

DIAGONALIZABLE DERIVATIONS OF FINITE-DIMENSIONAL ALGEBRAS II

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When an algebra is graded by a group, any additive character of the group induces a diagonalizable derivation of the ring. This construction is studied in detail for the case of a path algebra modulo relations and its fundamental group. We describe an injection of the character group into the first cohomology group following Assem-de la Peña. Rather general conditions are determined, in this context, which guarantee that a diagonalizable derivation is induced from the fundamental group.

This paper is the second installment in a series devoted to diagonalizable derivations. Suppose that R is a finite-dimensional algebra over the field k . We will denote by $\text{Der}(R)$ the space of k -algebra derivations from R to itself. Diagonalizable derivations arise naturally whenever R is graded by a group H . Indeed, every additive character $\chi \in \text{Hom}(H, k^+)$ can be assigned a derivation $D_\chi \in \text{Der}(R)$ according to the rule

$$D_\chi(r) = \chi(g)r$$

for every $r \in R$ in the homogeneous component of “degree” $g \in H$. Obviously, D_χ is diagonalizable. Conversely, if D is a diagonalizable derivation of the k -algebra S and H is the additive subgroup of k generated by the eigenvalues of D then, for the inclusion map $\iota : H \rightarrow k^+$, we have $D = D_\iota$.

In our first paper [FGGM], we proved that the span of all diagonalizable derivations of R comprise a Lie ideal of $\text{Der}(R)$ whenever k has characteristic zero or is algebraically closed of positive characteristic. This result turned out to be a powerful tool in describing what we called *spanned-by-split* derivations (i.e., those which are sums of diagonalizable derivations) in several classes of algebras. In what follows, we shall use the notation

$$\text{SPDer}(R)$$

for the subspace of $\text{Der}(R)$ consisting of spanned-by-split derivations.

This paper describes some of the examples which motivated our original paper. Consider R presented as $k\Gamma/I$, a path algebra modulo relations, graded by the fundamental group $\pi_1(\Gamma, I)$. In section [one](#), we review and

clarify notions of fundamental group. The [second](#) section is devoted to tightening two results of Assem and de la Peña:

- The map which assigns to each character in $\text{Hom}(\pi_1(\Gamma, I), k^+)$ a derivation in $\text{SPDer}(k\Gamma/I)$ is injective.
- The induced map of $\text{Hom}(\pi_1(\Gamma, I), k^+)$ to $H^1(k\Gamma/I)$ is injective.

Next we give a partial characterization of those diagonalizable derivations which arise in the form D_χ . Essentially, we require that the underlying algebra have its radical stabilized by the derivation and that the algebra of constants for the derivation be indecomposable.

We close with a short [section](#) placing our constructions in the context of Hopf algebras.

1. Fundamental Groups.

Let Γ be a finite connected directed graph. Temporarily forget about the orientation of arrows, obtaining the undirected graph Γ^{un} . If α is a walk from vertex x to vertex y and β is a walk from vertex y to vertex z then the concatenation $\alpha\beta$ is a walk from x to z and the reverse α^{-1} is a walk from y to x . Consider the “homotopy relation”, the smallest equivalence relation on walks which is compatible with right and left concatenation (whenever they make sense) and for which $\sigma\sigma^{-1}$ is equivalent to the trivial walk at w for any walk σ beginning at w . For a fixed vertex x , the classes of closed walks from x to itself comprise the fundamental group, $\pi_1(\Gamma)$. (Different choices of x yield isomorphic groups.)

We will need a more traditional description of the fundamental group. Fix a vertex v , the “base point”. For each vertex $w \in \Gamma$, choose a walk $\gamma_{v,w}$ from v to w in the underlying undirected graph Γ^{un} ; we require that $\gamma_{v,v}$ be the empty walk from v to itself. The set

$$\gamma = \{\gamma_{v,w} \mid w \text{ is a vertex}\}$$

will be referred to as a choice of *parade data*. If f is any walk in Γ^{un} from x to y then we define

$$c_\gamma(f) = \gamma_{v,x} f \gamma_{v,y}^{-1},$$

the walk which begins at v , takes the parade route to x , follows f from x to y , and then reverses the parade route to return to v from y . Observe that if g is already a closed walk from v to v then $c_\gamma(g) = g$. Also, every $c_\gamma(f)$ lies in the subgroup generated by

