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STARLIKE MAPPINGS ON BOUNDED BALANCED DOMAINS WITH C^1-PLURISUBHARMONIC DEFINING FUNCTIONS

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Let D be a bounded balanced domain with C^1 plurisubharmonic defining functions in $\mathbf{C}^n$. First, we give a necessary and sufficient condition that a locally biholomorphic mapping from D to $\mathbf{C}^n$ is starlike. Next, we give a growth theorem for normalized starlike mappings on D. We also give a quasiconformal extension of some strongly starlike mapping on D.

1. Introduction.

Let f be a univalent mapping in the unit disk Δ with $f(0) = 0$ and $f'(0) = 1$. Then the classical growth theorem is as follows:

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}.$$

Barnard, FitzGerald and Gong [1] and Chuaqui [2] extended this to normalized starlike mappings on the unit ball $\mathbf{B}^n$ in $\mathbf{C}^n$. Their proof uses the characterization of the starlikeness due to Suffridge [11]. Chuaqui [2] also obtained a quasiconformal extension of some strongly starlike mapping on $\mathbf{B}^n$.

In this paper, we will extend the above results to (strongly) starlike mappings on bounded balanced domains with C^1 plurisubharmonic defining functions in $\mathbf{C}^n$. Since we cannot use the characterization of the starlikeness due to Suffridge [11], we first give a necessary and sufficient condition that a locally biholomorphic mapping on such domains is starlike using the idea of Gong, Wang and Yu [4]. To prove that condition, a Schwarz type lemma on balanced pseudoconvex domains [5], [6] is needed.

2. A Schwarz type lemma.

In this section, we recall a Schwarz type lemma on balanced pseudoconvex domains [5], [6]. The Lempert function $\tilde{k}_D$ for a domain D in $\mathbf{C}^n$ is defined as follows:

$$\tilde{k}_D(x,y) = \inf\{\rho(\xi,\eta) \mid \xi,\eta \in \Delta, \exists\varphi \in H(\Delta,D) \text{ such that } \varphi(\xi)=x, \varphi(\eta)=y\},$$

where ρ is the Poincaré distance on the unit disk Δ.

Let D be a balanced pseudoconvex domain in $\mathbf{C}^n$. The Minkowski function h of D is defined as follows:

$$h(z) = \inf\left\{t > 0 \mid \frac{z}{t} \in D\right\}.$$

Then we have (Proposition 3.1.10. of Jarnicki and Pflug [7]),

$$\tilde{k}_D(0,x) = \rho(0,h(x)) \text{ for any } x \text{ in } D. \tag{2.1}$$

Using (2.1) and the fact that the Lempert functions are contractible with respect to holomorphic mappings, we have the following theorem [5], [6].

Theorem 1. *Let F be a holomorphic mapping from D to D such that $F(0)=0$. Then*

$$h(F(z)) \le h(z)$$

holds for all $z \in D$.

3. A necessary and sufficient condition for a locally biholomorphic mapping to be starlike.

Let D be a domain in $\mathbf{C}^n$ which contains 0. A holomorphic mapping from D to $\mathbf{C}^n$ is said to be starlike if f is biholomorphic, $f(0)=0$ and $f(D)$ is starlike with respect to the origin.

We say that D has C^1 plurisubharmonic defining functions, if for any $\zeta \in \partial D$, there exist a neighborhood U of ζ in $\mathbf{C}^n$ and a C^1 plurisubharmonic function r on U such that $D \cap U = \{z \in U \mid r(z) < 0\}$. Then D is pseudoconvex. From now on, let D be a bounded balanced pseudoconvex domain with C^1 plurisubharmonic defining functions. In this section, we give a necessary and sufficient condition for a locally biholomorphic mapping on D to be starlike.

Let

$$u(z_1,z_2,\ldots,z_n) = \sum_{i=1}^{n} |z_i|^{p_i}$$

and let

$$B(p_1,\ldots,p_n) = \{z \in \mathbf{C}^n \mid u(z) < 1\},$$

where $2p_n > p_1 \ge p_2 \ge \ldots \ge p_n > 1$. Gong, Wang and Yu [4] gave a necessary and sufficient condition that a locally biholomorphic mapping from $B(p_1,\ldots,p_n)$ to $\mathbf{C}^n$ is starlike.

Theorem 2. *Suppose that* $f : B(p_1, \dots , p_n) \to \mathbf{C}^n$ *is a locally biholomorphic mapping with* $f(0) = 0$*. Then* f *is starlike if and only if*

$$(du \cdot f^{-1}) \bullet (d\rho)|_{w=f(z)} \geq 0 \quad \text{for any} \quad z \in B(p_1, \dots , p_n) \setminus \{0\},$$

where $a \bullet b$ *is the inner product in* $\mathbf{R}^{2n}$ *and* $\rho(w)$ *is the distance function from the origin in* $\mathbf{R}^{2n}$*.*

Their proof uses the following properties of u.

