PACIFIC JOURNAL OF MATHEMATICS
Vol. 191, No. 1, 1999

SOME SUMMATIONS OF q-SERIES BY TELESCOPING

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Summation formulae for q-series with independent bases are obtained and used to derive transformation and expansion of q-series involving independent bases.

1. Introduction.

The sum of the first $(n+1)$-terms of the non-terminating very-well-poised ${}_6\phi_5[q][1]$

$$(1.1)\qquad \sum_{k=0}^{n} \frac{(1-aq^{2k})}{(1-a)} \frac{(a,b,c,\frac{a}{bc};q)_k}{(q,\frac{aq}{b},\frac{aq}{c},bcq;q)_k} q^k = \frac{(aq,bq,cq,\frac{aq}{bc};q)_n}{(\frac{aq}{b},\frac{aq}{c},bcq,q;q)_n}$$

follows from Jacksons q-analgue of Whipple's summation formula for a terminating very-well-poised balanced ${}_8\phi_7[q]$ (in [4, 2.6.2] setting $e = aq^{n+1}$). A bibasic analogue of (1.1) was obtained by Gasper [3].

$$(1.2)\qquad \sum_{k=0}^{n} \frac{(1-aq^kp^k)(1-bp^kq^{-k})}{(1-a)(1-b)} \frac{(a,b;p)_k(c,\frac{a}{bc};q)_k}{(q,\frac{aq}{b},q)_k(\frac{ap}{c},bcp;p)_k} q^k$$
$$= \frac{(ap,bp;p)_n(cq,\frac{aq}{bc};q)_n}{(q,\frac{aq}{b};q)_n(\frac{ap}{c},bcp;p)_n}, \quad n=0,1,2,\dots$$

and he used it for obtaining quadratic and cubic summation and transformations formulae for q-hypergeometric series. A little later Gasper and Rahman [5] obtained a bilateral extension of Gasper's bibasic summation formula (1.2):

If m,n are non-negative integers, then

$$(1.3)\qquad \sum_{k=-m}^{n} \frac{(1-adp^kq^k)}{(1-ad)} \frac{(1-\frac{bp^kq^{-k}}{d})}{(1-\frac{b}{d})} \frac{(a,b;p)_k}{(dq,\frac{adq}{b};q)_k} \frac{(c,\frac{ad^2}{bc};q)_k}{(\frac{adp}{c},\frac{bcp}{d};p)_k} q^k$$
$$= \frac{(1-a)(1-b)(1-c)(1-\frac{ad^2}{bc})}{d(1-ad)(1-\frac{b}{d})(1-\frac{c}{d})(1-\frac{ad}{bc})}$$
$$\times \left\{ \frac{(ap,bp;p)_n}{(dq,\frac{adq}{b};q)_n} \frac{(cq,\frac{ad^2q}{bc};q)_n}{(\frac{adp}{c},\frac{bcp}{d};q)_n} - \frac{(\frac{c}{ad},\frac{d}{bc};p)_{m+1}(\frac{1}{d},\frac{b}{ad};q)_{m+1}}{(\frac{1}{c},\frac{bc}{ad^2};q)_{m+1}(\frac{1}{a},\frac{1}{b};p)_{m+1}} \right\}.$$

Jain and Verma [6] used transformations of q-hypergeometric series to obtain a summation formula involving three independent bases:

(1.4)
$$\sum_{k=-m}^{n} \frac{(\beta;p)_k(c;q)_k(y;p)_k(\frac{\beta yc}{d^2};\frac{pP}{q})_k[(1-\frac{\beta cy}{d}p^kP^k)(1-\frac{y}{d}P^kq^{-k})(1-\frac{\beta}{d}p^kq^{-k})]q^k}{(dq;q)_k(\frac{\beta cp}{d};p)_k(\frac{\beta y}{d}\frac{pP}{q};\frac{pP}{q})_k(\frac{cyP}{d^2};P)_k}$$
$$= \frac{(1-\beta)(1-c)(1-y)(1-\frac{\beta cy}{d^2})}{(c-d)} \left\{ \frac{(\frac{d}{\beta c};p)_{m+1}(\frac{1}{d};q)_{m+1}(\frac{d}{cy};P)_{m+1}(\frac{d}{\beta y};\frac{pP}{q})_{m+1}}{(\frac{1}{\beta};p)_{m+1}(\frac{1}{c};q)_{m+1}(\frac{1}{y};P)_{m+1}(\frac{d^2}{\beta cy};\frac{pP}{q})_{m+1}} \right.$$
$$\left. - \frac{(\beta p;p)_n(cq;q)_n(yP;P)_n(\frac{\beta cy}{d^2}\frac{pP}{q};\frac{pP}{q})_n}{(\frac{\beta cp}{d};p)_n(dq;q)_n(\frac{cyP}{d};P)_n(\frac{\beta y}{d}\frac{pP}{q};\frac{pP}{q})_n} \right\},$$

which for $P=q$ reduces to the Gasper-Rahman's summation formula (1.3). The proof of (1.4) could be given by considering

$$\beta_k = \frac{(\beta p;p)_k(cq;q)_k(yP;P)_k(\frac{\beta cy}{d^2}\frac{pP}{q};\frac{pP}{q})_k}{(\frac{\beta c}{d}p;p)_k(dq;q)_k(\frac{cy}{d}P;P)_k(\frac{\beta y}{d}\frac{pP}{q};\frac{pP}{q})_k}$$

