

ON CENTRAL EXTENSIONS OF GYROCOMMUTATIVE GYROGROUPS

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Central extensions of gyrocommutative gyrogroups (K-loops) are studied in order to clarify the status of a cocycle equation introduced by Smith and Ungar. A sufficient and necessary conditions under which a central invariant extension is a gyrocommutative gyrogroup are formulated in terms of a 2-cochain $f(x, y)$. In particular, it is shown that for central invariant extensions of gyrocommutative gyrogroups defined by Cartan decompositions of simple Lie algebras, the corresponding $f(x, y)$ satisfies the cocycle equation, provided an extension is a gyrocommutative gyrogroup.

1. Introduction.

There has been a renewal of an interest in loop theory in recent years, concerning a special non-associative loop structure called a *gyrocommutative gyrogroup*, known also under the name of a K-loop. It began with a paper by A. Ungar [15], who pointed it out that the addition law of relativistic velocities leads to an interesting algebraic structure on a unit ball in R^n , which he originally called a K-loop. Further properties of these structures under the name of *gyrogroups* and *gyrocommutative gyrogroups* were summarized in [16]. Concerning terminology, see also remarks in [8].

Independent studies by A. Kreuzer [9] and H. Karzel, and H. Wefelscheid [4] clarified the status of *gyrocommutative gyrogroups* within the framework of loop theory and provided some important constructions generalizing an example of Ungar.

In fact, *gyrocommutative gyrogroups* or similar structures were contemplated, although not explicitly under that name, by M. Kikkawa [5], in relation to symmetric spaces and by P. Miheev and L. Sabinin in relation to so called odular structures, [10]. That first aspect of *gyrocommutative gyrogroups* has been discussed by Y. Friedman and A. Ungar in [2] and recently by W. Krammer and H.K. Urbantke in [8].

One can obtain more examples of *gyrocommutative gyrogroups* by means of their extensions. Such extensions are in fact extensions of loops. These were discussed a long time ago by R.H. Bruck in [1]. They generalize extensions of groups (see for example [12]). As for groups, the simplest among

their extensions are the ones with an abelian kernel, which include central extensions as well [12]. They are constructed, in principle, via a 2-cocycle, i.e. a 2-cochain $f(x, y)$ subject to a cocycle equation,

$$(1) \quad f(x_1, x_2) \cdot f(x_1 \cdot x_2, x_3) = f(x_2, x_3) \cdot f(x_1, x_2 \cdot x_3),$$

under the assumption that the group under consideration acts trivially on an abelian group.

Central invariant extensions of *gyrocommutative gyrogroups*, in a narrower sense, were defined in [14] and then employed with a particular purpose of reconstructing from a *gyrocommutative gyrogroup* of relativistic velocities, the Lorentz group, i.e. $SO_0(1, n)$; more specifically a standard matrix representation of that group. As far as central extensions are concerned it was assumed that the operators generating the left associant of an extension (in terminology of [10]) or the structure group of an extension (in terminology of [9]) are central (in the sense of [14]). That assumption resulted in an equation for the corresponding 2-cochain $f(x, y)$ which the authors called a cocycle equation. Thus, in principal the cocycle equation was incorporated into a definition of a central extension of a *gyrocommutative gyrogroup*. It reads,

$$(2) \quad f(x_1, x_2) \cdot f(x_2 \cdot x_1, x_3) = f(x_2, x_3) \cdot f(x_1, x_2 \cdot x_3).$$

That brings us to a question whether such an extension always has to arise from a 2-cochain satisfying the cocycle equation of [14]?

To answer that question we discuss matters in a context of central extensions of loops according to R.H. Bruck [1]. Basic definitions and properties of relevant concepts are presented in Sections 2 and 3. A multiplicative notation instead of an additive one, preferred by the authors of [14], is employed. Definitions concerning extensions of loops are, except of small modifications, faithful copies of the ones in [1]. Definitions of a *gyrogroup* and a *gyrocommutative gyrogroup* (K -loop) are equivalent to the ones of [16]; see Appendix. They emphasize the role of three identities, known as a Bol identity, an A -loop identity and an inverse automorphic identity (see for example [10]). In Section 4 a discussion is restricted to invariant, in the sense of [14], central extensions of a *gyrocommutative gyrogroup* M with a trivial action of M on an abelian group G . Also there, necessary and sufficient conditions under which a 2-cochain $f(x, y)$ determines an extension of M which is a *gyrogroup* or a *gyrocommutative gyrogroup* (Theorems 16 and 17) are provided. Theorem 20 of Section 5 is a generalization of the fact pointed out in [14], that central extensions of *gyrocommutative gyrogroups* corresponding to symmetric 2-cochains are *gyrocommutative gyrogroups* again.

In Section 6 we discuss *gyrocommutative gyrogroups* determined by Cartan decompositions of noncompact semisimple and in particular simple Lie

algebras. It turns out that any central invariant extension of the latter structure, if it is a *gyrocommutative gyrogroup*, then it arises from a 2-cochain which satisfies a cocycle equation of [14], (2), (Theorem 29).

2. Definitions of basic concepts.

Let G be an abelian group, M a loop, [11], and χ a function, $\chi : M \rightarrow \text{Aut}(G)$, which satisfies the following properties,

$$(3) \quad 1\chi = \text{id}_G,$$

where 1 and id_G are units of M and $\text{Aut}(G)$ respectively, and

$$(4) \quad (x \cdot y)\chi = x\chi \cdot y\chi,$$

for all $x, y \in M$.

Given $g \in G$ and $x \in M$, we denote the value of an automorphism $x\chi$ at g by gx .