(i) $u(z) = 0$ if and only if $z = 0$,
(ii) u is C^1-smooth on $B(p_1, \dots , p_n) \setminus \{0\}$,
(iii) u is continuous on $B(p_1, \dots , p_n)$,
(iv) $\overline{B_a} = \{z \in B(p_1, \dots , p_n) \mid u(z) \leq a\}$ for any $0 < a < 1$, where $B_a = \{z \in B(p_1, \dots , p_n) \mid u(z) < a\}$,
(v) $\overline{B_a}$ is compact for any $0 < a < 1$,
(vi) $u(F(z)) \leq u(z)$ for any $z \in B(p_1, \dots , p_n)$, where F is a holomorphic mapping from $B(p_1, \dots , p_n)$ into itself with $F(0) = 0$ and $DF(0) = \nu I$, $0 < \nu \leq 1$, where I denotes the identity matrix.

We will prove that the Minkowski function h of D satisfies the above properties.

Proposition 1. *Let* h *be the Minkowski function of* D*, where* D *is a bounded balanced pseudoconvex domain in* $\mathbf{C}^n$ *with* C^1 *plurisubharmonic defining functions. Then:*

(i) $h(z) = 0$ *if and only if* $z = 0$*,*
(ii) h *is* C^1*-smooth on* $\mathbf{C}^n \setminus \{0\}$*,*
(iii) h *is continuous on* $\mathbf{C}^n$*,*
(iv) $\overline{D_a} = \{z \in D \mid h(z) \leq a\}$ *for any* $0 < a < 1$*, where* $D_a = \{z \in D \mid h(z) < a\}$*,*
(v) $\overline{D_a}$ *is compact for any* $0 < a < 1$*,*
(vi) $h(F(z)) \leq h(z)$ *for any* $z \in D$*, where* F *is a holomorphic mapping from* D *into itself with* $F(0) = 0$*.*

Proof. (i) Since D is bounded, $h(z) = 0$ if and only if $z = 0$.

(ii) There exists a $R > 0$ such that the Euclidean closed ball $\overline{\mathbf{B}}(0, R)$ centered at 0 of radius R is contained in D. Since $h(z) = R^{-1}|z|h(Rz/|z|)$ for $z \neq 0$, it suffices to prove that h is C^1 in a neighborhood of $z_0 \in D \setminus \{0\}$. Let $\zeta = z_0/h(z_0) \in \partial D$ and let r be a C^1 plurisubharmonic defining function of D near ζ. Let $g(z, s) = r(z/s)$. Since $g(z, h(z)) = 0$ in a neighborhood of z_0, it suffices to show that $\partial g/\partial s \neq 0$ at $(z_0, h(z_0))$ by the implicit function theorem. We use the idea of a proof of Hopf's lemma (cf. Krantz [8], p. 61). Let $D_0 = \{t \in \mathbf{C} \mid t\zeta \in D\}$. Then $D_0 = \{t \in \mathbf{C} \mid |t| < 1\}$. Let $r_0(t) = r(t\zeta)$. Let $\mathbf{B}^*$ be the ball in $\mathbf{C}$ centered at $c(0 < c < 1)$ of radius $1 - c$. Let $\mathbf{B}_1$ be a ball in $\mathbf{C}$ centered at 1 of sufficiently small radius. Let $\mathbf{B}' = \mathbf{B}^* \cap \mathbf{B}_1$. Let $\psi(t) = \exp(-\alpha|t - c|^2) - \exp(-\alpha(1 - c)^2)$. Then ψ is subharmonic on a neighborhood of $\overline{\mathbf{B}'}$ for sufficiently large α. Since $r_0 < 0$ on $\overline{\partial\mathbf{B}' \cap \mathbf{B}^*}$, there exists an $\varepsilon > 0$ such that $r_0 + \varepsilon\psi < 0$ on $\overline{\partial\mathbf{B}' \cap \mathbf{B}^*}$. Since $r_0 + \varepsilon\psi$ is subharmonic, $r_0 + \varepsilon\psi$ attains its maximum on $\overline{\mathbf{B}'}$ at 1. Therefore,

$$\frac{\partial(r_0 + \varepsilon\psi)}{\partial x}(1) \geq 0,$$

where $x = \operatorname{Re} t$. Since $\partial\psi/\partial x(1) < 0$, we have $\partial r_0/\partial x(1) > 0$. Then

$$\frac{\partial g}{\partial s}(z_0, h(z_0)) = -\frac{1}{h(z_0)}\frac{\partial r_0}{\partial x}(1) \neq 0.$$

(iii) It suffices to show that h is continuous at 0. There exists a $R > 0$ such that the Euclidean closed ball $\overline{\mathbf{B}}(0, R)$ centered at 0 of radius R is contained in D. Let $M = \sup\{h(z) \mid z \in \partial\mathbf{B}(0, R)\}$. Then, for any $\varepsilon > 0$, $h < \varepsilon$ on $\mathbf{B}(0, \varepsilon R/M)$.

(iv) Since h is continuous, it suffices to show that $\{z \in D \mid h(z) \leq a\} \subset \overline{D_a}$. Let $h(z) \leq a$. Since $h(tz) = th(z) < a$ for $0 < t < 1$, $tz \in D_a$ and $tz \to z$ as $t \to 1$. This implies that $z \in \overline{D_a}$.

