PACIFIC JOURNAL OF MATHEMATICS
Vol. 192, No. 2, 2000

RESTRICTIONS OF $\Omega_m(q)$-MODULES TO ALTERNATING GROUPS

William J. Husen

We consider the restriction of an irreducible $\mathbf{F}\Omega_m(q)$-module M to a subgroup H where $F^*(H) \cong A_n$ and where $\mathbf{F}$ is algebraically closed with $(\mathrm{char}(\mathbf{F}), q) \neq 1$. Given certain restrictions on the highest weight of M, we show that if $m > n^6$, then $M{\downarrow_H}$ is reducible.

1. Introduction.

In the study of the maximal subgroups of classical groups, the following question arises: Given an absolutely irreducible module M for K and a subgroup H, when does $M{\downarrow_H}$ remain absolutely irreducible? In this article $K \cong \Omega_m(q)$ is the commutator subgroup of an m-dimensional orthogonal group over $\mathbf{F}_q$, and $F^*(H) \cong A_n$ is the alternating group of degree n. We treat the case that the field of definition of M has characteristic dividing q.

Let $\mathbf{F}$ be an algebraically closed field containing $\mathbf{F}_q$, the field with q elements, such that $\mathrm{char}(\mathbf{F}) > 3$. Then $K < \overline{K}$ where $\overline{K} \cong \Omega_m(\mathbf{F})$ and we may assume that M is a $\mathbf{F}K$-module. By [6, Theorem 43], every absolutely irreducible $\mathbf{F}K$-module is the restriction of an irreducible $\mathbf{F}\overline{K}$-module of the same weight. So we may assume that $M = M(\lambda)$ is an irreducible $\mathbf{F}\overline{K}$-module with highest weight λ. Let $\ell = \lfloor m/2 \rfloor$ be the Lie rank of $\overline{K}$ and let $\{\lambda_i\}$ be the fundamental dominant weights of $\overline{K}$. The labeling of these weights corresponds to the labeling of the Dynkin diagrams for $\overline{K}$ as given in [3].

Hypothesis 1.1. *Assume the following are true:*

(1) *If m is even, then* $\lambda = \left(\sum_{i=1}^{\ell-2} a_i\lambda_i\right) + a_{\ell-1}(\lambda_{\ell-1} + \lambda_\ell)$*;* $a_i \in \mathbf{Z}$*,* $a_i \geq 0$.

(2) *If m is odd, then* $\lambda = \left(\sum_{i=1}^{\ell-1} a_i\lambda_i\right) + 2a_\ell\lambda_\ell$*;* $a_i \in \mathbf{Z}$*,* $a_i \geq 0$.

(3) *If* $\mu_i = \sum_{j=i}^{\ell-1} a_j$*, m even or if* $\mu_i = \sum_{j=i}^{\ell} a_j$*, m odd then*
  (a) $\mu_1 < p = \mathrm{char}(\mathbf{F}_q)$*;*
  (b) $1 < \sum \mu_i = k < \ell$.

Conditions (1) and (2) imply that M is not a faithful module for any proper covering group of $\overline{K}$. We now state our main result:

Theorem 1.2. *Assume that H, K and $M = M(\lambda)$ are as above with $n, m \geq 10$ and $(q, 6) = 1$. Suppose further that λ satisfies Hypothesis 1.1. If $m > n^6$, then $M{\downarrow_H}$ is reducible.*

Our strategy is to produce a small subspace in M with a large stabilizer in H and then, using Frobenius reciprocity, produce an upper bound for $\dim(M)$. We produce a lower bound for $\dim(M)$ as an $\mathbf{F}\overline{K}$-module using the length of the Weyl group orbit of a subdominant weight in M. The result then follows by comparing these two bounds.

2. A construction of $\overline{W}(\lambda)$.

In this section we construct the Weyl module $\overline{W}(\lambda)$ of $\overline{K}$ with highest weight λ. Then M is a homomorphic image of this module. Our construction proceeds by first constructing the Weyl module $W(\lambda)$ for a complex Lie group G of the same type and rank as $\overline{K}$, then we use Kostant's $\mathbf{Z}$-form to produce $\overline{W}(\lambda)$. For notational convenience we assume that $\{\lambda_i\}$ are the fundamental dominant weights for G as well as for $\overline{K}$, and accordingly, assume that λ is a dominant weight of G.

Let V be a complex, m-dimensional vector space possessing a non-degenerate orthogonal form $\mathbf{f}(\ ,\ )$ and let $\mathcal{B}$ be a basis for V so that

$$\mathcal{B} = \begin{cases} \{e_i, f_i \mid 1 \leq i \leq \ell\} & \text{if } m \text{ is even} \\ \{e_i, f_i, x \mid 1 \leq i \leq \ell\} & \text{if } m \text{ is odd} \end{cases}$$

with $\mathbf{f}(e_i, e_j) = \mathbf{f}(f_i, f_j) = \mathbf{f}(x, e_i) = \mathbf{f}(x, f_i) = 0$, $\mathbf{f}(e_i, f_j) = \delta_{i,j}$ and $\mathbf{f}(x, x) = 2$. We then define $G = \Omega(V)$ and let T be the maximal torus of G with respect to $\mathcal{B}$. Set $V_e = \langle e_i \mid 1 \leq i \leq \ell\rangle$ and $V_f = \langle f_i \mid 1 \leq i \leq \ell\rangle$.