and observing that

$$\Delta\beta_k = (\beta p;p)_{k-1}(cq;q)_{k-1}(yP;P)_{k-1}\left(\frac{\beta yc}{d^2}\frac{pP}{q};\frac{pP}{q}\right)_{k-1} \times \frac{\left[(1-\frac{\beta cy}{d^2}p^kP^k)(1-\frac{y}{d}P^kq^{-k})(1-\frac{\beta}{d}p^kq^{-k})\right]}{(\frac{\beta cp}{d};p)_k(dq;q)_k(\frac{\beta y}{d}\frac{pP}{q};\frac{pP}{q})_k(\frac{cyP}{d};P)_k}, \tag{1.5}$$

and summing for k from $-m$ to n (m,n are non-negative integers) and using the usual convention:

$$\prod_{k=m}^{n} A_k = \begin{cases} A_mA_{m+1}\cdots A_n & m\le n \\ 1 & m=n-1 \\ (A_{n+1}A_{n+2}\cdots A_{m-1})^{-1} & m\ge n-2. \end{cases} \tag{1.6}$$

Chu [2] obtained a generalization of Gasper-Rahman's formula (after renaming suitably the sequences so as to remove redundant sequences)

(1.7)
$$\sum_{k=-m}^{n} \frac{(1-\alpha a_kb_k)(b_k-\frac{a_k}{\alpha d})}{(1-\alpha a_0b_0)(b_0-\frac{a_0}{d\alpha})} \frac{\prod_{j=0}^{k-1}[(1-a_j)(1-\frac{a_j}{d})(1-cb_j)(1-\frac{\alpha^2 d}{c}b_j)]}{\prod_{j=1}^{k}[(1-\alpha b_j)(1-\alpha db_j)(1-\frac{\alpha a_j}{c})(1-\frac{c}{d\alpha}a_j)]}$$
$$= \frac{(1-a_0)(1-\frac{a_0}{d})(1-b_0c)(1-\frac{\alpha^2 d}{c}b_0)}{\alpha(1-\alpha a_0b_0)(1-\frac{c}{\alpha})(b_0-\frac{a_0}{d\alpha})(1-\frac{\alpha d}{c})}$$

$$\times \left\{ \prod_{j=1}^{n} \left[\frac{(1-a_j)(1-\frac{a_j}{d})(1-b_jc)(1-\frac{\alpha^2 d}{c}b_j)}{(1-\alpha b_j)(1-\alpha b_j d)(1-\frac{\alpha a_j}{c})(1-\frac{c}{\alpha d}\,a_j)} \right] - \prod_{j=-m}^{0} \left[\frac{(1-\alpha b_j)(1-\alpha d b_j)(1-\frac{\alpha a_j}{c})(1-\frac{c}{\alpha d}\,a_j)}{(1-a_j)(1-\frac{a_j}{d})(1-cb_j)(1-\frac{\alpha^2 d}{c}\,b_j)} \right] \right\}$$

where $\langle a_j\rangle$ and $\langle b_j\rangle$ are arbitrary sequences such that none of the terms in the denominators vanish. This reduces to the Gasper-Rahman summation formula on setting $a_k = ap^k$, $b_k = q^k$ and replacing α and d by d and a/b, respectively.

In this paper we obtain in §2 a generalization of Chu's summation formula (1.7) involving four arbitrary sequences, which on specialization yields an extension of (1.1) to a summation formula with four independent bases p, q, P and Q and incorporating (1.7) as a special case. An expansion of the series $\sum_{n=0}^{\infty} A_n B_n (wx)^n$ into a series involving three independent bases is developed. A transformation of a series involving eight independent sequences is also developed. The note is concluded by obtaining in §3 some summation formulas which are different from the known ones by telescoping of series including q-Paff-Saalschütz's summation formula for a terminating balanced ${}_3\phi_2[q]$.

§2.

We begin this section by proving the summation formula:

If $\langle u_k\rangle$, $\langle v_k\rangle$, $\langle w_k\rangle$ and $\langle z_k\rangle$ are arbitrary sequences such that none of the terms in the denominators vanish and M, N are non-negative integers then

$$\sum_{k=-m}^{n} \frac{(1-u_kv_kw_kz_k)(1-\frac{w_kz_k}{u_kv_k})(1-\frac{v_kz_k}{u_kw_j})(1-\frac{u_kz_k}{v_kw_k})}{(1-u_0v_0w_0z_0)(1-\frac{w_0z_0}{u_0v_0})(1-\frac{v_0z_0}{u_0w_0})(1-\frac{u_0z_0}{v_0w_0})} \times \frac{\prod_{j=0}^{k-1}[(1-u_j^2)(1-v_j^2)(1-w_j^2)(1-z_j^2)](\frac{u_kv_kw_k}{z_k})}{\prod_{j=1}^{k}[(1-\frac{v_jw_jz_j}{u_j})(1-\frac{u_jw_jz_j}{v_j})(1-\frac{u_jv_jz_j}{w_j})(1-\frac{u_jv_jw_j}{z_j})]} \tag{2.1}$$