(v) Since h is continuous on $\mathbf{C}^n$, $\overline{D_a} = \{z \in D \mid h(z) \leq a\} = \{z \in \mathbf{C}^n \mid h(z) \leq a\}$. Then $\overline{D_a}$ is a bounded closed subset of $\mathbf{C}^n$.

(vi) See Theorem 1.

Using Proposition 1, we obtain the following theorem as in the proof of Theorem 2 due to Gong, Wang and Yu [**4**].

Theorem 3. *Let h be the Minkowski function of D, where D is a bounded balanced pseudoconvex domain in $\mathbf{C}^n$ with C^1 plurisubharmonic defining functions. Suppose that $f : D \to \mathbf{C}^n$ is a locally biholomorphic mapping with $f(0) = 0$. Then f is starlike if and only if*

$$(dh \cdot f^{-1}) \bullet (d\rho)|_{w=f(z)} \geq 0 \ \textit{for any} \ z \in D \setminus \{0\}, \tag{3.1}$$

where $a \bullet b$ is the inner product in $\mathbf{R}^{2n}$ and $\rho(w)$ is the distance function from the origin in $\mathbf{R}^{2n}$.

Remark 1. (i) It is mentioned in Gong, Wang and Yu [**4**] that FitzGerald pointed out that if the condition $2p_n > p_1$ is dropped, then the Schwarz type lemma does not hold for u. So, they cannot obtain Theorem 2 in the case that the condition $2p_n > p_1$ is dropped. However, Theorem 3 holds for all $B(p_1, \dots, p_n)$ with $p_1, \dots, p_n > 1$.

(ii) Let D and f be as in Theorem 3. Let $w(z) = (Df(z))^{-1}(f(z))$. Then the condition (3.1) can be written as follows:

$$\operatorname{Re}\left\langle \frac{\partial h^2}{\partial z}(z), \overline{w(z)} \right\rangle \geq 0 \ \text{ for any } \ z \in D \setminus \{0\}, \tag{3.2}$$

where $\partial h^2/\partial z = (\partial h^2/\partial z_1, \dots, \partial h^2/\partial z_n)$ and $\langle \cdot, \cdot \rangle$ denotes the Hermitian inner product in $\mathbf{C}^n$. In particular, Theorem 3 reduces to the Suffridge's theorem [**11**] when $D = B(p_1, \dots, p_n)$ with $p_1 = \cdots = p_n > 1$.

4. The growth and 1/4-theorems for normalized starlike mappings.

In this section, we give the growth and 1/4-theorems for normalized starlike mappings on bounded balanced pseudoconvex domains with C^1 plurisubharmonic defining functions using the ideas of Barnard, FitzGerald and Gong [**1**] and Chuaqui [**2**]. A holomorphic mapping f is said to be normalized if $f(0) = 0$ and $Df(0) = I$.

Let D be a bounded balanced pseudoconvex domain in $\mathbf{C}^n$ with C^1 plurisubharmonic defining functions and let f be a starlike mapping from D to $\mathbf{C}^n$. By the Remark after Theorem 3, we have

$$\mathrm{Re}\left\langle \frac{\partial h^2}{\partial z}(z), \overline{w(z)} \right\rangle \geq 0 \quad \text{for any} \quad z \in D \setminus \{0\},$$

where $\partial h^2/\partial z = (\partial h^2/\partial z_1, \ldots, \partial h^2/\partial z_n)$, $w(z) = (Df(z))^{-1}(f(z))$ and $\langle \cdot, \cdot \rangle$ denotes the Hermitian inner product in $\mathbf{C}^n$. Let $z \in \partial D$ and let $\zeta \in \Delta \setminus \{0\} = \{|\zeta| < 1\} \setminus \{0\}$. Then

$$0 \leq \mathrm{Re}\left\langle \frac{\partial h^2}{\partial z}(\zeta z), \overline{w(\zeta z)} \right\rangle = |\zeta|^2 \mathrm{Re}\left\langle \frac{\partial h^2}{\partial z}(z), \overline{\left(\frac{w(\zeta z)}{\zeta}\right)} \right\rangle. \tag{4.1}$$

Let

$$\phi_z(\zeta) = \left\langle \frac{\partial h^2}{\partial z}(z), \overline{\left(\frac{w(\zeta z)}{\zeta}\right)} \right\rangle.$$

Since $w(0) = 0$, ϕ_z is a holomorphic function on Δ and $\mathrm{Re}\phi_z \geq 0$ on Δ from (4.1). By differentiating $h^2(\zeta z) = \zeta\overline{\zeta} h^2(z)$ with respect to ζ, we have

$$\sum_{i=1}^{n} \frac{\partial h^2}{\partial z_j}(\zeta z) z_j = \overline{\zeta} h^2(z).$$

If $z \in \partial D$ and $\zeta = 1$,

$$\sum_{i=1}^{n} \frac{\partial h^2}{\partial z_j}(z) z_j = 1.$$

Since $Dw(0) = I$, this implies that $\phi_z(0) = 1$. If we put

$$\sigma(\zeta) = \frac{\phi_z(\zeta) - 1}{\phi_z(\zeta) + 1},$$

σ is a holomorphic function on Δ such that $\sigma(0) = 0$ and $|\sigma(\zeta)| \leq 1$. The mapping f is said to be strongly starlike if $\phi_z(\Delta)$ is contained in a compact subset of the right half-plane independent of $z \in \partial D$. This condition is equivalent to the condition that $|\sigma(\zeta)| \leq c < 1$ uniformly for $z \in \partial D$.