Suppose that λ satisfies hypothesis 1.1 and $d = \max\{i \mid \mu_i \neq 0\}$ so that $\mu = (\mu_1, \ldots, \mu_d)$ is a proper partition of k. Let $\mathcal{T}$ be the tableau of shape μ with entries $t_{i,j} = j + \sum_{s<i} \mu_s$. Define the following subgroups of the symmetric group $\mathcal{S}_k$:

$$\mathcal{R}_\mu = \{\sigma \in \mathcal{S}_k \mid \sigma(t_{i,j}) \text{ lies in the same row as } t_{i,j} \text{ for all } i, j\}$$

$$\mathcal{C}_\mu = \{\sigma \in \mathcal{S}_k \mid \sigma(t_{i,j}) \text{ lies in the same column as } t_{i,j} \text{ for all } i, j\}$$

and elements of $\mathbf{C}\mathcal{S}_k$:

$$r_\mu = \sum_{\sigma \in \mathcal{R}_\mu} \sigma \qquad \text{and} \qquad c_\mu = \sum_{\sigma \in \mathcal{C}_\mu} \operatorname{sgn}(\sigma)\sigma$$

Define $\kappa_{i,j} : V^{\otimes k} \to V^{\otimes(k-2)}$ by $\kappa_{i,j}(v_{l_1} \otimes \cdots \otimes v_{l_k}) = f(v_{l_i}, v_{l_j})(v_{l_1} \otimes \cdots \otimes \widehat{v_{l_i}} \otimes \cdots \otimes \widehat{v_{l_j}} \otimes \cdots \otimes v_{l_k})$ for $1 \le i < j \le k$ and set

$$\mathcal{K} = \bigcap_{i,j} \ker(\kappa_{i,j}).$$

$\mathcal{S}_k$ acts on $V^{\otimes k}$ by place permutation, specifically:

$$\sigma(v_{i_1} \otimes \cdots \otimes v_{i_k}) = v_{i_{\sigma^{-1}(1)}} \otimes \cdots \otimes v_{i_{\sigma^{-1}(k)}}.$$

This action commutes with the diagonal action of G on $V^{\otimes k}$.

Given $v \in V^{\otimes k}$, we define one additional element r^v_μ of the group algebra $\mathbf{C}\mathcal{S}_k$ as follows: Let $\mathcal{R}^v_\mu = \{\sigma \in \mathcal{R}_\mu \mid \sigma(v) = v\}$ and let $\{s_i\}$ be a left transversal for $\mathcal{R}^v_\mu$ in $\mathcal{R}_\mu$. Define $r^v_\mu = \sum_i s_i$ Notice that $r_\mu(v) = |\mathcal{R}^v_\mu| r^v_\mu(v)$.

By [**2**, Theorem 19.22], $W(\lambda) = c_\mu r_\mu \left(V^{\otimes k}\right) \cap \mathcal{K}$ is the Weyl module for G with highest weight λ. Since V is a complex vector space, $c_\mu r_\mu \left(V^{\otimes k}\right) = \langle c_\mu r^v_\mu(v) \mid v \in V^{\otimes k}\rangle$.

Define $V_{\mathbf{Z}} = \mathbf{Z}[\mathcal{B}]$ and let $\overline{V} = V_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{F}$. Then $\overline{\mathbf{f}}(\ ,\ ) = \mathbf{f}(\ ,\ ) \otimes 1_{\mathbf{F}}$ is a non-degenerate orthogonal form on $\overline{V}$. Without loss of generality, we may assume that $\overline{K} = \Omega(\overline{V})$. Moreover if $\overline{e}_i = e_i \otimes 1_{\mathbf{F}}$, $\overline{f}_i = f_i \otimes 1_{\mathbf{F}}$ and $\overline{x} = x \otimes 1_{\mathbf{F}}$, then

$$\overline{\mathcal{B}} = \begin{cases} \{\overline{e}_i, \overline{f}_i \mid 1 \le i \le \ell\} & \text{if } m \text{ is even} \\ \{\overline{e}_i, \overline{f}_i, \overline{x} \mid 1 \le i \le \ell\} & \text{if } m \text{ is odd} \end{cases}$$

is a standard basis for $\overline{V}$ with respect to $\overline{\mathbf{f}}(\ ,\ )$. We identify r_μ and c_μ with the elements $r_\mu \otimes 1_{\mathbf{F}}$ and $c_\mu \otimes 1_{\mathbf{F}}$ of $\mathbf{F}\mathcal{S}_k$.

Suppose that $L \subset \mathrm{End}(V)$ is the adjoint module for G so that L is a complex Lie algebra of type D_ℓ or B_ℓ. Let $\Delta = \{r_1, \ldots, r_\ell\}$ be the set of simple roots corresponding to the torus T and let Φ be the root system generated by Δ. Set $\Delta_0 = \{r_1, \ldots, r_{\ell-1}\}$ and let $\Phi_0 \subset \Phi$ be the subset generated by Δ_0. Using the setup of [**1**, §11.2], $\{\epsilon_r, h_{r_i} \mid r \in \Phi, 1 \le i \le \ell\}$ is a Chevalley basis for L and $\{\epsilon_r, h_{r_i} \mid r \in \Phi_0, 1 \le i \le \ell-1\}$ is a Chevalley basis for $L_0 \subset L$ where L_0 is a Lie algebra of type $A_{\ell-1}$. Let $G_0 < N_G(V_e \oplus V_f)$ such that $G_0 \cong SL_\ell(\mathbf{C})$. Then, by [**1**, Theorem 11.3.2], $G = \langle \exp(\zeta\epsilon_r) \mid r \in \Phi,\ \zeta \in \mathbf{C}\rangle$ and $G_0 = \langle \exp(\zeta\epsilon_r) \mid r \in \Phi_0,\ \zeta \in \mathbf{C}\rangle$. Note that neither G nor G_0 is the adjoint group for L or L_0, respectively. We may consider V_e to be the natural module for G_0. Under this identification, V_f is the dual of V_e.

