

HIGHER DIMENSIONAL LINKS IN A SIMPLICIAL COMPLEX EMBEDDED IN A SPHERE

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We show that any embedding of the n -skeleton of a $(2n + 3)$ -dimensional simplex into the $(2n + 1)$ -dimensional sphere contains a nonsplittable link of two n -dimensional spheres.

1. Introduction.

Throughout this paper we work in the piecewise linear category. Conway and Gordon showed in [1] that any embedding of the complete graph over six vertices into the 3-space contains a pair of nontrivially linked circles. We refer the reader to [6], [2], [4], [3] etc. for related works. In this paper we generalize the result of Conway and Gordon to higher dimensions.

Let σ_j^i be the i -skeleton of a j -dimensional simplex $\sigma_j = \langle v_1, v_2, \dots, v_{j+1} \rangle$ where v_1, v_2, \dots, v_j and v_{j+1} are the 0-simplices of σ_j . Let S^k be the k -dimensional unit sphere. Let X and Y be disjoint n -dimensional spheres embedded in S^{2n+1} . Then the linking number $lk(X, Y) \in \mathbb{Z}$ is defined up to sign, see for example [7]. Then the modulo 2 reduction $lk_2(X, Y) \in \mathbb{Z}/2\mathbb{Z}$ of $lk(X, Y)$ is well-defined. We note that $lk_2(X, Y) \equiv lk_2(Y, X) \pmod{2}$. Let \mathcal{L}^n be the set of all unordered pairs of disjoint subcomplexes of σ_{2n+3}^n each of which is homeomorphic to an n -dimensional sphere. We note that each element (J, K) of \mathcal{L}^n can be written as

$$(J, K) = (\partial\langle v_{a_1}, v_{a_2}, \dots, v_{a_{n+2}} \rangle, \partial\langle v_{b_1}, v_{b_2}, \dots, v_{b_{n+2}} \rangle)$$

where ∂ denotes the boundary and $\{a_1, a_2, \dots, a_{n+2}\} \cup \{b_1, b_2, \dots, b_{n+2}\} = \{1, 2, \dots, 2n + 4\}$. Therefore the number of the elements of \mathcal{L}^n is $\binom{2n+4}{n+2}/2$.

Theorem 1.1. *Let n be a non-negative integer. Let $f : \sigma_{2n+3}^n \rightarrow S^{2n+1}$ be an embedding. Then*

$$\sum_{(J,K) \in \mathcal{L}^n} lk_2(f(J), f(K)) \equiv 1 \pmod{2}.$$

We note that σ_5^1 is the complete graph over six vertices and the case $n = 1$ of Theorem 1.1 is what Conway and Gordon actually proved in [1]. By Theorem 1.1 we have that there is at least one $(J, K) \in \mathcal{L}^n$ with $lk(f(J), f(K)) \equiv 1 \pmod{2}$. Thus we have that any embedding of σ_{2n+3}^n into S^{2n+1} contains a nonsplittable link of two n -spheres.

2. Proof of Theorem 1.1.

The idea of the following proof is essentially the same as that of Conway and Gordon in [1].

Lemma 2.1. *For any embeddings $f, g : \sigma_{2n+3}^n \rightarrow S^{2n+1}$,*

$$\sum_{(J,K) \in \mathcal{L}^n} lk_2(f(J), f(K)) \equiv \sum_{(J,K) \in \mathcal{L}^n} lk_2(g(J), g(K)) \pmod{2}.$$

Proof. Since $n < 2n + 1$ we have that both f and g are homotopic to a constant map. Therefore f and g are homotopic. By a standard general position argument we can modify the homotopy between f and g and we may suppose that f and g are connected by a finite sequence of ‘crossing changes’ of n -simplices of σ_{2n+3}^n . Namely we have a homotopy $H : \sigma_{2n+3}^n \times [0, 1] \rightarrow S^{2n+1} \times [0, 1]$ with $H(x, 0) = (f(x), 0)$, $H(x, 1) = (g(x), 1)$ whose multiple points are only finitely many transversal double points of the product of n -simplices and $[0, 1]$ and no two of them have the same second entry. Then it is enough to show the case that H has just one double point. If the first entries of the preimage of the double point do not lie in disjoint n -simplices of σ_{2n+3}^n then we have $lk_2(f(J), f(K)) \equiv lk_2(g(J), g(K)) \pmod{2}$ for each $(J, K) \in \mathcal{L}^n$. Thus we may suppose without loss of generality that the first entries of the preimage lie in n -simplices $\langle v_1, v_2, \dots, v_{n+1} \rangle$ and $\langle v_{n+2}, v_{n+3}, \dots, v_{2n+2} \rangle$. Let