$$\{c_\gamma(a) \mid a \text{ is an arrow in } \Gamma\}.$$

Suppose the parade walk $\gamma_{v,w}$ is a sequence

$$a_1^{\varepsilon(1)}, a_2^{\varepsilon(2)}, \dots, a_t^{\varepsilon(t)}$$

of edges where each a_j is an arrow in Γ and $\varepsilon(j) = \pm 1$ according to whether the original orientation is preserved or reversed in the walk. Then

$$c_\gamma(a_1)^{\varepsilon(1)}c_\gamma(a_2)^{\varepsilon(2)} \dots c_\gamma(a_t)^{\varepsilon(t)} = 1.$$

We call the word on the left-hand side of the last equation a *parade walk relator*.

Given fixed parade data γ , the earlier remark about closed walks through v implies that $\pi_1(\Gamma)$ is generated by $\{c_\gamma(a) \mid a \text{ is an arrow}\}$. It is less obvious that $\pi_1(\Gamma)$ is the free group on the formal symbols $c_\gamma(a)$ modulo the parade walk relators. (Let F_γ be the free group on arrows modulo the parade walk relations for γ . The obvious map from walks in Γ^{un} to F_γ respects the homotopy equivalence relation. If we regard $\pi_1(\Gamma)$ as the group of equivalence classes of closed walks through v then the restriction of the factored map is a group homomorphism $\phi : \pi_1(\Gamma) \rightarrow F_\gamma$. For any arrow a from x to y ,

$$\begin{aligned} \phi(c_\gamma(a)) &= \phi(\gamma_{v,x}a\gamma_{v,y}^{-1}) \\ &= \phi(\gamma_{v,x})\phi(a)\phi(\gamma_{v,y})^{-1} \\ &= \phi(a) \\ &= \bar{a} \end{aligned}$$

where \bar{a} is the image of the symbol a in F_γ . Thus ϕ is surjective. But $\pi_1(\Gamma)$ is generated by the collection of all such $c_\gamma(a)$ and they are subject to the parade walk relations. It follows that ϕ is an isomorphism.) When the context is clear, we drop the subscript γ .

Suppose that I is an ideal of the path algebra $k\Gamma$ and that I is generated as an ideal by a set of relations ρ . The fundamental group $\pi_1(\Gamma, \rho)$ will turn out to be a certain image of $\pi_1(\Gamma)$. While it is possible to describe $\pi_1(\Gamma, \rho)$ abstractly ([S]), we will assume that $\pi_1(\Gamma)$ is already described using parade data γ . Let $N(\rho)$ be the normal subgroup of $\pi_1(\Gamma)$ generated by $c_\gamma(p)c_\gamma(q^{-1})$ as p and q range over all paths in the support of the same member of ρ . Then

$$\pi_1(\Gamma, \rho) = \pi_1(\Gamma)/N(\rho).$$

We denote the canonical homomorphism from $\pi_1(\Gamma)$ to $\pi_1(\Gamma, \rho)$ (which depends on γ) by ξ .

A choice of parade data γ induces a $\pi_1(\Gamma, \rho)$ -grading on $k\Gamma/I$. Explaining this gives us the opportunity to introduce the useful notion of weight ([G]). Suppose that Γ is a finite directed graph and H is a group. A *weight function* for Γ with values in H is an assignment W from the arrows of Γ to H . If we extend W multiplicatively so that vertices have weight $1 \in H$ then the domain of the extension (also called W) consists of all directed paths in Γ . The weight function now induces an H -grading on $k\Gamma$. We say that an ideal I of $k\Gamma$ is *homogeneous for W* provided it is homogeneous with respect to this grading. For such an ideal, the weight induces a grading on $k\Gamma/I$. In

the case of fundamental groups, we can consider the weight with values in $\pi_1(\Gamma, \rho)$ which sends an arrow a to $\xi(c_\gamma(a))$. The ideal I is homogeneous by the construction of $N(\rho)$.

