PACIFIC JOURNAL OF MATHEMATICS
Vol. 194, No. 2, 2000

COMMUTING ANALYTIC SELF-MAPS OF THE BALL

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Under broad conditions, two analytic self-maps of the disk fixing 0 commute under composition precisely when they have the same Schroeder map, where the Schroeder map for an analytic $\varphi : D \to D$ with $\varphi(0) = 0$ is the unique analytic function σ on D solving Schroeder's equation $\sigma \circ \varphi = \varphi'(0)\sigma$ and satisfying $\sigma'(0) = 1$. For analytic self-maps of the ball in C^N fixing 0 we may still seek analytic C^N-valued solutions σ to Schroeder's equation with $\sigma'(0) = I$, but considerable complications for existence and uniqueness of σ may ensue. Nevertheless, we show that there are reasonably general hypotheses under which it will still be the case that two analytic self-maps of the ball fixing 0 commute if and only if they share a common Schroeder map σ with $\sigma'(0) = I$.

1. Introduction.

If φ is an analytic map of the unit disk D into itself which fixes the origin and has derivative there satisfying $0 < |\varphi'(0)| < 1$ then there exists an analytic map σ on D that satisfies Schroeder's functional equation

$$\sigma \circ \varphi = \varphi'(0)\sigma.$$

This "Schroeder map" σ is unique up to constant multiples; its existence and uniqueness was proved by Koenigs in 1884 ([3]). It is usually convenient to require that σ satisfy $\sigma'(0) = 1$. Koenigs showed that in this case σ can be obtained as the almost uniform limit of normalized iterates of φ:

$$\sigma = \lim_{n\to\infty} \frac{\varphi_n}{\varphi'(0)^n},$$

where $\varphi_1 = \varphi$ and $\varphi_{k+1} = \varphi \circ \varphi_k$. When φ is univalent in D, σ will be also, so that φ is conjugate to multiplication by $\varphi'(0)$ on $\sigma(D) : \varphi = \sigma^{-1}\varphi'(0)\sigma$. Suppose ψ is an analytic self-map of D which commutes with φ under composition. Then necessarily $\psi(0) = 0$. Moreover, φ and ψ will have the same Schroeder maps, and conversely if $\psi : D \to D$ is analytic, fixes 0 and has the same Schroeder map as φ, then $\varphi \circ \psi = \psi \circ \varphi$. These results follow from the existence and (essential) uniqueness of the Schroeder map in one variable. ([1], [4].)

If φ is an analytic self-map of the unit ball B_N in C^N which fixes the origin, then by a Schroeder map for φ we will mean an analytic map $\sigma : B_N \to C^N$ which satisfies the functional equation

$$\sigma \circ \varphi = \varphi'(0)\sigma \tag{1}$$

where $\varphi'(0)$ is the linear map from C^N to C^N given by the matrix whose ij^{th} entry is $D_j\varphi_i(0)$. By analogy to the one variable case we restrict to the case that the eigenvalues of $\varphi'(0)$ are non-zero and of modulus strictly less than 1. In addition we exclude maps which are "unitary on a slice" of the ball; that is, maps φ for which there exists ζ, η in ∂B_N so that $\varphi(\lambda\zeta) = \lambda\eta$ for all $\lambda \in D$. We are chiefly interested in Schroeder maps σ which are locally univalent near 0. This is equivalent to requiring that $\sigma'(0)$ be invertible ([**5**, 1.3.7 and 15.1.8]). In fact when there is a solution to Equation (1) with $\sigma'(0)$ invertible, there will be a solution with $\sigma'(0) = I$. Precise conditions under which such a solution exists for a given φ are known ([**2**]; see also Theorem 1 and Corollary 2 below), but are somewhat complicated. A basic issue is whether any algebraic relationships of the form

$$\lambda_j = \lambda_1^{k_1}\lambda_2^{k_2}\cdots\lambda_N^{k_N}$$

hold between the eigenvalues λ_k of $\varphi'(0)$, where $k_i \geq 0$ and $\sum k_i \geq 2$, and if any such relationships do hold, whether they in fact prevent the existence of a locally univalent Schroeder map. Such an algebraic relationship for an eigenvalue of φ will be called a *resonance* of φ.

The results which make this precise are as follows. As a convenient normalization we may assume, by a unitary change of variables, that $\varphi'(0)$ is upper triangular.

Theorem 1 ([**2**]). *Suppose φ is an analytic map of B_N into B_N with $\varphi(0) = 0$ and $A = \varphi'(0)$ an upper triangular diagonalizable matrix, with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_N$ such that $0 < |\lambda_j| < 1$. Assume further that φ is not unitary on any slice. Suppose that $\lambda_j = \lambda_1^{k_1}\cdots\lambda_N^{k_N}$ is the longest expression (maximal $\sum k_i$) for one eigenvalue of A as a product of any number of the eigenvalues of A. Set $m = k_1 + \cdots + k_N$ and $M =$ the number of multi-indices for C^N of total order less than or equal to m. Let $\mathcal{M}$ be the upper left $M \times M$ corner of the matrix for the composition operator C_φ with respect to the standard (non-normalized) basis for any weighted Hardy space $H^2_\beta(B_N)$, ordered in the usual way. If $\mathcal{M}$ is diagonalizable, then Schroeder's Equation (1) has a solution σ with $\sigma'(0) = I$.*