Assume that $\mathcal{U}(L)$ is the universal enveloping algebra of L. From [**3**, §26], Kostant's $\mathbf{Z}$-form $\mathcal{U}_{\mathbf{Z}}(L)$ is the $\mathbf{Z}$-span of $\{\epsilon_r^m/m! \mid r \in \Phi, m \in \mathbf{Z}^+\}$. Given any vector v of weight λ in $W(\lambda)$, $\mathcal{U}_{\mathbf{Z}}(L)v \bigotimes_{\mathbf{Z}} \mathbf{F} = \overline{W}(\lambda)$ where $\overline{W}(\lambda)$ is the Weyl module for $\overline{K}$ with highest weight λ. By the previous remarks, $\mathcal{U}_{\mathbf{Z}}(L_0) \subset \mathcal{U}_{\mathbf{Z}}(L)$, which implies that $\mathcal{U}_{\mathbf{Z}}(L_0)v \bigotimes_{\mathbf{Z}} \mathbf{F} \subset \overline{W}(\lambda)$.

Define $v_{\mu_i} = \bigotimes_{j=1}^{\mu_i} e_i$ and $v_\mu = \bigotimes_{i=1}^{d} v_{\mu_i}$.

Lemma 2.1. *We have:*

(1) $c_\mu(v_\mu)$ *is a vector of weight* λ *in* $W(\lambda)$*;*
(2) $\mathcal{U}_{\mathbf{Z}}(L_0)c_\mu(v_\mu) = c_\mu r_\mu\left(V_e^{\otimes k}\right) \cap \mathbf{Z}[e_1, \dots, e_\ell]^{\otimes k}$.

Proof. First note that $\mathcal{R}_\mu^{v_\mu} = \mathcal{R}_\mu$ so that $r_\mu^{v_\mu}(v_\mu) = v_\mu$ and that $c_\mu(v_\mu) \neq 0$. This implies that $c_\mu(v_\mu) \in c_\mu r_\mu\left(V^{\otimes k}\right)$. It is clear that $c_\mu(v_\mu) \in \mathcal{K}$ so we have $c_\mu(v_\mu) \in W(\lambda)$. Let $t \in T$ and write $t = \operatorname{diag}(t_1, \dots, t_\ell, t_1^{-1}, \dots, t_\ell^{-1})$ or $t = \operatorname{diag}(t_1, \dots, t_\ell, t_1^{-1}, \dots, t_\ell^{-1}, t')$ depending on the parity of m. Then

$$tv = c_\mu(tc_\mu(v_\mu)) = c_\mu\left(\bigotimes_{i=1}^{d} t_i^{\mu_i} v_{\mu_i}\right) = \left(\prod_{i=1}^{d} t_i^{\mu_i}\right) c_\mu(v_\mu).$$

From the definition of μ it follows that $c_\mu(v_\mu)$ is a vector of weight λ and so (1) follows. With the identification of V_e with the natural module of G_0, we see by [**2**, Theorem 15.15] that $c_\mu r_\mu\left(V_e^{\otimes k}\right)$ is the Weyl module for G_0 corresponding to the partition μ of k via the Schur functor. The argument above restricted to $t \in T \cap G_0$ shows that $c_\mu(v_\mu)$ is a highest weight vector in $c_\mu r_\mu\left(V_e^{\otimes k}\right)$. In particular $\mathcal{U}(L_0)c_\mu(v_\mu) = c_\mu r_\mu\left(V_e^{\otimes k}\right)$. Using the proof of [**4**, Theorem 8.3.1], we have

$$\mathcal{U}_{\mathbf{Z}}(L_0)c_\mu(v_\mu) = c_\mu r_\mu\left(V_e^{\otimes k}\right) \cap \mathbf{Z}[e_1, \dots, e_\ell]^{\otimes k}$$

which completes our proof. □

Lemma 2.2. *Suppose* $\overline{v} = \overline{v}_{i_1} \otimes \cdots \otimes \overline{v}_{i_k}$ *where* $\{\overline{v}_i\}$ *is a collection of mutually orthogonal, linearly independent singular vectors. Then:*

(1) *If* $\operatorname{sgn}(\sigma_c)\sigma_c\sigma_r(\overline{v}) \neq -\overline{v}$ *for all* $\sigma_c \neq 1 \in \mathcal{C}_\mu, \sigma_r \in \mathcal{R}_\mu$*, then* $c_\mu r_\mu^{\overline{v}}(\overline{v}) \neq 0$*;*
(2) $c_\mu r_\mu^{\overline{v}}(\overline{v}) \in \overline{W}(\lambda)$*.*

Proof. Since $\overline{v}$ is a summand of $c_\mu r_\mu^{\overline{v}}(\overline{v})$ and all other summands of $c_\mu r_\mu^{\overline{v}}(\overline{v})$ have the form $\operatorname{sgn}(\sigma_c)\sigma_c\sigma_r(\overline{v})$, part (1) must hold. There is $g \in \overline{K}$ such that $g(\overline{v}_{i_j}) = \alpha_{i_j}\overline{e}_{i_j}$ such that $\alpha_{i_j} \neq 0$ for all $1 \leq i \leq k$. If $w = e_{i_1} \otimes \cdots \otimes e_{i_k}$, then $r_\mu^{\overline{v}} = r_\mu^w$. As

$$c_\mu r_\mu^w(w) \in c_\mu r_\mu\left(V_e^{\otimes k}\right) \cap \mathbf{Z}[e_1, \dots, e_\ell]^{\otimes k},$$