Unfortunately, $\pi_1(\Gamma, \rho)$ is dependent on the choice of relations for I . This can be remedied as follows. We say that a nonzero element $r \in I$ is *support minimal* if it cannot be written as a sum of two elements of I , each of whose supports are proper subsets of the support of r . (If $\sum \alpha_p p \in I$ is support minimal where the sum runs over paths p with scalars $\alpha_p \neq 0$ then we cannot erase any summands and maintain the subsum in I .) It is an immediate consequence of the next proposition that the fundamental group is the same for any two choices of ρ which consist of support minimal relations; this common group is denoted $\pi_1(\Gamma, I)$.

Proposition 1.1. *Let ρ be any generating set for I and suppose that $s \in I$ is support-minimal. Then $c(p)c(q)^{-1} \in N(\rho)$ for all p and q in the support of s .*

Proof. By definition, if $r \in \rho$ and σ is a path in the support of r then

$$c(\tau) \equiv c(\sigma) \pmod{N(\rho)}$$

for all τ in the support of r . We will abuse notation and write

$$\text{supp}(r) \equiv c(\sigma) \pmod{N(\rho)}.$$

For any two paths α and β , we then have

$$\text{supp}(\alpha r \beta) \equiv c(\alpha \sigma \beta) \pmod{N(\rho)}.$$

An arbitrary $s \in I$ is a linear combination of expressions $\alpha r \beta$ for paths α, β and for $r \in \rho$. Given $d \in \pi_1(\Gamma, \rho)$, set $s(d)$ to be the subcombination of all those $\alpha r \beta$ whose support lies in d , regarded as a coset. Then

$$s = \sum_d s(d)$$

with $s(d) \in I$ and the supports of the $s(d)$ pair-wise disjoint. Hence if s is support-minimal it must be equal to a single $s(d)$. \square

We mention one last time that we will only be able to speak about a $\pi_1(\Gamma, I)$ -grading of $k\Gamma/I$ in the presence of parade data γ . Thus if $\Psi \in \text{Hom}(\pi_1(\Gamma, I), k^+)$, then the induced derivation defined in the introduction depends on some choice of γ and, so, will frequently be written $D_{\Psi, \gamma}$.

2. Injectivity Theorems.

In this section, make the standing assumption that I is an admissible ideal of $k\Gamma$. (That is, we assume that $k\Gamma/I$ is finite-dimensional and I lies inside the square of the ideal generated by all arrows.)

Proposition 2.1. *Choose parade data for the connected directed graph Γ . Then D_Ψ is diagonalizable for every $\Psi \in \text{Hom}(\pi_1(\Gamma, I), k^+)$. Moreover the map*

$$\text{Hom}(\pi_1(\Gamma, I), k^+) \rightarrow \text{SPDer}(k\Gamma/I)$$

is injective.

Proof. It is obvious that D_Ψ is diagonalizable, so $\text{Hom}(\pi_1(\Gamma, I), k^+)$ maps into $\text{SPDer}(k\Gamma)$. Thus the issue is injectivity. By the standing assumption, every arrow in Γ survives modulo I . Since $\pi_1(\Gamma, I)$ is generated by

$$\{\xi(c(a)) \mid a \text{ is an arrow}\},$$

we see that $\pi_1(\Gamma, I)$ is generated by degrees which genuinely occur. (In the literature, the grading is sometimes referred to as *full*.)

However if R is any H -graded algebra and

$$\{h \in H \mid \text{the } h\text{-component of } R \text{ is not } 0\}$$

generates H then the map $\text{Hom}(H, k^+) \rightarrow \text{Der}(R)$ is always injective. \square

The next theorem provides a more significant injectivity result, which is based on a similar statement of Assem-de la Peña ([AP]).

Theorem 2.1. *Suppose $\chi \in \text{Hom}(\pi_1(\Gamma, I), k^+)$ and fix parade data for $\pi_1(\Gamma)$. If the associated derivation D_χ on $k\Gamma/I$ is inner then $\chi = 0$. Hence the induced map*

$$\text{Hom}(\pi_1(\Gamma, I), k^+) \rightarrow H^1(k\Gamma/I)$$

is injective.

Proof. We shall write $\Lambda = k\Gamma/I$. Then $\Lambda = \Lambda_0 \oplus \text{rad}\Lambda$ where we identify Λ_0 with $(k\Gamma)_0$: A commutative subalgebra with basis consisting of orthogonal idempotents $e(w)$, one for each vertex w of Γ .

