

RESTRICTIONS OF RANK-2 SEMISTABLE VECTOR
BUNDLES ON SURFACES IN POSITIVE
CHARACTERISTIC

ATSUSHI NOMA

We work over an algebraically closed field of positive characteristic. Let E be a semistable rank-2 vector bundle with respect to a very ample line bundle $\mathcal{O}(1)$ on a smooth projective surface. The purpose here is to give an effective bound d_0 such that if $d \geq d_0$ the restriction of E to a general member $C \in |\mathcal{O}(d)|$ is semistable.

1. Introduction.

Let E be a rank- r torsion free sheaf on a normal projective variety of dimension $n \geq 2$ defined over an algebraically closed field k . Assume that E is semistable with respect to a very ample line bundle $\mathcal{O}(1)$: Namely, if we set $\mu(F) = (c_1(F) \cdot \mathcal{O}(1)^{n-1})$ for a subsheaf F of E , $\mu(E) \geq \mu(F)$ holds for all subsheaf F of E .

A problem of finding a condition when the restriction $E|_C$ to a member $C \in |\mathcal{O}(d)|$ is semistable on C has been considered by several authors ([1], [3], [6], [7], [8]): Maruyama [6] proved that if $r < n$ then $E|_C$ is semistable for general $C \in |\mathcal{O}(d)|$ for every $d \geq 1$; Mehta and Ramanan [7] proved that there exists an integer d_0 such that if $d \geq d_0$ then $E|_C$ is semistable for general $C \in |\mathcal{O}(d)|$; Flenner [3] proved that if k is of characteristic 0 and d satisfies $\frac{\binom{d+n}{d} - d - 1}{d} > (\mathcal{O}(1)^n) \cdot \max(\frac{r^2-1}{4}, 1)$ then $E|_C$ is semistable for general $C \in |\mathcal{O}(d)|$. In other direction, in characteristic 0, Bogomolov [1] and Moriwaki [8] obtained an effective bound d_0 for some special restriction $E|_C$ to be semistable.

The purpose here is to give an effective bound d_0 in positive characteristic when E is a rank-2 vector bundle on a surface: If $d \geq d_0$ the restriction $E|_C$ of E to a general member $C \in |\mathcal{O}(d)|$ is semistable.

Our result is the following.

Theorem. *Let S be a smooth projective surface over an algebraically closed field k of characteristic $\text{char}(k) = p > 0$ and $\mathcal{O}_S(1)$ a very ample line bundle on S . Let E be a rank-2 semistable vector bundle with respect to $\mathcal{O}_S(1)$ on S . Set $\deg S = (\mathcal{O}_S(1)^2)$, $\Delta(E) = c_2(E) - (1/4)c_1^2(E)$, and $\nu = \min\{\mathcal{M} \cdot$*

$\mathcal{O}_S(1) > 0 : \mathcal{M} \in \text{Pic } S$. Let d be an integer with

$$d > \begin{cases} \frac{\Delta(E)}{\nu} + \frac{\sqrt{\Delta(E)}}{2\sqrt{3 \deg S}}, & \text{if } \Delta(E) > 0, \\ 0, & \text{if } \Delta(E) \leq 0. \end{cases}$$

Then the restriction $E|_C$ to a general $C \in |\mathcal{O}_S(d)|$ is semistable on C .

The theorem is proved based on ideas of Ein [2] and Flenner [3].

2. Proof of Theorem.

Set $L = \mathcal{O}_S(d)$. Let \mathbb{P}^{2*} be the projective space of lines in \mathbb{P}^2 and \mathbb{F} the incidence correspondence $\{(x, \ell) \in \mathbb{P}^2 \times \mathbb{P}^{2*} : x \in \ell\}$, namely

$$\mathbb{F} = \mathbb{P}_{\mathbb{P}^2}(\Omega_{\mathbb{P}^2}^1(2)) \subseteq \mathbb{P}_{\mathbb{P}^2}(\wedge^2 H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes \mathcal{O}_{\mathbb{P}^2}) = \mathbb{P}^2 \times \mathbb{P}^{2*}.$$

Let $\phi: S \rightarrow \mathbb{P}^2$ be a (separable) finite morphism defined by a 2-dimensional, base-point-free, linear subsystem \mathfrak{d} of $|L|$ containing a general curve $C \in |L|$. Pulling-back the correspondence \mathbb{F} by ϕ , we have the following diagram:

$$\begin{array}{ccccc} X & \longrightarrow & \mathbb{F} & \xrightarrow{\rho_0} & \mathbb{P}^{2*} \\ \pi \downarrow & \square & \downarrow \pi_0 & & \\ S & \xrightarrow{\phi} & \mathbb{P}^2 & & \end{array} .$$

We denote the composite $X \rightarrow \mathbb{F} \rightarrow \mathbb{P}^{2*}$ by ρ .