Let f be a starlike mapping on D with $|\sigma(\zeta)| \leq c \leq 1$ uniformly for $z \in \partial D$. Since

$$\mathrm{Re}\left\langle \frac{\partial h^2}{\partial z}(z), \overline{w(z)} \right\rangle = h^2(z)\mathrm{Re}\phi_{\tilde{z}}(h(z)) \quad \text{for } z \in D,$$

where $\tilde{z} = z/h(z)$, we obtain the following lemma by applying the Schwarz lemma to σ as in Lemma 2.1 of Pfaltzgraff [9].

Lemma 1.

$$h^2(z)\frac{1-ch(z)}{1+ch(z)} \leq \mathrm{Re}\left\langle \frac{\partial h^2}{\partial z}(z), \overline{w(z)} \right\rangle \leq h^2(z)\frac{1+ch(z)}{1-ch(z)} \quad \textit{for } \ z \in D \setminus \{0\}.$$

Let $v(z, s, t)$ be defined by

$$v(z, s, t) = f^{-1}(e^{s-t} f(z)) \tag{4.2}$$

for $0 \leq s \leq t$. Let $z \in D \setminus \{0\}$. Since

$$\frac{\partial}{\partial t} h(v) = -h(v)^{-1}\mathrm{Re}\left\langle \frac{\partial h^2}{\partial z}(v), \overline{w(v)} \right\rangle,$$

we have

$$\frac{\partial}{\partial t} h(v) \leq -h(v)\frac{1-ch(v)}{1+ch(v)} < 0 \tag{4.3}$$

by Lemma 1. Then we have $h(v(z, s, t)) \leq h(v(z, s, s)) = h(z)$. Moreover, we obtain the following inequalities by Lemma 1 as in Lemma 2.2 of Pfaltzgraff [9].

$$e^t \frac{h(v)}{(1-ch(v))^2} \leq e^s \frac{h(z)}{(1-ch(z))^2} \qquad \text{on } \ D \tag{4.4}$$

and

$$e^s \frac{h(z)}{(1+ch(z))^2} \leq e^t \frac{h(v)}{(1+ch(v))^2} \qquad \text{on } \ D. \tag{4.5}$$

Since $D = \{z \in \mathbf{C}^n \mid h(z) < 1\}$ is bounded with respect to the Euclidean distance, a bounded set with respect to h is bounded with respect to the Euclidean distance. By (4.4), we have

$$h(e^t v) \leq e^s \frac{h(z)}{(1-ch(z))^2}.$$

Then $\{e^t v\}_{t \geq s}$ forms a normal family on D. If f is normalized, we can show that there exists a sequence $\{t_m\}$ such that $t_m \to \infty$ and $e^{t_m} v(z, s, t_m) \to e^s f(z)$ on D as $m \to \infty$ as in Theorem 2.3 of Pfaltzgraff [9]. Substituting $t = t_m$ in (4.4) and (4.5) and letting $m \to \infty$, we have the following theorem.

Theorem 4. *Let D be a bounded balanced pseudoconvex domain in $\mathbf{C}^n$ with C^1 plurisubharmonic defining functions and let f be a normalized starlike mapping from D to $\mathbf{C}^n$ with $|\sigma(\zeta)| \leq c \leq 1$ uniformly for $z \in \partial D$. Let h be the Minkowski function of D. Then*

$$\frac{h(z)}{(1+ch(z))^2} \leq h(f(z)) \leq \frac{h(z)}{(1-ch(z))^2}.$$

For $D = B(p_1, \dots, p_n)$ with $p_1, \dots, p_n > 1$, we can show that the estimates are sharp as in Theorem 2.1 of Barnard, FitzGerald and Gong [1].

Theorem 5. *Let $p_1, \dots, p_n > 1$. Let f be a normalized starlike mapping from $B(p_1, \dots, p_n)$ to $\mathbf{C}^n$ with $|\sigma(\zeta)| \leq c \leq 1$ uniformly for $z \in \partial B(p_1, \dots, p_n)$. Let h be the Minkowski function of $B(p_1, \dots, p_n)$. Then*

$$\frac{h(z)}{(1+ch(z))^2} \leq h(f(z)) \leq \frac{h(z)}{(1-ch(z))^2}.$$

Furthermore the estimates are sharp.