$$= \frac{(1-u_0^2)(1-v_0^2)(1-w_0^2)(1-z_0^2)}{(1-u_0v_0w_0z_0)(1-\frac{w_0z_0}{u_0v_0})(1-\frac{v_0z_0}{u_0w_0})(1-\frac{u_0z_0}{v_0w_0})} \times \left\{ \prod_{j=1}^{n} \left[\frac{(1-u_j^2)(1-v_j^2)(1-w_j^2)(1-z_j^2)}{(1-\frac{v_jw_jz_j}{u_j})(1-\frac{u_jw_jz_j}{v_j})(1-\frac{u_jv_jz_j}{w_j})(1-\frac{u_jv_jw_j}{z_j})} \right] \right.$$

$$- \prod_{j=-m}^{0} \left[\frac{(1-\frac{v_jw_jz_j}{u_j})(1-\frac{u_jw_jz_j}{v_j})(1-\frac{u_jv_jz_j}{w_j})(1-\frac{u_jv_jw_j}{z_j})}{(1-u_j^2)(1-v_j^2)(1-w_j^2)(1-z_j^2)} \right] \Bigg\}.$$

Proof. Let

$$\tau_k = \prod_{j=1}^{k} \left[\frac{(1-u_j^2)(1-v_j^2)(1-w_j^2)(1-z_j^2)}{(1-\frac{v_jw_jz_j}{u_j})(1-\frac{u_jw_jz_j}{v_j})(1-\frac{u_jv_jz_j}{w_j})(1-\frac{u_jv_jw_j}{z_j})} \right].$$

Then by straight forward calculations, we get

$$\Delta\tau_k = \frac{\prod_{j=1}^{k-1} \Big[(1-u_j^2)(1-v_j^2)(1-w_j^2)(1-z_j^2)\Big]}{\prod_{j=1}^{k} \Big[(1-\frac{v_jw_jz_j}{u_j})(1-\frac{u_jw_jz_j}{v_j})(1-\frac{u_jv_jz_j}{w_j})(1-\frac{u_jv_jw_j}{z_j})\Big] (\frac{z_k}{u_kv_kw_k})}$$
$$\times \left[(1-u_kv_kw_kz_k)\left(1-\frac{w_kz_k}{u_kv_k}\right)\left(1-\frac{v_kz_k}{u_kw_k}\right)\left(1-\frac{u_kz_k}{v_kw_k}\right)\right]$$

where $\Delta\tau_k = \tau_k - \tau_{k-1}$. Now summing with respect to k from $-m$ to n, and using the fact that $\sum_{k=-m}^{n} \tau_k = \tau_n - \tau_{-m-1}$ and keeping in mind (1.6), we get (2.1) on simplification.

By setting $u_j = \sqrt{a}\, p^j$, $v_j = \sqrt{c}\, q^j$, $w_j = \sqrt{b}\, P^j$, $z_j = d\sqrt{\frac{a}{bc}}\, Q^j$ in (2.1), we get a summation formula involving four independent bases:

$$(2.2) \qquad \sum_{k=-m}^{n} \frac{(1-adp^kq^kP^kQ^k)(c-\frac{dP^kQ^k}{p^kq^k})(1-\frac{bp^kP^k}{dq^kQ^k})(1-\frac{ad}{bc}\,\frac{p^kQ^k}{q^kP^k})q^{2k}}{(1-ad)(c-d)(1-\frac{b}{d})(1-\frac{ad}{bc})}$$
$$\times \frac{(a;p^2)_k(c;q^2)_k(b;P^2)_k(\frac{ad^2}{bc};Q^2)_k}{(d\,\frac{qPQ}{p};\frac{qPQ}{p})_k(\frac{ad}{c}\,\frac{pPQ}{q};\frac{pPQ}{q})_k(\frac{ad}{b}\,\frac{pqQ}{P};\frac{pqQ}{P})_k(\frac{bc}{d}\,\frac{pqP}{Q};\frac{pqP}{Q})_k}$$
$$= \frac{(1-a)(1-b)(1-c)(1-\frac{ad^2}{bc})}{(1-ad)(c-d)(1-\frac{b}{d})(1-\frac{ad}{bc})}$$
$$\times \left\{ \frac{(ap^2;p^2)_n(cq^2;q^2)_n(bP^2;P^2)_n(\frac{ad^2}{bc}\,Q^2;Q^2)_n}{(\frac{dqPQ}{p};\frac{qPQ}{p})_n(\frac{ad}{c}\,\frac{pPQ}{q};\frac{pPQ}{q})_n(\frac{ad}{b}\,\frac{pqQ}{P};\frac{pqQ}{P})_n(\frac{bc}{d}\,\frac{pqP}{Q};\frac{pqP}{Q})_n} \right.$$
$$\left. - \frac{(\frac{1}{d};\frac{qPQ}{p})_{m+1}(\frac{c}{ad};\frac{pPQ}{q})_{m+1}(\frac{b}{ad};\frac{pqQ}{P})_{m+1}(\frac{d}{bc};\frac{pqP}{Q})_{m+1}}{(\frac{1}{a};p^2)_{m+1}(\frac{1}{c};q^2)_{m+1}(\frac{1}{b};P^2)_{m+1}(\frac{bc}{ad};Q^2)_{m+1}} \right\}.$$

Summation formula (2.2), on setting $Q = \frac{pP}{q}$ and replacing p^2, q^2, P^2, a, b, d by p, q, P, β, y, $\frac{cy}{d}$, respectively, reduces to the summation formula (1.4), which in turn incorporates (1.3) and (1.2) as special cases.

