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PERIODIC FLAT MODULES, AND FLAT MODULES FOR FINITE GROUPS

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If R is a ring of coefficients and G a finite group, then a flat RG-module which is projective as an R-module is necessarily projective as an RG-module. More generally, if H is a subgroup of finite index in an arbitrary group Γ, then a flat $R\Gamma$-module which is projective as an RH-module is necessarily projective as an $R\Gamma$-module. This follows from a generalization of the first theorem to modules over strongly G-graded rings. These results are proved using the following theorem about flat modules over an arbitrary ring S: If a flat S-module M sits in a short exact sequence $0 \to M \to P \to M \to 0$ with P projective, then M is projective. Some other properties of flat and projective modules over group rings of finite groups, involving reduction modulo primes, are also proved.

1. Introduction.

In the representation theory of finite groups, a great deal of attention has been given to the problem of determining whether a module over a group ring is projective. For example, a well known theorem of Chouinard [**13**] states that a module is projective if and only if its restriction to each elementary abelian subgroup is projective. A theorem of Dade [**16**] states that over an algebraically closed field of characteristic p, a finitely generated module for an elementary abelian p-group is projective if and only if its restriction to each cyclic shifted subgroup is projective, where a cyclic shifted subgroup is a certain kind of cyclic subgroup of the group algebra. For an infinitely generated module, the statement is no longer valid, but in [**8**] it is proved that an infinitely generated module is projective if and only if its restriction to each cyclic shifted subgroup defined over each extension field is projective. These theorems have formed the basis for the development of the theory of varieties for modules [**2**, **7**, **8**, **10**].

The purpose of this paper is to develop further ways of recognizing projective and flat modules over group rings. We begin with a general theorem about flat modules over an arbitrary ring.

Theorem 1.1 (= Theorem 2.5)**.** *Let R be a ring and M a flat R-module. If there exists a short exact sequence*

$$0 \to M \to P \to M \to 0$$

with P a projective R-module, then M itself is projective.

We then concentrate on flat modules over a group ring. Our main theorem on this subject is the following characterization of when a flat module over a group ring of a finite group is projective. This answers the main question posed in [**6**].

Theorem 1.2 (= Theorem 3.4)**.** *Let R be a ring of coefficients (not necessarily commutative) and G a finite group. If M is a flat RG-module which is projective as an R-module, then M is projective as an RG-module.*

The proof is relatively short, using Theorem 2.5, and can be found in Section 3. The proof, in fact, gives rather more, and generalizes naturally to strongly group graded rings, as follows.

Theorem 1.3 (= Theorem 4.5)**.** *Let G be a finite group, S a strongly G-graded ring, and R the identity component of S. If M is a flat S-module which is projective as an R-module, then M is projective as an S-module.*

A particular class of strongly G-graded rings consists of the crossed products $R{*}G$, so the theorem applies in that case. The case of crossed products applies in particular when G is a finite quotient of another group, which allows us to deduce the following result.

Corollary 1.4 (= Corollary 4.8)**.** *Let R be a ring of coefficients (not necessarily commutative) and H a subgroup of finite index in a group Γ (not necessarily finite). If M is a flat $R\Gamma$-module which is projective as an RH-module, then M is projective as an $R\Gamma$-module.*

The proof of Theorem 1.2 also generalizes to infinite groups, to give a theorem which is most easily stated in the terminology of [**4**, **5**].

Theorem 1.5 (= Theorem 5.2)**.** *Let R be a ring of coefficients (not necessarily commutative) and Γ a group (not necessarily finite). If M is a flat cofibrant $R\Gamma$-module, then M is projective.*

A secondary purpose of this paper is to collect various other facts about flat and projective modules over group rings of finite groups. The following two theorems occupy Sections 6 and 7. In fact, the proofs we give work in the context of subgroups of finite index in arbitrary groups, but we only state the restricted forms here.

Theorem 1.6 (See Theorems 6.1 and 7.3)**.** *Let R be a ring of coefficients and G a finite group. An RG-module M is projective (respectively, flat) if and only if*

(i) *M is projective (respectively, flat) as an R-module, and*
(ii) *M/pM is projective (respectively, flat) as an $(R/pR)G$-module for each prime number p dividing $|G|$.*

On the basis of this, it is tempting to suppose that it might be true that an RG-module M is projective if and only if it is projective as an R-module, and $K \otimes_R M$ is projective as a KG-module for every field K containing $R/\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subseteq R$. In Section 8, we give an example to show that this is false in general, even if R is a discrete valuation ring. In Section 9, on the other hand, we show that the statement is true if R is a Dedekind domain whose field of fractions has characteristic prime to $|G|$, such as a ring of algebraic integers, or $\mathfrak{p}$-local integers, or $\mathfrak{p}$-adic integers.

We use the notation $\mathrm{p.dim}_R M$ to denote the projective dimension of a module M over a ring R, and $\mathrm{w.dim}_R M$ to denote the weak dimension, namely the smallest length of a resolution of M by flat modules.

2. Periodic flat modules.

The purpose of this section is to prove that periodic flat modules do not exist, over any ring. Throughout this section, all modules are modules over a ring R which has an identity element, but which is not necessarily commutative.

Lemma 2.1 (Villamayor)**.** *Let M be a submodule of a projective R-module P. The following conditions are equivalent:*

(a) *P/M is flat.*
(b) *For each $x \in M$, there exists $f \in \mathrm{Hom}_R(P, M)$ such that $f(x) = x$.*
(c) *For each $x_1, \dots, x_n \in M$, there exists $f \in \mathrm{Hom}_R(P, M)$ such that $f(x_i) = x_i$ for $i = 1, \dots, n$.*

Proof. See Chase [**12**], Proposition 2.2 (where the result first appeared), or Rotman [**27**], Theorem 3.57. □

The following proposition and lemma are, with slight modifications, special cases of a result of Osofsky [**26**], Theorem 1.3; we provide direct proofs for the reader's convenience. In particular, the proposition implies the well known fact that countably related flat modules have projective dimension at most one ([**19**], Lemma 2).

Proposition 2.2. *Let M be a countably generated submodule of a projective R-module P. If P/M is flat, then M is projective.*

Proof. Let $x_1, x_2, \dots$ be a countable sequence of generators for M. By induction and Lemma 2.1, there exist $f_1, f_2, \dots$ in $\mathrm{Hom}_R(P, M)$ such that

$$(1 - f_n)(1 - f_{n-1}) \cdots (1 - f_1)(x_i) = 0$$

for $n \geq i$. It follows that the union of the kernels of the homomorphisms

$$g_n = (1 - f_n)(1 - f_{n-1}) \cdots (1 - f_1)|_M$$

equals M. Hence, there is a homomorphism $g : M \to P^{(\omega)} = P \oplus P \oplus \cdots$ given by the rule

$$g(x) = (x, g_1(x), g_2(x), \dots).$$

Let $f : P^{(\omega)} \to M$ be the homomorphism given by

$$f(a_1, a_2, \dots) = f_1(a_1) + f_2(a_2) + \cdots .$$

For all n, we have $g_n = (1 - f_n)g_{n-1}$ (where $g_0 = 1$), and so $f_n g_{n-1} = g_{n-1} - g_n$. Hence, for $x \in M$ we see that

$$\begin{aligned} fg(x) &= f_1(x) + f_2 g_1(x) + f_3 g_2(x) + \dots \\ &= x - g_1(x) + g_1(x) - g_2(x) + g_2(x) - g_3(x) + \cdots = x. \end{aligned}$$

Thus $fg = \mathrm{id}_M$, and therefore M is projective. □

Lemma 2.3. *Let M be a submodule of a projective R-module P, and suppose that P/M is flat. Then any countably generated submodule of M is contained in a countably generated submodule K of M such that P/K is flat.*

Proof. Since it is harmless to replace M and P by $M \oplus 0$ and $P \oplus Q$ for any projective module Q, there is no loss of generality in assuming that P is free, say with basis X.