Proof. We will show that the estimates are sharp. Let

$$f(z) = \left(\frac{z_1}{(1-cz_1)^2}, \frac{z_2}{(1-cz_2)^2}, \dots, \frac{z_n}{(1-cz_n)^2} \right).$$

Then f is a normalized biholomorphic mapping on $B(p_1, \dots, p_n)$ and

$$\phi_z(\zeta) = \sum_{j=1}^{n} \frac{\partial h^2}{\partial z_j}(z) z_j \frac{1 - c\zeta z_j}{1 + c\zeta z_j}$$

for any $z \in \partial B(p_1, \dots, p_n)$. Since $(\partial h^2/\partial z_j)(z) z_j \geq 0$ and $\sum_{j=1}^{n} (\partial h^2/\partial z_j)(z) z_j = 1$, we have $|\sigma(\zeta)| \leq c$ for any $\zeta \in \Delta$. Therefore, f is a normalized starlike mapping with $|\sigma(\zeta)| \leq c$ uniformly for $z \in \partial B(p_1, \dots, p_n)$. Since

$$h(f(z_1, 0, \dots, 0)) = \frac{1}{|1 - cz_1|^2} h((z_1, 0, \dots, 0))$$

and

$$h((z_1, 0, \dots, 0)) = |z_1|,$$

the estimates are sharp.

Corollary 1. *Let D be a bounded balanced pseudoconvex domain in $\mathbf{C}^n$ with C^1 plurisubharmonic defining functions and let f be a normalized starlike mapping from D to $\mathbf{C}^n$ with $|\sigma(\zeta)| \leq c \leq 1$ uniformly for $z \in \partial D$. Then the image of f contains $1/(1+c)^2 D$. If $D = B(p_1, \dots, p_n)$ with $p_1, \dots, p_n > 1$, the value $1/(1+c)^2$ is best possible.*

Let k be a positive integer. We say that f has a k-fold symmetric image if the image of f is unchanged when multiplied by the scalar complex number $\exp(2\pi i/k)$. If k-fold symmetry of f is assumed, then Theorems 4, 5 and Corollary 1 can be strengthened as follows as in Barnard, FitzGerald and Gong [**1**].

Corollary 2. *Let D be a bounded balanced pseudoconvex domain in $\mathbf{C}^n$ with C^1 plurisubharmonic defining functions and let f be a normalized starlike mapping from D to $\mathbf{C}^n$ with $|\sigma(\zeta)| \le c \le 1$ uniformly for $z \in \partial D$ and with a k-fold symmetric image for some positive integer k. Let h be the Minkowski function of D. Then*

$$\frac{h(z)}{(1+ch(z)^k)^{2/k}} \le h(f(z)) \le \frac{h(z)}{(1-ch(z)^k)^{2/k}}.$$

Therefore, the image of D under f contains $(1/(1+c)^{2/k})D$. Furthermore, these estimates are sharp when $D = B(p_1, \dots, p_n)$ with $p_1, \dots, p_n > 1$.

Corollary 3. *The only balanced domain which is the image of a bounded balanced pseudoconvex domain D in $\mathbf{C}^n$ with C^1 plurisubharmonic defining functions under a normalized biholomorphic mapping is D.*

5. Quasiconformal extensions.

In this section, we will show that a quasiconformal strongly starlike mapping with $|w|$ uniformly bounded on a bounded balanced pseudoconvex domain D in $\mathbf{C}^n$ with C^1 plurisubharmonic defining functions admits a quasiconformal extension to $\mathbf{C}^n$ using the idea of Chuaqui [**2**].

Let Ω, Ω' be domains in $\mathbf{R}^m$. A homeomorphism $f : \Omega \to \Omega'$ is said to be quasiconformal if it is differentiable a.e., ACL(absolutely continuous on lines) and

$$\|D(f;x)\|^m \le K|\det D(f;x)| \quad \text{a.e. in } \Omega,$$

where $D(f;x)$ denotes the (real) Jacobian matrix of f, K is a constant and

$$\|D(f;x)\| = \sup\{|D(f;x)(a)| \mid |a| = 1\}.$$

Theorem 6. *Let D be a bounded balanced pseudoconvex domain in $\mathbf{C}^n$ with C^1 plurisubharmonic defining functions, and let f be a quasiconformal, strongly starlike mapping with $|w|$ uniformly bounded on D. Then f extends to a quasiconformal homeomorphism of $\mathbf{R}^{2n}$ onto itself.*

Proof. We may assume that f is normalized. Let $f_i = u_i + \sqrt{-1}v_i$ and $z_i = x_i + \sqrt{-1}y_i$. We first show that $\|D(u,v;x,y)\|$ is uniformly bounded in D. Let $1/2 < h(z) < 1$. By Lemma 1, we have

$$h^2(z)\frac{1-ch(z)}{1+ch(z)} \le \left|\frac{\partial h^2}{\partial z}\right| \cdot |w|. \tag{5.1}$$

Using $Df(w) = f$, Theorem 4 and (5.1), we have

$$h\left(Df\left(\frac{w}{|w|}\right)\right) \le \left|\frac{\partial h^2}{\partial z}\right| \frac{1+ch(z)}{h(z)(1-ch(z))^3} \le 2\left|\frac{\partial h^2}{\partial z}\right| \frac{1+c}{(1-c)^3}.$$