The "standard basis" referred to in this theorem consists of the monomials $1, z_1, z_2, \ldots, z_n, z_1^2, z_1z_2, \ldots$ ordered as follows: z^α precedes z^γ where $\alpha = (\alpha_1, \ldots, \alpha_N)$ and $\gamma = (\gamma_1, \ldots, \gamma_N)$ are multi-indices, if either $|\alpha| < |\gamma|$ or, in the case $|\alpha| = |\gamma|$, if there is a j_0 so that $\alpha_j = \gamma_j$ for $j < j_0$ and $\alpha_{j_0} > \gamma_{j_0}$. The matrix for the composition operator C_φ with respect to this basis has as its j^{th} column the coefficients of φ^α with respect to this basis, where z^α is the j^{th} monomial in the prescribed ordering. A weighted Hardy space $H^2_\beta(B_N)$ is a Hilbert space of analytic functions on B_N for which the monomials form a complete orthogonal set of non-zero vectors satisfying

$$\beta(|\alpha_1|) \equiv \frac{\|z^{\alpha_1}\|}{\|z^{\alpha_1}\|_2} = \frac{\|z^{\alpha_2}\|}{\|z^{\alpha_2}\|_2}$$

whenever $|\alpha_1| = |\alpha_2|$, where $\|\cdot\|$ denotes the norm in $H^2_\beta(B_N)$ and $\|\cdot\|_2$ denotes the norm in $L^2(\sigma_N)$, σ_N being normalized Lebesgue measure on B_N. When $\varphi(0) = 0$ and φ is not unitary on any slice there exist weighted Hardy spaces on which the composition operator C_φ (defined by $C_\varphi(f) = f \circ \varphi$) is a compact operator ([2]).

There is a converse to Theorem 1 which says, under the same hypotheses on φ, that if φ has a Schroeder map with invertible derivative at the origin, then *every* upper left corner of the matrix for C_φ is diagonalizable.

While we won't have direct need for the full strength of Theorem 1 here, the following corollary will play a crucial role in our study of commuting analytic self-maps of B_N. It gives a description of all Schroeder maps (locally univalent or not) for φ, based on the presence or absence of resonances for φ.

Corollary 2 ([2])**.** *Suppose the hypotheses of Theorem 1 hold and that in addition $A = \varphi'(0)$ is diagonal. Then all solutions to Schroeder's Equation (1) can be described as $f = g \circ \sigma$ where σ is a Schroeder map with $\sigma'(0) = I$, as given in Theorem 1 and $g = (g_1, g_2, \dots, g_N)$ is a mapping on C^N with polynomial coordinate functions. Moreover, if $g_k = \sum c(\gamma) z^\gamma$, then the coefficients $c(\gamma), \gamma = (\gamma_1, \dots, \gamma_N)$ are 0 unless $\lambda_k = \lambda_1^{\gamma_1}\lambda_2^{\gamma_2}\cdots\lambda_N^{\gamma_N}\,(\gamma_i \geq 0)$, in which case $c(\gamma)$ can be chosen arbitrarily.*

If $A = \varphi'(0)$ is merely diagonalizable, with $SAS^{-1} = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$, then an arbitrary Schroeder map has the form $S^{-1} \circ g \circ S \circ \sigma$ with σ and g as just described.

Note that g_k always includes a linear term $b_k z_k$ (b_k arbitrary), and if λ_k is a repeated eigenvalue of $\varphi'(0)$ there will be other linear terms with arbitrary coefficients. The terms of g_k with order at least two correspond to the resonance relations for λ_k. When no resonance relations hold, g is linear. We emphasize that a resonance relation expresses an eigenvalue λ_j as a product $\lambda_1^{k_1}\lambda_2^{k_2}\cdots\lambda_N^{k_N}$ where $\sum k_i \geq 2$; a relation $\lambda_j = \lambda_k$ is not a resonance relation.

The goal of this paper is to study commuting analytic self-maps of B_N and to see, by analogy with known results in one variable, to what extent it still is the case that commuting maps are those which share a locally univalent Schroeder map. Our main results (Theorems 3 and 7) will show that under natural hypotheses on φ, a map ψ which commutes with φ and has no resonances in common with φ will share a locally univalent Schroeder map with φ. Examples will be give to show that this can fail if φ and ψ have resonances in common.

2. Non-resonant maps.

In studying the Schroeder maps for commuting self-maps φ, ψ of B_N, the easiest situation arises when at least one of φ, ψ has no resonances. This means, say, that no eigenvalue of $\varphi'(0)$ can be written as a product of two or more of the other eigenvalues, although repeated eigenvalues are allowed.

Theorem 3. *Suppose $\varphi : B_N \to B_N$ is analytic, with $\varphi(0) = 0$. Assume that $A = \varphi'(0)$ is upper triangular diagonalizable with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_N, 0 < |\lambda_j| < 1$. Assume further that φ is not unitary on any slice and that no resonance relations hold for any of the λ_j's. If $\psi : B_N \to B_N$ is analytic and $\psi \circ \varphi = \varphi \circ \psi$ then φ and ψ share a common Schroeder map which is locally univalent near* 0.