It may be pointed out that (2.2) reduces to Chu's summation formula (1.7) on setting $u_j=\sqrt{a_j}$, $v_j=\sqrt{a_j}/\sqrt{d}$, $w_j=\sqrt{cb_j}$, $z_j=\alpha\sqrt{d/c}\,\sqrt{b_j}$.

Setting $m=0$ in (2.2), replacing z_i by αz_i and setting $\alpha=\frac{u_0}{v_0w_0z_0}$, we get that

(2.3)

$$\sum_{j=0}^{n}\frac{(\frac{v_0w_0z_0}{u_0}-u_jv_jw_jz_j)(\frac{v_0w_0z_0}{u_0}-\frac{w_jz_j}{u_jv_j})(\frac{v_0w_0z_0}{u_0}-\frac{v_jz_j}{u_jw_j})(\frac{v_0w_0z_0}{u_0}-\frac{u_jz_j}{v_jw_j})\frac{u_jv_jw_j}{z_j}}{(1-u_0^2)(1-v_0^2)(1-w_0^2)(v_0^2w_0^2-u_0^2)}$$
$$\times\frac{\prod_{i=0}^{j-1}[(1-u_i^2)(1-v_i^2)(1-w_i^2)(\frac{v_0^2w_0^2z_0^2}{u_0^2}-z_i^2)]}{\prod_{i=1}^{k}[(\frac{v_0w_0z_0}{u_0}-\frac{v_iw_iz_i}{u_i})(\frac{v_0w_0z_0}{u_0}-\frac{u_iw_iz_i}{v_i})(\frac{v_0w_0z_0}{u_0}-\frac{u_iv_iz_i}{w_i})(\frac{u_0}{v_0w_0z_0}-\frac{u_iv_iw_i}{z_i})]}$$
$$=\frac{z_0^3}{v_0w_0u_0^3}$$
$$\times\prod_{i=1}^{n}\left[\frac{(1-u_i^2)(1-v_i^2)(1-w_i^2)(\frac{v_0^2w_0^2z_0^2}{u_0^2}-z_i^2)}{(\frac{v_ow_0z_0}{u_0}-\frac{v_iw_iz_i}{u_i})(\frac{v_0w_0z_0}{u_0}-\frac{u_iw_iz_i}{v_i})(\frac{v_0w_0z_0}{u_0}-\frac{u_iv_iz_i}{w_i})(\frac{u_0}{v_0w_0z_0}-\frac{u_iv_iw_i}{z_i})}\right].$$

Next, using (2.3), the following transformation involving eight arbitrary sequences is obtained:

(2.4)

$$\sum_{k=0}^{n}\prod_{i=1}^{n-k}\left[\frac{(1-U_i^2)(1-V_i^2)(1-W_i^2)(\frac{V_0^2W_0^2Z_0^2}{U_0^2}-Z_i^2)}{(\frac{V_0W_0Z_0}{U_0}-\frac{V_iW_iZ_i}{U_i})(\frac{V_0W_0Z_0}{U_0}-\frac{U_iW_iZ_i}{V_i})(\frac{V_0W_0Z_0}{U_0}-\frac{U_iV_iZ_i}{W_i})(\frac{U_0}{V_0W_0Z_0}-\frac{U_iV_iW_i}{Z_i})}\right]$$
$$\times\frac{(\frac{v_0w_0z_0}{u_0}-u_kv_kw_kz_k)(\frac{v_0w_0z_0}{u_0}-\frac{w_kz_k}{u_kv_k})(\frac{v_0w_0z_0}{u_0}-\frac{v_kz_k}{u_kw_k})(\frac{v_0w_0z_0}{u_0}-\frac{u_kz_k}{v_kw_k})\frac{u_kv_kw_k}{z_k}}{(1-u_0^2)(1-v_0^2)(1-w_0^2)(v_0^2w_0^2-u_0^2)}$$
$$\times\frac{\prod_{i=0}^{k-1}[(1-u_i^2)(1-v_i^2)(1-w_i^2)(\frac{v_0^2w_0^2z_0^2}{u_0^2}-z_i^2)]}{\prod_{i=1}^{k}[(\frac{v_0w_0z_0}{u_0}-\frac{v_iw_iz_i}{u_i})(\frac{v_0w_0z_0}{u_0}-\frac{u_iw_iz_i}{v_i})(\frac{v_0w_0z_0}{u_0}-\frac{u_iv_iz_i}{w_i})(\frac{u_0}{v_0w_0z_0}-\frac{u_iv_iw_i}{z_i})]}$$
$$=\frac{V_0W_0U_0^3}{Z_0^3}\,\frac{z_0^3}{v_0w_0u_0^3}$$
$$\times\prod_{i=1}^{n}\left[\frac{(1-u_i^2)(1-v_i^2)(1-w_i^2)(\frac{v_0^2w_0^2z_0^2}{u_0^2}-z_i^2)}{(\frac{v_0w_0z_0}{u_0}-\frac{v_iw_iz_i}{u_i})(\frac{v_0w_0z_0}{u_0}-\frac{u_iw_iz_i}{v_i})(\frac{v_0w_0z_0}{u_0}-\frac{u_iv_iz_i}{w_i})(\frac{u_0}{v_0w_0z_0}-\frac{u_iv_iw_i}{z_i})}\right]$$
$$\times\sum_{j=0}^{n}\frac{(\frac{V_0W_0Z_0}{U_0}-U_jV_jW_jZ_j)(\frac{V_0W_0Z_0}{U_0}-\frac{W_jZ_j}{U_jV_j})(\frac{V_0W_0Z_0}{U_0}-\frac{V_jZ_j}{U_jW_j})(\frac{V_0W_0Z_0}{U_0}-\frac{U_jZ_j}{V_jW_j})}{(1-U_0^2)(1-V_0^2)(1-W_0^2)(V_0^2W_0^2-U_0^2)}$$

