

SIMPLE CONNECTIVITY OF THE MARKOV PARTITION SPACE

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In Wagoner, 1987 the simplicial complex P_A of Markov partitions was introduced as a tool for studying the group of automorphisms of a subshift of finite type (X_A, σ_A) built from a zero-one transition matrix A . Triangles in P_A led to the matrix Triangle Identities in Wagoner, Pac. Journal, 1990 which have been used in Wagoner, 1990, 1990, 1990, 1992, Kim, Roush & Wagoner, 1992, and the Williams Conjecture counterexample paper Kim & Roush, to appear.

A key fact about P_A is that it is contractible. See Wagoner, 1987. The purpose of this note is to correct the proof on pp. 99-100 in Wagoner, 1987 that P_A is simply connected and in the process to improve the bound in Proposition 2.13 of Wagoner, 1987.

Proposition. *A closed path in P_A with L edges can be spanned by a (possibly singular) triangulated 2-disc in P_A having at most $8L^2 + L$ triangles.*

The difficulty with the proof of (2.13) in [W1] occurs in the diagram of Step 2 on p. 99, because it may not be the case that $V_{i-1} \cap V_{i+1}$ is a Markov partition.

To correct this, it is better to change to a more straightforward notation and let $U \xrightarrow{-} V$ rather than $V \xrightarrow{-} U$ mean $V < U < \sigma_A(V) \cap V$. Recall from [W1] that $U \xrightarrow{+} V$ means $U < V < U \cap \sigma_A^{-1}(U)$. Then Definition 2.10 of [W1] becomes

$$U \rightarrow V \quad \text{iff} \quad U \xrightarrow{+} U \cap V \xrightarrow{-} V.$$

In particular, now $U \xrightarrow{-} V$ implies that $U \rightarrow V$ in P_A but with extra information, whereas in [W1] the notation $V \xrightarrow{-} U$ implied $U \rightarrow V$, which is somewhat contrary. Here are some properties of the arrows $U \xrightarrow{+} V$ and $U \xrightarrow{-} V$. See [W1].

- 1) If $U \xrightarrow{+} V$ and U is a Markov partition, then V is a Markov partition.
 If $U \xrightarrow{-} V$ and V is a Markov partition, then U is a Markov partition.

- 2) If $U \rightarrow V$ and U and V are Markov partitions, then $U \cap V$ is a Markov partition.
- 3) If $U \xrightarrow{+} V$, W and U is a Markov partition, then $U \xrightarrow{+} V \cap W$ and $V \cap W \xrightarrow{+}$ is a Markov partition. If $U, V \xrightarrow{-} W$ and $W \xrightarrow{+}$ is a Markov partition, then $U \cap V \xrightarrow{-} W$ and $U \cap V$ is a Markov partition.
- 4) If $U \xrightarrow{-} X \xrightarrow{+} V$ and U , X , and V are Markov partitions, then $U \xrightarrow{+} U \cap V \xrightarrow{-} V$ and $U \cap V$ is a Markov partition.

For completeness, we recall Definition 2.11 of [W1] giving the simplicial structure on P_A . Namely, an n -simplex of P_A is an ordered $(n + 1)$ -tuple $\langle V_0, V_1, \dots, V_n \rangle$ such that $V_i \rightarrow V_j$ whenever $i \leq j$.

The next step is to replace the diagram in Step 2 on p. 99 with the following diagram

$$\begin{array}{ccccc}
 \vdots & & & & \vdots \\
 V_{i+2} & \xrightarrow{+} & & & V_{i+1} \cap V_{i+2} \cap V_{i+3} \\
 & \searrow + & & & \uparrow \\
 \downarrow & & V_{i+1} \cap V_{i+2} & & \\
 & \swarrow - & & & \\
 V_{i+1} & \xleftarrow{-} & & & V_i \cap V_{i+1} \cap V_{i+2} \\
 & & - & & \\
 \uparrow & \swarrow - & & & \downarrow \\
 & \nearrow + & V_i \cap V_{i+1} & & \\
 V_i & \xrightarrow{+} & & & V_{i-1} \cap V_i \cap V_{i+1} \\
 & \searrow + & & & \uparrow \\
 \downarrow & & V_{i-1} \cap V_i & & \\
 & \swarrow - & & & \\
 V_{i-1} & \xleftarrow{-} & & & V_{i-2} \cap V_{i-1} \cap V_i \\
 & & - & & \\
 \uparrow & \swarrow - & & & \downarrow \\
 & \nearrow + & V_{i-2} \cap V_{i-1} & & \\
 V_{i-2} & \xrightarrow{+} & & & V_{i-3} \cap V_{i-2} \cap V_{i-1} \\
 & & + & & \\
 \vdots & & & & \vdots
 \end{array}$$

We can now deform a closed path of length L to a constant path as follows: Step 1 on p. 99 deforms the closed path of length L to an alternating

closed path of length $2L$ with vertices V_0, V_1, \dots, V_{2L} . The number of triangles in this deformation is at most L . Then the above diagram deforms the alternating closed path of length $2L$ on the left to an alternating closed path of length $2L$ on the right with vertices of the form $V_{i-1} \cap V_i \cap V_{i+1}$. Repeating the deformation in the diagram $L-1$ more times produces an alternating closed path of length $2L$ with vertices of the form

$$V_{i-L} \cap V_{i-L+1} \cap \dots \cap V_{i-1} \cap V_i \cap V_{i+1} \dots \cap V_{i+L-1} \cap V_{i+L}.$$

Thus all the vertices in this path are equal to

$$V_0 \cap V_1 \cap \dots \cap V_{2L}.$$

The total number of triangles in this deformation is at most $8L^2 + L$.

Remark. The argument in [W1] that $H_n(P_A) = 0$ for $n \geq 2$ avoids the $V_{i-1} \cap V_{i+1}$ type difficulty, because all intersections of Markov partitions encountered in the proof are Markov partitions as a consequence of properties (1) through (4) above. There is a typographical change on p. 102, l.10: $V_{p_s} \cap V_{p_s}$ should read $V_{p_s} \cap V_{q_s}$.

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