Since h is C^1 on $\mathbf{C}^n \setminus \{0\}$, $h(Df(w/|w|))$ is bounded for $1/2 < h(z) < 1$. Since $D = \{h(z) < 1\}$ is bounded, $|Df(w/|w|)|$ is uniformly bounded for $1/2 < h(z) < 1$. By the Cauchy-Riemann equations, this implies that $D(u,v;x,y)^t(\mathrm{Re}\ w/|w|, \mathrm{Im}\ w/|w|)$ is uniformly bounded for $1/2 < h(z) < 1$. Since f is quasiconformal, $\|D(u,v;x,y)\|$ is uniformly bounded for $1/2 < h(z) < 1$. Then $\|D(u,v;x,y)\|$ is uniformly bounded in D.

Next we will show that f admits a continuous extension to $\overline{D}$, and the extension is univalent in $\overline{D}$. For $a \in \partial D$, let $f(a) = \lim_{j\to\infty} f(t_j a)$, where $t_j < 1$ and $t_j \to 1$. This is well-defined, since $\|D(u,v;x,y)\|$ is uniformly bounded in D. Let g be the Riemannian metric induced on ∂D by the Euclidean metric on $\mathbf{R}^{2n}$, and let d_g be the distance function on ∂D with respect to g. For any positive ε, let $U_g(a) = \{z \in \partial D \mid d_g(a,z) < \varepsilon/2M\}$, where $M = \sup\{\|D(u,v;x,y)\| \mid (x,y) \in D\}$. Since the topology on ∂D defined by d_g coincides with the topology induced on ∂D by the Euclidean topology on $\mathbf{C}^n$, there exists a $\delta > 0$ such that $U(a) = \{z \in \partial D \mid |z-a| < \delta\} \subset U_g(a)$. Let

$$V = \{z \in \mathbf{C}^n \mid |z-a| < \delta/2\} \cap \left\{z \in \overline{D} \mid L\left(\frac{1}{h(z)} - 1\right) < \min\left(\frac{\delta}{2}, \frac{\varepsilon}{2M}\right)\right\},$$

where $L = \sup\{|z| \mid z \in \overline{D}\}$. Then V is an open neighborhood of a in $\overline{D}$. Let $z \in V$. Then $z/h(z) \in U(a)$, since

$$|a - z/h(z)| \le |a-z| + |z|\left(\frac{1}{h(z)} - 1\right) < \delta.$$

Then there exists a piecewise C^1-curve $\gamma : [0,1] \to \partial D$ such that $\gamma(0) = a$, $\gamma(1) = z/h(z)$ and $L_g(\gamma) < \varepsilon/2M$, where $L_g(\gamma)$ denotes the length of γ with respect to g. Let $\iota : \partial D \to \mathbf{R}^{2n}$ be the natural inclusion mapping. Then, we have

$$\begin{aligned}
|f(a) - f(z/h(z))| &= \lim_{j\to\infty}\left|\int_0^1 \frac{d}{ds} f(t_j(\iota\circ\gamma)(s))ds\right| \\
&\le \lim_{j\to\infty}\int_0^1 \left|\frac{d}{ds} f(t_j(\iota\circ\gamma)(s))\right| ds \\
&\le \lim_{j\to\infty}\int_0^1 M\left|\left(t_j\frac{d}{ds}(\iota\circ\gamma)(s)\right)\right| ds \\
&= M\int_0^1 \sqrt{g(\dot\gamma(s),\dot\gamma(s))}ds \\
&< \frac{\varepsilon}{2}.
\end{aligned}$$

Then $|f(z)-f(a)| \le |f(a)-f(z/h(z))| + |f(z/h(z))-f(z)| \le \varepsilon/2 + M|z - z/h(z)| < \varepsilon$ This implies that f is continuous on $\overline{D}$. Since

$$h(v)^{-1}\frac{\partial}{\partial t}h(v) \le \frac{-(1-c)}{1+c}$$

for $z \neq 0$ by (4.3), we have

$$h(v) \le h(z)\exp\left\{-\frac{1-c}{1+c}(t-s)\right\}$$

as in Pfaltzgraff [**10**]. This implies that

$$\overline{v(D,s,t)} \subset D \quad \text{for } 0 \le s < t. \tag{5.2}$$

Let $f_t(z) = e^t f(z)$ for $t \ge 0$. By (4.2), we have $f_s(z) = f_t(v(z,s,t))$ for $z \in D$. Then by (5.2), $f_s(\overline{D}) \subset f_t(D)$ for $0 \le s < t$. Therefore

$$v(z,s,t) = f_t^{-1}(f_s(z)) \quad (0 \le s < t)$$

defines a continuous extension of v to $\overline{D}$. For $z \in D$, we have

$$\begin{aligned} |v(z,s,t)-z| &\le \int_s^t \left|\frac{\partial}{\partial \tau}v(z,s,\tau)\right| d\tau \\ &= \int_s^t |-w(v(z,s,\tau))| d\tau \\ &\le C(t-s) \end{aligned} \tag{5.3}$$