Lemma 2.1 implies that $c_\mu r_\mu^w(w) \in \mathcal{U}_{\mathbf{Z}}(L)v$. Writing $\overline{w} = \alpha_{i_1}\overline{e}_{i_1} \otimes \cdots \otimes \alpha_{i_k}\overline{e}_{i_k}$, we then have

$$c_\mu r_\mu^{\overline{w}}(\overline{w}) \in \mathcal{U}_{\mathbf{Z}}(L)v \bigotimes_{\mathbf{Z}} \mathbf{F} = \overline{W}(\lambda).$$

Finally, as $\overline{W}(\lambda)$ is a $\mathbf{F}\overline{K}$-module, $g^{-1}c_\mu r_\mu^{\overline{w}}(\overline{w}) = c_\mu r_\mu^{\overline{v}}(\overline{v}) \in \overline{W}(\lambda)$. □

Though $W(\lambda)$ is a irreducible module for G, $\overline{W}(\lambda)$ may not be an irreducible module for $\overline{K}$; however, it does possess a unique maximal submodule by [**6**, Lemma 80] which we denote by $\operatorname{Rad}(\overline{W}(\lambda))$. Moreover, $M \cong \overline{W}(\lambda)/\operatorname{Rad}(\overline{W}(\lambda))$.

We now discuss the orthogonal forms on $V^{\otimes k}$ and $W(\lambda)$. Suppose $v, w \in V^{\otimes k}$ where $v = v_1 \otimes \cdots \otimes v_k$ and $w = w_1 \otimes \cdots \otimes w_k$. We define $\mathbf{f}^k(\ ,\ )$ by

$$\mathbf{f}^k(v, w) = \prod_{i=1}^{k} \mathbf{f}(v_i, w_i).$$

$\mathbf{f}^k(\ ,\ )$ is a non-degenerate, G-invariant orthogonal form on $V^{\otimes k}$. This form is also invariant under the action of $\mathcal{S}_k$. Note that

$$\begin{aligned}
\mathbf{f}^k[c_\mu(v), c_\mu(w)] &= \sum_{\sigma \in \mathcal{C}_\mu} \operatorname{sgn}(\sigma)\mathbf{f}^k[\sigma(v), c_\mu(w)] \\
&= \sum_{\sigma \in \mathcal{C}_\mu} \operatorname{sgn}(\sigma)\mathbf{f}^k[v, \sigma^{-1}c_\mu(w)] \\
&= \sum_{\sigma \in \mathcal{C}_\mu} \mathbf{f}^k[v, c_\mu(w)] \\
&= |\mathcal{C}_\mu|\mathbf{f}^k[v, c_\mu(w)].
\end{aligned}$$

We define $\mathbf{f}^k_\mu(\ ,\ )$ on $c_\mu\left(V^{\otimes k}\right)$ by

$$\mathbf{f}^k_\mu[c_\mu(v), c_\mu(w)] = \mathbf{f}^k[v, c_\mu(w)].$$

By a similar argument as above, we see that $\mathbf{f}^k[v, c_\mu(w)] = \mathbf{f}^k[w, c_\mu(v)]$, so this form is symmetric. Since $\mathbf{f}^k(\ ,\ )$ is bilinear and G-invariant, $\mathbf{f}^k_\mu(\ ,\ )$ is also bilinear and G-invariant. Therefore $\mathbf{f}^k_\mu(\ ,\ )$ is a G-invariant orthogonal form on $W(\lambda) \subset c_\mu\left(V^{\otimes k}\right)$. As before, $\overline{\mathbf{f}}^k(\ ,\ ) = \mathbf{f}^k(\ ,\ ) \otimes 1_{\mathbf{F}}$ is a $\overline{K}$-invariant orthogonal form on $\overline{V}^{\otimes k}$ and $\overline{\mathbf{f}}^k_\mu(\ ,\ ) = \mathbf{f}^k_\mu(\ ,\ ) \otimes 1_{\mathbf{F}}$ is a $\overline{K}$-invariant orthogonal form on $\overline{W}(\lambda)$. This form is possibly degenerate. We denote the radical of this form as $\overline{W}(\lambda)^\perp$. The following lemma is generally known, although we present a proof:

Lemma 2.3. $\operatorname{Rad}(\overline{W}(\lambda)) = \overline{W}(\lambda)^\perp$.