Definition 1. A (G, M, χ) -extension (E, θ) is a pair consisting of a loop E and a homomorphism θ of E onto M , such that,

- (i) $\ker(\theta) \subset A(E)$, where $A(E)$ is the associator of E , i.e. a subset $A(E)$ of E , such that $(e_1 \cdot e_2) \cdot e_3 = e_1 \cdot (e_2 \cdot e_3)$ if at least one of e_1, e_2, e_3 is in $A(E)$,
- (ii) the center of $\ker(\theta)$ equals to G ,
- (iii) $g \cdot e = e \cdot (g x)$ for all $g \in G, e \in E$ and $x = e\theta$.

A (G, M, χ) -extension, (E, θ) , is called central if $\ker(\theta) = G$.

Definition 2. A normalized 2-cochain f is a function $f : M^2 \rightarrow G$, with values $f(x_1, x_2)$, taking the value 1 whenever one of x_1 or x_2 is 1.

The 3-coboundary δf of f is the following normalized 3-cochain,

$$(5) \quad \delta f(x_1, x_2, x_3) = [f(x_1, x_2)x_3] \cdot [f(x_2, x_3)]^{-1} \cdot f(x_1 \cdot x_2, x_3) \cdot [f(x_1, x_2 \cdot x_3)]^{-1}.$$

Definition 3. Let f be a normalized 2-cochain. We define a central (G, M, χ) extension, (E, θ) , as follows.

- (i) E is the set of all ordered pairs (x, g) , where $x \in M$ and $g \in G$,
- (ii) $(x, g) \cdot (y, h) = (x \cdot y, f(x, y) \cdot (g y) \cdot h)$,
- (iii) $(x, g)\theta = x$.

We denote this central extension by (G, M, χ, f) .

Remark 1. In order to fulfill the conditions of Definition 1 G has to be identified with its homomorphic image in E under a natural injective homomorphism which sends $g \in G$ into $(1, g)$.

It is known, [1], that each central (G, M, χ) -extension is equivalent to at least one (G, M, χ, f) -extension.

Further discussion will concern extensions corresponding to a trivial function χ , i.e. the one that assigns to each $x \in M$ the identity automorphism of G . Such extensions will be referred to as (G, M) or (G, M, f) -extensions respectively.

To define a *gyrogroup* and a *gyrocommutative gyrogroup* we point out certain important identities.

Definition 4. Let M be a loop. The following identities,

$$(6) \quad L_x \circ L_y \circ L_x = L_{x \cdot (y \cdot x)},$$

$$(7) \quad \ell(x, y) \circ L_z = L_{\ell(x, y)z} \circ \ell(x, y),$$

where $\ell(x, y) = L_{x \cdot y}^{-1} \circ L_x \circ L_y$, and

$$(8) \quad (x \cdot y)^{-1} = x^{-1} \cdot y^{-1},$$

are called a B , an A and an I -identity respectively. B stands for Bol, A for an A -loop, and I for inverse automorphic. (For any $z \in M$, z^{-1} means its right inverse.)

For purposes of this paper we introduce the following terminology. A loop will be called a B -loop, a BA -loop or a BAI -loop, if a B , a B and an A or B , A and I -identities hold.

Remark 2. There are other equivalent forms of those identities. In particular, (7) means that the mapping $\ell(x, y)$ is an automorphism of M for all $x, y \in M$. We list below two identities equivalent to (6) and (7) correspondingly,

$$(9) \quad x \cdot (y \cdot (x \cdot z)) = (x \cdot (y \cdot x)) \cdot z,$$

$$(10) \quad l(x, y)(z \cdot w) = (l(x, y)z) \cdot l(x, y)w.$$

Now we define a *gyrogroup* and a *gyrocommutative gyrogroup*.

Definition 5. Any BA -loop is called a *gyrogroup* and any BAI -loop is called a *gyrocommutative gyrogroup*.

We have also this natural definition.

Definition 6. A central (G, M) extension, (E, θ) , of a *gyrocommutative gyrogroup* is called a B , a BA or a BAI -extension if E is a B , a BA or a BAI -loop respectively.

3. Central extensions of gyrocommutative gyrogroups.

We study now (G, M) central extensions of a *gyrocommutative gyrogroup* M . We follow the method of Bruck, [1].

Let $u(x)$ be a normalized transversal function of E . Thus, for each $x \in M$, $u(x)$ is an element of E , such that

$$(11) \quad u(x)\theta = x,$$

and

$$(12) \quad u(1) = 1 .$$

Then every $e \in E$ has a unique representation of the form $e = u(x) \cdot g$, with $g \in G$ and $x = e\theta$. Moreover,

$$(13) \quad u(x) \cdot g \cdot u(y) \cdot h = u(x) \cdot u(y) \cdot g \cdot h = u(x \cdot y) \cdot f(x, y) \cdot g \cdot h,$$

where f is a normalized, because of (12), 2-cochain. (There is no need for additional parentheses in that identity since $g, h \in G \subset A(E)$, and G is abelian.) Then the mapping $u(x) \cdot g \mapsto (x, g)$ gives the equivalence of E and (G, M, f) . (See [1].)

Thus, for all $x, y \in M$,

$$(14) \quad u(x) \cdot u(y) = u(x \cdot y) \cdot f(x, y).$$

We make now few simple observations.

Lemma 7. *Let (E, θ) be a central (G, M) extension and $u(x)$ a normalized transversal function of E . Then for all $g \in G$ and $e \in E$,*

$$(15) \quad g \cdot e = e \cdot g,$$

and

$$(16) \quad (e \cdot g)^{-1} = g^{-1} \cdot e^{-1} = e^{-1} \cdot g^{-1}.$$

If E is a B -loop, then

$$(17) \quad (u(x))^{-1} = u(x^{-1}) \cdot [f(x, x^{-1})]^{-1} = [f(x, x^{-1})]^{-1} \cdot u(x^{-1}),$$

$$(18) \quad f(x, x^{-1}) = f(x^{-1}, x),$$

$$(19) \quad \ell(u(a), u(b))u(c) = u(\ell(a, b)c) \cdot g_{a,b,c},$$

where

$$(20) \quad g_{a,b,c} = [f(a, b) \cdot f((a \cdot b)^{-1}, a \cdot b)]^{-1} \\ \cdot f((a \cdot b)^{-1}, a \cdot (b \cdot c)) \cdot f(a, b \cdot c) \cdot f(b, c)$$

and

$$(21) \quad \ell(u(a), u(b))g = g,$$

for all $g \in G$.