Assume that the restriction $E|_C$ to a general curve $C \in \mathfrak{d} \subset |L|$ is not semistable. In other words, the restriction $\pi^*E|_{\rho^{-1}(\ell)}$ to $\rho^{-1}(\ell)$ for a general $\ell \in \mathbb{P}^{2*}$ is not semistable, since $\rho^{-1}(\ell)$ is isomorphic to a divisor $C \in \mathfrak{d}$ and $\pi^*E|_{\rho^{-1}(\ell)} \cong E|_C$ under this isomorphism. Consider a relative Harder-Narasimhan filtration (HN-filtration) of π^*E over ρ , which has a property that its restriction to $\rho^{-1}(\ell)$ for a general $\ell \in \mathbb{P}^{2*}$ is the HN-filtration of $\pi^*E|_{\rho^{-1}(\ell)}$ (see [4, (3.2)]). By assumption, the relative HN-filtration is $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 = \pi^*E$ and we may assume that \mathcal{E}_1 is locally free of rank 1 on X . Hence if W denotes the class of the tautological bundle of $X = \mathbb{P}_S(\phi^*(\Omega_{\mathbb{P}^2}^1(2)))$, we have $\mathcal{E}_1 \cong \mathcal{O}_X(aW) \otimes \pi^*\mathcal{M}$ for some $a \in \mathbb{Z}$ and $\mathcal{M} \in \text{Pic } S$, since $\text{Pic } X \cong \mathbb{Z}W \oplus \pi^*\text{Pic } S$ (see [5, Ch. III Ex. 12.5]). Since $\mathcal{M}|_{\pi(\rho^{-1}(\ell))} \subset E|_{\pi(\rho^{-1}(\ell))}$ is the HN-filtration of $E|_{\pi(\rho^{-1}(\ell))}$ for a general $\ell \in \mathbb{P}^{2*}$, we have

$$(1) \quad (c_1(E) - 2\mathcal{M} \cdot L) < 0.$$

Consequently $H^0(\rho^{-1}(\ell), \pi^*(E \otimes \mathcal{M}^\vee)|_{\rho^{-1}(\ell)}) \cong k$ for a general $\ell \in \mathbb{P}^{2*}$, and hence $\rho_*\pi^*(E \otimes \mathcal{M}^\vee)$ is of rank 1 and reflexive. Therefore we have $\rho_*\pi^*(E \otimes \mathcal{M}^\vee) = \mathcal{O}_{\mathbb{P}^{2*}}(-t)$ for some $t \in \mathbb{Z}$. The semistability of E implies

$$(2) \quad t > 0$$

since $H^0(\mathbb{P}^{2*}, \rho_* \pi^*(E \otimes \mathcal{M}^\vee)) = H^0(X, \pi^*(E \otimes \mathcal{M}^\vee)) = H^0(S, E \otimes \mathcal{M}^\vee)$ and (1) holds. The natural map $\mathcal{O}_X(-tW) = \rho^* \rho_* \pi^*(E \otimes \mathcal{M}^\vee) \rightarrow \pi^*(E \otimes \mathcal{M}^\vee)$ induces an exact sequence

$$(3) \quad 0 \rightarrow \mathcal{O}_X(-tW) \otimes \pi^* \mathcal{M} \rightarrow \pi^* E \rightarrow \mathcal{O}_X(tW) \otimes \pi^* \mathcal{O}_S(c_1(E) - \mathcal{M}) \otimes \mathcal{I}_Z \rightarrow 0$$

with a closed subscheme Z of codimension 2 in X .