Given a countably generated submodule $K_0 \subseteq M$, choose a countable subset $Y \subseteq X$ such that K_0 is contained in the submodule P_0 of P generated by Y. Then P_0 is a direct summand of P. Let $\pi : P \to P_0$ be a projection, so that π is an idempotent endomorphism of P with countably generated image containing K_0.

We claim that M has a countably generated submodule $K_1 \supseteq K_0$ such that for each $x \in K_0$, there exists $f \in \mathrm{Hom}_R(P, K_1)$ with $f(x) = x$.

Let $x_1, x_2, \dots$ be a countable sequence of generators for K_0. By Lemma 2.1, there exist $f_1, f_2, \dots$ in $\mathrm{Hom}_R(P, M)$ such that $f_n(x_i) = x_i$ for $n \geq i$. Since $\pi(x_i) = x_i$ for all i, we may replace f_n by $f_n\pi$. Now each $f_n P$ is countably generated, and so the module

$$K_1 = K_0 + \sum_{n=1}^{\infty} f_n P$$

is a countably generated submodule of M. Any $x \in K_0$ lies in the submodule generated by $x_1, \dots, x_n$ for some n, whence $f_n(x) = x$. Since $f_n P \subseteq K_1$, this establishes the claim.

Now repeat this process ω times, obtaining countably generated submodules $K_0 \subseteq K_1 \subseteq \dots$ of M such that for each $x \in K_t$, there is some

$f \in \operatorname{Hom}_R(P, K_{t+1})$ with $f(x) = x$. Then $K = \bigcup_{t=0}^{\infty} K_t$ is a countably generated submodule of M, and P/K is flat by Lemma 2.1. ☐

Lemma 2.4. *Let $0 \to M \xrightarrow{\subseteq} \bigoplus_{i\in I} P_i \xrightarrow{g} M \to 0$ be a short exact sequence with M flat and the P_i countably generated projective. Given any countable subset $J_0 \subseteq I$ and any countably generated submodule $K_0 \subseteq M$, there exists a countable subset $J \subseteq I$ such that $J_0 \subseteq J$, the submodule $P_J = \bigoplus_{j\in J} P_j$ contains K_0, the image gP_J equals $M \cap P_J$, and gP_J is projective.*

Proof. Set $P_J = \bigoplus_{j\in J} P_j$ and $M_J = M \cap P_J$ for all subsets $J \subseteq I$.

We first show that given J_0 and K_0, there exists a countable subset $J \subseteq I$, containing J_0, such that $K_0 \subseteq gP_J$ and gP_J is projective.

In view of Lemma 2.3, after enlarging K_0 we may assume that P_I/K_0 is flat. Then, after enlarging J_0, we may also assume that $K_0 \subseteq gP_{J_0}$. Now apply Lemma 2.3 again to obtain a countably generated submodule $K_1 \subseteq M$ such that $gP_{J_0} \subseteq K_1$ and P_I/K_1 is flat. Continuing in this manner, we obtain countable subsets $J_0 \subseteq J_1 \subseteq \dots$ in I and countably generated submodules $K_0 \subseteq K_1 \subseteq \dots$ in M such that $K_i \subseteq gP_{J_i} \subseteq K_{i+1}$ and P_I/K_i is flat for all i. Then $J = \bigcup_{i=0}^{\infty} J_i$ is a countable subset of I, and $K_0 \subseteq gP_J = \bigcup_{i=0}^{\infty} K_i$. Since the P_I/K_i are flat, so is P_I/gP_J, and thus gP_J is projective by Proposition 2.2. This establishes the first claim.

We next observe that any countable subset J_0 of I is contained in a countable subset J such that M_J is countably generated. To see this, use the first claim to find a countable subset J of I such that $J_0 \subseteq J$ and gP_J is projective. Hence, the short exact sequence

$$0 \to M_J \to P_J \to gP_J \to 0$$

splits, and so M_J is countably generated.

Now return to arbitrary J_0 and K_0 as in the hypotheses of the lemma. After enlarging J_0, we may assume that $K_0 \subseteq P_{J_0}$. By the second claim above, there exists a countable subset $J_1 \subseteq I$ such that $J_0 \subseteq J_1$ and M_{J_1} is countably generated. By the first claim, there exists a countable subset $J_2 \subseteq J$, containing J_1, such that $M_{J_1} \subseteq gP_{J_2}$ and gP_{J_2} is projective. Since gP_{J_2} is countably generated, it is contained in P_{J_3} for some countable subset $J_3 \subseteq I$, and thus $gP_{J_2} \subseteq M_{J_3}$. It is harmless to enlarge J_3 enough to contain J_2, and the second claim allows us, after a further enlargement, to assume that M_{J_3} is countably generated.

Continuing in this manner, we obtain countable subsets $J_0 \subseteq J_1 \subseteq \dots$ in I such that

$$M_{J_{2i-1}} \subseteq gP_{J_{2i}} \subseteq M_{J_{2i+1}}$$

and $gP_{J_{2i}}$ is projective for all i. Then $J = \bigcup_{i=1}^{\infty} J_i$ is a countable subset of I such that $K_0 \subseteq P_J$ and $gP_J = M_J$. Since M_J is the ascending union of the projectives $gP_{J_{2i}}$, it must be flat. In view of the short exact sequence

$$0 \to M_J \to P_J \to M_J \to 0,$$

we conclude from Proposition 2.2 that M_J is projective. □

We are now ready to prove the main theorem. In case M is $\aleph_n$-generated for some nonnegative integer n, the conclusion follows immediately from the theorem of Osofsky quoted above ([**26**], Theorem 1.3), which in this case implies that p.dim $M \leq n$. D. Simson has pointed out that this case also follows from his Theorem 1.5 in [**28**], and that it is possible to base a proof of the general case on a transfinite induction argument involving [**28**], Proposition 1.4.

Theorem 2.5. *If $0 \to M \to P \to M \to 0$ is a short exact sequence of R-modules with M flat and P projective, then M must be projective.*

Proof. We may assume that the homomorphism $M \to P$ is an inclusion map, and we let g denote the homomorphism $P \to M$.

By a theorem of Kaplansky [**20**], the projective module P is a direct sum of countably generated submodules, say $P = \bigoplus_{i \in I} P_i$. Set $P_J = \bigoplus_{j \in J} P_j$ and $M_J = M \cap P_J$ for all subsets $J \subseteq I$.

Consider the set $\mathcal{P}$ of all pairs (J, L) where J is a subset of I such that $gP_J = M_J$ and L is a submodule of P_J such that $P_J = M_J \oplus L$. In particular, $(\emptyset, 0) \in \mathcal{P}$. Order $\mathcal{P}$ by componentwise inclusion:

$$(J_1, L_1) \leq (J_2, L_2) \quad \iff \quad J_1 \subseteq J_2 \ \text{ and } \ L_1 \subseteq L_2.$$

If $\{(J_\alpha, L_\alpha)\}$ is any nonempty chain in $\mathcal{P}$, then $(\bigcup_\alpha J_\alpha, \bigcup_\alpha L_\alpha)$ clearly belongs to $\mathcal{P}$. Therefore Zorn's Lemma gives us a maximal element of $\mathcal{P}$, say (J, L).

If $J = I$, then $P = M \oplus L$ and we are done. Hence, assume that $J \neq I$.