for some positive constant C, since $|w|$ is uniformly bounded. Since v is continuous on $\overline{D}$, this estimate holds for $z \in \overline{D}$. Suppose that $f(z_1) = f(z_2)$ for $z_1, z_2 \in \overline{D}$. Then for $t > 0$, we have

$$f_t(v(z_1,0,t)) = f_t(v(z_2,0,t)).$$

Since f_t is univalent in D, we obtain $v(z_1,0,t) = v(z_2,0,t)$. Letting $t \to 0$, we have $z_1 = z_2$ by (5.3). Therefore, f is univalent in $\overline{D}$.

Let

$$F(z) = \begin{cases} f(z) & z \in \overline{D} \\ h(z) f(\frac{z}{h(z)}) & z \notin \overline{D}. \end{cases}$$

We will show that F is the quasiconformal extension of f. It is easy to show that F is continuous and univalent on $\mathbf{R}^{2n}$. Let $\mathbf{R}^{2n} \cup \{\infty\} = S^{2n}$ be a one point compactification of $\mathbf{R}^{2n}$. We extend F to S^{2n} by $F(\infty) = \infty$. By Theorem 4, F is a continuous bijective mapping from S^{2n} onto itself. Therefore, F is a homeomorphism from S^{2n} onto itself. Thus F is a homeomorphism from $\mathbf{R}^{2n}$ onto itself. For $0 < r < 1$, let

$$F^r(z) = \begin{cases} f(rz) & z \in \overline{D} \\ h(z) f(r\frac{z}{h(z)}) & z \notin \overline{D}. \end{cases}$$

Then

$$F^r(z/r) = \begin{cases} f(z) & z \in \overline{D_r} \\ r^{-1}h(z)f(\frac{z}{r^{-1}h(z)}) & z \notin \overline{D_r}. \end{cases}$$

Since $r^{-1}h(z)$ is the Minkowski function of D_r, F^r is a homeomorphism from $\mathbf{R}^{2n}$ onto itself. We will show that $F^r \to F$ uniformly on compact subsets of $\mathbf{R}^{2n}$, F^r is differentiable a.e., F^r is ACL and

$$\|D(u^r, v^r; x, y)\|^{2n} \leq K|\det D(u^r, v^r; x, y)| \quad \text{a.e. in } \mathbf{R}^{2n},$$

where $F^r_i = u^r_i + \sqrt{-1}v^r_i$ and K is independent of r and x. Then by Corollary 21.3 and Corollary 37.4 of Väisälä [**12**], F is quasiconformal. Since f is continuous on $\overline{D}$, $F^r \to F$ uniformly on compact subsets of $\mathbf{R}^{2n}$. Since h is C^1 on $\mathbf{R}^{2n} \setminus \{0\}$, F^r is differentiable on $\mathbf{R}^{2n} \setminus \partial D$.

Since f is quasiconformal in D, there exists a positive constant K_1 such that

$$\|D(u, v; x, y)\|^{2n} \leq K_1|\det D(u, v; x, y)| \text{ in } D. \tag{5.4}$$

Then we have

$$\|D(u^r, v^r; x, y)\|^{2n} \leq K_1|\det D(u^r, v^r; x, y)| \text{ in } D, \tag{5.5}$$

since

$$D(u^r, v^r; x, y) = rD(u, v; rx, ry) \text{ on } D. \tag{5.6}$$

For $z \notin \overline{D}$, let $\zeta = rh(z)^{-1}z \in D \setminus \{0\}$ and let $\zeta = \xi + \sqrt{-1}\eta$. Then

$$D(u^r, v^r; x, y) = rD(u, v; \xi, \eta)(I + M(\xi, \eta)),$$

where

$$M(\xi, \eta) = r^{-1}\begin{pmatrix} \mathrm{Re}(w(\zeta) - \zeta) \\ \mathrm{Im}(w(\zeta) - \zeta) \end{pmatrix} \mathrm{grad}h(\xi, \eta).$$

Since h is C^1 on $\mathbf{C}^n \setminus \{0\}$ and $\|M(\xi, \eta)\| = r^{-1}|w(\zeta) - \zeta||\mathrm{grad}h(\xi, \eta)|$, $\|M(\xi, \eta)\|$ is uniformly bounded for r near 1. Then

$$\begin{aligned} \|D(u^r, v^r; x, y)\| &\leq r\|D(u, v; \xi, \eta)\|\|I + M(\xi, \eta)\| \\ &\leq r\|D(u, v; \xi, \eta)\|(1 + \|M(\xi, \eta)\|) \\ &\leq K_2\|D(u, v; \xi, \eta)\|. \end{aligned} \tag{5.7}$$