Proof. Indeed, (15) follows directly from (iii) of Definition 1. (16) is a consequence of the fact that $G \subset A(E)$ and (15). e^{-1} is understood here as the right inverse of e in the loop E .

For B -loops right and left inverses coincide [11]. Due to (14) we have,

$$(22) \quad u(x) \cdot u(x^{-1}) = f(x, x^{-1}),$$

and consequently

$$(23) \quad u(x^{-1}) = [u(x)]^{-1} \cdot f(x, x^{-1}).$$

Next,

$$(24) \quad u(x^{-1}) \cdot u(x) = f(x^{-1}, x),$$

$$(25) \quad u(x) = [u(x^{-1})]^{-1} \cdot f(x^{-1}, x)$$

and due to (16),

$$(26) \quad [u(x)]^{-1} = [f(x^{-1}, x)]^{-1} \cdot u(x^{-1}).$$

Then from (23) and (26) we get (18) and (17). To prove (19) we employ (15) and the other just proven identities. The method of proving (21) is similar. \square

We prove now this fact.

Proposition 8. *Let M be a gyrocommutative gyrogroup. The central extension $E = (G, M, f)$ is a B -extension, if and only if,*

$$(27) \quad f(x, y \cdot (x \cdot z)) \cdot f(y, x \cdot z) \cdot f(x, z) = f(x \cdot (y \cdot x), z) \cdot f(x, y \cdot x) \cdot f(y, x).$$

Proof. It is clear, due to (9) and (13), (see also [1]), that a B -identity for E is equivalent to,

$$(28) \quad u(x) \cdot (u(y) \cdot (u(x) \cdot u(z))) = (u(x) \cdot (u(y) \cdot u(x))) \cdot u(z).$$

Now, using (14), (15), the fact that $G \subset A(E)$ and commutativity of G , one obtains,

$$(29) \quad \begin{aligned} u(x) \cdot (u(y) \cdot (u(x) \cdot u(z))) \\ = u(x \cdot (y \cdot (x \cdot z))) \cdot f(x, y \cdot (x \cdot z)) \cdot f(y, x \cdot z) \cdot f(x, z) \end{aligned}$$

and

$$(30) \quad \begin{aligned} (u(x) \cdot (u(y) \cdot u(x))) \cdot u(z) \\ = u((x \cdot (y \cdot x)) \cdot z) \cdot f(x \cdot (y \cdot x), z) \cdot f(x, y \cdot x) \cdot f(y, x). \end{aligned}$$

That shows equivalence of (28) and (27), since M is a B -loop. \square

As far as BA -extensions are concerned we have,

Proposition 9. *Let M be a gyrocommutative gyrogroup. The central extension $E = (G, M, f)$ is a BA-extension if and only if it is a B-extension and,*

$$(31) \quad f(z, w) \cdot g_{x,y,z \cdot w} = f(\ell(x, y)z, \ell(x, y)w) \cdot g_{x,y,z} \cdot g_{x,y,w}.$$

Proof. An A -identity for E is equivalent to (see (10)),

$$(32) \quad \ell(u(x), u(y)) \cdot (u(z) \cdot u(w)) = [\ell(u(x), u(y))u(z)] \cdot \ell(u(x), u(y))u(w).$$

To put it into an equivalent form of (31), it suffices to apply (19), (20) and (21). Indeed, one can rewrite then (32) into,

$$(33) \quad \begin{aligned} u(\ell(x, y)(z \cdot w)) \cdot f(z, w) \cdot g_{x,y,z \cdot w} \\ = u(\ell(x, y)z \cdot \ell(x, y)w) \cdot f(\ell(x, y)z, \ell(x, y)w) \cdot g_{x,y,z} \cdot g_{x,y,w}, \end{aligned}$$

which due to the fact that M is an A -loop is equivalent to (31). \square

Finally we have a Proposition concerning central BAI-extensions.

Proposition 10. *Let M be a gyrocommutative gyrogroup. The central extension $E = (G, M, f)$ is a BAI-extension if and only if it is a BA-extension and,*

$$(34) \quad f(x^{-1}, x) \cdot f(y^{-1}, y) = f(x^{-1}, y^{-1}) \cdot f(x, y) \cdot f((x \cdot y)^{-1}, x \cdot y).$$

Proof. An I -identity for E is equivalent to,

$$(35) \quad (u(x) \cdot u(y))^{-1} = (u(x))^{-1} \cdot (u(y))^{-1}.$$

Next one derives, by a rather straightforward process, which employs Lemma 7, these identities,

$$(36) \quad [u(x) \cdot u(y)]^{-1} = u((x \cdot y)^{-1}) \cdot [f(x, y) \cdot f((x \cdot y)^{-1}, x \cdot y)]^{-1},$$

and

$$(37) \quad [u(x)]^{-1} \cdot [u(y)]^{-1} = u(x^{-1} \cdot y^{-1}) \cdot f(x^{-1}, y^{-1}) \cdot [f(x^{-1}, x) \cdot f(y^{-1}, y)]^{-1}.$$

Feeding them back into (35) and making use of the fact that M satisfies an I -identity, one infers equivalence of (35) and (34). \square

4. Central invariant extensions of gyrocommutative gyrogroups.

Definition 11. A central extension $E = (G, M, f)$ of a loop M is called invariant if

$$(38) \quad f(\ell(a, b)x, \ell(a, b)y) = f(x, y)$$

for all $a, b, x, y \in M$.