Since $gP_J = M_J$, it follows from the Nine Lemma or the Snake Lemma that there is a short exact sequence

$$0 \to M/M_J \xrightarrow{f} P/P_J \xrightarrow{\overline{g}} M/M_J \to 0$$

where f is induced by the inclusion map $M \to P$ and $\overline{g}$ is induced by g. Then f maps M/M_J isomorphically onto the submodule $\overline{M} = (M + P_J)/P_J \subseteq P/P_J$, and we obtain a short exact sequence

$$0 \to \overline{M} \xrightarrow{\subseteq} P/P_J \xrightarrow{h} \overline{M} \to 0$$

where $h = f\overline{g}$. Note that $P/P_J \cong P_{I \setminus J}$; in particular, P/P_J is projective, and is a direct sum of countably generated submodules $(P_i + P_J)/P_J$ for $i \in I \setminus J$. Since M_J is a direct summand of P_J, it is also a direct summand of P. Then from $M_J \subseteq M \subseteq P$ it follows that M_J is a direct summand of M. Hence, M/M_J, and thus $\overline{M}$, is flat.

Now apply Lemma 2.4 to the situation above, with initial data corresponding to a nonempty subset of $I \setminus J$. The lemma then gives us a subset

$K \subseteq I$, properly containing J, such that $h(P_K/P_J) = \overline{M} \cap (P_K/P_J)$ and $h(P_K/P_J)$ is projective. Since

$$h(P_K/P_J) = f(gP_K/M_J) = (gP_K + P_J)/P_J$$
$$\overline{M} \cap (P_K/P_J) = [(M + P_J) \cap P_K]/P_J = (M_K + P_J)/P_J,$$

we obtain $gP_K + P_J = M_K + P_J$. Intersecting with M and using the modular law, we see that $gP_K = M_K$.

Since the kernel of $h|_{P_K/P_J}$ is the submodule $\overline{M} \cap (P_K/P_J) = (M_K + P_J)/P_J$, we find that $h(P_K/P_J) \cong P_K/(M_K + P_J)$. The latter quotient is thus projective, and so $P_K = (M_K + P_J) \oplus T$ for some T. Since $P_J = M_J \oplus L$ and

$$M_K \cap L = M_K \cap P_J \cap L = M_J \cap L = 0,$$

we have $M_K + P_J = M_K \oplus L$ and so $P_K = M_K \oplus L \oplus T$. But then $(K, L \oplus T) \in \mathcal{P}$, contradicting the maximality of (J, L).

Therefore $J = I$ and $P = M \oplus L$. □

3. R-projective and RG-flat implies RG-projective.

We present the most basic version of our main theorem – for modules over group rings of finite groups – in this section. The additional results needed to develop versions for modules over group rings of infinite groups are worked out in the following two sections.

Let R be a ring of coefficients and G a group. If M is a left RG-module and N an RG-R-bimodule, we can make $N \otimes_R M$ into a left RG-module via $r(n \otimes m) = (rn) \otimes m$ and $g(n \otimes m) = (gn) \otimes (gm)$ for $r \in R$, $n \in N$, $m \in M$, $g \in G$. This is called the *diagonal* RG-module structure on $N \otimes_R M$. There is also the *basic* RG-module structure, where $s(n \otimes m) = (sn) \otimes m$ for $s \in RG$, $n \in N$, $m \in M$. We shall assume that $N \otimes_R M$ is equipped with the diagonal structure unless otherwise specified.

In our first lemma, G can be arbitrary, but later in the section we shall assume that G is finite.

Lemma 3.1.

(a) *If M is a left RG-module which is projective as an R-module, then $RG \otimes_R M$ is a projective left RG-module.*

(b) *Let M be a projective left RG-module and N an RG-R-bimodule. If N is a free right R-module with a basis of elements centralized by R, then $N \otimes_R M$ is a projective left RG-module.*

(c) *Let M be a flat left RG-module and N an RG-R-bimodule. If N is a free right R-module with a basis of elements centralized by R, then $N \otimes_R M$ is a flat left RG-module.*

Proof. (a) It is clear that with respect to the basic RG-module structure, $RG \otimes_R M$ is a projective left RG-module. Hence, it suffices to show that the diagonal structure is isomorphic to the basic structure.

As an R-module, $RG \otimes_R M = \bigoplus_{x \in G}(x \otimes M)$. There is an R-module automorphism ϕ on $RG \otimes_R M$ such that $\phi(x \otimes m) = x \otimes x^{-1}m$ for $x \in G$, $m \in M$. Since

$$\phi(gx \otimes gm) = gx \otimes x^{-1}m$$

for g, $x \in G$ and $m \in M$, we see that ϕ maps $RG \otimes_R M$ with the diagonal structure isomorphically onto the same tensor product with the basic structure.

(b) There is a left RG-module M' such that $M \oplus M'$ is free, and $N \otimes_R M$ is isomorphic to a direct summand of $N \otimes_R (M \oplus M')$. Hence, we may assume that M is a free left RG-module. Now M is isomorphic to a direct sum of copies of RG, and so $N \otimes_R M$ is isomorphic to a direct sum of copies of $N \otimes_R RG$. Thus, we need only consider the case when $M = RG$.

Let $\{n_i \mid i \in I\}$ be an R-centralizing basis for N_R. Then there is an R-module decomposition $N \otimes_R RG = \bigoplus_i (n_i \otimes RG)$. There is a nonstandard left RG-module structure on $N \otimes_R RG$, under which $r(n \otimes s) = rn \otimes s$ but $g(n \otimes s) = n \otimes (gs)$ for $r \in R$, $n \in N$, $s \in RG$, $g \in G$. With respect to this structure, each $n_i \otimes RG$ is an RG-submodule of $N \otimes_R RG$, isomorphic to RG. Hence, $N \otimes_R RG$ with the nonstandard structure is a free left RG-module. Thus, it suffices to show that the diagonal structure on $N \otimes_R RG$ is isomorphic to this nonstandard structure.

As an R-module, $N \otimes_R RG = \bigoplus_{x \in G}(N \otimes x)$. There is an R-module automorphism ψ on $N \otimes_R RG$ such that $\psi(n \otimes x) = x^{-1}n \otimes x$ for $n \in N$ and $x \in G$. Since

$$\psi(gn \otimes gx) = x^{-1}n \otimes gx$$

for g, $x \in G$ and $n \in N$, we see that ψ maps the diagonal structure of $N \otimes_R RG$ isomorphically onto the nonstandard structure.

(c) Since M is flat, it can be written as a filtered colimit of finitely generated projective modules, by the Lazard–Govorov theorem [**18**, **22**]. Since $N \otimes_R -$ commutes with filtered colimits, the statement follows from (b). $\square$

For the rest of this section, we suppose that the group G is finite. In Section 5, we shall show how to generalize the techniques to infinite groups. View R as the trivial RG-RG-bimodule. There is a bimodule isomorphism $R \cong Ru$, where $u = \sum_{g \in G} g \in RG$, and so we get a short exact sequence $0 \to R \to RG \to \overline{B} \to 0$ of RG-RG-bimodules, where $\overline{B} = RG/Ru$. Note that this sequence splits when viewed as a short exact sequence of right or left R-modules.

Lemma 3.2. (a) *If M is a left RG-module with finite projective dimension, and M is projective as an R-module, then M is projective as an RG-module.*

(b) *If M is a left RG-module with finite weak dimension, and M is projective as an R-module, then M is flat as an RG-module.*

Proof. (a) For R commutative, this can be found in Corollary 5.5 of [**4**] or Lemma 2.3 of [**9**]. The proof we give for R not necessarily commutative follows the same lines, but more care is needed in the details.

There is a short exact sequence

$$0 \to R \otimes_R M \to RG \otimes_R M \to \overline{B} \otimes_R M \to 0$$

of left RG-modules. Note that $R \otimes_R M \cong M$ as RG-modules, and that $RG \otimes_R M$ is a projective left RG-module by Lemma 3.1(a).

Since RG is a free right R-module with a basis $\{u\} \cup (G \setminus \{1\})$ that centralizes R, it follows that $\overline{B}$ is a free right R-module with an R-centralizing basis. Hence, Lemma 3.1(b) shows that $\overline{B} \otimes_R P$ is a projective left RG-module for any projective left RG-module P. On tensoring $\overline{B}$ with a projective RG-resolution for M, we thus see that $\mathrm{p.dim}_{RG}(\overline{B} \otimes_R M) \leq \mathrm{p.dim}_{RG} M$. If $\mathrm{p.dim}_{RG}(\overline{B} \otimes_R M)$ were positive, it would follow from the above short exact sequence that $\mathrm{p.dim}_{RG} M < \mathrm{p.dim}_{RG}(\overline{B} \otimes_R M)$, which is impossible. Therefore $\overline{B} \otimes_R M$ is a projective RG-module, and thus M is also.