For $s = \sum_w \lambda_w e(w) \in \Lambda_0$ we compute $\text{ad } s$. If $\bar{m} \in \Lambda$ is the image of a path m in Γ from vertex x to vertex y then

$$(\text{ad } s)(\bar{m}) = (\lambda_x - \lambda_y)\bar{m}.$$

Thus $\text{ad } s$ is always diagonalizable and all images of paths are among its eigenvectors.

Now suppose that $\chi \in \text{Hom}(\pi_1(\Gamma, I), k^+)$ and $D_\chi = \text{ad } b$ for some $b \in \Lambda$. Set $b = s + n$ for $s \in \Lambda_0$ and $n \in \text{rad}\Lambda$. The image \bar{m} of every path is an eigenvector for D_χ corresponding to eigenvalue $(\chi \circ \xi)(c(m))$. Thus $\text{ad } b$ is diagonalizable with a basis of eigenvectors which are images of paths. It follows that $\text{ad } b$ and $\text{ad } s$ must commute. But then $(\text{ad } b) - (\text{ad } s)$ is diagonalizable at the same time that it is equal to $\text{ad } n$, which is nilpotent. We conclude that $D_\chi = \text{ad } s$ for $s \in \Lambda_0$.

If $s = \sum_w \lambda_w e(w)$ then $(\chi \circ \xi)(c(m)) = \lambda_x - \lambda_y$ for every path m beginning at x and ending at y , whose image \bar{m} is nonzero. In particular, if a is an

arrow in Γ from x to y and $\varepsilon = \pm 1$ the $(\chi \circ \xi)(c(\bar{a})) = \varepsilon \cdot (\lambda_x - \lambda_y)$. It follows that if w is an arbitrary vertex in Γ and

$$a_1^{\varepsilon(1)}, \dots, a_t^{\varepsilon(t)}$$

is the parade walk from the base point v to w then

$$(\chi \circ \xi) \left(c \left(a_1^{\varepsilon(1)} \cdots c(a_t)^{\varepsilon(t)} \right) \right) = \lambda_v - \lambda_w.$$

On the other hand, $(\chi \circ \xi)(1) = 0$. We conclude that

$$s = \lambda_v \left(\sum_w e(w) \right) = \lambda_v \cdot 1.$$

Therefore $D_\chi = \text{ad } s = 0$. □

Corollary 2.1 ([BM]). *Let $\Lambda = k\Gamma/I$ be a path-monomial algebra which is finite dimensional. If $H^1(\Lambda, \Lambda) = 0$ then Γ is a tree.*

Proof. We are assuming that the generating set ρ for I consists of monomials. As a consequence, $\pi_1(\Gamma, I)$ is a free group; it is trivial if and only if Γ is a tree. Thus if Γ is not a tree then $\text{Hom}(\pi_1(\Gamma, I), k^+)$ is nonzero. By the theorem, H^1 is nonzero. □

3. Fundamental Derivations.

The argument presented in the previous theorem rests on the following property of the diagonalizable derivation D_χ . For any pair of vertices x and y there exists an undirected walk $a_1^{\varepsilon(1)}, \dots, a_t^{\varepsilon(t)}$ such that $\sum_j \varepsilon(j) D_\chi(\bar{a}_j) = 0$. This property turns out to be crucial in trying to characterize those diagonalizable derivations which arise from a fundamental group.

Definition 3.1. Let R be a finite-dimensional k -algebra. We say that a derivation $D \in \text{Der}(R)$ is **fundamental** provided that there exists a finite directed graph Γ and an admissible ideal I of $k\Gamma$ such that $R \simeq k\Gamma/I$ and there is parade data γ together with some $\Psi \in \text{Hom}(\pi_1(\Gamma, I), k^+)$ so that

$$D = D_{\Psi, \gamma}$$

under the identification given by the isomorphism.

