\sigma) + l(w)$, we obtain

$$\begin{aligned} & \left(q^{l(\sigma)} \sum_{\tau \in [W_1 / W_1^{\sigma W_2}]} q^{l(\tau)} \right) \left(\sum_{w \in W_2} q^{l(w)} \right) \\ &= \sum_{\tau \in [W_1 / W_1^{\sigma W_2}]} \sum_{w \in W_2} q^{l(\tau) + l(\sigma) + l(w)} \\ &= \sum_{\tau \in [W_1 / W_1^{\sigma W_2}]} \sum_{w \in W_2} q^{l(\tau\sigma w)}. \end{aligned}$$

□

Example 5.9. Let G be the group PGSp_4 . The rank of G is 2, and we choose a set of simple roots consisting of a short root α_1 and a long root α_2 . We also let α_0 be the highest root corresponding to this basis. The Weyl group W_0 is generated by the reflections $w_1 = w_{\alpha_1}$ and $w_2 = w_{\alpha_2}$, while W_{af} is generated by these reflections and $w_0 = t(\alpha_0^\vee)w_{\alpha_0}$. The group Ω is cyclic of order 2. We denote the generator of Ω by ρ and note that it interchanges w_0 and w_2 but fixes w_1 .

We consider the Coxeter groups $W_1 = \langle w_1 \rangle$ and $W_2 = \langle w_0, w_2 \rangle$ inside W_{af} . The reflections w_0 and w_2 commute so that W_2 consists of the four elements e, w_0, w_2 and $w_0 w_2$. Let $W'_i = W_i \Omega$ and let P'_i be the compact subgroup $IW'_i I$ of $G(F)$ ($i = 1, 2$). We will use the results of this to decompose double cosets in $P'_1 \backslash G(F) / P'_2$ and $P'_2 \backslash G(F) / P'_1$.

Let $\sigma = w_1 w_0$. It is easily shown that $\sigma \in [W'_1 \backslash \tilde{W} / W'_2]$. Consider the double coset $P'_1 \sigma P'_2$. The group $W_1^{\sigma W_2} = W_1 \cap \sigma W_2 \sigma^{-1}$ is trivial so that $[W_1 / W_1^{\sigma W_2}] = [W_1 / W_1^{\sigma W_2}] = \{e, w_1\}$. The stabilizer $\Omega_{P'_1, P'_2}^\sigma$ is trivial as well

which means that $J_{P'_1, P'_2}^\sigma = \{e, \rho\}$. By Lemma 5.6, $P'_1\sigma P'_2$ is the disjoint union of the four double cosets

$$\begin{aligned} Iw_0w_1P' \\ Iw_1w_0w_1P'_2 \\ I\rho w_0w_1P'_2 &= Iw_2w_1P'_2 \\ I\rho w_1w_0w_1P'_2 &= Iw_1w_2w_1P'_2. \end{aligned}$$

Thus by Theorem 5.7, $P'_1\sigma P'_2$ is the disjoint union of the $2q^2 + 2q^3$ cosets

$$\begin{aligned} gw_0w_1(\nu_1)P'_2 & \quad (\nu_1 \in R^2) \\ gw_1w_0w_1(\nu_2)P'_2 & \quad (\nu_2 \in R^3) \\ gw_2w_1(\nu_3)P'_2 & \quad (\nu_3 \in R^2) \\ gw_1w_2w_1(\nu_4)P'_2 & \quad (\nu_4 \in R^3). \end{aligned}$$

Now let $\sigma = w_1w_0w_1 \in [W'_2 \backslash \tilde{W} / W'_2]$ and consider the double coset $P'_2\sigma P'_2$. Here the group $W_2^{\sigma W_2} = W_2 \cap \sigma W_2 \sigma^{-1} = \langle w_0 \rangle$ since we have the braid relation $w_1w_0w_1w_0w_1w_0w_1 = w_0$. Thus $[W_2 / W_2^{\sigma W_2}] = \{e, w_2\}$. Also, $J_{P'_2, P'_2}^\sigma = \{e, \rho\}$. By Lemma 5.6, $P'_2\sigma P'_2$ is the disjoint union of the four double cosets

$$\begin{aligned} Iw_1w_0w_1P'_2 \\ Iw_2w_1w_0w_1P'_2 \\ I\rho w_1w_0w_1P'_2 &= Iw_1w_2w_1P'_2 \\ I\rho w_2w_1w_0w_1P'_2 &= Iw_0w_1w_2w_1P'_2. \end{aligned}$$