Since $M(\xi, \eta)$ has rank 1,

$$\begin{aligned} \det(I + M(\xi, \eta)) &= 1 + \mathrm{tr}\ M(\xi, \eta) \\ &= r^{-2}\mathrm{Re}\left\langle \frac{\partial h^2}{\partial z}(\zeta), \overline{w(\zeta)} \right\rangle \\ &\geq r^{-2}h^2(\zeta)\frac{1 - ch(\zeta)}{1 + ch(\zeta)} \\ &\geq \frac{1 - c}{1 + c} \end{aligned}$$

by Lemma 1. Then

$$(5.8)\quad \begin{aligned} |\det D(u^r,v^r;x,y)| &= r^{2n}|\det D(u,v;\xi,\eta)||\det(I+M(\xi,\eta))| \\ &\geq r^{2n}\frac{1-c}{1+c}|\det D(u,v;\xi,\eta)|. \end{aligned}$$

By (5.4), (5.7) and (5.8), we have

$$(5.9)\quad \begin{aligned} \|D(u^r,v^r;x,y)\|^{2n} &\leq K_2^{2n}\|D(u,v;\xi,\eta)\|^{2n} \\ &\leq K_1K_2^{2n}|\det D(u,v;\xi,\eta)| \\ &\leq r^{-2n}\frac{1+c}{1-c}K_1K_2^{2n}|\det D(u^r,v^r;x,y)|. \end{aligned}$$

By (5.5) and (5.9), we have

$$\|D(u^r,v^r;x,y)\|^{2n} \leq K|\det D(u^r,v^r;x,y)| \quad \text{a.e. in } \mathbf{R}^{2n},$$

where K is independent of r and x.

Let $\mathbf{R}_i^{2n-1}=\{x\in\mathbf{R}^{2n}\mid x_i=0\}$ and let P_i be the orthogonal projection of $\mathbf{R}^{2n}$ onto $\mathbf{R}_i^{2n-1}$. Let Q be a closed $2n$-interval. Let $J_y=Q\cap\mathrm{P}_i^{-1}(y)$. We will show that F^r is absolutely continuous on J_y for almost every $y\in\mathrm{P}_iQ$. Let $A=\{y\in\mathrm{P}_iQ\mid J_y\cap\partial D \text{ is uncountable }\}$. By Theorem 30.16 of Väisälä [**12**], $m_{2n-1}(A)=0$. For any $y\in\mathrm{P}_iQ\setminus A$, $F^r|_{J_y}$ is an injective path, and $J_y\cap\partial D$ is countable. By (5.6) and (5.7), $|\partial_iF^r|$ is bounded on $U\setminus\partial D$ for $1\leq i\leq 2n$, where U is a neighborhood of $\partial D\cup J_y$, since $\|D(u,v;x,y)\|$ is uniformly bounded in D. Then F^r is absolutely continuous on every closed subinterval of $J_y\setminus(J_y\cap\partial D)$ and

$$\int_{J_y}|\partial_iF^r|dm_1<\infty.$$

By Theorem 30.12 of Väisälä [**12**], $F^r|_{J_y}$ is absolutely continuous.

This completes the proof.

References

[1] R.W. Barnard, C.H. FitzGerald and S. Gong, *The growth and* $1/4$*-theorems for starlike mappings in* $\mathbf{C}^n$, Pacific J. Math., **150** (1991), 13-22.

[2] M. Chuaqui, *Application of subordination chains to starlike mappings in* $\mathbf{C}^n$, Pacific J. Math., **168** (1995), 33-48.

[3] S. Gong, *Biholomorphic mappings in several complex variables*, Contemporary Mathematics, **142** (1993), 15-48.

[4] S. Gong, S. Wang and Q. Yu, *A necessary and sufficient condition that biholomorphic mappings are starlike on a class of Reinhardt domains*, Chin. Ann. of Math., **13B**(**1**) (1992), 95-104.

[5] H. Hamada, *A Schwarz lemma in several complex variables* II, Proceedings of the Fourth International Colloquium on Finite or Infinite Dimensional Complex Analysis, Kumamoto, Japan, July 29-31, 1996, Kyushu Univ. Co-op (Fukuoka, Japan, December 1996), 61-70.

[6] ———, *A Schwarz lemma on complex ellipsoids*, Ann. Pol. Math., **67** (1997), 269-275.

[7] M. Jarnicki and P. Pflug, *Invariant distances and metrics in complex analysis*, de Gruyter, Berlin-New York, 1993.

[8] S.G. Krantz, *Function theory of several complex variables*, Wiley, 1982.

[9] J.A. Pfaltzgraff, *Subordination chains and univalence of holomorphic mappings in* $\mathbf{C}^n$, Math. Ann., **210** (1974), 55-68.

[10] ———, *Subordination chains and quasiconformal extension of holomorphic mappings in* $\mathbf{C}^n$, Ann. Acad, Scie. Fenn. Ser. A I Math., **1** (1975), 13-25.

[11] T.J. Suffridge, *The principle of subordination applied to functions of several complex variables*, Pacific J. Math., **33** (1970), 241-248.

[12] J. Väisälä, *Lectures in n-dimensional quasiconformal mappings*, Lecture Notes in Math., **229**, Springer Verlag, New York, 1971.

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