Proof. Define $\overline{v}_{-\mu_i} = \bigotimes_{j=1}^{\mu_i} \overline{f}_i$ and $\overline{v}_{-\mu} = \bigotimes_{i=1}^{d} \overline{v}_{-\mu_i}$. Noting that $r_\mu^{\overline{v}_{-\mu}} = 1$, $c_\mu(v_{-\mu}) \neq 0 \in \overline{W}(\lambda)$ by Lemma 2.2. A similar argument as in the proof of Lemma 2.1 shows that $c_\mu(v_{-\mu})$ is a vector of weight $-\lambda$. Hypothesis 1.1 implies that $d < \ell$. In particular, there is an element ω_0 of the Weyl group of $\overline{K}$ such that $\omega_0[c_\mu(v_{-\mu})] = c_\mu(v_\mu)$. This means that $M = M(\lambda)$ must be self-dual. Clearly we have that $\overline{W}(\lambda)^\perp \subset \operatorname{Rad}(\overline{W}(\lambda))$ and that $\overline{W}(\lambda)/\overline{W}(\lambda)^\perp$ is non-degenerate, so this latter module is also self-dual. Since M is self-dual and is a homomorphic image of $\overline{W}(\lambda)/\overline{W}(\lambda)^\perp$, $\overline{W}(\lambda)/\overline{W}(\lambda)^\perp$ must possess a submodule isomorphic to M. Since $M \cong \overline{W}(\lambda)/\operatorname{Rad}(\overline{W}(\lambda))$ and $\operatorname{Rad}(\overline{W}(\lambda))$ does not possess a constituent which is isomorphic to M, we must have $\operatorname{Rad}(\overline{W}(\lambda)) = \overline{W}(\lambda)^\perp$ and our result follows. $\square$

Lemma 2.4. *Let* $\{\overline{v}_i, \overline{w}_i \mid 1 \leq i \leq k\}$ *be a hyperbolic basis for some* $2k$*-dimensional subspace of* $\overline{V}$*. Set* $\overline{v} = \overline{v}_1 \otimes \cdots \otimes \overline{v}_k$ *and* $\overline{w} = \overline{w}_1 \otimes \cdots \otimes \overline{w}_k$*. Then:*

(1) $c_\mu r_\mu(\overline{v}) \neq 0,\ c_\mu r_\mu(\overline{w}) \neq 0$;
(2) $c_\mu r_\mu(\overline{v}),\ c_\mu r_\mu(\overline{w}) \in \overline{W}(\lambda)$;
(3) $\overline{\mathbf{f}}^k_\mu[c_\mu r_\mu(\overline{v}), c_\mu r_\mu(\overline{w})] \neq 0$.

Proof. Parts (1) and (2) follow from Lemma 2.2 since $r^{\overline{v}}_\mu = r^{\overline{w}}_\mu = r_\mu$ and the $\overline{v}_i$ are distinct, similarly for $\overline{w}_i$. If $\sigma_1, \sigma_2 \in \mathcal{S}_k$, then

$$\overline{\mathbf{f}}^k[\sigma_1(\overline{v}), \sigma_2(\overline{w})] = \prod_{i=1}^{k} \overline{\mathbf{f}}[\overline{v}_{\sigma_1^{-1}(i)}, \overline{w}_{\sigma_2^{-1}(i)}] = \begin{cases} 1 & \text{if } \sigma_1 = \sigma_2 \\ 0 & \text{otherwise.} \end{cases}$$

Recall that $\mathcal{R}_\mu \cap \mathcal{C}_\mu = 1$. Then we have

$$\begin{aligned} \overline{\mathbf{f}}^k_\mu[c_\mu r_\mu(\overline{v}), c_\mu r_\mu(\overline{w})] &= \overline{\mathbf{f}}^k[r_\mu(\overline{v}), c_\mu r_\mu(\overline{w})] \\ &= \sum_{\sigma \in \mathcal{R}_\mu} \overline{\mathbf{f}}^k[\sigma(\overline{v}), c_\mu r_\mu(\overline{w})] \\ &= \sum_{\sigma \in \mathcal{R}_\mu} \overline{\mathbf{f}}^k[\sigma(\overline{v}), \sigma(\overline{w})] \\ &= |\mathcal{R}_\mu|. \end{aligned}$$

Part (3) then follows as $|\mathcal{R}_\mu| = \prod_{i=1}^d \mu_i!$ and $\mu_i < \mathrm{char}(\mathbf{F}_q)$ for all i. □

Lemma 2.5. *M possesses a vector of weight λ_k.*

Proof. Let $\{\overline{e}_i, \overline{f}_i \mid 1 \leq i \leq k\}$ be a subset of our standard basis $\overline{\mathcal{B}}$ for $\overline{V}$. By part (2) of Lemma 2.4, $c_\mu r_\mu(\overline{e}_1 \otimes \cdots \otimes \overline{e}_k) \in \overline{W}(\lambda)$. An argument similar to that used in Lemma 2.1 shows that $c_\mu r_\mu(\overline{e}_1 \otimes \cdots \otimes \overline{e}_k)$ is a vector of weight λ_k. Hence λ_k is a subdominant weight of λ. Condition (3) of Hypothesis 1.1 insures that λ is p-restricted. Therefore using the results of [**5**], M possesses a vector of weight λ_k. □

3. Elementary abelian 3-subgroup E_k.

Assume that $k \leq n/3 - 2$ and recall that $F^*(H)$ possesses a subgroup H_0 isomorphic to S_{n-2}. Let

$$E_k \cong \langle (123), (456), \ldots, (3k-2,\ 3k-1,\ 3k) \rangle < A_n$$

be a subgroup of H_0 generated by commuting 3-cycles in $F^*(H)$ so that E_k is an elementary abelian 3-group of rank k. Then

$$N_k = N_{H_0}(E_k) \cong S_3 \wr S_k \times S_{n-3k-2}$$

$$C_k = C_{H_0}(E_k) \cong E_k \times S_{n-3k-2}$$

and let $H_k < C_k$ so that $H_k \cong S_{n-3k-2}$. Note that $C_{N_k}(H_k) \cong S_3 \wr S_k$ and this subgroup controls fusion in E_k. Let $\sigma \neq 1 \in E_k$ and assume that σ is the product of k_1 disjoint 3-cycles. Then $C_{N_k}(\sigma) \cong \mathbf{Z}_3 \wr S_{k_1} \times S_3 \wr S_{k-k_1} \times S_{n-3k-2}$ which implies $|\sigma^{N_k}| = 2^{k_1}\binom{k}{k_1}$.