Proof. Since φ is not unitary on any slice and $\varphi\psi(0) = \psi\varphi(0) = \psi(0)$ we must have $\psi(0) = 0$, since the fixed point set of φ in B_N is affine ([5, 8.2.3]). By the $m = 1, M = N+1$ case of Theorem 1 we know that φ has a Schroeder map σ_φ with $\sigma'_\varphi(0) = I$. Moreover,

$$(\sigma_\varphi \circ \psi) \circ \varphi = \sigma_\varphi \circ \varphi \circ \psi = \varphi'(0)(\sigma_\varphi \circ \psi)$$

so $\sigma_\varphi \circ \psi$ is a Schroeder map for φ. By Corollary 2 this tells us that $\sigma_\varphi \circ \psi = S^{-1}BS\sigma_\varphi$ where S diagonalizes $\varphi'(0)$ and B is linear. Differentiation of this equation gives $\sigma'_\varphi(0)\psi'(0) = S^{-1}BS\sigma'_\varphi(0)$ so that in fact $S^{-1}BS = \psi'(0)$ and σ_φ is a Schroeder map for both φ and ψ, with derivative at 0 equal to I. ☐

It need not be the case that φ and ψ have the same *set* of Schroeder maps; see Example 1 in the next section.

As a converse to this result we have the following theorem.

Theorem 4. *Suppose φ, ψ are analytic self-maps of B_N, each fixing* 0, *with $\varphi'(0)\psi'(0) = \psi'(0)\varphi'(0)$. Suppose further that there exists an analytic $\sigma : B_N \to C^N$ with $\sigma'(0)$ invertible and both $\sigma \circ \varphi = \varphi'(0)\sigma$ and $\sigma \circ \psi = \psi'(0)\sigma$. Then $\varphi \circ \psi = \psi \circ \varphi$.*

Proof. Since σ is locally univalent near 0 we may write

$$\varphi = \sigma^{-1}\varphi'(0)\sigma$$

and

$$\psi = \sigma^{-1}\psi'(0)\sigma$$

in an open neighborhood of 0. Thus near 0 we have

$$\varphi \circ \psi = \sigma^{-1}\varphi'(0)\sigma\sigma^{-1}\psi'(0)\sigma = \sigma^{-1}\varphi'(0)\psi'(0)\sigma = \sigma^{-1}\psi'(0)\varphi'(0)\sigma = \psi \circ \varphi.$$

Since $\varphi \circ \psi = \psi \circ \varphi$ in an open neighborhood of 0 and the compositions are defined on B_N we must have $\varphi \circ \psi = \psi \circ \varphi$ in B_N. $\square$

The last result need not hold if the hypothesis on the commutability of the derivatives at 0 is omitted: Take φ, ψ to be linear maps which do not commute. They share $\sigma(z) = z$ as a common locally univalent Schroeder map.

3. Resonances.

We begin with several examples which will help set the stage for Theorem 7, the main result of this section.

Example 1. Let $\varphi(z_1, z_2) = (c_1 z_1, c_1^3 z_2 + c_2 z_1^2)$ where c_1, c_2 are sufficiently small non-zero constants so that $\varphi(B_2) \subset B_2$. Note that

$$\varphi'(0) = \begin{pmatrix} c_1 & 0 \\ 0 & c_1^3 \end{pmatrix}$$

and the resonance $\lambda_2 = \lambda_1^3$ holds for the eigenvalues $\lambda_1 = c_1, \lambda_2 = c_1^3$ of $\varphi'(0)$. It is easy to check that

$$\sigma_\varphi = \left(z_1, z_2 + \frac{c_2}{c_1^3 - c_1^2} z_1^2 \right)$$

is a Schroeder map for φ with derivative at 0 equal to I (this example is also discussed in [**2**]). By Corollary 2 all Schroeder maps for φ are of the form $g \circ \sigma_\varphi$ where g is a polynomial map $(b_1 z_1, b_2 z_2 + b_3 z_1^3)$ for arbitrary constants b_1, b_2 and b_3, and thus have the form

$$\left(b_1 z_1, b_2 z_2 + \frac{b_2 c_2}{c_1^3 - c_1^2} z_1^2 + b_3 z_1^3 \right).$$

Now suppose that ψ commutes with φ. We know from the calculations in Theorem 3 that $\sigma_\varphi \circ \psi$ is a Schroeder map for φ and hence $\sigma_\varphi \circ \psi = g \circ \sigma_\varphi$ for g as above. From this we easily determine that ψ must be of the form

$$\left(b_1 z_1, b_2 z_2 + \frac{c_2}{c_1^3 - c_1^2} (b_2 - b_1^2) z_1^2 + b_3 z_1^3 \right)$$

for some constants b_1, b_2, b_3, and moreover any choice of these constants will give a map which commutes with φ. If these constants are chosen sufficiently small, $\psi(B_2) \subset B_2$. Note that whenever $b_3 \neq 0$ we have a commuting map which is not an iterate of φ, so the set of maps which commute with φ is considerably larger than just the natural iterates of φ.

If $b_2 \neq b_1^3$ then

$$\left(z_1, z_2 + \frac{c_2}{c_1^3 - c_1^2} z_1^2 + \frac{b_3}{b_2 - b_1^3} z_1^3 \right)$$

is a common Schroeder map for φ and ψ with derivative at 0 equal to I. We remark that

$$\left(z_1, z_2 + \frac{c_2}{c_1^3 - c_1^2} z_1^2 + \alpha z_1^3\right)$$

where $\alpha \neq b_3/(b_2 - b_1^3)$ is a Schroeder map for φ but not for ψ, so that while φ and ψ have a locally univalent Schroeder map in common, their sets of Schroeder maps are not the same.