$$\times \frac{U_jV_jW_j}{Z_j}$$

$$\times \prod_{i=0}^{j-1}\Big[(1-U_i^2)(1-V_i^2)(1-W_i^2)\left(\frac{V_0^2W_0^2Z_0^2}{U_0^2}-Z_i^2\right)$$

$$\times\left(\frac{v_0w_0z_0}{u_0}-\frac{v_{n-i}w_{n-i}z_{n-i}}{u_{n-i}}\right)\left(\frac{v_0w_0z_0}{u_0}-\frac{u_{n-i}w_{n-i}z_{n-i}}{v_{n-i}}\right)\Big]$$

$$\times \frac{1}{\prod_{i=0}^{j}\left[\left(\frac{V_0W_0Z_0}{U_0}-\frac{U_iW_iZ_i}{U_i}\right)\left(\frac{V_0W_0Z_0}{U_0}-\frac{U_iW_iZ_i}{V_i}\right)\left(\frac{V_0W_0Z_0}{U_0}-\frac{U_iV_iZ_i}{W_i}\right)\left(\frac{U_0}{V_0W_0Z_0}-\frac{U_iV_iW_i}{Z_i}\right)\right]}$$

$$\times \frac{\prod_{i=0}^{j-1}\left[\left(\frac{v_0w_0z_0}{u_0}-\frac{u_{n-i}v_{n-i}z_{n-i}}{w_{n-i}}\right)\left(\frac{u_0}{v_0w_0z_0}-\frac{u_{n-i}v_{n-i}w_{n-i}}{z_{n-i}}\right)\right]}{\prod_{i=0}^{j}\left[(1-u_{1+n-i}^2)(1-v_{1+n-i}^2)(1-w_{1+n-i}^2)\left(\frac{v_0^2w_0^2z_0^2}{u_0^2}-z_{1+n-i}^2\right)\right]}.$$

Transformation (2.4) can be proved by expanding the first product on the left hand side by using (2.3) (with n replaced by $n-k$), interchanging the order of summations and evaluating the inner sum by using (2.3) once again and simplifying to get the right hand side of (2.4).

Transformation (2.4), on replacing $u_i, v_i, w_i, z_i, U_i, V_i, W_i, Z_i$ by $\sqrt{a}\,p^i$, $\sqrt{c}\,q^i$, $\sqrt{b}\,P^i$, Q^i, $\sqrt{A}\,\widetilde{p}^i$, $\sqrt{C}\,\widetilde{q}^i$, $\sqrt{B}\,\widetilde{P}^i$, $\widetilde{Q}^i$, respectively, reduces on some simplification to the following transformation of q-series involving eight independent bases:

$$(2.5)\quad \sum_{k=0}^{n}\frac{(1-ap^kq^kP^kQ^k)(\frac{q^kP^kQ^k}{p^k}-cq^{2k})(1-b\,\frac{p^kP^k}{q^kQ^k})(1-\frac{a}{bc}\,\frac{p^kQ^k}{q^kP^k})}{(1-a)(1-c)(1-b)(1-\frac{a}{bc})}$$

$$\times\frac{(a;p^2)_k(c;q^2)_k(b;P^2)_k(\frac{a}{bc};Q^2)_k}{(\frac{qPQ}{p};\frac{qPQ}{p})_k(\frac{a}{c}\,\frac{pPQ}{q};\frac{pPQ}{q})_k(\frac{a}{b}\,\frac{pqQ}{P};\frac{pqQ}{P})_k(bc\,\frac{pqP}{Q};\frac{pqP}{Q})_k}$$

$$\times\frac{(\frac{\widetilde{q}^{-n}\widetilde{P}^{-n}\widetilde{Q}^{-n}}{\widetilde{p}^{-n}};\frac{\widetilde{q}\widetilde{P}\widetilde{Q}}{\widetilde{p}})_k(\frac{C}{A}\,\frac{\widetilde{p}^{-n}\widetilde{P}^{-n}\widetilde{Q}^{-n}}{\widetilde{q}^{-n}};\frac{\widetilde{p}\widetilde{P}\widetilde{Q}}{\widetilde{q}})_k(\frac{B}{A}\,\frac{\widetilde{p}^{-n}\widetilde{q}^{-n}\widetilde{Q}^{-n}}{\widetilde{P}^{-n}};\frac{\widetilde{p}\widetilde{q}\widetilde{Q}}{\widetilde{P}})_k}{(\frac{\widetilde{p}^{-2n}}{A};\widetilde{p}^2)_k(\frac{\widetilde{q}^{-2n}}{C};\widetilde{q}^2)_k(\frac{\widetilde{P}^{-2n}}{B};\widetilde{P}^2)_k(\frac{BC}{A}\,\widetilde{Q}^{-2n};\widetilde{Q}^2)_k}$$

