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CORRECTION TO "ON A THEOREM OF KOCH"

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As was pointed out to the author by E. Benjamin, the proof of Theorem 1 is incomplete. The argument in the penultimate sentence in that proof fails to take into account, for example, the case

$$p_1 \equiv 3 \bmod 4, \quad p_j \equiv 1 \bmod 4 \qquad (j \geq 2),$$
$$\left(\frac{p_i}{p_j}\right) = \begin{cases} +1 & \text{if } \ i = 1, \ j \neq i \\ -1 & \text{if } \ i > 1, \ j \neq i. \end{cases}$$

A treatment of some, but not all, missing cases can be found in a preprint by Benjamin ("On Imaginary Quadratic Number Fields with 2-Class Group of Rank 4 and infinite 2-Class Field Tower," 1999). We describe here a revision of the original argument which treats all cases simultaneously, thus completing the proof of Theorem 1.

In the sequel, *discriminant* will mean discriminant of a quadratic field. A *prime discriminant* is a discriminant which is divisible by only one prime. A discriminant factors uniquely into a product of prime discriminants. A *factorization* of d is an unordered pair (d_1, d_2) of discriminants whose product is d.

A factorization (d_1, d_2) of d is called a C_4-*factorization* if $(\frac{d_1}{p_2}) = (\frac{d_2}{p_1}) = 1$ for all primes $p_j | d_j, j = 1, 2$. Let e_4 be the 4-rank of the class group of $K = \mathbb{Q}(\sqrt{d})$. The criterion of Rédei [**10**] states that 2^{e_4} is the number of C_4-factorizations of d. There is a natural group law on the set of factorizations of d. Namely, write $d = \prod_{j=1}^t q_j$ as a product of t prime discriminants. If $d_1 = \prod_{j=1}^t q_j^{a_j}$ and $d_1' = \prod_{j=1}^t q_j^{b_j}$ are discriminants dividing d, the product of $(d_1, d/d_1)$ and $(d_1', d/d_1')$ is defined to be $(d_1'', d/d_1'')$, where $d_1'' = \prod_j q_j^{a_j+b_j}$, the exponents $a_j + b_j$ being considered modulo 2. The factorization $(1, d)$ is the identity of the group of factorizations, which is an elementary abelian 2-group of rank $t - 1$. The subset of C_4-factorizations of d is a subgroup isomorphic to $\mathbb{F}_2^{e_4}$.

With these preliminaries in place, we replace the final two sentences of the proof of Theorem 1 with the following.

Suppose the discriminant d of the imaginary quadratic field K has 5 prime divisors. By assumption, $3 \leq e_4 \leq 4$.

Claim. There exists a C_4-factorization (d_1, d_2) of d with d_1 a prime discriminant.

Proof of claim. If $e_4 = 4$, every factorization of d is a C_4-factorization, so we are done. Suppose $e_4 = 3$. There are 8 C_4-factorizations, one of which is the trivial one $(1, d)$. We proceed by contradiction: Suppose the other seven C_4-factorizations are all of type $(q_\alpha q_\beta, q_\gamma q_\delta q_\eta)$. Without loss of generality, we may assume $(q_1q_2, q_3q_4q_5)$ is a C_4-factorization. Suppose $(q_\alpha q_\beta, q_\gamma q_\delta q_\eta)$ is another C_4-factorization. If $\{\alpha, \beta\} \cap \{1, 2\}$ were empty, then the product of these two C_4-factorizations would be $(q_1q_2q_\alpha q_\beta, q_\gamma)$, a contradiction. So we may assume without loss of generality that $\alpha = 1$ and $\beta = 3$. Choose another C_4-factorization $(q_\sigma q_\tau, q_\lambda q_\mu q_\nu)$ not in the span of $(q_1q_2, q_3q_4q_5)$ and $(q_1q_3, q_2q_4q_5)$. Repeating the previous argument twice, we have $\{\sigma, \tau\} \cap \{1, 2\}$ and $\{\sigma, \tau\} \cap \{1, 3\}$ are non-empty, and therefore must both equal $\{1\}$. Hence, we may assume $\sigma = 1$, $\tau = 4$. Thus the group of C_4-factorizations is generated by $(q_1q_2, q_3q_4q_5)$, $(q_1q_3, q_2q_4q_5)$ and $(q_1q_4, q_2q_3q_5)$. But the product of these is $(q_1q_2q_3q_4, q_5)$ a contradiction. This completes the proof of the claim.

By the claim, we may assume, after a possible renumbering, that $(q_1, q_2q_3q_4q_5)$ is a C_4-factorization. Let $F = \mathbb{Q}(\sqrt{q_1})$ and $E = F(\sqrt{d})$. In E/F, 8 prime ideals of F (those dividing q_2, q_3, q_4, and q_5) ramify. By genus theory, the 2-rank of the ideal class group of E is at least 6, which implies (by the Golod-Shafarevich theorem) that E has an infinite unramified 2-class field tower (see, for example, the last sentence in Section 2 of Martinet [**8**]). Since E/K is an unramified quadratic extension, the 2-class field tower of K is also infinite. This completes the proof of Theorem 1.

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