On the other hand, if $b_2 = b_1^3, b_3 \neq 0$ and

$$\psi(z_1, z_2) = \left(b_1 z_1, b_2 z_2 + \frac{c_2}{c_1^3 - c_1^2}(b_2 - b_1^2) z_1^2 + b_3 z_1^3\right)$$

then ψ commutes with φ but ψ has *no* locally univalent Schroeder map by the converse of Theorem 1. One can check that the upper left 7×7 corner of the matrix for C_ψ has diagonal entries $1, b_1, b_1^3, b_1^2, b_1^4, b_1^6, b_1^3$ and three non-zero off diagonal entries: $c_2(b_1^3 - b_1^2)/(c_1^3 - c_1^2)$ in the 4-3 position, $c_2(b_1^3 - b_1^2) b_1/(c_1^3 - c_1^2)$ in the 7-5 position, and $b_3 \neq 0$ in the 7-3 position. This matrix is not diagonalizable. Note that the situation being considered here is that of the resonances of φ also being resonances of ψ, where ψ is not a natural iterate of φ.

We also note that this example shows that two self-maps of the ball which each commute with φ need not commute with each other as

$$\psi_1(z_1, z_2) = \left(b_1 z_1, b_2 z_2 + \frac{c_2}{c_1^3 - c_1^2}(b_2 - b_1^2) z_1^2 + b_3 z_1^3\right)$$

and

$$\psi_2(z_1, z_2) = \left(b_1 z_1, b_2 z_2 + \frac{c_2}{c_1^3 - c_1^2}(b_2 - b_1^2) z_1^2 + \frac{1}{2} b_3 z_1^3\right)$$

both commute with φ but fail to commute with each other if b_1, b_2 and b_3 are chosen to be sufficiently small non-zero values with $b_1^3 \neq b_2$.

In two variables only one resonance relation is possible (either $\lambda_1 = \lambda_2^n$ or $\lambda_2 = \lambda_1^m$), but as the number of dimensions increases so do the possible variety of resonance equations. The next example, describing a general situation in C^3 will be instructive for formulating a general theorem.

Example 2. Consider any analytic mapping $\varphi : B_3 \to B_3$, fixing 0 and not unitary on any slice, where $\varphi'(0)$ is diagonal, with diagonal entries λ_j satisfying $1 > |\lambda_1| > |\lambda_2| > |\lambda_3| > 0$ where the resonances

$$\lambda_2 = \lambda_1^n, (n \geq 2) \quad \text{and}$$

$$\lambda_3 = \lambda_1^m \lambda_2^k = \lambda_1^{m+nk} (m, k \geq 0, m + k \geq 2, \text{ and } m < n)$$

hold. Notice that we, in fact, have $k+1$ different resonances for λ_3:

$$\lambda_3 = \lambda_1^{m+nk} = \lambda_1^{r_1}\lambda_2 = \lambda_1^{r_2}\lambda_2^2 = \cdots = \lambda_1^{r_k}\lambda_2^k$$

where

$$r_j + jn = m + nk \tag{2}$$

for $j = 1, \ldots, k$. If φ satisfies the hypotheses of Corollary 2 then all Schroeder maps are of the form $g \circ \sigma_\varphi$ where σ_φ is a Schroeder map satisfying $\sigma_\varphi'(0) = I$ and g is a polynomial mapping with

$$g_1 = b_1 z_1, g_2 = b_2 z_2 + c_1 z_1^n$$

and

$$g_3 = b_3 z_3 + c_2 z_1^{m+nk} + c_3 z_1^{r_1} z_2 + c_4 z_1^{r_2} z_2^2 + \cdots + c_{k+2} z_1^{r_k} z_2^k$$

for arbitrary choice of the coefficients. Denote the collection of all such polynomial maps $\mathcal{G}_\varphi$.

Now suppose $\psi : B_3 \to B_3$ commutes with φ and that no resonance of φ is also a resonance of ψ. We know $\sigma_\varphi \circ \psi$ is a Schroeder map for φ so $\sigma_\varphi \circ \psi = g \circ \sigma_\varphi$ for some $g \in \mathcal{G}_\varphi$. Taking derivatives, we see that $\sigma_\varphi'(0)\psi'(0) = g'(0)\sigma_\varphi'(0)$ and thus $\psi'(0) = g'(0) = \operatorname{diag}(b_1, b_2, b_3)$. Our hypothesis on the resonances of ψ implies that $b_2 \neq b_1^n, b_3 \neq b_1^{m+nk}, b_3 \neq b_1^{r_1} b_2, \ldots, b_3 \neq b_1^{r_k} b_2^k$.

We claim that there exists $\hat{g}$ in $\mathcal{G}_\varphi$ with $\hat{g}'(0) = I$ solving $\hat{g} \circ g = g'(0)\hat{g}$. Once the claim is verified we see the following holds in a neighborhood of 0:

$$\begin{aligned}(\hat{g} \circ \sigma_\varphi) \circ \psi &= (\hat{g} \circ \sigma_\varphi) \circ \sigma_\varphi^{-1} \circ g \circ \sigma_\varphi = \hat{g} \circ g \circ \sigma_\varphi \\ &= g'(0) \circ \hat{g} \circ \sigma_\varphi = \psi'(0)(\hat{g} \circ \sigma_\varphi)\end{aligned}$$

since $\psi = \sigma_\varphi^{-1} g \sigma_\varphi$ near 0. If $(\hat{g} \circ \sigma_\varphi) \circ \psi = \phi'(0)(\hat{g} \circ \sigma_\varphi)$ holds near 0, then it holds in B_3 since $\hat{g}$ is defined on C^3. This shows that $\hat{g} \circ \sigma_\varphi$ is a Schroeder map for ψ with derivative at 0 equal to I; it is also a Schroeder map for φ by Corollary 2.