$$\times\left(\frac{\widetilde{p}^{-n}\widetilde{q}^{-n}\widetilde{P}^{-n}}{BC\widetilde{Q}^{-n}};\frac{\widetilde{p}\widetilde{q}\widetilde{P}}{\widetilde{Q}}\right)_k$$

$$=\frac{(ap^2;p^2)_n(cq^2;q^2)_n(bP^2;P^2)_n(\frac{aQ^2}{bc};Q^2)_n}{(\frac{qPQ}{p};\frac{qPQ}{p})_n(\frac{a}{c}\,\frac{pPQ}{q};\frac{pPQ}{q})_n(\frac{a}{b}\,\frac{pqQ}{P};\frac{pqQ}{P})_n(bc\,\frac{pqP}{Q};\frac{pqP}{Q})_n}$$

$$\times \frac{(\frac{\widetilde{q}\widetilde{P}\widetilde{Q}}{\widetilde{p}};\frac{\widetilde{q}\widetilde{P}\widetilde{Q}}{\widetilde{p}})_n(\frac{A}{C}\,\frac{\widetilde{p}\widetilde{P}\widetilde{Q}}{\widetilde{q}};\frac{\widetilde{p}\widetilde{P}\widetilde{Q}}{\widetilde{q}})_n(\frac{A}{B}\,\frac{\widetilde{p}\widetilde{q}\widetilde{Q}}{\widetilde{P}};\frac{\widetilde{p}\widetilde{q}\widetilde{Q}}{\widetilde{P}})_n(BC\,\frac{\widetilde{p}\widetilde{q}\widetilde{P}}{\widetilde{Q}};\frac{\widetilde{p}\widetilde{q}\widetilde{P}}{\widetilde{Q}})_n}{(a\widetilde{p}^2;\widetilde{p}^2)_n(C\widetilde{q}^2;\widetilde{q}^2)_n(B\widetilde{P}^2;\widetilde{P}^2)_n(\frac{A\widetilde{Q}^2}{BC};\widetilde{Q}^2)_n}$$

$$\times \sum_{j=0}^{n} \frac{(1-A\widetilde{p}^j\widetilde{q}^j\widetilde{P}^j\widetilde{Q}^j)(\frac{\widetilde{q}^j\widetilde{P}^j\widetilde{Q}^j}{\widetilde{p}^j}-C\widetilde{q}^{2j})(1-\frac{B\widetilde{p}^j\widetilde{P}^j}{\widetilde{q}^j\widetilde{Q}^j})(1-\frac{A}{BC}\,\frac{\widetilde{p}^j\widetilde{Q}^j}{\widetilde{q}^j\widetilde{P}^j})}{(1-A)(1-C)(1-B)(1-\frac{A}{BC})}$$

$$\times \frac{(A;\widetilde{p}^2)_j(C;\widetilde{q}^2)_j(B;\widetilde{P}^2)_j(\frac{A}{BC};\widetilde{Q}^2)_j}{(\frac{\widetilde{q}\widetilde{P}\widetilde{Q}}{\widetilde{p}};\frac{\widetilde{q}\widetilde{P}\widetilde{Q}}{\widetilde{p}})_j(\frac{A}{C}\,\frac{\widetilde{p}\widetilde{P}\widetilde{Q}}{\widetilde{q}};\frac{\widetilde{p}\widetilde{P}\widetilde{Q}}{\widetilde{q}})_j(\frac{A}{B}\,\frac{\widetilde{p}\widetilde{q}\widetilde{Q}}{\widetilde{P}};\frac{\widetilde{p}\widetilde{q}\widetilde{Q}}{\widetilde{P}})_j(BC\,\frac{\widetilde{p}\widetilde{q}\widetilde{P}}{\widetilde{Q}};\frac{\widetilde{p}\widetilde{q}\widetilde{P}}{\widetilde{Q}})_j}$$

$$\times \frac{(\frac{q^{-n}P^{-n}Q^{-n}}{p^{-n}};\frac{qPQ}{P})_j(\frac{c}{a}\,\frac{p^{-n}P^{-n}Q^{-n}}{q^{-n}};\frac{pPQ}{q})_j(\frac{b}{a}\,\frac{p^{-n}q^{-n}Q^{-n}}{P^{-n}};\frac{pqQ}{P})_j}{(\frac{p^{-2n}}{a};p^2)_j(\frac{q^{-2n}}{c};q^2)_j(\frac{P^{-2n}}{b};P^2)_j(\frac{bcQ^{-2n}}{a};Q^2)_j}$$

$$\times \left(\frac{p^{-n}q^{-n}P^{-n}}{bcQ^{-n}};\frac{pqP}{Q}\right)_j.$$

Transformation (2.5) is a generalization of Gasper's [4, Ex. 3.21] quadbasic transformation (from which Gasper deduces a transformation of a half-poised ${}_{10}\phi_9$ into another half-poised ${}_{10}\phi_9$ [4, Ex. 3.24]) to which it reduces on setting $Q=q,\ P=p,\ \widetilde{P}=\widetilde{p},\ \widetilde{Q}=\widetilde{q}$ and then replacing $p^2,\ q^2,\ \widetilde{p}^2$ and $\widetilde{q}^2$ by p,q,P and Q, respectively.