The surjection in (3) induces a unique morphism $\sigma: X \setminus Z \rightarrow \mathbb{P}_S(E)$ with

$$\begin{aligned} \sigma^* \mathcal{O}_{\mathbb{P}(E)}(1) &= \mathcal{O}_X(tW) \otimes \pi^* \mathcal{O}_S(c_1(E) - \mathcal{M})|_{(X \setminus Z)} \\ \sigma^* \Omega_{\mathbb{P}(E)/S}^1 &= \mathcal{O}_X(-2tW) \otimes \pi^* \mathcal{O}_S(2\mathcal{M} - c_1(E))|_{(X \setminus Z)}, \end{aligned}$$

by the universal property of projective bundle $\tau: \mathbb{P}_S(E) \rightarrow S$. If the differential

$$d\sigma: \sigma^* \Omega_{\mathbb{P}(E)/S}^1 \rightarrow \Omega_{X/S}^1|_{(X \setminus Z)}$$

is zero, then S -morphism σ factors through the relative Frobenius $F_{(X \setminus Z)/S}: X \setminus Z \rightarrow (X \setminus Z)^{(1)}$ of $X \setminus Z$ over S (see [2, (1.4)]). Namely there exists an S -morphism $\sigma_1: (X \setminus Z)^{(1)} \rightarrow \mathbb{P}_S(E)$ such that $\sigma = \sigma_1 \circ F_{(X \setminus Z)/S}$. Here for an S -scheme Y , by $Y^{(r)}$ we denote the base change of the structure morphism $\eta: Y \rightarrow S$ by the r th (absolute) Frobenius $F_S^r: S \rightarrow S$; $F_{Y/S}^r: Y \rightarrow Y^{(r)}$ is the S -morphism induced by the (absolute) Frobenius $F_Y^r: Y \rightarrow Y$ of Y and the structure morphism η by the property of products. Furthermore, if $d\sigma_1 = 0$, then there exists a morphism $\sigma_2: (X \setminus Z)^{(2)} \rightarrow \mathbb{P}_S(E)$ such that $\sigma = \sigma_2 \circ F_{(X \setminus Z)/S}^2$. Proceeding in this way with [2, (1.4)], we claim that there exists a morphism $\sigma_r: (X \setminus Z)^{(r)} \rightarrow \mathbb{P}_S(E)$ such that $\sigma = \sigma_r \circ F_{(X \setminus Z)/S}^r$ and the relative differential

$$d\sigma_r: \sigma_r^* \Omega_{\mathbb{P}(E)/S}^1 \rightarrow \Omega_{X^{(r)}/S}^1|_{(X \setminus Z)^{(r)}}$$

is nonzero for some $r \geq 0$. In fact, suppose that we have a morphism $\sigma_r: (X \setminus Z)^{(r)} \rightarrow \mathbb{P}_S(E)$ such that $\sigma = \sigma_r \circ F_{(X \setminus Z)/S}^r$ for some $r \geq 0$. Here we set $\sigma_0 = \sigma$ if $r = 0$. Then we have the following diagram:

$$\begin{array}{ccc} X \setminus Z & \xrightarrow{F_{(X \setminus Z)/S}^r} & (X \setminus Z)^{(r)} \\ \downarrow & & \downarrow \searrow \sigma_r \\ X & \xrightarrow{F_{X/S}^r} & X^{(r)} & \mathbb{P}_S(E) \\ \downarrow \pi & & \downarrow \pi_r \swarrow \tau \\ S & = & S. \end{array}$$

Since $X \cong \mathbb{P}_S(\phi^*(\Omega_{\mathbb{P}^2}^1(2)))$, we have $X^{(r)} \cong \mathbb{P}_S(F_S^{r*}\phi^*(\Omega_{\mathbb{P}^2}^1(2)))$. If W' is the class of the tautological line bundle of $X^{(r)}$ over S , then $F_{X/S}^{r*}\mathcal{O}_{X^{(r)}}(W') \cong \mathcal{O}_X(p^r W)$ and $\Omega_{X^{(r)}/S}^1 \cong \pi_r^*(p^r L) \otimes \mathcal{O}_{X^{(r)}}(-2W')$. On the other hand, $F_{X/S}^{r*}\pi_r^*\mathcal{A} \cong \pi^*\mathcal{A}$ for every $\mathcal{A} \in \text{Pic } S$. Since $\sigma = \sigma_r \circ F_{(X \setminus Z)/S}^r$ and $\text{Pic } X^{(r)} \cong \mathbb{Z}W' \oplus \pi_r^*\text{Pic } S$, the morphism σ_r induces an exact sequence

$$\begin{aligned} 0 \rightarrow \mathcal{O}_{X^{(r)}}\left(-\frac{t}{p^r}W'\right) \otimes \pi_r^*\mathcal{M} &\rightarrow \pi_r^*E \\ \rightarrow \mathcal{O}_{X^{(r)}}\left(\frac{t}{p^r}W'\right) \otimes \pi_r^*\mathcal{O}_S(c_1(E) - \mathcal{M}) \otimes \mathcal{I}_{Z'} &\rightarrow 0, \end{aligned}$$

and we have $t/p^r \in \mathbb{Z}$, where Z' is a codimension 2 closed subscheme of $X^{(r)}$. Since $t > 0$, the latter implies that σ factors through the relative Frobenius over S only in finite times. Therefore for some $r \geq 0$, the morphism σ_r must have the nonzero relative differential $d\sigma_r$, as required.