Definition 12. For any 2-cochain f we define a 3-cochain Δf ,

$$(39) \quad \Delta f(x, y, z) = f(x, y) \cdot [f(y, z)]^{-1} \cdot f(y \cdot x, z) \cdot [f(x, y \cdot z)]^{-1}.$$

The expression for Δf is different from the one for δf , (5). Indeed, those two are related by,

$$(40) \quad \delta f(x, y, z) = \Delta f(x, y, z) \cdot f(x \cdot y, z) \cdot [f(y \cdot x, z)]^{-1}.$$

However, it is Δf rather than δf , which is important in analysis of identities of Propositions 8-10. We have now a sequence of a Proposition and two Theorems that correspond to Propositions 8-10 in case of central invariant extensions.

Proposition 13. *Let M be a gyrocommutative gyrogroup. The central invariant extension $E = (G, M, f)$ is a B-extension if and only if,*

$$(41) \quad \Delta f(x, y \cdot x, \ell(y, x)z) \cdot \Delta f(y, x, z) = 1.$$

Proof. We rewrite (27) as,

$$(42) \quad f(v, u \cdot (v \cdot w)) \cdot f(u, v \cdot w) \cdot f(v, w) = f(v \cdot (u \cdot v), w) \cdot f(v, u \cdot v) \cdot f(u, v).$$

From (39) we infer that

$$(43) \quad f(u, v \cdot w) \cdot f(v, w) \cdot \Delta f(u, v, w) = f(u, v) \cdot f(v \cdot u, w).$$

Hence (42) can be put into,

$$(44) \quad f(v, u \cdot (v \cdot w)) \cdot f(v \cdot u, w) = f(v \cdot (u \cdot v), w) \cdot f(v, u \cdot v) \cdot \Delta f(u, v, w).$$

However,

$$(45) \quad u \cdot (v \cdot w) = (u \cdot v) \cdot \ell(u, v)w$$

and

$$(46) \quad f(v \cdot u, w) = f(\ell(u, v)(v \cdot u), \ell(u, v)w) = f(u \cdot v, \ell(u, v)w),$$

since M is a gyrocommutative gyrogroup and (38) holds. Consequently, (44) can be rewritten as,

$$(47) \quad f(x, (y \cdot x) \cdot \ell(y, x)z) \cdot f(y \cdot x, \ell(y, x)z) \\ = f(x \cdot (y \cdot x), z) \cdot f(x, y \cdot x) \cdot \Delta f(y, x, z).$$

We simplify (47) further by means of (43) applied to the left-hand side of (47) with $u = x$, $v = y \cdot x$ and $w = \ell(y, x)z$. The result is,

$$(48) \quad f((y \cdot x) \cdot x, \ell(y, x)z) \\ = f(x \cdot (y \cdot x), z) \cdot \Delta f(y, x, z) \cdot \Delta f(x, y \cdot x, \ell(y, x)z).$$

Finally, we observe that

$$(49) \quad f(x \cdot (y \cdot x), z) = f(\ell(y, x)\ell(x, y \cdot x)((y \cdot x) \cdot x), \ell(y, x)z) \\ = f((y \cdot x) \cdot x, \ell(y, x)z),$$

where the last inference is based on the identity,

$$(50) \quad \ell(y, x)\ell(x, y \cdot x) = \ell(y, x)\ell(x, y) = id_M,$$

satisfied in any *gyrocommutative gyrogroup* M (see [16], [14]). Feeding (49) back into (48) reduces (48) to a desired form of (41). It is clear that an outlined process of inference of (41) from (27) can be reversed. Therefore (41) and (27) are indeed equivalent. \square

Before we discuss BA -extensions, we need an identity.

Lemma 14. *Let M be a gyrocommutative gyrogroup and $E = (G, M, f)$ an invariant central B -extension. Then,*

$$(51) \quad f((a \cdot b)^{-1}, a \cdot (b \cdot c)) \cdot f(b \cdot a, c) = f(a \cdot b, (a \cdot b)^{-1}).$$

Proof. We employ (42) with $v = (a \cdot b)^{-1}$, $u = a \cdot b$ and $w = (a \cdot b) \cdot \ell(a, b)c$. That leads to,

$$(52) \quad f((a \cdot b)^{-1}, (a \cdot b) \cdot \ell(a, b)c) \cdot f(a \cdot b, \ell(a, b)c) = f(a \cdot b, (a \cdot b)^{-1}).$$

However,

$$(53) \quad f((a \cdot b)^{-1}, (a \cdot b) \cdot \ell(a, b)c) = f((a \cdot b)^{-1}, a \cdot (b \cdot c)),$$

and

$$(54) \quad f(a \cdot b, \ell(a, b)c) = f(\ell(a, b)(b \cdot a), \ell(a, b)c) = f(b \cdot a, c).$$

Feeding (53) and (54) back into (52) one arrives at (51). \square

Now we prove this Lemma.

Lemma 15. *Let $\Phi : M \rightarrow G$ be a homomorphism of a loop M into a group G . Then for all $x, y, z \in M$,*

$$(55) \quad \Phi(\ell(x, y)z) = \Phi(z).$$

Proof. Indeed,

$$(56) \quad \Phi(x \cdot (y \cdot z)) = \Phi(x) \cdot \Phi(y) \cdot \Phi(z)$$

and

$$(57) \quad \Phi(x \cdot (y \cdot z)) = \Phi((x \cdot y) \cdot \ell(x, y)z) = \Phi(x) \cdot \Phi(y) \cdot \Phi(\ell(x, y)z).$$

Hence (55) follows. \square

Theorem 16. *Let M be a gyrocommutative gyrogroup. The central invariant extension $E = (G, M, f)$ is a BA -extension if and only if*

$$(58) \quad \Delta f(x, y \cdot x, z) \cdot \Delta f(y, x, z) = 1$$

and

$$(59) \quad \Delta f(x, y, z \cdot w) = \Delta f(x, y, z) \cdot \Delta f(x, y, w).$$