Let $\varphi \in E_k^* = \mathrm{Hom}(E_k, \mathbf{F}^*)$. The group N_k acts on this group by $\varphi^g : \sigma \mapsto \varphi(g^{-1}\sigma g)$ for $g \in N_k$, $\sigma \in E_k$. We abuse notation slightly and define φ^{-1} by $\varphi^{-1} : \sigma \mapsto \varphi(\sigma^{-1})$ for all $\sigma \in E_k$. Recall that $\mathrm{In}_{N_k}(\varphi) = \{g \in N_k | \varphi^g = \varphi\}$ is the inertia group of φ in N_k and note that $H_k \in \mathrm{In}_{N_k}(\varphi)$.

If $\varphi \in E_k^*$ is non-trivial, then the previous remarks concerning the action of N_k on E_k imply that $[N_k : \mathrm{In}_{N_k}(\varphi)] = 2^{k_1}\binom{k}{k_1}$ for some k_1, $1 \leq k_1 \leq k$ and that $\varphi^{-1} \in \varphi^{N_k}$. Since $\binom{k}{k_1} \geq k$ unless $k = k_1$, in which case $2^{k_1} \geq 2k$, we have $[N_k : \mathrm{In}_{N_k}(\varphi)] \geq 2k$.

4. Decomposition of $\overline{V}\downarrow_{E_k}$ and C_k-invariant subspace of $\overline{W}(\lambda)$.

We continue to assume that $k \leq n/3 - 2$ and we now consider the restriction of $\overline{V}$ to E_k. Since $\mathrm{char}(\mathbf{F}) \neq 3$, we have $\overline{V}\downarrow_{E_k} \cong \bigoplus_{\varphi \in E_k^*} \overline{V}_\varphi$ where $\overline{V}_\varphi$ is the homogeneous component of φ. Let $\overline{v}_1 \in \overline{V}_{\varphi_1}$ and $\overline{v}_2 \in \overline{V}_{\varphi_2}$. Then $(g\overline{v}_1, g\overline{v}_2) = \varphi_1(g)\varphi_2(g)(\overline{v}_1, \overline{v}_2)$ for all $g \in E_k$. If $\varphi_1^{-1} \neq \varphi_2$ then $(\overline{v}_1, \overline{v}_2) = 0$ which implies $\overline{V}_{\varphi_1} \perp \overline{V}_{\varphi_2}$ when $\varphi_1^{-1} \neq \varphi_2$. Since $\overline{V}$ is non-degenerate, $\dim(\overline{V}_{\varphi_1}^{\perp}) = \dim(\overline{V}) - \dim(\overline{V}_{\varphi_1})$ and it follows that $\overline{V}_\varphi \oplus \overline{V}_{\varphi^{-1}}$ must be non-degenerate and therefore possesses a hyperbolic basis.

Pick $\varphi \neq 1$ so that $\overline{V}_\varphi \neq 0$. Since $g\overline{V}_\varphi = \overline{V}_{\varphi^g}$ for $g \in N_k$, we may consider $\overline{V}_\varphi$ to be an $\mathbf{F}\mathrm{In}_{N_k}(\varphi)$-module. Let E_{k-1}^* be a maximal subgroup of E_k^* which does not contain φ. Define $\mathcal{O}_+ = \varphi E_{k-1}^* \cap \varphi^{N_k}$ and $\mathcal{O}_- = \varphi^{-1} E_{k-1}^* \cap \varphi^{N_k}$ so that $\mathcal{O}_+ \cup \mathcal{O}_- = \varphi^{N_k}$ and $|\mathcal{O}_+| = |\mathcal{O}_-| \geq k$. Moreover $\varphi_i \in \mathcal{O}_+$ if and only if $\varphi_i^{-1} \in \mathcal{O}_-$. We assume that $\mathcal{O}_+ = \{\varphi_i\}$ and that $\mathcal{O}_- = \{\varphi_i^{-1}\}$. Then $\left(\bigoplus_{\varphi_i \in \mathcal{O}_+} \overline{V}_{\varphi_i}\right) \bigoplus \left(\bigoplus_{\varphi_i^{-1} \in \mathcal{O}_-} \overline{V}_{\varphi_i^{-1}}\right)$ is an $\mathbf{F}N_k$-submodule of $\overline{V}\downarrow_{N_k}$. If $\varphi' \in \varphi^{N_k}$ then, as $C_{N_k}(H_k)$ also controls fusion in E_k^*, there is a $g \in C_{N_k}(H_k)$ such that $g\overline{V}_\varphi = \overline{V}_{\varphi'}$. In particular $\overline{V}_\varphi \cong \overline{V}_{\varphi'}$ as $\mathbf{F}H_k$-modules. Define $D = \dim(\overline{V}_\varphi)$ so that $D = \dim(\overline{V}_{\varphi_i})$ for all i.