To verify the claim we will show that coefficients $\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_{k+2}$ may be determined so that $\hat{g}$ given by

$$\hat{g}_1 = z_1, \hat{g}_2 = z_2 + \hat{c}_1 z_1^n$$

and

$$\hat{g}_3 = z_3 + \hat{c}_2 z_1^{m+nk} + \hat{c}_3 z_1^{r_1} z_2 + \hat{c}_4 z_1^{r_2} z_2^2 + \cdots + \hat{c}_{k+2} z_1^{r_k} z_2^k$$

solves $\hat{g} \circ g = g'(0)\hat{g}$. Notice that $\hat{g}_1 \circ g = g_1 = b_1 z_1 = b_1 \hat{g}_1$ and that $\hat{g}_2 \circ g = b_2 \hat{g}_2$ provided $\hat{c}_1 = c_1/(b_2 - b_1^n)$; the hypothesis $b_2 \neq b_1^n$ being used here.

Finally, we turn to

$$\hat{g}_3 \circ g = b_3 \hat{g}_3. \tag{3}$$

Using the forms of $\hat{g}_3$ and g, we expand the left side of Equation (3) into a sum of monomials and observe that each of these monomials is a scalar multiple of a monomial which also appears in $b_3\hat{g}_3$, the right side of Equation (3). To see this, observe that when we expand $g_1^{r_j} g_2^j = (b_1 z_1)^{r_j}(b_2 z_2 + c_1 z_1^n)^j$ we get terms which are scalar multiples of the monomials $z_1^{r_j} z_2^s (z_1^n)^{j-s} = z_1^{r_j+n(j-s)} z_2^s (0 \leq s \leq j)$. Since $r_j + n(j-s) = r_s$, this monomial, with some scalar coefficient, appears in $b_3\hat{g}_3$.

By equating in turn the coefficients of

$$z_1^{r_k} z_2^k, z_1^{r_{k-1}} z_2^{k-1}, \dots, z_1^{r_1} z_2, z_1^{m+nk},$$

we obtain equations for the unknown coefficients $\hat{c}_{k+2}, \hat{c}_{k+1}, \dots, \hat{c}_2$. The equation obtained from the coefficients of $z_1^{r_k} z_2^k$ is

$$c_{k+2} + \hat{c}_{k+2} b_2^k b_1^{r_k} = b_3 \hat{c}_{k+2}$$

which may be solved for $\hat{c}_{k+2}$ provided $b_3 \neq b_1^{r_k} b_2^k$; this is guaranteed by the hypothesis on the resonances of ψ. Continuing, suppose that by comparing the coefficients of $z_1^{r_k} z_2^k, z_1^{r_{k-1}} z_2^{k-1}, \dots, z_1^{r_{j+1}} z_2^{j+1}$ the coefficients $\hat{c}_{k+2}, \hat{c}_{k+1}, \dots, \hat{c}_{j+3}$ have been determined. Next we compare coefficients of $z_1^{r_j} z_2^j$ on both sides of Equation (3). None of the terms

$$\hat{c}_2 g_1^{m+nk}, \hat{c}_3 g_1^{r_1} g_2, \dots, \hat{c}_{j+1} g_1^{r_{j-1}} g_2^{j-1}$$

contribute any terms of the form $z_1^{r_j} z_2^j$. The expansion of $\hat{c}_{j+2} g_1^{r_j} g_2^j$ contributes a term $\hat{c}_{j+2} b_1^{r_j} b_2^j z_1^{r_j} z_2^j$. The expansions of

$$\hat{c}_{j+3} g_1^{r_{j+1}} g_2^{j+1}, \dots, \hat{c}_{k+2} g_1^{r_k} g_2^k$$

contribute terms $z_2^{r_j} z_2^j$ all of whose coefficients involve the previously determined coefficients $\hat{c}_{j+3}, \dots, \hat{c}_{k+2}$ (and b_1, b_2). Thus equating the coefficients of $z_1^{r_j} z_2^j$ on both sides of Equation (3) leads to an equation of the form

$$c_{j+2} + \hat{c}_{j+2} b_1^{r_j} b_2^j + \text{ known terms } = b_3 \hat{c}_{j+2}$$

where "known terms" refers to a sum involving the known values $\hat{c}_{j+3}, \dots, \hat{c}_{k+2}$ and the b_i's. This equation may be solved for $\hat{c}_{j+2}$ provided $b_3 \neq b_1^{r_j} b_2^j$, which is part of our hypothesis. Continuing this process we determine all of the coefficients of the second and higher order terms of $\hat{g}_j$. Note that the only first order term in $\hat{g}_3 \circ g$ is $b_3 z_3$ and this is the only first order term on the right side of Equation (3). Thus we have found a choice of coefficients so that $\hat{g} \circ g = g'(0)\hat{g}$, verifying the claim.