Next, we obtain an expansion of $\sum_{r=0}^{\infty} A_rB_r(xw)^r/(q;q)_r$ terms of q-series having three independent bases. In the summation formula (2.2), setting $m=0,\ c=q^{-2n}$, where n is a non-negative integer and letting $d\to 1$, we get

$$\sum_{j=0}^{n} \frac{(1-ap^jq^jP^jQ^j)(\frac{q^jP^jQ^j}{p^j}-q^{2j-2n})(1-\frac{bp^jP^j}{q^jQ^j})(1-\frac{a}{b}\,\frac{p^jQ^j}{q^{j-2n}P^j})}{(1-a)(1-q^{-2n})(1-b)(1-\frac{a}{b}\,q^{2n})}$$
$$\times \frac{(a;p^2)_j(q^{-2n};q^2)_j(b;P^2)_j(\frac{a}{b}\,q^{2n};Q^2)_j}{(\frac{qPQ}{p};\frac{qPQ}{p})_j(\frac{apPQ}{q^{1-2n}};\frac{pPQ}{q})_j(\frac{a}{b}\,\frac{pqQ}{P};\frac{pqQ}{P})_j(\frac{bpq^{1-2n}P}{Q};\frac{pqP}{Q})_j}=\delta_{n,0}.$$

Replacing a and b by $ap^rq^rP^rQ^r$ and $bp^rq^{-r}P^rQ^{-r}$ respectively, where r is a non-negative integer, setting $Q=\frac{pq}{P}$ and then replacing $p^2,\ q^2,\ P^2,\ n$ by $p,\ q,\ P,\ m$, respectively, we get

(2.6)
$$\delta_{m,0}=\sum_{j=0}^{m} \frac{(1-ap^{j+r}q^{j+r})(1-bP^{j+r}q^{-j-r})(1-\frac{a}{b}\,p^{r+j}q^{2r+m}P^{-r-j})q^j}{(1-ap^rq^r)(1-bP^rq^{-r})(1-\frac{a}{b}\,p^rq^{2r+m}P^{-r})}$$
$$\times \frac{(ap^rq^r;p)_j(q^{-m};q)_j(bP^rq^{-r};P)_j(\frac{a}{b}\,p^rq^{2r+m}P^{-r};\frac{pq}{P})_j}{(q;q)_j(ap^{1+r}q^{r+m};p)_j(\frac{a}{b}\,p^{r+1}q^{1+2r}P^{-r-1};\frac{pq}{P})_j(bq^{-m-r}P^{r+1};P)_j}.$$

But, we know that

$$B_r x^r = \sum_{m=0}^{\infty} \frac{(1-\frac{a}{b}\,p^{r+m}q^{2r+2m}P^{-r-m})(\frac{a}{b}\,p^r q^{2r}P^{-r};\frac{pq}{P})_m(apq^r;p)_r(bPq^{-r};P)_r}{(1-\frac{a}{b}\,p^r q^{2r}P^{-r})(q;q)_m(apq^{r+m};p)_r(bPq^{-m-r};P)_r} \times \{q^{-rm}B_{r+m}C_{r,m}x^{r+m}\delta_{m,0}\},$$

where $\langle B_r\rangle$ and $\langle C_{r,m}\rangle$ are arbitrary sequences of complex numbers such that $C_{r,0}=1$ for $r=0,1,2,\ldots$ Substituting for $\delta_{m,0}$ from (2.6), interchanging the order of summation and setting $j=n-r$ and $m=n+k-r$, we get

$$\begin{aligned} B_r x^r = \sum_{k=0}^{\infty}\sum_{n=r}^{\infty} &(-)^n(1-ap^nq^n)(apq^r;p)_{n-1}(1-bP^nq^{-n})(bPq^{-r};P)_{n-1} \\ &\times \frac{(1-\frac{a}{b}\,p^{n+k}q^{2n+2k}P^{-n-k})x^{n+k}(q^{-n};q)_r}{(apq^{n+k};p)_n(bPq^{-n-k};P)_n(q;q)_n(q;q)_k} \\ &\left\{\left(1-\frac{a}{b}\,p^nq^{n+k+r}P^{-n}\right)\left(\frac{a}{b}\,\frac{p^{n+1}q^{n+r+1}}{P^{n+1}};\frac{pq}{P}\right)_{k-1}\right. \\ &\left.\times\left(\frac{a}{b}\,p^{r+1}q^{n+k+r+1}P^{-r-1};\frac{pq}{P}\right)_{n-r-1}B_{n+k}C_{r,n+k-r}q^{n(r-k-n)+\frac{n^2}{2}+\frac{n}{2}}\right\}. \end{aligned}$$