Proof. Let E be a BA -extension. We show first that the condition (31) of Proposition 9 implies (59). Indeed, due to (38), (31) reads,

$$(60) \quad g_{x,y,z \cdot w} = g_{x,y,z} \cdot g_{x,y,w}.$$

Making use of (39), with $u = a$, $v = b$, and $w = c$ in (20), one obtains,

$$(61) \quad f((a \cdot b)^{-1}, a \cdot b) \cdot \Delta f(a, b, c) \cdot g_{a,b,c} = f(b \cdot a, c) \cdot f((a \cdot b)^{-1}, a \cdot (b \cdot c)).$$

Employing now (61) and (51) in (60), one arrives at (59). Now, having established that, we employ Lemma 15 to prove that (41) implies (58). Conversely, if E is a central invariant extension and satisfies (58) and (59), then by Lemma 15 it satisfies (41) as well. Thus E is a B -extension. Consequently, (61) holds and because of (51) it reads $g_{a,b,c} = [\Delta f(a, b, c)]^{-1}$. Next, (59) can be rewritten into (60) which for invariant extensions is (31). \square

Theorem 17. *Let M be a gyrocommutative gyrogroup. The central invariant extension $E = (G, M, f)$ is a BAI -extension if and only if,*

$$(62) \quad \Delta f(x, y \cdot x, z) \cdot \Delta f(y, x, z) = 1,$$

$$(63) \quad \Delta f(x, y, z \cdot w) = \Delta f(x, y, z) \cdot \Delta f(x, y, w)$$

and

$$(64) \quad f(x, y) \cdot \Delta f(y, x, x^{-1}) = f(y, x) \cdot \Delta f(x \cdot y, x^{-1}, y^{-1}).$$

Proof. Due to Theorem 16 it suffices to prove that a central invariant BA -extension is a BAI -extension if and only if (64) holds. We prove that (64) is equivalent to (34).

From (43) with $u = x^{-1} \cdot y^{-1}$, $v = x$ and $w = y$, we get,

$$(65) \quad f(x^{-1} \cdot y^{-1}, x \cdot y) \cdot f(x, y) \cdot \Delta f(x^{-1} \cdot y^{-1}, x, y) \\ = f(x^{-1} \cdot y^{-1}, x) \cdot f(y^{-1}, y).$$

Next, using (65) and an I -identity for M in (34), one arrives at,

$$(66) \quad f(x^{-1}, x) \cdot \Delta f(x^{-1} \cdot y^{-1}, x, y) = f(x^{-1}, y^{-1}) \cdot f(x^{-1} \cdot y^{-1}, x).$$

Employing again (43), with $u = y^{-1}$, $v = x^{-1}$, $w = x$, we arrive at,

$$(67) \quad f(y^{-1}, x^{-1}) \cdot f(x^{-1} \cdot y^{-1}, x) = f(x^{-1}, x) \cdot \Delta f(y^{-1}, x^{-1}, x).$$

Feeding it back into (66) leads to,

$$(68) \quad f(x^{-1}, y^{-1}) \cdot \Delta f(y^{-1}, x^{-1}, x) = f(y^{-1}, x^{-1}) \cdot \Delta f(x^{-1} \cdot y^{-1}, x, y),$$

which is equivalent to (64). \square

5. Central symmetric and invariant extensions of a gyrocommutative gyrogroup.

Definition 18. A central extension $E = (G, M, f)$ is called symmetric if,

$$(69) \quad f(x, y) = f(y, x).$$

We prove this Proposition.

Proposition 19. *Let M be a gyrocommutative gyrogroup and let $E = (G, M, f)$ be a central, symmetric, invariant, BA-extension. Then, E is a BAI-extension. In particular,*

$$(70) \quad \Delta f(z, y, x) = (\Delta f(x, y, z))^{-1},$$

$$(71) \quad \Delta f(x, y, y) = 1,$$

and

$$(72) \quad \Delta f(a, b, c) = \Delta f(c, a, b) = \Delta f(b, c, a).$$

Proof. We prove (70)-(72) first. In order to prove (70) it suffices to apply a definition of Δf , and (69). Next we employ identities, (58) and (59) of Theorem 16. In particular, in (58) we substitute $x = a, y = b/a$ and $z = c$. Since M is a B -loop (even a gyrocommutative gyrogroup), then (see [10], [13]),

$$(73) \quad b/a = a^{-1} \cdot ((a \cdot b) \cdot a^{-1}).$$

Feeding that back into (58), applying (70) and (59), one arrives at,

$$(74) \quad \Delta f(a, b, c) = \Delta f(c, a, b) \cdot \Delta f(c, a, a^{-1}).$$

Putting $b = 1$, one infers that $\Delta f(c, a, a^{-1}) = 1$, which due to (59) is equivalent to $\Delta f(c, a, a) = 1$. That in turn is equivalent to (71). Now, (74) reads,

$$(75) \quad \Delta f(a, b, c) = \Delta f(c, a, b),$$

which implies (72).

Next, it is not difficult to prove the principal assertion of the Proposition. Indeed, it suffices to prove the identity (64) of Theorem 17. That identity is a straightforward consequence of (69), (59) and (70)-(72). \square

We close this section with a Theorem.

Theorem 20. *Let M be a gyrocommutative gyrogroup and $E = (G, M, f)$ a symmetric, invariant, BA-extension. E is a BAI-extension if and only if,*

$$(76) \quad \Delta f(x, y, z \cdot w) = \Delta f(x, y, z) \cdot \Delta f(x, y, w),$$

$$(77) \quad \Delta f(z, y, x) = (\Delta f(x, y, z))^{-1},$$

$$(78) \quad \Delta f(x, y, y) = 1,$$

and

$$(79) \quad \Delta f(x, y, z) = \Delta f(z, x, y) = \Delta f(y, z, x).$$

Proof. Indeed, necessity of these conditions follows from Theorem 16 and Proposition 19. Conversely, it is not difficult to infer from them (58). Thus, according to Theorem 16, E is a BA -extension and by Proposition 19, it is a BAI -extension. \square

6. Gyrocommutative gyrogroups determined by Cartan decompositions of noncompact semisimple Lie algebras.

For definitions of related concepts the reader is referred to [3].