We set some notation and terminology which will be useful in the main result. We now restrict attention to the case that the eigenvalues $\varphi'(0)$ are distinct, non-zero, and of modulus less than 1. There is no loss of generality in assuming that $\varphi'(0)$ is upper triangular, with diagonal entries $\lambda_1, \lambda_2, \dots, \lambda_N$ satisfying $1 > |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_N| > 0$, since there is a unitary map U so that $U^*\varphi'(0)U$ is upper triangular with the eigenvalues of $\varphi'(0)$ appearing in the prescribed order. Moreover, if φ and ψ commute, then so do $U^*\varphi U$ and $U^*\psi U$, and φ and ψ have a common locally univalent Schroeder map if and only if $U^*\varphi U$ and $U^*\psi U$ do. This ordering on the eigenvalues of $\varphi'(0)$ implies that λ_1 has no resonance relations, and in general a resonance for λ_j is of the form

$$\lambda_j = \lambda_1^{k_1}\lambda_2^{k_2}\cdots\lambda_{j-1}^{k_{j-1}}$$

where $k_i \geq 0$ and $\sum k_i \geq 2$.

For $j \geq 2$ we say that a monomial $cz_1^{k_1}z_2^{k_2}\cdots z_{j-1}^{k_{j-1}}$ (c any non-zero scalar) is *j-permissible* (for φ) if

$$\lambda_j = \lambda_1^{k_1}\lambda_2^{k_2}\cdots\lambda_{j-1}^{k_{j-1}};$$

call the corresponding multi-index $(k_1, k_2, \ldots, k_{j-1}, 0, \ldots, 0)$ j-permissible as well. There is a one-to-one correspondence between a resonance for λ_j and a j-permissible monomial with scalar coefficient 1 (or a j-permissible multi-index). For a given φ, let Γ_j denote the j-permissible multi-indices, so that $(k_1, k_2, \ldots, k_{j-1}, 0, \ldots, 0) \in \Gamma_j$ if and only if $\lambda_j = \lambda_1^{k_1}\lambda_2^{k_2}\cdots\lambda_{j-1}^{k_{j-1}}$ and Γ_j is empty if λ_j has no resonance relations. We order the multi-indices in Γ_j according to the following rule: A multi-index α in Γ_j preceedes a multi-index β if either the k_{j-1} entry of α is greater than the k_{j-1} entry of β, or if the entries in the k_i through k_{j-1} positions agree for some $i < j$, then the k_{i-1} entry of α is greater than the k_{i-1} entry of β. This is not the "usual" ordering on multi-indices. For example, if φ has resonance relations $\lambda_2 = \lambda_1^2$, $\lambda_3 = \lambda_1^3 = \lambda_1\lambda_2$ and

$$\lambda_4 = \lambda_3^2\lambda_1 = \lambda_2^2\lambda_3 = \lambda_1^2\lambda_2\lambda_3 = \lambda_2^3\lambda_1 = \lambda_1^4\lambda_3 = \lambda_2^2\lambda_1^3 = \lambda_1^5\lambda_2 = \lambda_1^7 \tag{4}$$

then the ordering on Γ_4 is

$$(1,0,2,0), (0,2,1,0), (2,1,1,0), (4,0,1,0)$$

$$(1,3,0,0), (3,2,0,0), (5,1,0,0), (7,0,0,0).$$

Recall the notation $\mathcal{G}_\varphi$ is used for the collection of all polynomial mappings $g = (g_1, g_2, \ldots, g_N)$ where

$$g_j(z_1, \ldots, z_N) = b_j z_j + \sum_{\gamma\in\Gamma_j} c^j(\gamma) z^\gamma$$

where the coefficients b_j and $c^j(\gamma)$ are arbitrary.

Lemma 5. *With φ as just described, suppose $g \in \mathcal{G}_\varphi$ and $\hat{g} \in \mathcal{G}_\varphi$ with $\hat{g}'(0) = I$. Then the monomials of order at least two in the expansion of $\hat{g}_j \circ g$ are all j-permissible, for $j \geq 2$.*

Proof. The coordinate functions $\hat{g}_j$ are of the form

$$\hat{g}_j = z_j + \sum \hat{c}^j(\gamma) z^\gamma$$

where the sum is over all multi-indices γ in Γ_j. Thus

$$\hat{g}_j \circ g = g_j + \sum_{\Gamma_j} \hat{c}^j(\gamma) g^\gamma$$

and it suffices to show that each monomial in the expansion of g^γ is j-permissible. Consider g^γ where $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_{j-1}, 0, \dots, 0)$, $\gamma_i \geq 0$, $\sum \gamma_i \geq 2$. Computing a term of $g^\gamma = g_1^{\gamma_1} g_2^{\gamma_2} \cdots g_{j-1}^{\gamma_{j-1}}$ involves making a choice of γ_1 terms from g_1 (necessarily each of these will be $b_1 z_1$), γ_2 terms from g_2, γ_3 terms from g_3, etc. Since $\lambda_j = \lambda_1^{\gamma_1} \lambda_2^{\gamma_2} \cdots \lambda_{j-1}^{\gamma_{j-1}}$, making these successive choices produces a j-permissible monomial. □

As an example, again suppose as above that φ has resonances $\lambda_2 = \lambda_1^2$, $\lambda_3 = \lambda_1^3 = \lambda_1\lambda_2$ and λ_4 has the resonance relations in Equation (4). In the expansion of $\hat{g}_4 \circ g$ we obtain, for example, the terms from $g_2^2 g_3$ since $(0,2,1,0) \in \Gamma_4$. The monomials obtained by choosing two terms from g_2 (either $b_2 z_2$ or a multiple of z_1^2) and one from g_3 (either $b_3 z_3$, a multiple of z_1^3 or a multiple of $z_1 z_2$) are all in Γ_4.