(b) This follows in the same way, using Lemma 3.1(c) instead of 3.1(b). □

Lemma 3.3. *If M is a flat left RG-module which is projective as an R-module, then there exists a short exact sequence*

$$0 \to M \to P \to N \to 0$$

of left RG-modules such that P is projective, N is flat, and also N is projective as an R-module.

Proof. Consider the short exact sequence

$$0 \to R \otimes_R M \to RG \otimes_R M \to \overline{B} \otimes_R M \to 0$$

of left RG-modules, from the proof of Lemma 3.2. Then $R \otimes_R M \cong M$ as RG-modules, and $RG \otimes_R M$ is a projective RG-module. Further, $\overline{B}$ is a free right R-module with an R-centralizing basis. Hence, Lemma 3.1(c) shows that $\overline{B} \otimes_R M$ is a flat RG-module. If $\{b_i \mid i \in I\}$ is an R-centralizing basis for $\overline{B}_R$, then $\overline{B} \otimes_R M = \bigoplus_i (b_i \otimes M)$ where each $b_i \otimes M$ is an R-submodule of $\overline{B} \otimes_R M$ isomorphic to M. Thus, $\overline{B} \otimes_R M$ is projective as an R-module. □

Theorem 3.4. *Let R be a ring of coefficients* (*not necessarily commutative*) *and G a finite group. If M is a flat RG-module which is projective as an R-module, then M is projective as an RG-module.*

Proof. Let us consider the case of left modules. Set $M_0 = M$, and for $i = 0, -1, -2, \dots$ choose short exact sequences

$$0 \to M_{i-1} \to P_i \to M_i \to 0$$

of left RG-modules with P_i projective. Note that these M_i are all flat. In view of Lemma 3.3, there exist short exact sequences

$$0 \to M_i \to P_{i+1} \to M_{i+1} \to 0$$

of left RG-modules for $i = 0, 1, 2, \dots$ such that P_{i+1} is projective, M_{i+1} is flat, and also M_{i+1} is projective as an R-module.

Taking the direct sum of all these short exact sequences, we obtain a short exact sequence

$$0 \to \bigoplus_{i\in\mathbb{Z}} M_i \to \bigoplus_{i\in\mathbb{Z}} P_i \to \bigoplus_{i\in\mathbb{Z}} M_i \to 0$$

of left RG-modules with projective middle term and flat end term. By Theorem 2.5, $\bigoplus_{i\in\mathbb{Z}} M_i$ is a projective RG-module, and therefore so is M. □

4. Strongly G-graded rings.

Easy examples show that Theorem 3.4 does not remain valid as stated if G is allowed to be infinite – e.g., take R to be a field and G an infinite cyclic group. It is, however, natural to ask for a version of the theorem in which projectivity over R is replaced by projectivity over RH where H is a subgroup of finite index in G. This context reduces easily to the case that H is normal in G, and then we can express RG as a crossed product $(RH) * (G/H)$. Thus, what is needed is a generalization of Theorem 3.4 in which RG is replaced by a crossed product $R * G$. In fact, we can replace $R * G$ by any strongly G-graded ring with identity component R, and the theorem is proved in that generality in this section.

Throughout the section, let G be a group, S a G-graded ring, and $R = S_1$ (the identity component of S). Some of our proofs are adapted from Hopf algebra methods, which apply in the following way. If we view $\mathbb{Z}G$ as a Hopf algebra over $\mathbb{Z}$, then S is a left $\mathbb{Z}G$-comodule algebra, which means that it is simultaneously a $\mathbb{Z}G$-comodule and a $\mathbb{Z}$-algebra, in such a way that the comodule structure map is a $\mathbb{Z}$-algebra homomorphism. The structure map here is the map $\lambda : S \to \mathbb{Z}G \otimes_{\mathbb{Z}} S$ given by $s \mapsto \sum_{g\in G} g \otimes s_g$. The statement that λ is a $\mathbb{Z}$-algebra homomorphism is equivalent to the statement that $S_g S_h \subseteq S_{gh}$ for $g, h \in G$. The subring of $\mathbb{Z}G$-coinvariants, namely the elements $s \in S$ such that $\lambda(s) = 1 \otimes s$, is just R.

Given any left $\mathbb{Z}G$-module N and any left S-module M, one can make $N \otimes_{\mathbb{Z}} M$ into a left S-module so that $s \cdot (n \otimes m) = \sum_{g\in G} gn \otimes s_g m$. This

S-module action is called the *semi-diagonal action* (Cornick and Kropholler [**14**], p. 44).

Lemma 4.1. *Let S be a G-graded ring, N a left $\mathbb{Z}G$-module, and M a left S-module. Assume that N is free as a $\mathbb{Z}$-module. If M is free, or projective, or flat as an S-module, then so is $N \otimes_{\mathbb{Z}} M$.*

Remark. The free and projective cases are done in Cornick and Kropholler [**14**], Corollary 3.3 with a different proof.

Proof. We claim that it suffices to consider the case $M = {}_SS$. First of all, the free and projective conclusions follow directly from this case and the fact that the functor $N \otimes_{\mathbb{Z}} -$ commutes with direct sums. Secondly, if M is flat, then by the Lazard-Govorov Theorem [**18**, **22**], it is a filtered colimit of free modules. Since the functor $N \otimes_{\mathbb{Z}} -$ also commutes with filtered colimits, it then follows from the previous cases that $N \otimes_{\mathbb{Z}} M$ is flat.

Thus, we restrict attention to $N \otimes_{\mathbb{Z}} S$. Observe that $S \otimes_{\mathbb{Z}} N$, made into a left S-module in the usual way, is a free left S-module. Hence, it suffices to show that $N \otimes_{\mathbb{Z}} S \cong S \otimes_{\mathbb{Z}} N$ as left S-modules.

Define $\phi : S \otimes_{\mathbb{Z}} N \to N \otimes_{\mathbb{Z}} S$ as the unique group homomorphism such that $s \otimes n \mapsto \sum_{g\in G} gn \otimes s_g$ for $s \in S$, $n \in N$. Observe that

$$\begin{aligned}\phi\big(t \cdot (s \otimes n)\big) &= \phi(ts \otimes n)\\ &= \sum_{g\in G} gn \otimes (ts)_g = \sum_{g,h\in G} ghn \otimes t_g s_h\\ &= t \cdot \sum_{h\in G} hn \otimes s_h = t \cdot \phi(s \otimes n)\end{aligned}$$

for $s, t \in S$ and $n \in N$. Thus, ϕ is an S-module homomorphism.

Similarly, define $\psi : N \otimes_{\mathbb{Z}} S \to S \otimes_{\mathbb{Z}} N$ so that $n \otimes s \mapsto \sum_{g\in G} s_g \otimes g^{-1}n$ for $n \in N$, $s \in S$, and observe that

$$\begin{aligned}\psi\big(t \cdot (n \otimes s)\big) &= \sum_{h\in G} \psi(hn \otimes t_h s) = \sum_{g,h\in G} (t_h s)_g \otimes g^{-1}hn\\ &= \sum_{g,h\in G} t_h s_{h^{-1}g} \otimes g^{-1}hn = \sum_{g,h\in G} t_h s_g \otimes g^{-1}n = t \cdot \psi(n \otimes s)\end{aligned}$$

for $s, t \in S$ and $n \in N$. Thus, ψ is an S-module homomorphism.