Let \mathfrak{g} be a noncompact semisimple Lie algebra over \mathbf{R} and let $\mathfrak{g} = \mathfrak{t} + \mathfrak{p}$, be a Cartan decomposition of \mathfrak{g} . Then the mapping $\theta : T + X \mapsto T - X$, where $T \in \mathfrak{t}$ and $X \in \mathfrak{p}$ is an involutive automorphism of \mathfrak{g} .

We say that a pair (G, H) is associated with (\mathfrak{g}, θ) if G is a connected Lie group with Lie algebra \mathfrak{g} and H is a Lie subgroup of G with Lie algebra \mathfrak{t} . We shall refer to (G, H) as to a pair of a noncompact type.

Theorem 21 (Helgason [3]). *Let (G, H) be a pair of a noncompact type associated with (\mathfrak{g}, θ) . Then:*

- (i) *There exists an involutive, analytic automorphism Θ of G whose fixed point set is H and whose differential at the identity of G is θ . In particular H is closed.*
- (ii) *The mapping $\varphi : \mathfrak{p} \times H \rightarrow G$, defined by*

$$(80) \quad \varphi(X, h) = (\exp X)h$$

is a diffeomorphism.

Remark 3. According to that Theorem, $G = PH$, where $P = \exp \mathfrak{p}$, is an exact decomposition of G , i.e. any element $g \in G$ has a unique representation $g = ph$, where $p \in P$ and $h \in H$. We refer to those p and h as to P - and H -factors of g respectively.

Next, there is a natural binary operation \star on P determined by the exact decomposition of G . Indeed, $\forall p_1, p_2 \in P$, one defines $p_1 \star p_2$ to be a unique P -factor of $p_1 p_2$. Thus,

$$(81) \quad p_1 p_2 = (p_1 \star p_2) h(p_1, p_2),$$

where $h(p_1, p_2)$ is an H -factor of $p_1 p_2$. It turns out that this operation provides a *gyrocommutative gyrogroup* structure on P . The origins of this fact can be traced in papers of M. Kikkawa [5]; see also [10], [2] and for the most recent discussion of the subject, [8]. Notice also that the inverse in the group G of an element $p \in P$, which we denote by p^{-1} , is an element of P and

it is the inverse of p in the loop (P, \star) . Indeed, $1 = pp^{-1} = (p\star p^{-1})h(p, p^{-1})$. Consequently, $p\star p^{-1} = 1$. The same is true for $p^{-1}\star p$.

Theorem 22. *Let (G, H) be a pair of a noncompact type associated with (\mathfrak{g}, θ) , $G = PH$ the corresponding exact decomposition of G and \star a binary operation on P determined by the decomposition. Then (P, \star) is a smooth gyrocommutative gyrogroup.*

The following fact will be useful in a sequel.

Lemma 23. *Let n be a positive integer, $n \geq 2$ and $p_1, \dots, p_n \in P$. Then,*

$$(82) \quad p_n p_{n-1} \cdots p_2 p_1 = q_n h_n,$$

where q_n, h_n are elements of P and H respectively, determined recursively by,

$$(83) \quad q_1 = p_1, h_1 = 1,$$

and (see (81)),

$$(84) \quad q_{k+1} = p_{k+1} \star q_k, h_{k+1} = h(p_{k+1}, q_k)h_k,$$

for $k = 1, \dots, n - 1$.

Proof. By induction. For $n = 2$ it follows from (81). Suppose the assertion is true for $n = m$. Take $n = m + 1$. Then,

$$p_{m+1} p_m \cdots p_1 = p_{m+1} q_m h_m = (p_{m+1} \star q_m)h(p_{m+1}, q_m)h_m,$$

where the last equality is inferred from (81). Hence $q_{m+1} = p_{m+1} \star q_m$, and $h_{m+1} = h(p_{m+1}, q_m)h_m$. Thus the assertion is true for $n = m + 1$. \square

Now we can prove the following Proposition.

Proposition 24. *Let (G, H) be a pair of noncompact type associated with (\mathfrak{g}, θ) , where \mathfrak{g} is simple. Let $G = PH$ be the corresponding exact decomposition of G . Then, the group G is generated by the set P and the subgroup H is generated by the set*

$$(85) \quad S = \{h(p_1, p_2) : p_1, p_2 \in P\},$$

in the sense that any element of G or H is a product of a finite number of elements of P or S respectively.

Proof. Indeed, the homogeneous space G/H of the pair (G, H) is reductive (see [7], p. 27), because $Ad(h)\mathfrak{p} \subset \mathfrak{p}$, for all $h \in H$ (see [3]). Then, ([7], p. 27), $\mathfrak{l} = \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]$ is a nontrivial ideal of \mathfrak{g} and the corresponding connected normal subgroup of G is generated by the set $P = \{\exp(X) : X \in \mathfrak{p}\}$. However, since \mathfrak{g} is simple $\mathfrak{l} = \mathfrak{g}$ and G itself is generated by the set P .

To prove the second part of this Proposition, assume that $h \in H$. Then according to the first part of the Proposition there exists a finite sequence p_1, \dots, p_n of elements of P , such that $h = p_n \cdots p_1$. Employing Lemma

23 we infer that $h = q_n h_n$, where $q_n \in P, h_n \in H$ and h_n is a product of elements from S . Moreover, since $G = PH$ is an exact decomposition, we must have $q_n = 1$ and $h = h_n$. \square

We need yet another definition ([10]).