There are some obvious things we can say about a fundamental derivation D of R . First, D must be diagonalizable. Indeed, the images of paths in R are all eigenvectors. As another consequence, $D(\text{rad}R) \subseteq \text{rad}R$. Notice that the algebra R is k -elementary, which means that $R/\text{rad}R$ is a finite product of copies of k . More is true: The algebra complement in $k\Gamma$ to the ideal generated by all arrows, which coincides with the span of the vertex

idempotents, survives as an algebra complement to $\text{rad}R$ in R . Thus $\text{rad}R$ has an algebra complement which lies inside the “subalgebra of constants”

$$R^D = \{r \in R \mid D(r) = 0\}.$$

We shall see that these properties come close to characterizing fundamental derivations.

Lemma 3.1. *Let W be an H -valued weight on the arrows of Γ and let I be a W -homogeneous ideal of $k\Gamma$. Suppose that for a fixed vertex x and every other vertex y there exists a walk in Γ^{un} ,*

$$\gamma_{x,y} : b_1^{\varepsilon(1)}, \dots, b_t^{\varepsilon(t)}$$

from x to y such that

$$W(b_1)^{\varepsilon(1)} \dots W(b_t)^{\varepsilon(t)} = 1$$

in H . Then there is a homomorphism $\theta : \pi_1(\Gamma, I) \rightarrow H$ such that

$$(\theta \circ \xi)(c_\gamma(a)) = W(a)$$

for all arrows a .

Proof. As we remarked earlier, $\pi_1(\Gamma)$ is isomorphic to the free group on $\{c_\gamma(a) \mid a \text{ is an arrow of } \Gamma\}$ modulo the parade walk relators

$$\{c_\gamma(\gamma_{x,y}) \mid x \neq y\}.$$

Hence W induces a group homomorphism $\theta : \pi_1(\Gamma) \rightarrow H$ such that

$$\theta(c_\gamma(a)) = W(a)$$

for all arrows a .

Suppose that ρ is a support-minimal set of relations for the homogeneous ideal I . We claim that the elements of ρ are homogeneous. If $r \in \rho$ write $r = \sum_h r_h$ where each r_h is a nontrivial linear combination of paths with weight h . By homogeneity, each r_h lies in I . But the support of r_h is clearly a subset of the support of r . Hence $r = r_h$ for some choice of h .

It follows that if p and q are paths in the support of some r in ρ then $W(p) = W(q)$. Therefore θ is the identity on $N(\rho)$, the normal subgroup generated by all possible $c_\gamma(p)c_\gamma(q)^{-1}$ of this sort. We conclude that θ factors through $\pi_1(\Gamma, I)$. \square

Theorem 3.1. *Assume that I is an admissible ideal of $k\Gamma$. Suppose that*

- (a) *E is a diagonalizable derivation of $k\Gamma/I$ which vanishes on images of vertices and for which the images of arrows are eigenvectors, i.e., for each arrow a in Γ there is a scalar $\omega(\bar{a})$ such that $E(\bar{a}) = \omega(\bar{a})\bar{a}$;*
- (b) *there is a vertex x such that for every other vertex y there exists a walk*

$$\gamma_{x,y} : b_1^{\varepsilon(1)}, \dots, b_t^{\varepsilon(t)}$$

from x to y such that

$$\sum_j \varepsilon(j)\omega(\bar{b}_j) = 0.$$

Then there is some $\Psi \in \text{Hom}(\pi_1(\Gamma, I), k^+)$ with $E = D_{\Psi, \gamma}$ for $\gamma = \{\gamma_{x,y}\}$.

Proof. We apply the lemma with $H = k^+$. Since I is admissible, we see that different arrows cannot have the same images in $k\Gamma/I$. Thus it makes sense to define a function W on arrows via $W(a) = \omega(\bar{a})$, thereby lifting the k^+ -grading to $k\Gamma$. We conclude from the lemma that there is an additive character Ψ on $\pi_1(\Gamma, I)$ such that

$$(\Psi \circ \xi)(c_\gamma(a)) = \omega(\bar{a}).$$

Thus the two derivations D_Ψ and E agree on the images of arrows and vertices. But these elements generate $k\Gamma/I$ as an algebra. \square

The next result is a close relative to Theorem 3.4 in [G].