Finally, we compute that

$$\phi\psi(n \otimes s) = \sum_{g\in G} \phi(s_g \otimes g^{-1}n) = \sum_{g,h\in G} hg^{-1}n \otimes (s_g)_h = \sum_{g\in G} n \otimes s_g = n \otimes s$$

$$\psi\phi(s \otimes n) = \sum_{g\in G} \psi(gn \otimes s_g) = \sum_{g,h\in G} (s_g)_h \otimes h^{-1}gn = \sum_{g\in G} s_g \otimes n = s \otimes n$$

for $s \in S$ and $n \in N$. Therefore $\phi\psi$ and $\psi\phi$ are identity maps. $\square$

Now assume that S is strongly G-graded. This means that for g, $h \in G$ we have $S_g S_h = S_{gh}$, and not just $S_g S_h \subseteq S_{gh}$.

Lemma 4.2. *If S is a strongly G-graded ring and $R = S_1$, then S is a projective left (or right) R-module.*

Proof. This is well known (see for example Lemma 2.1 of Yi [**29**] or Remark 2 on pages 1035-1036 of Nastasescu [**24**]), but the proof is short, so we include it. If $g \in G$, then there exist elements $x_i \in S_{g^{-1}}$ and $y_i \in S_g$ (for $1 \leq i \leq n$) such that $\sum_{i=1}^n x_i y_i = 1$. The elements x_i and y_i then give left R-module homomorphisms from S_g to R and R to S_g displaying S_g as a direct summand of a direct sum of n copies of ${}_R R$. Thus each S_g is a projective left R-module, and hence so is S. □

Lemma 4.3. *Let S be a strongly G-graded ring and M a left S-module. If M is projective as a left R-module, then $\mathbb{Z}G \otimes_{\mathbb{Z}} M$ is projective as a left S-module.*

Proof. First consider $\mathbb{Z}G \otimes_{\mathbb{Z}} S$, made into a left S-module with the semi-diagonal action. This is also a right S-module in the standard way, and the two module actions commute, so it is an S-S-bimodule. We claim that $\mathbb{Z}G \otimes_{\mathbb{Z}} S \cong S \otimes_R S$ as S-S-bimodules. (This is a special case of Cornick and Kropholler [**14**], Lemma 5.1.) We can argue as in Montgomery [**23**], Theorem 8.1.7, as follows.

There is a "Hopf-Galois map" $\beta : S \otimes_R S \to \mathbb{Z}G \otimes_{\mathbb{Z}} S$ such that $s \otimes t \mapsto \sum_{g \in G} g \otimes s_g t$ for $s, t \in S$; observe that β is an S-S-bimodule map. Since S is strongly graded, for each $g \in G$ there exist finitely many elements $x_{g,i} \in S_g$ and $y_{g,i} \in S_{g^{-1}}$ such that $\sum_i x_{g,i} y_{g,i} = 1$. We can define a right S-module homomorphism $\alpha : \mathbb{Z}G \otimes_{\mathbb{Z}} S \to S \otimes_R S$ such that $g \otimes u \mapsto \sum_i x_{g,i} \otimes y_{g,i} u$ for $g \in G$, $u \in S$. It can be checked that α is a bimodule homomorphism, but that falls out of proving that $\alpha = \beta^{-1}$. Now

$$\begin{aligned} \beta\alpha(g \otimes u) &= \sum_i \beta(x_{g,i} \otimes y_{g,i} u) = \sum_i \sum_{h \in G} h \otimes (x_{g,i})_h y_{g,i} u \\ &= \sum_i g \otimes x_{g,i} y_{g,i} u = g \otimes u \end{aligned}$$

for $g \in G$, $u \in S$, and so $\beta\alpha$ is an identity map. Also,

$$\begin{aligned} \alpha\beta(s \otimes t) &= \sum_{g \in G} \alpha(g \otimes s_g t) = \sum_{g \in G} \sum_i x_{g,i} \otimes y_{g,i} s_g t \\ &= \sum_{g \in G} \sum_i x_{g,i} y_{g,i} s_g \otimes t = s \otimes t \end{aligned}$$

for $s, t \in S$ (because $y_{g,i} s_g \in R$), and so $\alpha\beta$ is an identity map. Therefore β is a bimodule isomorphism, establishing the claim.

We now have left S-module isomorphisms

$$S \otimes_R M \cong (S \otimes_R S) \otimes_S M \cong (\mathbb{Z}G \otimes_{\mathbb{Z}} S) \otimes_S M \cong \mathbb{Z}G \otimes_{\mathbb{Z}} M.$$

(One must observe that the standard abelian group isomorphism in the last step is in fact an S-module map.) Since ${}_RM$ is projective, so is ${}_S(S \otimes_R M)$, and the proof is complete. □

Continue to assume that S is strongly G-graded, where now G is assumed to be finite. Set $u = \sum_{g \in G} g \in \mathbb{Z}G$. Then $\mathbb{Z}u$ is an ideal of $\mathbb{Z}G$, and $\mathbb{Z}u$ is isomorphic to the trivial $\mathbb{Z}G$-$\mathbb{Z}G$-bimodule $\mathbb{Z}$. We have a short exact sequence

$$0 \to \mathbb{Z}u \to \mathbb{Z}G \to \mathbb{Z}G/\mathbb{Z}u \to 0$$

of $\mathbb{Z}G$-$\mathbb{Z}G$-bimodules, and we note that all three of these bimodules are free as $\mathbb{Z}$-modules. In particular, the sequence splits as a $\mathbb{Z}$-module sequence, and so it remains exact after tensoring over $\mathbb{Z}$ with any $\mathbb{Z}$-module. We write $\overline{B}$ for the $\mathbb{Z}G$-$\mathbb{Z}G$-bimodule $\mathbb{Z}G/\mathbb{Z}u$.

Lemma 4.4. *Let S be a strongly G-graded ring, where G is a finite group, and let M be a left S-module.*

(a) *If ${}_RM$ is projective and* $\text{p.dim}_S M < \infty$*, then ${}_SM$ is projective.*
(b) *If* $\text{p.dim}_S M < \infty$*, then* $\text{p.dim}_S M = \text{p.dim}_R M$.
(c) *Suppose that M is a flat left S-module such that ${}_RM$ is projective. Then there exists a short exact sequence of left S-modules*

$$0 \to M \to P \to N \to 0$$

such that P is projective, N is flat, and ${}_RN$ is projective.

Proof. We have the short exact sequence of left S-modules

$$0 \to \mathbb{Z}u \otimes_{\mathbb{Z}} M \to \mathbb{Z}G \otimes_{\mathbb{Z}} M \to \overline{B} \otimes_{\mathbb{Z}} M \to 0.$$

Since $\mathbb{Z}u$ is the trivial $\mathbb{Z}G$-module, $\mathbb{Z}u \otimes_{\mathbb{Z}} M \cong M$. If ${}_RM$ is projective, then so is ${}_S(\mathbb{Z}G \otimes_{\mathbb{Z}} M)$ by Lemma 4.3.

(a) Apply $\overline{B} \otimes_{\mathbb{Z}} -$ to a projective resolution of ${}_SM$. In view of Lemma 4.1 and the exactness of $\overline{B} \otimes_{\mathbb{Z}} -$, we obtain a projective resolution of ${}_S(\overline{B} \otimes_{\mathbb{Z}} M)$. Thus, $\text{p.dim}_S(\overline{B} \otimes_{\mathbb{Z}} M) \le \text{p.dim}_S M$, and we conclude that the short exact sequence above must split.

(b) Since ${}_RS$ is projective by Lemma 4.2, $\text{p.dim}_R M \le \text{p.dim}_S M$, and so $\text{p.dim}_R M = n < \infty$. Choose an exact sequence of S-modules

$$0 \to K \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

where the P_i are projective. Since the P_i are also projective R-modules, ${}_RK$ must be projective. Further, $\text{p.dim}_S K < \infty$ because $\text{p.dim}_S M < \infty$, and so ${}_SK$ is projective by part (a). Thus $\text{p.dim}_S M \le n$.