Since Theorem 3 applies when φ has no resonances, in the next two results we consider the resonant case.

Theorem 6. *Let φ be an analytic self-map of B_N fixing 0 and not unitary on any slice with $\varphi'(0)$ upper triangular with distinct diagonal entries λ_j satisfying*

$$1 > |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_N| > 0.$$

Suppose φ has at least one resonance relation. Let $g \in \mathcal{G}_\varphi$ with

$$g'(0) = \operatorname{diag}\{b_1, b_2, \dots, b_N\}$$

and assume that whenever a resonance relation

$$\lambda_j = \lambda_1^{k_1} \cdots \lambda_{j-1}^{k_{j-1}}$$

holds then

$$b_j \neq b_1^{k_1} \cdots b_{j-1}^{k_{j-1}}.$$

Then there exists $\hat{g} \in \mathcal{G}_\varphi$ with $\hat{g}'(0) = I$ and $\hat{g} \circ g = g'(0)\hat{g}$.

Proof. By hypotheses the coordinate functions of g are

$$g_j(z_1, \dots, z_N) = b_j z_j + \sum_{\gamma \in \Gamma_j} c^j(\gamma) z^\gamma.$$

Set

$$\hat{g}_j(z_1, \dots, z_N) = z_j + \sum_{\gamma \in \Gamma_j} \hat{c}^j(\gamma) z^\gamma$$

so that $\hat{g}'(0) = I$. We need only show that the coefficients $\hat{c}^j(\gamma)$ may be determined so that

$$\hat{g} \circ g = g'(0)\hat{g} \tag{5}$$

holds. We will determine these coefficients in the order of the multi-indices in Γ_j. For $\gamma = (\gamma_1, \dots, \gamma_{j-1}, 0, \dots, 0) \in \Gamma_j$, write b^γ for $b_1^{\gamma_1} b_2^{\gamma_2} \cdots b_{j-1}^{\gamma_{j-1}}$.

If $\Gamma_j = \emptyset$, then $\hat{g}_j \circ g = b_j \hat{g}_j$ holds automatically. If $\Gamma_j \neq \emptyset$, let τ_1 be the first multi-index in Γ_j, and compare the coefficients of z^{τ_1} on both sides of

$$\hat{g}_j \circ g = b_j \hat{g}_j \tag{6}$$

to obtain

$$c^j(\tau_1) + \hat{c}^j(\tau_1) b^{\tau_1} = b_j \hat{c}^j(\tau_1)$$

which can be solved for the unknown $\hat{c}^j(\tau_1)$ since $b_j \neq b^{\tau_1}$.

Next compare the coefficients of z^{τ_2} in Equation (6), where τ_2 is the second multi-index of Γ_j. Only for $\gamma = \tau_1, \tau_2$ can g^γ contribute a z^{τ_2} term. Thus we are led to the equation

$$c^j(\tau_2) + b^{\tau_2} \hat{c}^j(\tau_2) + \cdots = b_j \hat{c}^j(\tau_2)$$

where $\cdots$ indicates terms depending only on coefficients of g and/or the just determined value $\hat{c}^j(\tau_1)$. This can be solved for $\hat{c}^j(\tau_2)$ since $b^{\tau_2} \neq b_j$. Proceeding in this way through the multi-indices of Γ_j in the prescribed order we obtain equations

$$c^j(\tau_k) + b^{\tau_k} \hat{c}^j(\tau_k) + \cdots = b_j \hat{c}^j(\tau_k)$$

where the omitted terms on the left are known quantites, possibly involving the coefficients $\hat{c}^j(\tau_i)$ where τ_i preceedes τ_k.

At this point we have determined $\hat{c}^j(\gamma), \gamma \in \Gamma_j$ so that in Equation (6) the coefficients of any $z^\tau, \tau \in \Gamma_j$ agree on both sides. Recall that by Lemma 5 the monomials of order at least two which appear in the expansion of the left side of Equation (6) are all j-permissible, so in fact we have shown that the coefficients of z^τ for any multi-index τ of total order at least two on both sides of the equation agree. The only non-zero first order terms on either side of Equation (6) are $b_j z_j$. Hence with the determined values of $\hat{c}^j(\gamma)$, Equation (5) holds. □

Theorem 7. *Let $\varphi : B_N \to B_N$ be analytic such that $\varphi(0) = 0$, φ is not unitary on any slice, and $A = \varphi'(0)$ is upper triangular with distinct diagonal entries λ_j satisfying $1 > |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_N| > 0$. Assume that φ has resonances so Γ_j is non-empty for at least one j. Suppose φ has a Schroeder map σ_φ with $\sigma_\varphi'(0) = I$. If $\varphi \circ \psi = \psi \circ \varphi$ for some analytic self-map ψ of*

B_N, and the resonances of φ are not also resonances of ψ, then φ and ψ have a common Schroeder map which is locally univalent near 0.

