

HARDY'S UNCERTAINTY PRINCIPLE ON SEMISIMPLE GROUPS

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A theorem of Hardy states that, if f is a function on \mathbb{R} such that $|f(x)| \leq C e^{-\alpha|x|^2}$ for all x in \mathbb{R} and $|\hat{f}(\xi)| \leq C e^{-\beta|\xi|^2}$ for all ξ in \mathbb{R} , where $\alpha > 0$, $\beta > 0$, and $\alpha\beta > 1/4$, then $f = 0$. Sitaram and Sundari generalised this theorem to semisimple groups with one conjugacy class of Cartan subgroups and to the K -invariant case for general semisimple groups. We extend the theorem to all semisimple groups.

1. Introduction.

The Uncertainty Principle states, roughly speaking, that a nonzero function f and its Fourier transform \hat{f} cannot both be sharply localised. This fact may be manifested in different ways. The version of this phenomenon described in the abstract is due to Hardy [3]; we call it Hardy's Uncertainty Principle. Considerable attention has been devoted recently to discovering new forms of and new contexts for the Uncertainty Principle (see [2] for a recent comprehensive survey). In particular, Sitaram and Sundari [4] generalised Hardy's Uncertainty Principle to connected semisimple Lie groups with one conjugacy class of Cartan subgroups and to the K -invariant case for general connected semisimple Lie groups. We extend the theorem of Sitaram and Sundari [4], and establish a form of Hardy's Uncertainty Principle for all connected semisimple Lie groups with finite centre.

2. The theorem.

Let G be a connected real semisimple Lie group with finite centre. Let KAN be an Iwasawa decomposition of G , and let MAN be the associated minimal parabolic subgroup of G . The Lie algebras of G and A are denoted by \mathfrak{g} and \mathfrak{a} . The Killing form of \mathfrak{g} induces an inner product on \mathfrak{a} and hence on the dual \mathfrak{a}^* ; in both cases the corresponding norms are denoted by $|\cdot|$. Haar measures on K and G are fixed; that on K is normalised so that the total mass of K is 1. Integrals over G and K are relative to these Haar measures.

Any irreducible unitary representation μ of M may be realised as the left-translation representation on a finite-dimensional subspace \mathcal{H}_μ of $C(M)$, the space of continuous complex-valued functions on M . For such a μ , and λ in

the complexification $\mathfrak{a}_\mathbb{C}^*$ of \mathfrak{a}^* , we define the space $\mathcal{H}_{\mu,\lambda}^0$ to be the subspace of $C(G)$ of all functions ξ with the properties that

$$\xi(gan) = \xi(g) \exp((i\lambda - \rho) \log a) \quad \forall g \in G \quad \forall a \in A \quad \forall n \in N$$

and

$$m \mapsto \xi(gm) \in \mathcal{H}_\mu \quad \forall g \in G.$$

Note that such functions are determined by their restrictions to K , i.e., effectively we are dealing with a subspace of $C(K)$. The representation $\pi_{\mu,\lambda}^0$ of G is the left-translation representation of G on this space. We define the inner product $\langle \xi, \eta \rangle$ of ξ and η in $\mathcal{H}_{\mu,\lambda}^0$ to be

$$\int_K \xi(k) \bar{\eta}(k) dk;$$

$\| \cdot \|$ denotes the associated norm.

Denote by $\mathcal{H}_{\mu,\lambda}$ the completion of $\mathcal{H}_{\mu,\lambda}^0$ with this norm, and by $\pi_{\mu,\lambda}$ the extension of $\pi_{\mu,\lambda}^0$ to $\mathcal{H}_{\mu,\lambda}$. The space $\mathcal{H}_{\mu,\lambda}$ may be identified with a subspace of $L^2(K)$, and $\mathcal{H}_{\mu,\lambda}^0$ with the space of continuous functions in $\mathcal{H}_{\mu,\lambda}$.

For μ in \widehat{M} and λ in \mathfrak{a}^* , the representation $\pi_{\mu,\lambda}$ is unitary. This representation lifts to a representation of $L^1(G)$ by integration, as follows. First, for f in $L^1(G)$ and ξ and η in $\mathcal{H}_{\mu,\lambda}$, the integral

$$\int_G f(g) \langle \pi_{\mu,\lambda}(g)\xi, \eta \rangle dg$$

converges, to $B_f(\xi, \eta)$ say. Next, B_f is a sesquilinear form on $\mathcal{H}_{\mu,\lambda}$. Thus there exists a unique bounded operator, denoted $\pi_{\mu,\lambda}(f)$, such that

$$\langle \pi_{\mu,\lambda}(f)\xi, \eta \rangle = \int_G f(g) \langle \pi_{\mu,\lambda}(g)\xi, \eta \rangle dg \quad \forall \xi, \eta \in \mathcal{H}_{\mu,\lambda}.$$

We denote by $\| \cdot \|$ the operator norm of such operators, relative to the given norm on $\mathcal{H}_{\mu,\lambda}$. If $\lambda \in \mathfrak{a}_\mathbb{C}^* \setminus \mathfrak{a}^*$, then the matrix coefficients $g \mapsto \langle \pi_{\mu,\lambda}(g)\xi, \eta \rangle$ need not be bounded, and for general f in $L^1(G)$ it may not be possible to define $\pi_{\mu,\lambda}(f)$. However, for f which decays sufficiently rapidly at infinity in G , in particular for f in the theorem below, $\pi_{\mu,\lambda}(f)$ may still be defined by the procedure above.