(c) The S-module $\mathbb{Z}G \otimes_{\mathbb{Z}} M$ is projective by Lemma 4.3, and $N := \overline{B} \otimes_{\mathbb{Z}} M$ is a flat left S-module by Lemma 4.1. Finally, observe that the semi-diagonal

S-module action on N, when restricted to R, is given by $r \cdot (b \otimes m) = b \otimes rm$ for $r \in R$, $b \in \overline{B}$, $m \in M$. Since $\overline{B}$ is a free $\mathbb{Z}$-module, it follows that N as an R-module is isomorphic to a direct sum of copies of M and thus is projective. □

Theorem 4.5. *Let S be a strongly G-graded ring, where G is a finite group, and let $R = S_1$. If M is a flat S-module which is projective as an R-module, then M is projective as an S-module.*

Proof. In the proof of Theorem 3.4, replace Lemma 3.3 by Lemma 4.4(c). □

Corollary 4.6. *Let S be a strongly G-graded ring, where G is a finite group, and let $R = S_1$. If M is any S-module, then*

$$\text{p.dim}_S M = \max\{\text{p.dim}_R M,\ \text{w.dim}_S M\}.$$

Proof. Suppose that M is a left module. Obviously neither $\text{p.dim}_R M$ nor $\text{w.dim}_S M$ is greater than $\text{p.dim}_S M$. Now suppose that

$$\max\{\text{p.dim}_R M,\ \text{w.dim}_S M\} = n < \infty.$$

Choose an exact sequence of S-modules

$$0 \to K \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

where the P_i are projective. Since the P_i are also projective R-modules, ${}_RK$ must be projective and ${}_SK$ must be flat. Hence, ${}_SK$ is projective by Theorem 4.5, and therefore $\text{p.dim}_S M \leq n$. □

Suppose now that S is a ring (strongly) graded by an arbitrary group Γ. If H is a subgroup of Γ, we set $S_H = \sum_{h \in H} S_h$, which is a subring of S, (strongly) graded by H.

Corollary 4.7. *Let S be a strongly Γ-graded ring, where Γ is an arbitrary group, and let H be a subgroup of finite index in Γ. If M is a left S-module which is projective as an S_H-module, then it is projective as an S-module.*

Proof. There is a normal subgroup $N \lhd \Gamma$ of finite index contained in H, and we may view S_H as an (H/N)-graded ring, where $(S_H)_{hN} = \sum_{x \in hN} S_x$ for $hN \in H/N$. Observe that since S is strongly Γ-graded, S_H is strongly (H/N)-graded. Hence, it follows from Lemma 4.2 that S_H is projective as a left or right S_N-module, and so M is projective as an S_N-module. We now view S as a strongly (Γ/N)-graded ring and apply Theorem 4.5 to obtain the desired conclusion. □

Since group rings are examples of strongly graded rings, Corollary 4.7 specializes to the following result.

Corollary 4.8. *Let R be a coefficient ring and H a subgroup of finite index in a group Γ. If M is a flat $R\Gamma$-module which is projective as an RH-module, then it is projective as an $R\Gamma$-module.*

5. Group rings of infinite groups.

Another direction in which Theorem 3.4 can be extended is to allow G to be infinite but to strengthen the assumption of projectivity over R. In order to use the methods of Section 3, then, the main point is to find a replacement for the short exact sequence

$$0 \to R \to RG \to \overline{B} \to 0.$$

For this purpose, we use the module B introduced by Kropholler [**21**], see also Cornick and Kropholler [**15**], Benson [**4**].

Given any set Σ, and any coefficient ring R, we denote by $B(\Sigma, R)$ the set of functions from Σ to R which take only finitely many different values in R. We make this into a ring with pointwise operations. If R is commutative, then so is $B(\Sigma, R)$. Since R may be identified with the subring of constant functions in $B(\Sigma, R)$, the latter naturally obtains the structure of an R-R-bimodule.

Let Γ be a group, and write B for $B(\Gamma, R)$. This is an $R\Gamma$-$R\Gamma$-bimodule in a standard way via left and right multiplication in Γ. Namely, for $g, h \in \Gamma$, and $f \in B$, the function gfh is defined by the rule $(gfh)(x) = f(hxg)$. The image of R in B is an $R\Gamma$-$R\Gamma$-subbimodule, on which Γ acts trivially. We write $\overline{B}$ for the quotient B/R, so that there is a short exact sequence of $R\Gamma$-$R\Gamma$-bimodules

$$0 \to R \to B \to \overline{B} \to 0.$$

This is what we use as a replacement for the short exact sequence used in Section 3. Note that this sequence splits as a short exact sequence of right or left R-modules.

The following lemma is essentially due to Kropholler.

Lemma 5.1. *The bimodule B is a free right R-module with an R-centralizing basis. This basis may be chosen to include the constant function* 1 *as one element. Hence, $\overline{B}$ is also a free right R-module with an R-centralizing basis.*

Proof. It follows from a result of Nöbeling [**25**] (see Fuchs [**17**], Lemma 97.2 for the simplified proof due to G. Bergman) that $B(\Sigma, \mathbb{Z})$ is a free abelian group for any set Σ. The inclusion of $\mathbb{Z}$ into $B(\Sigma, \mathbb{Z})$ (as constant functions) is a pure monomorphism, and therefore splits. Any complementary direct summand, being a subgroup of a free abelian group, is itself free abelian. Hence, the constant function 1 may be chosen as part of a free basis for $B(\Sigma, \mathbb{Z})$. Tensoring with R, and using the fact that $B(\Sigma, \mathbb{Z}) \otimes_{\mathbb{Z}} R \cong B(\Sigma, R)$ as R-R-bimodules, we obtain an R-centralizing right R-module basis for $B(\Sigma, R)$. Now apply this in the case $\Sigma = \Gamma$. $\square$

A left $R\Gamma$-module M is said to be *cofibrant* if $B \otimes_R M$ is a projective left $R\Gamma$-module. (As in Section 3, we assume that tensor products of $R\Gamma$-R-bimodules with $R\Gamma$-modules are equipped with the diagonal $R\Gamma$-module structure.) The motivation for this definition may be found in [**4**, **5**]. If Γ happens to be finite, this is the same as saying that M is projective as an R-module.

Theorem 5.2. *Let R be a ring of coefficients* (*not necessarily commutative*) *and Γ a group* (*not necessarily finite*). *If M is a flat cofibrant $R\Gamma$-module then M is projective.*

Proof. Set $M_0 = M$. As in the proof of Theorem 3.4, it suffices to find short exact sequences
$$0 \to M_i \to P_{i+1} \to M_{i+1} \to 0$$
of left $R\Gamma$-modules for $i = 0, 1, 2, \dots$ such that P_{i+1} is projective and M_{i+1} is flat.

First, we have a short exact sequence
$$0 \to R \otimes_R M \to B \otimes_R M \to \overline{B} \otimes_R M \to 0$$
of left $R\Gamma$-modules where $R \otimes_R M \cong M$, and where $B \otimes_R M$ is projective because M is cofibrant. By Lemmas 5.1 and 3.1(c), $\overline{B} \otimes_R M$ is flat. Thus, we obtain a short exact sequence
$$0 \to M_0 \to P_1 \to M_1 \to 0$$
of the desired form, where additionally $M_1 \cong \overline{B} \otimes_R M_0$.

Since $\overline{B}_R$ is free, there is a short exact sequence
$$0 \to \overline{B} \otimes_R M_0 \to \overline{B} \otimes_R P_1 \to \overline{B} \otimes_R M_1 \to 0$$
of left $R\Gamma$-modules, where $\overline{B} \otimes_R M_0 \cong M_1$ is given, $\overline{B} \otimes_R P_1$ is projective by Lemmas 5.1 and 3.1(b), and $\overline{B} \otimes_R M_1$ is flat by Lemmas 5.1 and 3.1(c). This provides us with a short exact sequence
$$0 \to M_1 \to P_2 \to M_2 \to 0$$
of the desired form, where additionally $M_2 \cong \overline{B} \otimes_R M_1$.