We take such $r \geq 0$ and $\sigma_r: (X \setminus Z)^{(r)} \rightarrow \mathbb{P}_S(E)$. If $C \in \mathfrak{d} \subset |L|$, since $C \cong \rho^{-1}(\ell) \cong F_{X/S}^r(\rho^{-1}(\ell))$ for some $\ell \in \mathbb{P}^{2*}$, we can consider $C \subset X^{(r)}$. Then we have $\mathcal{O}_{X^{(r)}}(W')|_C \cong \mathcal{O}_C$ and $\pi_r^*\mathcal{A}|_C \cong \mathcal{A}|_C$ for every $\mathcal{A} \in \text{Pic } S$. The restriction $d\sigma_r|_C$ to general $C \in \mathfrak{d}$ is nonzero by the choice of r . This implies that

$$(4) \quad (L \cdot 2\mathcal{M} - c_1(E)) \leq p^r(L^2) \leq t(L^2),$$

since

$$\begin{aligned} \sigma_r^*\Omega_{\mathbb{P}(E)/S}^1|_C &= \mathcal{O}_C(2\mathcal{M} - c_1(E)) \\ \Omega_{X^{(r)}/S}^1|_C &= \mathcal{O}_C(p^r L), \end{aligned}$$

and since $t/p^r \in \mathbb{Z}$.

Restricting the exact sequence (3) to a general member $W \in |\mathcal{O}_X(W)|$ not containing any associate points of Z , we have an exact sequence

$$\begin{aligned} 0 \rightarrow \mathcal{O}_W(-tW) \otimes \pi^*\mathcal{M}|_W &\rightarrow \pi^*E|_W \\ \rightarrow \mathcal{O}_W(tW) \otimes \pi^*\mathcal{O}_S(c_1(E) - \mathcal{M})|_W \otimes \mathcal{I}_{Z \cap W} &\rightarrow 0. \end{aligned}$$

On the other hand, we note that $W^3 = 0$ and $W^2 \cdot \pi^*\mathcal{A} = (\mathcal{A} \cdot L)$ for any $\mathcal{A} \in \text{Pic } S$, since $W^2 - \pi^*L \cdot W + (\pi^*L^2) = W^2 - c_1(\phi^*(\Omega_{\mathbb{P}^2}^1(2))) \cdot W + c_2(\phi^*(\Omega_{\mathbb{P}^2}^1(2))) = 0$. Thus from the exact sequence above, noting that

$W \rightarrow S$ is birational via π , we have

$$\begin{aligned} c_2(E) &= c_2(\pi^*E|W) \\ &= -t^2(\mathcal{O}_W(W)^2) + t(\mathcal{O}_W(W) \cdot \pi^*\mathcal{O}_S(2\mathcal{M} - c_1(E))|W) \\ &\quad + (\pi^*\mathcal{M}|W \cdot \pi^*\mathcal{O}_S(c_1(E) - \mathcal{M})|W) + \deg(Z \cap W) \\ &= t(L \cdot 2\mathcal{M} - c_1(E)) - (\mathcal{M} \cdot \mathcal{M} - c_1(E)) + \deg(Z \cap W) \\ &\geq t(L \cdot 2\mathcal{M} - c_1(E)) - (\mathcal{M} \cdot \mathcal{M} - c_1(E)), \end{aligned}$$

and hence

$$\Delta(E) \geq 2t(L \cdot \mathcal{M} - (1/2)c_1(E)) - ((\mathcal{M} - (1/2)c_1(E))^2).$$

By the Hodge index theorem for L and $\mathcal{M} - (1/2)c_1(E)$, we have

$$(5) \quad \Delta(E) \geq 2t(L \cdot \mathcal{M} - (1/2)c_1(E)) - \frac{(L \cdot \mathcal{M} - (1/2)c_1(E))^2}{(L^2)}.$$

From (4) and (5), it follows that

$$\Delta(E) \geq 3 \frac{(L \cdot \mathcal{M} - (1/2)c_1(E))^2}{(L^2)}.$$

When $\Delta(E) \leq 0$, this contradicts (1). When $\Delta(E) > 0$, we have

$$(6) \quad (L \cdot \mathcal{M} - (1/2)c_1(E)) \leq \sqrt{\frac{\Delta(E)(L^2)}{3}}.$$