Given the above decomposition, we form the following:

$$\overline{V}_+ = \bigotimes_{i=1}^{k} \overline{V}_{\varphi_i} \qquad \text{and} \qquad \overline{V}_- = \bigotimes_{i=1}^{k} \overline{V}_{\varphi_i^{-1}}.$$

Recall that $D = \dim(\overline{V}_{\varphi_i})$ and assume that $\{\overline{v}_{i,j}, \overline{w}_{i,j} \mid 1 \leq j_i \leq D\}$ is a hyperbolic basis for $\overline{V}_{\varphi_i} \oplus \overline{V}_{\varphi_i^{-1}}$. Define $\overline{v}^{j_1,\cdots,j_k} = \bigotimes_{i=1}^k \overline{v}_{i,j_i}$ and $\overline{w}^{j_1,\cdots,j_k} = \bigotimes_{i=1}^k \overline{w}_{i,j_i}$. Then $\{\overline{v}^{j_1,\cdots,j_k}, \overline{w}^{j_1,\cdots,j_k} \mid 1 \leq j_i \leq D\}$ forms a hyperbolic basis for $\overline{V}_+ \oplus \overline{V}_-$. If $\sigma \in \mathcal{S}_k$, then $\sigma(\overline{v}^{j_1,\cdots,j_k}) = \overline{v}^{j_1,\cdots,j_k}$ if and only if $\sigma = 1$

since the V_{φ_i} are distinct. Moreover, $r_{\mu}^{\overline{v}^{j_1,\cdots,j_k}} = r_\mu$ for all $\overline{v}^{j_1,\cdots,j_k} \in \overline{V}_+$. Similarly for $\overline{w}^{j_1,\cdots,j_k} \in V_-$.

By parts (1) and (2) of Lemma 2.4, and as $\overline{V}_\pm$ are both totally singular, $c_\mu r_\mu\left(\overline{V}_\pm\right) \subset \overline{W}(\lambda)$. By part (3) of Lemma 2.4, $\overline{\mathbf{f}}_{\mu}^{k}[c_\mu r_\mu(\overline{v}^{j_1,\cdots,j_k}), c_\mu r_\mu(\overline{w}^{j_1,\cdots,j_k})] \neq 0$. Whenever $(j_1,\ldots,j_k) \neq (j_1',\ldots,j_k')$, we have that $\overline{\mathbf{f}}_{\mu}^{k}[c_\mu r_\mu(\overline{v}^{j_1,\cdots,j_k}), c_\mu r_\mu(\overline{w}^{j_1',\cdots,j_k'})] = 0$. Therefore $\{c_\mu r_\mu(\overline{v}^{j_1,\cdots,j_k}), c_\mu r_\mu(\overline{w}^{j_1,\cdots,j_k}) \mid 1 \leq j_i \leq D\}$ is a hyperbolic basis for

$$c_\mu r_\mu\left(\overline{V}_+\right) \bigoplus c_\mu r_\mu\left(\overline{V}_-\right).$$

Lemma 4.1. *We have:*

(1) $\overline{V}_\pm \cong c_\mu r_\mu(\overline{V}_\pm)$ *as* $\mathbf{F}C_k$*-modules;*
(2) *If* k *is even, then* C_k *stabilizes a* 1*-dimensional subspace of* M*;*
(3) *If* k *is odd, then* C_k *stabilizes a* D*-dimensional subspace of* M*.*

Proof. Given the hyperbolic basis $\{\overline{v}^{j_1,\cdots,j_k}, \overline{w}^{j_1,\cdots,j_k} \mid 1 \leq j_i \leq D\}$ for $\overline{V}_+ \oplus \overline{V}_-$, it is clear that the map $\overline{v}^{j_1,\cdots,j_k} \mapsto c_\mu r_\mu(\overline{v}^{j_1,\cdots,j_k})$ is a C_k-invariant bijection. Therefore $\overline{V}_+ \cong c_\mu r_\mu(\overline{V}_+)$ as $\mathbf{F}C_k$-modules. The case for $\overline{V}_-$ follows by a similar argument, proving part (1). Suppose that k is even and recall that $\overline{V}_{\varphi_i} \cong \overline{V}_{\varphi_j}$ and $\overline{V}_{\varphi_i^{-1}} \cong \overline{V}_{\varphi_j^{-1}}$ as $\mathbf{F}H_k$-modules. As $H_k \cong S_{n-3k-2}$ and all irreducible $\mathbf{F}S_{n-2k-2}$ are self-dual, H_k stabilizes a 1-dimensional subspace of $\overline{V}_{\varphi_i} \otimes \overline{V}_{\varphi_j}$. It follows by induction that H_k stabilizes a 1-dimensional subspace of $\overline{V}_+$. If k is odd, then the same argument leads to a D-dimensional subspace being stabilized by H_k. As E_k acts as scalars on $\overline{V}_\pm$, these spaces are, in fact, stabilized by C_k. Using part (1), C_k stabilizes a subspace $\overline{W}_0$ of one of these dimensions in $\overline{W}(\lambda)$. Since $c_\mu r_\mu\left(\overline{V}_+\right) \bigoplus c_\mu r_\mu\left(\overline{V}_-\right)$ possesses a hyperbolic basis, $\overline{W}_0 \cap \overline{W}(\lambda)^\perp = 0$. If we let

$$M_0 = \left(\overline{W}_0 + \overline{W}(\lambda)^\perp\right)/\overline{W}(\lambda)^\perp$$

then Lemma 2.3 implies that $M_0 \subset \overline{W}(\lambda)/\overline{W}(\lambda)^\perp \cong M$, hence (2) and (3). □

5. Proof of Theorem 1.2.

We are now in a position to prove Theorem 1.2:

Since M possesses a vector $\overline{v}_{\lambda_k}$ of weight λ_k by Lemma 2.5, we can produce a lower bound for $\dim(M)$ as follows: Let $\mathrm{Weyl}(\overline{K})$ be the Weyl group of $\overline{K}$ and recall that ℓ is the Lie rank of $\overline{K}$. We compute $C_{\mathrm{Weyl}(\overline{K})}(\lambda_k)$ using [**3**, §13.1], and compute $|\lambda_k^{\mathrm{Weyl}(\overline{K})}|$, whence

$$\dim(M) \geq |\lambda_k^{\mathrm{Weyl}(\overline{K})}| = 2^k \binom{\ell}{k}. \tag{1}$$

Case 1. First suppose that $k \geq n/3 - 1$. We assume that $\dim(\overline{V}) \geq 2n^4$, so $\ell \geq n^4$. Since $\dim(M) \leq \sqrt{|H|} \leq \sqrt{n!}$, Eq. (1) implies that $2^k \binom{\ell}{k} \leq \sqrt{n!}$. Trivially, $2^{n^4/2} > \sqrt{n!}$ for all $n \geq 1$, so that $k < n^4/2 \leq \ell/2$. Using the fact that $\binom{\ell}{k_1} < \binom{\ell}{k_2}$ if $k_1 < k_2 < \ell/2$, $\binom{\ell}{k}$ will be minimal when $k = n/3 - 1$ and $\ell = n^4$. Note also that $\binom{\ell}{k} = \prod_{i=1}^{k} \frac{(\ell-i+1)}{(k-i+1)} \geq \frac{(\ell-k+1)^k}{k^k}$. We have:

$$2^{n/3-1}\binom{n^4}{n/3-1} < \sqrt{n!},$$

$$2^{n/3-1}\frac{(n^4 - n/3 + 2)^{n/3-1}}{(n/3)^{n/3-1}} < (n^{1/2})^{n-1},$$

$$2^{n/3-1}(n^3-1)^{n/3-1} < n^{(n-1)/2},$$

$$n^{n-3} < n^{(n-1)/2},$$

$$n - 3 < (n-1)/2,$$

$$n < 5.$$

This contradicts our assumption that $n \geq 10$, so that $\dim(\overline{V}) \leq 2n^4$ or $k < n/3 - 1$.

Case 2. We assume that $k < n/3 - 1$ and that k is odd. Lemma 4.1 and Frobenius reciprocity imply $\dim(M) \leq D[H : C_k]$. Since $D \geq \frac{\ell}{2k}$ and $[H : C_k] = \frac{n!}{2(3^k)(n-3k-2)!}$, we have $\dim(M) \leq \frac{\ell}{2k}\ \frac{n!}{3^k(n-3k-2)!}$. Combining with (1) we get:

$$2^k\binom{\ell}{k} \leq \frac{\ell}{2k}\ \frac{n!}{2(3^k)(n-3k-2)!},$$

$$2^k\binom{\ell-1}{k-1} < \frac{n^{3k+2}}{3^{k-1}},$$

$$2^k\frac{(\ell-k+1)^{k-1}}{(k-1)^{k-1}} < \frac{n^{3(k-1)}n^5}{3^{k-1}},$$

$$2\frac{\ell-k}{k-1} < \frac{n^3}{3}n^{5/(k-1)}.$$

Observing that $(k-1)n^{5/(k-1)} < n^3$ when $k \geq 3$ and $n \geq 10$, we have

$$2\ell < \frac{n^6 + 2n}{3} < n^6.$$

Case 3. Finally we assume that $k < n/3 - 1$ and that k is even. Again Lemma 4.1 and Frobenius reciprocity imply that $\dim(M) \leq [H : C_k] \leq \frac{n!}{2(3^k)(n-3k-2)!}$. Combining with (1) we get:

$$2^k\binom{\ell}{k} \leq \frac{n!}{3^k(n-3k-2)!},$$

$$2^k \frac{(\ell - k + 1)^k}{k^k} < \frac{n^{3k+2}}{3^k} = \frac{n^{3k}}{3^k} n^2,$$
$$2\frac{\ell - k}{k} < \frac{n^3}{3} n^{2/k},$$
$$2\ell < \frac{n^5 + 3n}{9}.$$

In all cases, $2\ell < n^6$, which implies that $\dim(\overline{V}) \leq n^6$. This completes the proof of Theorem 1.2. □

Acknowledgements. The author would like to thank Prof. Kay Magaard and Prof. Robert L. Griess for their help during the preparation of this article, and the referee for suggested improvements, particularly for Lemma 2.5.

References

[1] R.W. Carter, *Simple groups of Lie type*, Wiley-Interscience, 1989.

[2] W. Fulton and J. Harris, *Representation theory*, Springer-Verlag, 1991.

[3] J.E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, 1972.

[4] G.D. James and A. Kerber, *The Representation Theory of the Symmetric Groups*, Encyclopedia of Math. and its Appl., Vol. 16, Addison-Wesley, 1981.

[5] A. Premet, *Weights of infinitesimally irreducible representations of Chevalley groups over a field of prime characteristic*, Math. USSR Sbornik, **61** (1988), 167-183 (English translation).

[6] R. Steinberg, *Lectures on Chevalley Groups*, Yale University Mathematics Department, 1968.

Received June 16, 1998 and revised November 11, 1998. This research was done at Wayne State University, Detroit MI, 48202.

Ohio State University
Columbus, OH 43210
E-mail address: husen@math.ohio-state.edu

This paper is available via http://nyjm.albany.edu:8000/PacJ/2000/192-2-7.html.