An obvious induction now completes the proof. □

6. Reduction modulo p and projectivity.

In this section, we prove that projectivity for modules over a group ring is determined by the reduction modulo each prime dividing the group order. The proof works in the context of a group and a subgroup of finite index, so this is the context in which we present it.

Theorem 6.1. *Let R be a ring of coefficients and H a subgroup of finite index in a group G. An RG-module M is projective if and only if*

(i) *M is projective as an RH-module, and*

(ii) *M/pM is projective as an $(R/pR)G$-module for each prime number p dividing $[G : H]$.*

Proof. It is easy to see that a projective RG-module satisfies (i) and (ii). Conversely, we suppose that M is an RG-module satisfying (i) and (ii), and we shall prove that M is projective. Since M is projective as an RH-module, it suffices to prove that M is projective relative to H. By D. G. Higman's criterion (see for example Proposition 3.6.4 of [3]), this amounts to showing that the identity endomorphism can be written as

$$1_M = \mathrm{Tr}_{H,G}(\alpha)$$

for some $\alpha \in \mathrm{End}_{RH}(M)$, where $\mathrm{Tr}_{H,G}(\alpha)$ is the RG-endomorphism defined by

$$\mathrm{Tr}_{H,G}(\alpha)(m) = \sum_{g \in G/H} g(\alpha(g^{-1}(m))).$$

The sum runs over a set of left coset representatives of H in G.

Let $[G : H] = \prod_i p_i^{\gamma_i}$ where the p_i are distinct primes. Since M is projective as an RH-module, for each p_i, the natural map

$$\mathrm{End}_{RH}(M) \to \mathrm{End}_{(R/p_iR)H}(M/p_iM)$$

given by reduction modulo p_i is surjective, and the kernel of this map equals $p_i\mathrm{End}_{RH}(M)$. Since M/p_iM is a projective $(R/p_iR)G$-module, its identity endomorphism can be written as $\mathrm{Tr}_{H,G}(\overline{\alpha}_i)$ for some

$$\overline{\alpha}_i \in \mathrm{End}_{(R/p_iR)H}(M/p_iM).$$

In view of the observations above, this implies that there exist elements α_i, $\beta_i \in \mathrm{End}_{RH}(M)$ such that

$$1_M = \mathrm{Tr}_{H,G}(\alpha_i) + p_i\beta_i.$$

Note that $p_i\beta_i = 1_M - \mathrm{Tr}_{H,G}(\alpha_i)$ is an RG-homomorphism. Multiplying the above expressions, we get

$$1_M = \prod_i (\mathrm{Tr}_{H,G}(\alpha_i) + p_i\beta_i)^{\gamma_i} = \mathrm{Tr}_{H,G}(\alpha') + [G : H]\beta'$$

for some $\alpha', \beta' \in \mathrm{End}_{RH}(M)$. (Here, we have used the fact that if $a \in \mathrm{End}_{RH}(M)$ and $b \in \mathrm{End}_{RG}(M)$, then $\mathrm{Tr}_{H,G}(a)b = \mathrm{Tr}_{H,G}(ab)$ and $b\mathrm{Tr}_{H,G}(a) = \mathrm{Tr}_{H,G}(ba)$.) Now the map $[G : H]\beta' = 1_M - \mathrm{Tr}_{H,G}(\alpha')$ is an RG-homomorphism, so applying $\mathrm{Tr}_{H,G}$ we get

$$[G : H]\mathrm{Tr}_{H,G}(\beta') = [G : H]^2\beta',$$

and hence $\delta = \mathrm{Tr}_{H,G}(\beta') - [G : H]\beta'$ satisfies $[G : H]\delta = 0$. Hence,

$$\delta = \delta.1_M = \delta(\mathrm{Tr}_{H,G}(\alpha') + [G : H]\beta') = \mathrm{Tr}_{H,G}(\delta\alpha'),$$

and so

$$[G:H]\beta' = \mathrm{Tr}_{H,G}(\beta') - \delta = \mathrm{Tr}_{H,G}(\beta' - \delta\alpha').$$

Finally,

$$1_M = \mathrm{Tr}_{H,G}(\alpha') + [G:H]\beta' = \mathrm{Tr}_{H,G}(\alpha' + \beta' - \delta\alpha'). \qquad \square$$

7. Reduction modulo p and flatness.

In this section, we prove that flatness for modules over a group ring is determined by the reduction modulo each prime dividing the group order. Again, we prove the theorem in the generality of a subgroup of finite index.

Lemma 7.1. *Let R be a ring of coefficients and H a subgroup of finite index in a group G. If M is a left RG-module and N a right RG-module with M flat as a left RH-module, then $[G:H]$ annihilates $\mathrm{Tor}_n^{RG}(N,M)$ for all $n > 0$.*

Proof. Let $g_1, \dots, g_l$ be a set of left coset representatives of H in G, and let

$$0 \to M' \xrightarrow{\phi} F \to M \to 0$$

be a short exact sequence with F a free left RG-module. Then we have the long exact sequence

$$0 \to \mathrm{Tor}_1^{RG}(N,M) \to N \otimes_{RG} M' \xrightarrow{\phi_*} N \otimes_{RG} F \to N \otimes_{RG} M \to 0.$$

Consider an element $\sum_{i=1}^k n_i \otimes m_i'$ in the kernel of ϕ_*. By Lemma 2.1, there is an RH-homomorphism $\rho : F \to M'$ such that $\rho\phi(g_j^{-1}m_i') = g_j^{-1}m_i'$ for $1 \le j \le l$ and $1 \le i \le k$. Let $\theta = \mathrm{Tr}_{H,G}(\rho)$. Then $\theta : F \to M'$ is an RG-homomorphism such that $\theta\phi(m_i') = [G:H]m_i'$ for $1 \le i \le k$. So we have

$$0 = \theta_*\phi_*\left(\sum_{i=1}^k n_i \otimes m_i'\right) = \sum_{i=1}^k n_i \otimes \theta\phi(m_i') = [G:H]\sum_{i=1}^k n_i \otimes m_i'.$$

This proves that $[G:H]$ annihilates $\mathrm{Tor}_1^{RG}(N,M)$. Since M' is also flat as an RH-module and

$$\mathrm{Tor}_n^{RG}(N,M') \cong \mathrm{Tor}_{n+1}^{RG}(N,M)$$

for $n \ge 1$, an induction on n (i.e., dimension shifting) shows that

$$[G:H]\mathrm{Tor}_n^{RG}(N,M) = 0$$

for $n \ge 1$. $\square$

Lemma 7.2. *Let S be a ring, $p \in S$ a central element, and M a left S-module. Assume that M/pM is flat as a left S/pS-module, and that $\mathrm{Tor}_n^S(S/pS, M) = 0$ for all $n > 0$. If N is any right S-module, then $\mathrm{Tor}_n^S(N,M)$ is p-torsionfree for all $n > 0$ and p-divisible for all $n > 1$.*

Proof. For any right S/pS-module L, we have

$$\mathrm{Tor}_n^S(L, M) \cong \mathrm{Tor}_n^{S/pS}(L, M/pM) = 0$$

for all $n > 0$ by [**11**], Proposition VI.4.1.2. In particular,

$$\mathrm{Tor}_n^S(N/pN, M) = \mathrm{Tor}_n^S(N[p], M) = 0$$

for all $n > 0$, where $N[p] = \mathrm{ann}_N(p)$. Consider the short exact sequences

$$0 \to pN \xrightarrow{\phi} N \to N/pN \to 0$$

$$0 \to N[p] \to N \xrightarrow{\pi} pN \to 0$$

where ϕ is the inclusion map and π is multiplication by p. From the long exact sequences for $\mathrm{Tor}_*^S(-, M)$, we see that $\mathrm{Tor}_n^S(\phi, M)$ is an isomorphism for all $n > 0$, while $\mathrm{Tor}_n^S(\pi, M)$ is an isomorphism for all $n > 1$ and a monomorphism for $n = 1$.