On the other hand, from (5), it follows

$$\frac{\Delta(E)}{(L \cdot \mathcal{M} - (1/2)c_1(E))} + \frac{(L \cdot \mathcal{M} - (1/2)c_1(E))}{(L^2)} \geq 2t.$$

Since $L = \mathcal{O}_S(d)$, by using the assumption $(\mathcal{O}_S(1) \cdot \mathcal{M} - (1/2)c_1(E)) \geq \nu/2$ to the first term and (6) to the second term, we have

$$\frac{1}{d} \left(\frac{\Delta(E)}{\nu} + \frac{\sqrt{\Delta(E)}}{2\sqrt{3} \deg S} \right) \geq t.$$

By assumption of d , we have $t < 1$ hence $t \leq 0$, which contradicts (2). This completes the proof. \square

Remark. Let S , $\mathcal{O}_S(1)$, E and d be as in [Theorem](#).

(1) Let \mathfrak{d} be a 2-dimensional linear subsystem of $|\mathcal{O}_S(d)|$ defining a separable, finite morphism from S to \mathbb{P}^2 . The proof of [theorem](#) implies that the restriction $E|C$ is semistable for a general member $C \in \mathfrak{d}$.

(2) Assume that $\Delta(E) > 0$ and that the restriction $E|C$ to be a general member of $C \in |\mathcal{O}_S(1)|$ is not semistable with HN-filtration $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 = E|C$. Then it follows from (6) that

$$\deg \mathcal{E}_1 - (1/2) \deg(E|C) \leq \sqrt{\deg S \cdot \Delta(E)/3}$$

holds. This inequality is exactly that of Ein in [2, (4.1)] when $S = \mathbb{P}^2$ and $\mathcal{O}_S(1) = \mathcal{O}_{\mathbb{P}^2}(1)$.

(3) I do not know the bound in Theorem is optimal or not. For example, let E be the m th Frobenius pull-back $F^{m*}(\Omega_{\mathbb{P}^2}(2))$ of the twisted cotangent bundle on \mathbb{P}^2 , which plays an important role in the proof of Theorem. We know that E is semistable (see for example [2]) and $\Delta(E) = p^{2m}/4$. Thus Theorem implies that $E|_C$ is semistable on a general curve C of degree d if $d > p^{2m}/4 + p^m/(4\sqrt{3})$. On the other hand, from a calculation of $H^0(C, E(-(p^m + 1)/2)|_C)$ by using the Euler sequence, it follows that $E|_C$ is semistable for general C of degree d if $d > (3p^m + 5)/4$ for $p \neq 2$.

References

- [1] F.A. Bogomolov, *On stability of vector bundles on surfaces and curves*, in ‘Einstein Metrics and Yang-Mills Connections’, eds. T. Mabuchi and S. Mukai, Lecture Notes in Pure and App. Math., **145** (1993), 187-221, Dekker, New York.
- [2] L. Ein, *Stable vector bundles on projective spaces in char $p > 0$* , Math. Ann., **254** (1980), 53-72.
- [3] H. Flenner, *Restrictions of semistable bundles on projective varieties*, Comment. Math. Helvetici, **59** (1984), 635-650.
- [4] O. Forster, A. Hirschowitz and M. Schneider, *Type de scindage généralisé pour les fibrés stables*, in ‘Vector bundles and differential equations’, Proceedings, Nice, 1979, ed. A. Hirxhowitz, Progress in Math., **7** (1980), 65-81, Birkhäuser, Basel, Boston, Stuttgart.
- [5] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Math., **52** (1977), Springer-Verlag, Berlin/New York.
- [6] M. Maruyama, *Boundedness of semi-stable sheaves of small ranks*, Nagoya Math. J., **78** (1980), 65-94.
- [7] V. Mehta and A. Ramanathan, *Semi-stable sheaves on projective varieties and their restriction to curves*, Math. Ann., **258** (1982), 213-224.
- [8] A. Moriwaki, *Arithmetic Bogomolov-Gieseker’s inequality*, Amer. J. Math., **117** (1995), 1325-1347.

Received July 21, 1997 and revised March 10, 1998.

YOKOHAMA NATIONAL UNIVERSITY
 79-2 TOKIWADAI, HODOGAYA-KU
 YOKOHAMA 240-8501, JAPAN
E-mail address: noma@ed.ynu.ac.jp

This paper is available via <http://nyjm.albany.edu:8000/PacJ/1999/189-1-8.html>.