Definition 25. Let M be a loop. Then its left associant, $as_l(M)$ is a subgroup of the group of left multiplications of M , $LMlt(M)$, generated by the set, $\{\ell(x, y) : x, y \in M\}$.

A relation between $as_l(P)$ and H is clarified in the following Proposition.

Proposition 26. *Under the assumptions of Proposition 24, the group $as_l(P)$ is identical to the group of conjugations of P by elements of H .*

Proof. Indeed, for all $p_1, p_2, p_3 \in P$,

$$\ell(p_1, p_2)p_3 = (p_1 \star p_2)^{-1} \star (p_1 \star (p_2 \star p_3)).$$

However, employing (81) one obtains,

$$\begin{aligned} p_1 p_2 p_3 &= (p_1 \star (p_2 \star p_3))h, \\ (p_1 \star p_2)^{-1} &= h_{12} p_2^{-1} p_1^{-1}, \end{aligned}$$

where $h, h_{12} \in H$ and $h_{12} = h(p_1, p_2) \in S$. (Concerning the inverses of elements of P see Remark 3.) Hence,

$$h_{12} p_3 = (p_1 \star p_2)^{-1} (p_1 \star (p_2 \star p_3))h = [\ell(p_1, p_2)p_3]h_1,$$

where h_1 is yet another element of H .

Consequently, $\ell(p_1, p_2)p_3 = (h_{12} p_3 h_{12}^{-1})(h_{12} h_1^{-1})$. Since the left-hand side of the previous equation is an element of P , and the factors of its right-hand side are in P and H respectively, therefore $h_{12} h_1^{-1} = 1$ and,

$$\ell(p_1, p_2)p_3 = h_{12} p_3 h_{12}^{-1}.$$

Notice that h_{12} depends on p_1 and p_2 only. Therefore $\ell(p_1, p_2)$ is indeed a conjugation of P by $h_{12} \in H$. Consequently, any element of $as_l(P)$ is a conjugation of P by an element of H . Conversely, given $h \in H$, it can be represented as a product of elements of the form $h(p, q)$, where $p, q \in P$, (85). Therefore a conjugation by h is a product of conjugations by such elements. Since a conjugation by $h(p, q)$ equals to $\ell(p, q)$, therefore a conjugation by h is an element of $as_l(P)$. \square

Now we arrive at this result.

Proposition 27. *Under the assumptions of Proposition 24, let $\Phi : P \rightarrow A$, be a homomorphism of a gyrocommutative gyrogroup P into an abelian group A . Then there exists a homomorphism $\Psi : G \rightarrow A$, such that $\Psi|_P = \Phi$ and for all $h \in H$, $\Psi(h) = 1$.*

Proof. Indeed, we define the mapping $\Psi : G \rightarrow A$ by, $\Psi(g) = \Phi(p)$ for all $g \in G$, where p is a unique P -factor of g . Next we verify that Ψ is a homomorphism. Indeed, for $p_1, p_2 \in P$ and $h_1, h_2 \in H$, there exists, due to (81), $h \in H$ such that,

$$p_1 h_1 p_2 h_2 = p_1 p_2^{h_1} h_1 h_2 = (p_1 \star p_2^{h_1}) h,$$

where $p_2^{h_1} = h_1 p_2 h_1^{-1}$.

Consequently,

$$(86) \quad \Psi(p_1 h_1 p_2 h_2) = \Phi(p_1 \star p_2^{h_1}) = \Phi(p_1) \Phi(p_2^{h_1}).$$

However, according to Proposition 26, the conjugation of P by h_1 is an element of $as_i(P)$. Then, employing Lemma 15 we can rewrite (86) into,

$$\Psi(p_1 h_1 p_2 h_2) = \Phi(p_1) \Phi(p_2) = \Psi(p_1 h_1) \Psi(p_2 h_2).$$

□

Finally we point out the following fact.

Proposition 28. *Let $\psi : \mathfrak{g} \rightarrow \mathfrak{a}$, be a homomorphism of a simple Lie algebra \mathfrak{g} , with a Cartan decomposition $\mathfrak{g} = \mathfrak{t} + \mathfrak{p}$, into an abelian Lie algebra \mathfrak{a} , such that $\psi|_{\mathfrak{t}} = 0$. Then, $\psi = 0$.*

Proof. Indeed, $\ker \psi \neq \{0\}$ and $\ker \psi$ is an ideal of \mathfrak{g} . But \mathfrak{g} is simple. Therefore $\ker \psi = \mathfrak{g}$. □

We arrive now at the main assertion of this section.

Theorem 29. *Let (G, H) be a pair of a noncompact type associated with (\mathfrak{g}, θ) , where \mathfrak{g} is simple. Let $G = PH$ be the corresponding exact decomposition of G and \star a binary operation on P determined by that decomposition. Let $E = (A, P, f)$, where A is an abelian Lie group and $f(p, q)$ a smooth 2-cochain, be a central, invariant extension of P . Then:*

- (i) *E is a BA-extension if and only if $\Delta f(p, q, r) = 1$, for all $p, q, r \in P$.*
- (ii) *E is a BAI-extension if and only if $\Delta f(p, q, r) = 1$ and $f(p, q) = f(q, p)$, for all $p, q \in P$.*