Therefore $\mathrm{Tor}_n^S(\phi\pi, M)$ is an isomorphism for all $n > 1$ and a monomorphism for $n = 1$. Since $\phi\pi : N \to N$ is just multiplication by p, so is $\mathrm{Tor}_n^S(\phi\pi, M)$ (e.g., [**27**], Theorem 8.13), and the lemma follows. □

Theorem 7.3. *Let R be a ring of coefficients and H a subgroup of finite index in a group G. An RG-module M is flat if and only if*

(i) *M is flat as an RH-module, and*

(ii) *M/pM is flat as an $(R/pR)G$-module for each prime p dividing $[G : H]$.*

Proof. That flatness of M as an RG-module implies (i) and (ii) is well known and easy to see. Condition (i) holds because $(-) \otimes_{RH} M$ is naturally equivalent to the composition of the exact functors $(-) \otimes_{RH} RG$ and $(-) \otimes_{RG} M$. Condition (ii) holds because $(-) \otimes_{(R/pR)G} (M/pM)$ is naturally equivalent to the restriction of $(-) \otimes_{RG} M$ to the category of right $(R/pR)G$-modules.

For the converse, we begin by observing that (i) implies that M is flat as an R-module. Since RG is a flat R-module, it now follows from [**11**], Proposition VI.4.1.1 that

$$\mathrm{Tor}_n^{RG}(RG/pRG, M) = \mathrm{Tor}_n^{RG}((R/pR) \otimes_R RG, M) \cong \mathrm{Tor}_n^R(R/pR, M) = 0$$

for all $n > 0$ and any prime p.

Now let N be an arbitrary right RG-module. By (ii) and Lemma 7.2, $\mathrm{Tor}_n^{RG}(N, M)$ is p-torsionfree for all $n > 0$ and any prime p dividing $[G : H]$. On the other hand, Lemma 7.1 shows that $[G : H]$ annihilates $\mathrm{Tor}_n^{RG}(N, M)$, and therefore $\mathrm{Tor}_n^{RG}(N, M) = 0$ for all $n > 0$. □

8. Passage to fields of coefficients.

Let R be a commutative ring of coefficients and G a finite group. On the basis of what we have proved, it is tempting to suppose that it might be true that an RG-module M is projective if and only if

(i) M is projective as an R-module, and
(ii) $K \otimes_R M$ is projective as a KG-module for every field K containing $R/\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subseteq R$.

In this section, we give an example to show that this is false, and in the next section we show that it is true if R is a Dedekind domain of characteristic zero.

Let k be an algebraically closed field of characteristic 2 and let $G = \mathbb{Z}/2 = \langle g \mid g^2 = 1\rangle$. Let $R = k[[t]]$ be the ring of formal power series over k in a variable t. Then $X = k[t^{-1}]$ can be considered as an R-module in an obvious way. In fact, X is the injective hull of k as an R-module. Since R has global dimension one, there is an R-module projective resolution of X of the form

$$0 \to Q_1 \xrightarrow{\rho} Q_0 \to X \to 0.$$

Let M be the RG-module whose underlying R-module is $Q_0 \oplus Q_1$, with the group element g acting as the matrix $\left(\begin{smallmatrix} 1 & \rho \\ 0 & 1 \end{smallmatrix}\right)$. Then M is not even flat, let alone projective, since the image of $1 - g$ does not coincide with the kernel of $1 + g$ (see Theorem 2.1 of [6]). Since Q_0 and Q_1 are projective as R-modules, so is M, and so (i) is satisfied. To check condition (ii), we note that any candidate for the field K is an extension of either $k = R/(t)$ or of $k((t))$, the field of fractions of R, so it suffices to examine these two fields. Now $k \otimes_R X = 0$ because every element of X is in the image of multiplication by t, so $k \otimes_R M$ is a projective kG-module. (Here we use the criterion of [6], Theorem 2.1 again.) Similarly, $k((t)) \otimes_R X = 0$ because every element is killed by a suitably high power of t, so $k((t)) \otimes_R M$ is a projective $k((t))G$-module.

9. Dedekind domains of coefficients.

In this section, we examine what happens if R is a Dedekind domain of coefficients whose characteristic (i.e., the characteristic of the field of fractions of R) is prime to the order of the group. In this case, if n is a nonzero integer prime to the characteristic of R, then the principal ideal nR is nonzero, and so it can be written as a finite product of maximal ideals (cf. Atiyah and Macdonald [1]). All that we actually need is that nR contain a finite product of maximal ideals, and for this to hold it suffices that R be a commutative noetherian domain of Krull dimension one whose characteristic is prime to n.

Theorem 9.1. *Let H be a subgroup of finite index in a group G and R a commutative noetherian ring of coefficients such that the principal ideal generated by $[G : H]$ (times the identity element) contains a finite product of maximal ideals that contain $[G : H]$. Then an RG-module M is projective if and only if*

(i) *M is projective as an RH-module, and*

(ii) $(R/\mathfrak{m}) \otimes_R M$ *is projective as an* $(R/\mathfrak{m})G$*-module for each maximal ideal* $\mathfrak{m} \subseteq R$ *containing* $[G : H]$.

Proof. We modify the proof of Theorem 6.1 as follows.

Let M be an R-module satisfying (i) and (ii), and let $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$ be maximal ideals of R, containing $[G : H]$, such that $\prod_i \mathfrak{m}_i$ is contained in the ideal $[G : H]R$. Since R is noetherian, each $\mathfrak{m}_i$ is finitely generated. Thus, since $M/\mathfrak{m}_i M$ is a projective $(R/\mathfrak{m}_i)G$-module, it follows as in the previous proof that

$$1_M = \mathrm{Tr}_{H,G}(\alpha_i) + \beta_i$$

for some $\alpha_i \in \mathrm{End}_{RH}(M)$ and some $\beta_i \in \mathfrak{m}_i \mathrm{End}_{RH}(M)$. Multiplying these expressions, and noting that β_i is an RG-homomorphism, we get

$$1_M = \mathrm{Tr}_{H,G}(\alpha') + \prod_i \beta_i$$

for some $\alpha' \in \mathrm{End}_{RH}(M)$. Since R is commutative and $\prod_i \mathfrak{m}_i \subseteq [G : H]R$, we get $\prod_i \beta_i = [G : H]\beta'$ for some $\beta' \in \mathrm{End}_{RH}(M)$.

The remainder of the proof now proceeds as before. □

Theorem 9.2. *Let R be a commutative noetherian domain of coefficients of Krull dimension one, and G a finite group whose order is relatively prime to the characteristic of R. Then an RG-module M is projective if and only if*

(i) M *is projective as an* R*-module, and*
(ii) $M/\mathfrak{m}M$ *is projective as an* $(R/\mathfrak{m})G$*-module for each maximal ideal* $\mathfrak{m} \subseteq R$ *containing* $|G|$.

Proof. The ideal $I = |G|R$ is nonzero by hypothesis. Well-known results then imply that I contains a finite product of maximal ideals that contain I. For instance, R/I has Krull dimension zero and so is artinian (e.g., [**1**], Theorem 8.5), whence R/I has only finitely many prime ideals, all of which are maximal (e.g., [**1**], Propositions 8.1 and 8.3). Thus the prime radical (nilradical) of R/I contains a finite product of maximal ideals. Since this radical is nilpotent (e.g., [**1**], Proposition 8.4), the desired conditions on I hold. Therefore the theorem follows from Theorem 9.1. □

Remark. This theorem seems to be well known for *finitely generated* $\mathbb{Z}G$-modules, but does not seem to be well known in the generality described here.

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