$$(J_1, K_1) = (\partial\langle v_1, v_2, \dots, v_{n+1}, v_{2n+3} \rangle, \partial\langle v_{n+2}, v_{n+3}, \dots, v_{2n+2}, v_{2n+4} \rangle)$$

and

$$(J_2, K_2) = (\partial\langle v_1, v_2, \dots, v_{n+1}, v_{2n+4} \rangle, \partial\langle v_{n+2}, v_{n+3}, \dots, v_{2n+2}, v_{2n+3} \rangle).$$

Then we have $lk_2(f(J_i), f(K_i)) \equiv lk_2(g(J_i), g(K_i)) + 1 \pmod{2}$ for $i = 1, 2$ and $lk_2(f(J), f(K)) \equiv lk_2(g(J), g(K)) \pmod{2}$ for $(J, K) \in \mathcal{L}^n$, $(J, K) \neq (J_1, K_1), (J_2, K_2)$ as unordered pair. This completes the proof. \square

Lemma 2.2. *There is an embedding $f : \sigma_{2n+3}^n \rightarrow S^{2n+1}$ with*

$$\sum_{(J,K) \in \mathcal{L}^n} lk_2(f(J), f(K)) \equiv 1 \pmod{2}.$$

Proof. We use the fact that S^{2n+1} is homeomorphic to the join of two n -dimensional spheres, see Chapter 1 of [5]. Let P be the join of the two simplicial complices $J_0 = \partial\langle v_1, v_2, \dots, v_{n+2} \rangle$ and $K_0 = \partial\langle v_{n+3}, v_{n+4}, \dots, v_{2n+4} \rangle$. Since $\sigma_{2n+3} = \langle v_1, v_2, \dots, v_{2n+4} \rangle$ is the join of $\langle v_1, v_2, \dots, v_{n+2} \rangle$ and $\langle v_{n+3}, v_{n+4}, \dots, v_{2n+4} \rangle$ we have that P is a subcomplex of σ_{2n+3} . Then we have that σ_{2n+3}^n is a subcomplex of P . Since P is homeomorphic to S^{2n+1} we have an embedding, say f , of σ_{2n+3}^n into S^{2n+1} . Let $(J, K) \in \mathcal{L}^n$. Then

$$(J, K) = (\partial\langle v_{a_1}, v_{a_2}, \dots, v_{a_{n+2}} \rangle, \partial\langle v_{b_1}, v_{b_2}, \dots, v_{b_{n+2}} \rangle)$$

for some $\{a_1, a_2, \dots, a_{n+2}\}$ and $\{b_1, b_2, \dots, b_{n+2}\}$. If $(J, K) \neq (J_0, K_0)$ as unordered pair then we have that the $(n+1)$ -simplices $\langle v_{a_1}, v_{a_2}, \dots, v_{a_{n+2}} \rangle$ and $\langle v_{b_1}, v_{b_2}, \dots, v_{b_{n+2}} \rangle$ are contained in P . Therefore $f(J)$ and $f(K)$ bound disjoint $(n+1)$ -dimensional disks in S^{2n+1} and we have $\ell k_2(f(J), f(K)) \equiv 0 \pmod{2}$. It is clear that $\ell k_2(f(J_0), f(K_0)) \equiv 1 \pmod{2}$. This completes the proof. \square

Theorem 1.1 follows immediately from Lemma 2.1 and Lemma 2.2.

Remark 2.3. If we consider a general position map $f : \sigma_{j+k+3}^k \rightarrow S^{j+k+1}$ for $0 \leq j \leq k$ and consider all pair (J, K) of disjoint j -sphere and k -sphere in σ_{j+k+3}^k , then we have a result that is a generalization of Lemma 2.1. The proof is essentially the same. However it turns out that the sum of ℓk_2 is zero whenever $j < k$. In fact, for any finite simplicial complex Q and $j < k$, there is a general position map $f : Q \rightarrow S^{j+k+1}$ whose image is contained in the upper hemisphere and whose restriction to the j -skeleton of Q is an embedding into the equator $S^{j+k} \subset S^{j+k+1}$. Then it is easy to see that $\ell k_2(f(J), f(K)) = 0$ for any pair (J, K) of disjoint j -sphere and k -sphere in Q .

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