Lemma 3.2. *Let R be a finite-dimensional k -elementary algebra. Assume that D is a diagonalizable derivation of R such that $D(\text{rad}R) \subseteq \text{rad}R$ and R^D contains an algebra complement to $\text{rad}R$. Then there exists a finite directed graph Γ , an admissible ideal I of $k\Gamma$, and a derivation \tilde{D} of $k\Gamma$ such that*

- (a) $\tilde{D}(I) \subseteq I$;
- (b) \tilde{D} vanishes on the vertex idempotents of $k\Gamma$;
- (c) each arrow is an eigenvector for \tilde{D} ;
- (d) $R \simeq k\Gamma/I$ with \tilde{D} inducing D .

Proof. Let $e(1), \dots, e(n)$ be orthogonal idempotents whose sum is 1, which span a complement to $\text{rad}R$, and which satisfy $D(e(j)) = 0$ for $j = 1, \dots, n$. Then

$$D(e(i)\text{rad}R e(j)) \subseteq e(i)\text{rad}R e(j)$$

for all i and j . An elementary eigenspace argument using the diagonalizability of D implies that the pair of D -stable spaces

$$e(i)\text{rad}R e(j) \cap (\text{rad}R)^2 \subseteq e(i)\text{rad}R e(j)$$

splits with a D -stable vector space complement $A(i, j)$.

We argue that $\{A(i, j) \mid 1 \leq i, j \leq n\}$ generates $\text{rad}R$ as an algebra. Denote by A the algebra generated by these subspaces. Certainly $A \subseteq \text{rad}R$. If the algebras do not coincide, the nilpotence of $\text{rad}R$ implies that there must be a largest m such that $(\text{rad}R)^m$ does not lie in A . That is, there exist $r_1, \dots, r_m \in \text{rad}R$ such that

$$r_1 r_2 \cdots r_m \notin A.$$

We may assume that $r_j \in e_{f(j)}\text{rad}R e_{g(j)}$ for some choice of indices $f(j)$ and $g(j)$. Write $r_j = a_j + s_j$ with $a_j \in A(f(j), g(j))$ and $s_j \in (\text{rad}R)^2$. Then

$$r_1 r_2 \cdots r_m = a_1 a_2 \cdots a_m + z$$

where $z \in (\text{rad}R)^{m+1}$. In other words, $a_1 \cdots a_m + z \in A$. We have reached the contradiction that $r_1 \cdots r_m \in A$.

We can now describe Γ . Its vertices are the idempotents $e(1), \dots, e(n)$. Choose a basis for each $A(i, j)$ which consists of eigenvectors for D . These basis vectors comprise the set of arrows which begin at $e(i)$ and end at $e(j)$. There is an obvious algebra map from $k\Gamma$ onto R . If a is one of the designated eigenvectors for D in $A(i, j)$ and $D(a) = \lambda a$ then we define the function \tilde{D} on the arrow a by $\tilde{D}(a) = \lambda a$. It is easy to see that \tilde{D} extends uniquely to a derivation of $k\Gamma$ which vanishes on vertex idempotents.

The lemma follows with I the kernel of the map $k\Gamma \rightarrow R$. □

Theorem 3.2. *Let R be a finite-dimensional k -elementary algebra. Suppose that D is a diagonalizable derivation of R with $D(\text{rad}R) \subseteq \text{rad}R$. Suppose, further, that R^D contains an algebra complement to $\text{rad}R$ and that R^D is indecomposable as an algebra. Then D is fundamental.*

Proof. We carry over all of the notation in the previous lemma. Define a new graph G whose vertices are $e(1), \dots, e(n)$ (the vertices of Γ) and construct an arrow from $e(i)$ to $e(j)$ provided $e(i)R^D e(j) \neq 0$. The sum of the $e(h)$ over all those vertex idempotents in a connected component of G is a central idempotent of R^D . Since R^D is indecomposable, G is connected.

We claim that if $e(i)R^D e(j) \neq 0$ then there is a path m in Γ such that m begins at $e(i)$, ends at $e(j)$, and $D(\bar{m}) = 0$. To see this, suppose that $e(i)xe(j) \neq 0$ for $x \in R^D$. Write $x = \sum \alpha_p \bar{p}$ with α_p a nonzero scalar and \bar{p} the image in R of a path p in $k\Gamma$ which begins at $e(i)$ and ends at $e(j)$. We may assume that the \bar{p} which appear in the sum are linearly independent in R . Each such \bar{p} is an eigenvector for D . Since eigenvectors for distinct eigenvalues are linearly independent, we conclude that $D(\bar{p}) = 0$ for every \bar{p} which appears.