Before giving the proof, we clarify the meaning of the hypothesis "the resonances of φ are not also resonances of ψ". Since φ and ψ commute, so do $\varphi'(0)$ and $\psi'(0)$. Since $\varphi'(0)$ is assumed to have distinct eigenvalues, this means that $\varphi'(0)$ and $\psi'(0)$ may be simultaneously diagonalized, and we may find an $N \times N$ invertible matrix S so that

$$S\varphi'(0)S^{-1} = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$$

and

$$S\psi'(0)S^{-1} = \operatorname{diag}(\mu_1, \mu_2, \dots, \mu_N)$$

where the λ_j*'s* appear in non-increasing order, but there is no apriori ordering on the μ_j*'s*. To say that the resonances of φ are not also resonances of ψ means that if

$$\lambda_j = \lambda_1^{k_1} \cdots \lambda_{j-1}^{k_{j-1}}$$

then

$$\mu_j \neq \mu_1^{k_1} \cdots \mu_{j-1}^{k_{j-1}}$$

(with the given ordering on the μ_j*'s*).

Proof. If $\varphi \circ \psi = \psi \circ \varphi$ we have already observed that $\sigma_\varphi \circ \psi$ is a Schroeder map for φ. By Corollary 2, we must have

$$\sigma_\varphi \circ \psi = S^{-1} \circ g \circ S \circ \sigma_\varphi$$

where S diagonalizes $\varphi'(0)$ and $\psi'(0)$ as just described and $g \in \mathcal{G}_\varphi$ so that the coordinate functions of g are

$$g_j(z_1, \dots, z_N) = b_j z_j + \sum_{\gamma \in \Gamma_j} c^j(\gamma) z^\gamma.$$

Upon differentiation of the relation $\sigma_\varphi \circ \psi = S^{-1} g S \sigma_\varphi$ we see that $\psi'(0) = S^{-1} g'(0) S$ so that

$$\operatorname{diag}(\mu_1, \dots, \mu_N) = S\psi'(0)S^{-1} = g'(0) = \operatorname{diag}(b_1, \dots, b_N)$$

and by hypothesis $\lambda_j = \lambda_1^{k_1} \cdots \lambda_{j-1}^{k_{j-1}} \Rightarrow b_j \neq b_1^{k_1} \cdots b_{j-1}^{k_{j-1}}$. By Theorem 6, there exists $\hat{g} \in \mathcal{G}_\varphi$ with $\hat{g}'(0) = I$ and $\hat{g} \circ g = g'(0)\hat{g}$. By Corollary 2, $S^{-1}\hat{g}S\sigma_\varphi$ is a Schroeder map for φ; its derivative at 0 is I. The following calculation shows that it is also a Schroeder map for ψ:

$$\begin{aligned}(S^{-1}\hat{g}S\sigma_\varphi)\psi = S^{-1}\hat{g}SS^{-1}gS\sigma_\varphi &= S^{-1}\hat{g}gS\sigma_\varphi \\ &= S^{-1}g'(0)\hat{g}S\sigma_\varphi \\ &= (S^{-1}g'(0)S)S^{-1}\hat{g}S\sigma_\varphi \\ &= \psi'(0)(S^{-1}\hat{g}S\sigma_\varphi).\end{aligned}$$

□

In Example 1 we saw that Theorem 7 can fail if the resonances of φ are also resonances of ψ. Of course, if ψ is a natural iterate of φ, then φ and ψ will commute, have the same resonances, and have a common Schroeder map.

One application of Theorems 7 and 3 is to extract qualitative information about the maps which commute with a given map. Our next theorem is a result in this direction. It depends on the following result from [**2**].

Proposition 8 ([**2**])**.** *Let φ be an analytic map of B_N into itself with $\varphi(0) = 0$ and $A = \varphi'(0)$ invertible and suppose φ is not unitary on any slice of B_N. If σ_φ is an analytic map of B_N into $\mathbf{C}^N$ that solves Schroeder's functional equation $\sigma_\varphi \circ \varphi = Af$ and $\sigma'_\varphi(0)$ is invertible, then σ_φ is univalent on B_N if and only if φ is univalent on B_N.*

Corollary 9. *Suppose φ and ψ are commuting analytic self-maps of B_N, both fixing* 0*, not unitary on any slice, and having invertible derivative at* 0*. Suppose further that they satisfy the hypotheses of either Theorem* 3 *or Theorem* 7*. Then if φ is univalent in B_N so is ψ.*

Proof. There is a common locally univalent Schroeder map for φ and ψ which by the "if" direction of Proposition 8 is univalent in B_N. Now apply the "only if" direction of the proposition for ψ to conclude that ψ is univalent in B_N. □

The invertibility of $\varphi'(0)$ and $\psi'(0)$ is necessary in this corollary, as the maps $\varphi(z_1, z_2) = (1/2z_1, 1/3z_2)$ and $\psi(z_1, z_2) = (1/2z_1, 0)$ which share the Schroeder map $\sigma(z_1, z_2) = (z_1, z_2)$ show.

Finally, we observe that our proof of Theorem 7 depends on the hypothesis that the eigenvalues of $\varphi'(0)$ are distinct. This hypothesis plays a significant role in Theorem 6 as it means each coordinate function g_j has at most one non-zero linear term. We leave consideration of the repeated eigenvalue case for a later time.

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Received August 11, 1998 and revised July 6, 1999.

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