It follows from Theorem 5.7 that $P'_2\sigma P'_2$ is the disjoint union of the $2q^3 + 2q^4$ cosets

$$\begin{aligned} gw_1w_0w_1(\nu_1)P'_2 & \quad (\nu_1 \in R^3) \\ gw_2w_1w_0w_1(\nu_2)P'_2 & \quad (\nu_2 \in R^4) \\ gw_1w_2w_1(\nu_3)P'_2 & \quad (\nu_3 \in R^3) \\ gw_0w_1w_2w_1(\nu_4)P'_2 & \quad (\nu_4 \in R^4). \end{aligned}$$

6. Degrees of Spherical Hecke Operators.

We now consider the special case when the subgroups P_1 and P_2 containing I are both equal to the hyperspecial subgroup K . Using the results of the previous section, we can determine the left cosets of K occurring in a given double coset $K\sigma K$. The number of such cosets is by definition the degree of the function $\text{ch}_{K\sigma K}$ in the spherical Hecke algebra of K . With the coset decomposition of the preceding section, we will derive the formula for the degree given in [5, §5].

Since $W_K = W_0$, the elements of $W_0 \backslash \tilde{W} / W_0$ index the double coset space $K \backslash G(F) / K$. Therefore, as a result of the fact that $\tilde{W} \cong X_*(T) \rtimes W_0$, we see that $K \backslash G(F) / K$ can be identified with the set of orbits of W_0 on $X_*(T)$. (This is the Cartan decomposition—cf. [6, 2.5].) A set of representatives

in $X_*(T)$ for these orbits is the set X_+ of dominant co-characters, i.e., co-characters λ such that $\langle \alpha, \lambda \rangle \geq 0$ for all α in Φ^+ . The isomorphism $\tilde{W} \cong X_*(T) \times W_0$ can be chosen to satisfy the condition that the image in \tilde{W} of the element $\lambda(\pi)$ of $T(F)$ is $t(\lambda)$. Thus we have a one-to-one correspondence

$$X_+ \longrightarrow K \backslash G(F) / K$$

given by the map $\lambda \mapsto Kt(\lambda)K = K\lambda(\pi)K$.

Now given $\lambda \in X_*(T)$, there is a unique element w_λ in W_0 such that $l(t(\lambda)w_\lambda) = \min_{w \in W_0} l(t(\lambda)w)$ (see [6, §1.9]). It can be shown (see [7, §2.2]) that the representative of the double coset $W_0t(\lambda)W_0$ of shortest length is $t(\lambda)w_\lambda$. Thus $[W_0 \backslash \tilde{W} / W_0]$ is equal to the set $\{t(\lambda)w_\lambda \mid \lambda \in X_+\}$.

Since W_0 acts on $X_*(T)$, we may consider the stabilizer W_0^λ in W_0 of an element λ of $X_*(T)$. It is easily seen that $W_0^\lambda = W_0^{t(\lambda)w_\lambda W_0}$ so that $[W_0 / W_0^\lambda] = [W_0 / W_0^{t(\lambda)W_0}]$.

Fix λ in X_+ . Applying Theorem 5.2 with $P_1 = P_2 = K$ and $\sigma = t(\lambda)w_\lambda$ we arrive at the following results.

Proposition 6.1. *The double coset*

$$K\lambda(\pi)K = Kt(\lambda)K$$

is equal to the disjoint union

$$\coprod_{\tau \in [W_0 / W_0^\lambda]} \coprod_{\nu \in R^{l(\tau t(\lambda)w_\lambda)}} g_{\tau t(\lambda)w_\lambda}(\nu)K.$$

Corollary 6.2. *The number of left (or right) cosets of K in $K\lambda(\pi)K$ is*

$$\begin{aligned} q^{l(t(\lambda)w_\lambda)} \sum_{\tau \in [W_0 / W_0^\lambda]} q^{l(\tau)} &= q^{\min_{w \in W_0} l(t(\lambda)w)} \sum_{\tau \in [W_0 / W_0^\lambda]} q^{l(\tau)} \\ &= \sum_{\gamma \in W_0 t(\lambda)W_0} q^{l(\gamma)} \Big/ \sum_{w \in W_0} q^{l(w)}. \end{aligned}$$

Proof. This follows from Corollary 5.8 and the definition of w_λ . □

Example 6.3. In [8], for compact forms G of PGSp_4 over \mathbb{Q} corresponding to various quaternion algebras, we determined the actions of spherical Hecke algebras on $G(\mathbb{Q}_p)$ for certain primes p at which G is split. This necessitated computations of the above kind for the simple adjoint group $G = \text{PGSp}_4$ of type $C_2 = B_2$. Examples of such computations follow.