Multiplying both sides by $\frac{A_r w^r}{(q;q)_r}$ and summing from $r=0$ to ∞ and interchanging the order of summation on the right hand side, we get

$$\begin{aligned} &\sum_{r=0}^{\infty} A_r B_r\,\frac{(xw)^r}{(q;q)_r} \\ &= \sum_{n=0}^{\infty}\frac{(1-ap^nq^n)(1-bP^nq^{-n})(-x)^nq^{\binom{n}{2}+n}}{(q;q)_n} \\ &\quad\times\sum_{k=0}^{\infty}\frac{(1-\frac{a}{b}\,\frac{p^{n+k}q^{2n+2k}}{P^{n+k}})x^kB_{n+k}}{(q;q)_k(apq^{n+k};p)_n(bPq^{-n-k};P)_n} \\ &\quad\times\sum_{r=0}^{\infty}\frac{(q^{-n};q)_r(apq^r;p)_{n-1}(bPq^{-r};P)_{n-1}}{(q;q)_r} \\ &\quad\times\left\{\left(1-\frac{a}{b}\,\frac{p^nq^{n+k+r}}{P^r}\right)\left(\frac{a}{b}\,\frac{p^{n+1}q^{n+r+1}}{P^{n+1}};\frac{pq}{P}\right)_{k-1}\right. \\ &\quad\quad\left.\times\left(\frac{a}{b}\,\frac{p^{r+1}q^{n+k+r+1}}{P^{r+1}};\frac{pq}{P}\right)_{n-r-1}\right\}C_{r,n+k-r}w^nq^{n(r-k-n)}A_r, \end{aligned}$$

which is a generalization of Gasper's bibasic expansion formula [4, (3.7.6)] to which it reduces on setting $P=p$. It may be noted that on setting $P=p$ the terms in $\{\cdots\}$ of the above expression combine to yield $(\frac{a}{b}\,q^{n+r+1};q)_{n+k-r-1}$ as in the bibasic expansion formula of Gasper [4, (3.7.6)].

§3.

All the summation formulae proved so far are for one generalization of (1.1), a very-well-poised q-series. We next derive a summation formula which gives the sum of a balanced series.

Let $\langle x_i\rangle$, $\langle y_i\rangle$ and $\langle z_i\rangle$ be arbitrary sequences and a an indeterminate so that none of the terms in the denominators vanish and m, n are non-negative integers. Then

$$\sum_{k=-m}^{n} \frac{(1-\frac{y_k}{az_k})(1-\frac{x_k}{az_k})z_k}{(1-\frac{y_0}{az_0})(1-\frac{x_0}{az_0})} \frac{\prod_{i=0}^{k-1}[(1-x_i)(1-y_i)]}{\prod_{i=1}^{k}[(1-az_i)(1-\frac{x_iy_i}{az_i})]}$$

$$= \frac{(1-x_0)(1-y_0)}{a(1-\frac{y_0}{az_0})(1-\frac{x_0}{az_0})}$$

$$\times\left\{\prod_{i=1}^{n}\left[\frac{(1-x_i)(1-y_i)}{(1-az_i)(1-\frac{x_iy_i}{az_i})}\right]-\prod_{i=-m}^{0}\left[\frac{(1-az_i)(1-\frac{x_iy_i}{az_i})}{(1-x_i)(1-y_i)}\right]\right\}. \tag{3.1}$$

For proving (3.1) we consider

$$\tau_k=\prod_{i=1}^{k}\left[\frac{(1-x_i)(1-y_i)}{(1-az_i)(1-\frac{x_iy_i}{az_i})}\right].$$

Then by straight forward calculations we find that

$$\Delta\tau_k = az_k\left(1-\frac{y_k}{az_k}\right)\left(1-\frac{x_k}{az_k}\right)\frac{\prod_{i=1}^{k-1}[(1-x_i)(1-y_i)]}{\prod_{i=1}^{k}[(1-az_i)(1-\frac{x_iy_i}{z_i})]},$$

which on summing over k from $-m$ to n, gives (3.1) after using (1.6).

In view of this it is natural to look for a telescoping proof of the q-Paff-Salschütz summation formula [4, (1.7.2)]

$$S_n \equiv {}_3\phi_2\left[\begin{matrix}a, b, q^{-n}; q, q\\ c,\ \frac{ab}{c}\,q^{1-n}\end{matrix}\right]=\frac{(\frac{c}{a},\ \frac{c}{b};\ q)_n}{(c,\ \frac{c}{ab};\ q)_n}. \tag{3.2}$$

To this end define for non-negative integers n and r

$$F(n,r)=\frac{(a,b,q^{-n};q)_rq^r}{(q,c,\ \frac{ab}{c}\,q^{1-n};q)_r}$$

$$\text{and} \quad G(n,r) = \frac{(a,b;q)_{r+1}(q^{-n};q)_r}{(q,c,\frac{ab}{c}\,q^{1-n};q)_r}.$$

Notice that $S_0 = 1$, $G(n,n+1) = G(0,-1) = 0$. By straightforward calculations we can verify that

$$\left(1-\frac{c}{a}\,q^n\right)\left(1-\frac{c}{b}\,q^n\right)F(n,r) - (1-cq^n)\left(1-\frac{cq^n}{ab}\right)F(n+1,r) \\ - \frac{c}{ab}\,q^n[G(n,r)-G(n,r-1)] = 0.$$

Summing over r from 0 to $n+1$ and using $G(n,n+1) = 0 = G(0,-1)$, we get

$$\left(1-\frac{c}{a}\,q^n\right)\left(1-\frac{c}{a}\,q^n\right)S_n = (1-cq^n)\left(1-\frac{c}{ab}\right)S_{n+1}$$

which yields

$$S_n = \frac{(\frac{c}{a},\frac{c}{b};\,q)_n}{(c,\frac{c}{ab};q)_n}\,S_0.$$

By using $S_0 = 1$, we get (3.2).

Acknowledgment. The authors wish to thank the referee for helpful comments.

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Received January 2, 1998 and revised April 29, 1998. The first author was partially supported by an NSERC grant.

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