Proof. The conditions (59) of Theorem 16 or (63) of Theorem 17 are crucial. Indeed, let $\forall p, q \in P, \Phi_{p,q}$ be a mapping of P into A , defined by

$$\Phi_{p,q}(r) = \Delta f(p, q, r),$$

for all $r \in P$. Now, if that mapping is a homomorphism of a gyrocommutative gyrogroup P into a group A , then it can be extended to a homomorphism $\Psi_{p,q}$ of the group G into A such that $\Psi_{p,q}|_P = \Phi_{p,q}$ and $\Psi_{p,q}(h) = 1$, for all $h \in H$ (Proposition 27). $\Psi_{p,q}$ induces, in turn, a homomorphism $\psi_{p,q}$ of Lie algebras \mathfrak{g} and \mathfrak{a} ($\psi_{p,q}$ is a differential of $\Psi_{p,q}$ at $1 \in G$). It must be a trivial homomorphism, by Proposition 28; $\forall p, q \in P, \forall Z \in \mathfrak{g}, \psi_{p,q} Z = 0$. However, the group G is generated by elements of the form $\exp Z$, where $Z \in \mathfrak{g}$, and

$\Psi_{p,q}(\exp Z) = \exp \psi_{p,q}Z = 1$. Thus $\Psi_{p,q}$ is itself trivial. Consequently, if $\Delta f(p, q, r)$ satisfies (59) of Theorem 16, then $\Delta f = 1$, which already implies identity (58). Similarly, (63) of Theorem 16, implies $\Delta f = 1$, (62) and reduces (64) into, $f(p, q) = f(q, p)$.

Conversely, given that $\Delta f = 1$, one infers using Theorem 16 that E is a BA -extension. Likewise, assuming $\Delta f = 1$ and the identity $f(p, q) = f(q, p)$, one employs Theorem 17 to infer that E is a BAI -extension. \square

Remark 4. The equation, $\Delta f = 1$ (2) has been called in [14] a cocycle equation. Together with a condition of invariance for a cochain f , they were employed there to study central extensions of *gyrocommutative gyrogroups*. Our slightly more general treatment of central extensions shows that at least for a class of *gyrocommutative gyrogroups* discussed in this section (Theorem 29), a cocycle equation emerges in a natural way. This class of *gyrocommutative gyrogroups* is determined by Cartan decompositions of simple Lie algebras. Many of classical Lie algebras are of that type. (For their list see [6], Appendix C.) In particular, a *gyrocommutative gyrogroup*, which has been one of the main objects studied in ([14]), and which arises from the pair $(SO_0(1, n), SO(n))$, belongs to that class.

Appendix.

For completeness of presentation we clarify here how the definitions of *gyrogroup* and *gyrocommutative gyrogroup* used in this paper correspond to the original ones.

According to [16] a *gyrogroup* is a grupoid (M, \cdot) satisfying the following axioms.

(G1) There is an element $1 \in M$, such that,

$$1 \cdot x = x,$$

for all $x \in M$.

(G2) For each $x \in M$ there is an $a \in M$ such that

$$a \cdot x = 1.$$

(G3) For any $x, y, z \in M$ there exists a unique element $gyr[x, y]z \in M$ such that

$$x \cdot (y \cdot z) = (x \cdot y) \cdot gyr[x, y]z.$$

(G4) If $gyr[x, y]$ denotes the map of M into M , given by $z \mapsto gyr[x, y]z$ then $gyr[x, y] \in Aut(M, \cdot)$.

(G5) For all $x, y \in M$

$$gyr[x, y] = gyr[x \cdot y, y].$$

From these axioms one can infer (see [16]) that the equation $x \cdot y = z$ has a unique solution for x or y , given the other two elements. Thus (M, \cdot) is a loop, [11].

Proposition 30. *Let (M, \cdot) be a loop. (M, \cdot) is a gyrogroup if and only if it is an A -loop and it is a left Bol loop, [11].*

Proof. Indeed, suppose that (M, \cdot) is a gyrogroup. Then (G3) can be rewritten into $\text{gyr}[x, y]z = L_{x \cdot y}^{-1}L_xL_yz = \ell(x, y)z$. Hence $\text{gyr}[x, y] = \ell(x, y)$. Therefore (G3) and (G4) are equivalent to the statement that (M, \cdot) is an A -loop. To prove Bol-identity we calculate $L_xL_yL_xz = x \cdot (y \cdot (x \cdot z)) = x \cdot ((y \cdot x) \cdot \ell(y, x)z) = (x \cdot (y \cdot x)) \cdot \ell(x, y \cdot x)\ell(y, x)z$. However, (see [16]), $\ell(x, y \cdot x) = \ell(x, y)$ and $\ell(y, x) = (\ell(x, y))^{-1}$. Hence $L_xL_yL_x = L_{x \cdot (y \cdot x)}$, which is a Bol identity, (6).

Conversely, let (M, \cdot) be an A -loop and a Bol loop. Then (G1) – (G4) hold automatically. It suffices to prove (G5). We obtain, $L_{(x \cdot y) \cdot y} = L_{(x \cdot y) \cdot (x^{-1} \cdot (x \cdot y))} = L_{x \cdot y}L_{x^{-1}}L_{x \cdot y}$, where we have employed the fact that for Bol loops $\ell(x, x^{-1}) = \ell(x^{-1}, x) = L_{x^{-1}}L_x = \text{id}_M$, [11]. Therefore $L_{(x \cdot y) \cdot y}^{-1}L_{x \cdot y}L_y = L_{x \cdot y}^{-1}L_{x^{-1}}^{-1}L_{x \cdot y}^{-1}L_{x \cdot y}L_y = L_{x \cdot y}^{-1}L_xL_y$, which is equivalent to $\ell(x \cdot y, y) = \ell(x, y)$, which is (G5). \square

Now, according to [16], (M, \cdot) is a *gyrocommutative gyrogroup* if it is a *gyrogroup*, which satisfies an additional axiom,

$$(G6) \quad x \cdot y = \text{gyr}[x, y](y \cdot x).$$

As it has been indicated in [16], this identity in a *gyrogroup* is equivalent to an inverse automorphic identity, (8).

Therefore the following is true.

Proposition 31. *Let (M, \cdot) be a loop. (M, \cdot) is a gyrocommutative gyrogroup if and only if it is an A -loop, a left Bol loop and it satisfies an inverse automorphic identity.*

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