We carry over the notation from the example in the previous section. Since PGSp_4 is of adjoint type (i.e., $X_*(T)$ is the lattice dual to the lattice generated by Φ), the fundamental co-weights $\check{\omega}_1, \check{\omega}_2$ in Λ (which satisfy $\langle \alpha_i, \check{\omega}_j \rangle = \delta_{ij}$) are in $X_*(T)$. Furthermore, any λ in X_+ is of the form $a_1 \check{\omega}_1 + a_2 \check{\omega}_2$ for some non-negative integers a_1 and a_2 .

If λ in X_+ is 0 then W_0^λ is clearly all of W_0 and $[W_0/W_0^\lambda]$ is trivial. On the other hand, if $\lambda = a_1\check{\omega}_1 + a_2\check{\omega}_2$ for positive a_1, a_2 , then $W_0^\lambda = \langle e \rangle$ and $[W_0/W_0^\lambda] = W_0$. Now suppose that λ is the long co-weight $\check{\omega}_1$. We have $W_0^{\check{\omega}_1} = \langle w_2 \rangle$ and $[W_0/W_0^{\check{\omega}_1}]$ is the set $\{e, w_1, w_2w_1, w_1w_2w_1\}$. Also, $t(\check{\omega}_1)$ can be shown to have the reduced expression $w_0w_1w_2w_1$. A reduced expression for $t(\check{\omega}_1)w_{\check{\omega}_1}$ can be obtained from $t(\check{\omega}_1)$ by dropping the last three involutions in this expression (which are contained in W_0) to yield w_0 .

We therefore have that $K\check{\omega}_1(\pi)K$ is the disjoint union of the double cosets

$$\begin{aligned} &Iw_0K \\ &Iw_1w_0K \\ &Iw_2w_1w_0K \\ &Iw_1w_2w_1w_0K. \end{aligned}$$

It follows that $K\check{\omega}_1(\pi)K$ is the union of the $q + q^2 + q^3 + q^4 = q \cdot \frac{q^4-1}{q-1}$ cosets

$$\begin{aligned} &g_{w_0}(\nu_1)K && (\nu_1 \in R) \\ &g_{w_1w_0}(\nu_2)K && (\nu_2 \in R^2) \\ &g_{w_2w_1w_0}(\nu_3)K && (\nu_3 \in R^3) \\ &g_{w_1w_2w_1w_0}(\nu_4)K && (\nu_4 \in R^4). \end{aligned}$$

For the short co-weight $\check{\omega}_2$, we have $W_0^{\check{\omega}_2} = \langle w_1 \rangle$ and $[W_0/W_0^{\check{\omega}_2}]$ is the set $\{e, w_2, w_1w_2, w_2w_1w_2\}$. A reduced expression for $t(\check{\omega}_2)$ is $w_0w_1w_0\rho = \rho w_2w_1w_2$, and for $t(\check{\omega}_2)w_{\check{\omega}_2}$ is ρ . $K\check{\omega}_2K$ is the union of the cosets

$$\begin{aligned} &I\rho K \\ &Iw_2\rho K \\ &Iw_1w_2\rho K \\ &Iw_2w_1w_2\rho K. \end{aligned}$$

Therefore, $K\check{\omega}_2K$ is the union of the $1 + q + q^2 + q^3 = \frac{q^4-1}{q-1}$ left cosets

$$\begin{aligned} &\rho K \\ &g_{w_2\rho}(\nu_1)K && (\nu_1 \in R) \\ &g_{w_1w_2\rho}(\nu_2)K && (\nu_2 \in R^2) \\ &g_{w_2w_1w_2\rho}(\nu_3)K && (\nu_3 \in R^3). \end{aligned}$$

Remark 6.4. If \overline{G} is the group over k obtained by taking the special fiber of \underline{G} , denote by \overline{B}_λ the standard parabolic subgroup of \overline{G} corresponding to the co-character λ and the choice of positive roots Φ^+ . In [8], it is shown using the formula in Corollary 6.2 that the index $[K\lambda(\pi)K : K]$ is equal to

$$\frac{\#(\overline{G}/\overline{B}_\lambda)(k)}{q^{\dim(\overline{G}/\overline{B}_\lambda)(k)}} \cdot q^{\langle 2\delta, \lambda \rangle},$$

where $\delta = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ and $\langle \cdot, \cdot \rangle$ is the standard pairing between $X_*(T)$ and $X^*(T)$.

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