We put the previous two paragraphs together. For each $i \neq j$ there is a walk from $e(i)$ to $e(j)$ in G with edge sequence

$$g_1^{\varepsilon(1)}, g_2^{\varepsilon(2)}, \dots, g_v^{\varepsilon(v)} .$$

This walk gives rise to an “expanded” walk

$$m_1^{\varepsilon(1)}, m_2^{\varepsilon(2)}, \dots, m_v^{\varepsilon(v)}$$

from $e(i)$ to $e(j)$ in Γ , where each m_d is a *path* with the same endpoints as g_d and $D(\bar{m}_d) = 0$. If we rewrite the second walk as

$$a_1^{\eta(1)}, a_2^{\eta(2)}, \dots, a_t^{\eta(t)}$$

for arrows a_i in Γ then

$$\sum \eta(i)D(\bar{a}_i) = 0.$$

(The point is that if the sum of the eigenvalues along a path m_d is zero then the same is true for the sum of the negatives of those eigenvalues in reverse order along the path.) For each $2 \leq j \leq n$ pick such a walk $\gamma_{1,j}$ from the base point $e(1)$ to $e(j)$.

According to Theorem 3.1, there exists some $\Psi \in \text{Hom}(\pi_1(\Gamma, I), k^+)$ such that $D = D_{\Psi, \gamma}$. \square

Corollary 3.1. *Assume that k is an algebraically closed field of characteristic zero and R is a finite-dimensional local k -algebra. Then every diagonalizable derivation of R is fundamental.*

Proof. Since k is algebraically closed and R is local, we have

$$R/\text{rad}R \simeq k.$$

It follows that R^D is local, and so, indecomposable. Finally, it is well known that $D(\text{rad}R) \subseteq \text{rad}R$ for any $D \in \text{Der}(R)$, by virtue of $\text{char}k = 0$. \square

4. Hopf Algebras.

We end with a hint that there may be other classes of derivations beside diagonalizable ones for which an interesting theory exists. Every diagonalizable derivation of a k -algebra corresponds to a group grading by a subgroup of the additive group k^+ . Equivalently, every diagonalizable derivation has the form D_χ where $\chi \in \text{Hom}(G, k^+)$ for some group G which grades the algebra. A group grading for an algebra R is an example of an H -comodule algebra action on R , where H is a Hopf algebra. (In the special case, $H = kG$ with the standard Hopf structure.) In general, if H is any Hopf algebra then R is an H -comodule algebra provided that R is a left H -comodule, via

$$\lambda : R \rightarrow H \otimes R$$

(so $\lambda(a) = \sum a_0 \otimes a_1$ for $a \in R$) and

$$\begin{aligned} \lambda(ab) &= \sum a_0 b_0 \otimes a_1 b_1 \quad \text{for } a, b \in R; \\ \lambda(1) &= 1 \otimes 1. \end{aligned}$$

See [Mo], Section 4.1 for more details. In the particular case of the group algebra, R is a kG -comodule algebra if and only if it is G -graded as an algebra.

If ϵ is the augmentation for H then those functionals which are ϵ -derivations,

$$\text{Der}_k^\epsilon(H, k) = \{f \in H^* \mid f(ab) = \epsilon(a)f(b) + f(a)\epsilon(b)\},$$

comprise a Lie algebra under the commutator $[f, g] = f * g - g * f$. (Here $*$ is convolution on H^* .) When $H = kG$ then

$$\text{Der}_k^\epsilon(kG, k) = \text{Hom}(G, k^+).$$

Back to the general set-up, for each $f \in \text{Der}_k^\epsilon(H, k)$ define $D_f \in \text{Hom}_k(R, R)$ by

$$D_f(a) = \sum f(a_0)a_1$$

for all $a \in R$. We leave it as an exercise that $D_f \in \text{Der}(R)$ and the map $\text{Der}_k^\epsilon(H, k) \rightarrow \text{Der}(R)$ sending f to D_f is a Lie algebra homomorphism. This construction subsumes our earlier D_χ .

The subspace of derivations spanned by all D_f , as one runs over all comodule algebra actions of one or more Hopf algebras H , deserves future scrutiny.

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