Theorem. *Suppose that $C, \alpha, C_\mu, \beta_\mu$ are positive constants and $\alpha\beta_\mu > 1/4$ for all μ in \widehat{M} , and that f is a measurable function on G such that*

$$|f(kak')| \leq C \exp(-\alpha |\log a|^2) \quad \forall k, k' \in K \quad \forall a \in A$$

and

$$\| \pi_{\mu,\lambda}(f) \| \leq C_\mu \exp(-\beta_\mu |\lambda|^2) \quad \forall \mu \in \widehat{M} \quad \forall \lambda \in \mathfrak{a}^*.$$

Then $f = 0$.

Proof. Let σ and τ be irreducible representations of K , with characters χ_σ and χ_τ . Define $f_{\sigma,\tau}$ by the formula

$$f_{\sigma,\tau}(g) = \dim \sigma \dim \tau \int_K \int_K \bar{\chi}_\sigma(k) \bar{\chi}_\tau(k') f(kgk') dk dk'.$$

By a straightforward estimate,

$$|f_{\sigma,\tau}(kak')| \leq C (\dim \sigma \dim \tau)^2 \exp(-\alpha |\log a|^2) \quad \forall k, k' \in K \quad \forall a \in A.$$

Further, $\pi_{\mu,\lambda}(f_{\sigma,\tau})$ is the composition $P_\sigma \pi_{\mu,\lambda}(f) P_\tau$, where P_σ and P_τ are the projections of $L^2(K)$ onto the σ -isotypic and τ -isotypic subspaces, so that

$$\|\pi_{\mu,\lambda}(f_{\sigma,\tau})\| \leq C_\mu \exp(-\beta_\mu |\lambda|^2) \quad \forall \mu \in \widehat{M} \quad \forall \lambda \in \mathfrak{a}^*.$$

Now the arguments of Section 3 of [4] show that, if α_μ is chosen such that $0 < \alpha_\mu < \alpha$ and $\alpha_\mu \beta_\mu > 1/4$, then

$$\begin{aligned} \|\pi_{\mu,\lambda}(f_{\sigma,\tau})\| &\leq C_{\sigma,\tau,\mu} \int_G \Phi_{i \operatorname{Re}(\lambda)}(x) |f(x)| dx \\ &\leq C'_{\sigma,\tau,\mu} \exp\left(\frac{|\lambda|^2}{4\alpha_\mu}\right) \quad \forall \mu \in \widehat{M} \quad \forall \lambda \in \mathfrak{a}_\mathbb{C}^*, \end{aligned}$$

where $\Phi_{i \operatorname{Re}(\lambda)}$ denotes the usual elementary spherical function, and hence that

$$\pi_{\mu,\lambda}(f_{\sigma,\tau}) = 0 \quad \forall \mu \in \widehat{M} \quad \forall \lambda \in \mathfrak{a}_\mathbb{C}^*.$$

By Harish-Chandra's subquotient theorem (see G. Warner [5, p. 452]), if π is any irreducible unitary representation of G on a Hilbert space \mathcal{H}_π , then there exist μ in \widehat{M} and λ in $\mathfrak{a}_\mathbb{C}^*$ and closed subspaces S_0 and S_1 of $\mathcal{H}_{\mu,\lambda}$ such that π is Naïmark equivalent to the quotient representation $\dot{\pi}_{\mu,\lambda}$ of $\pi_{\mu,\lambda}$ on S_1/S_0 . This means that there is an intertwining operator A_π with dense domain and range between (π, \mathcal{H}_π) and $(\dot{\pi}_{\mu,\lambda}, S_1/S_0)$. Consequently $\pi(f_{\sigma,\tau}) = 0$, first on the domain of A_π by the intertwining property, and then on all \mathcal{H}_π by continuity. In summary,

$$\langle \pi(f_{\sigma,\tau})\xi, \eta \rangle = 0 \quad \forall \xi, \eta \in \mathcal{H}_\pi,$$

and therefore, summing over σ and τ , we see that

$$\langle \pi(f)\xi, \eta \rangle = 0 \quad \forall \xi, \eta \in \mathcal{H}_\pi.$$

It follows that $\pi(f) = 0$ for all π in \widehat{G} , the unitary dual of G , whence $f = 0$ by the Plancherel theorem. \square

The argument of this paper may also be applied in other contexts. For instance, we may show the following: if f is a measurable function on G , rapidly decreasing in the sense that for any B in \mathbb{R}^+ there exists A in \mathbb{R}^+ such that

$$|f(kak')| \leq A \exp(-\alpha B |\log a|) \quad \forall k, k' \in K \quad \forall a \in A,$$

and if on each principal series induced from the minimal parabolic subgroup, the group-theoretic Fourier transform vanishes on a set of positive Plancherel measure, then f is zero. This is a qualitative uncertainty principle related to [1].

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