

EQUIVARIANT EMBEDDINGS OF STEIN DOMAINS SITTING INSIDE OF COMPLEX SEMIGROUPS

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In this paper we prove an equivariant version of Hörmanders embedding theorem for Stein manifolds. More concretely, let G be a connected Lie group sitting in its complexification $G_{\mathbb{C}}$ and $D \subseteq G_{\mathbb{C}}$ a $G \times G$ -invariant Stein domain. Under slight obstructions on D we construct a Hilbert space \mathcal{H} equipped with a unitary $G \times G$ -action and a holomorphic equivariant closed embedding $e: D \rightarrow \mathcal{H}^* \setminus \{0\}$.

Introduction.

An interesting problem in the field of equivariant complex analysis is: Given a connected Lie group G sitting in its universal complexification $G_{\mathbb{C}}$, how do the $G \times G$ -invariant Stein domains in $G_{\mathbb{C}}$ look like. K.-H. Neeb has shown in [Ne98] that all domains of the form

$$D = G \exp_{G_{\mathbb{C}}}(iD_h),$$

where $D_h \subseteq \mathfrak{g}$ is a $\text{Ad}(G)$ -invariant convex domain consisting of elliptic elements, i.e., all operators $i \text{ad } X$, $X \in D_h$, are diagonalizable over the reals, are Stein manifolds. Moreover there is also strong evidence for that these D exhaust up to multiplication with $N_{G_{\mathbb{C}}}(G)$ all proper bi-invariant Stein domains in $G_{\mathbb{C}}$ (cf. [GG77], [Ne98]).

By Hörmander's Embedding Theorem one knows that every Stein manifold of dimension n can be embedded biholomorphically as a closed submanifold of \mathbb{C}^{2n+1} (cf. [Hö73]). Now the natural question is: Given a biinvariant Stein domain $D = G \exp_{G_{\mathbb{C}}}(iD_h)$ in $G_{\mathbb{C}}$, does there exist a $G \times G$ -equivariant embedding into some complex Hilbert space \mathcal{H} endowed with a unitary $G \times G$ -action. In this paper we show that under quite natural assumptions the answer is affirmative. More concretely, if $\text{Ad}(G)$ is closed in $\text{Aut}(\mathfrak{g})$, the center $Z(G)$ is compact and the convex domain D_h is pointed, then there exists a positive definite biinvariant holomorphic kernel K on D , such that the map

$$e_K: D \rightarrow \mathcal{H}_K^* \setminus \{0\}, \quad z \mapsto K_z$$

defines a $G \times G$ -equivariant closed embedding. Here \mathcal{H}_K denotes the reproducing kernel Hilbert space and $K_z: \mathcal{H}_K \rightarrow \mathbb{C}$, $f \mapsto f(z)$ the point evaluations corresponding to K .

Our method to construct such a kernel K is to sum up kernels K^λ associated to unitary highest weight representations $(\pi_\lambda, \mathcal{H}_\lambda)$ of G over a certain lattice $\Gamma \subseteq \mathfrak{it}^*$, where \mathfrak{t} denotes a compactly embedded Cartan subalgebra of \mathfrak{g} . More precisely, we set

$$K = \sum_{\lambda \in \Gamma} \|\lambda\|^N K^\lambda$$

with $\|\cdot\|$ denoting a norm on \mathfrak{it}^* and $N \in \mathbb{N}$. These kernels K have the important property of tending to infinity at the boundary of D , i.e.,

$$\lim_{z \rightarrow \partial D} K(z, z) = \infty;$$

a result which is crucial for verifying the closedness of the map e_K .

We think that our results are a little bit surprising and we do not really understand what is actually going on. For instance, what is the reason for that one has to exclude zero in \mathcal{H}_K^* to achieve the closedness of the map e_K , or, is it possible to find an equivariant closed holomorphic embedding $D \rightarrow E$ into a complex topological vector space endowed with a continuous $G \times G$ -action. We hope that our results give rise to a further discussion leading to a better understanding of these phenomena.

I. The boundary behaviour of bi-invariant kernels.

In this first section we characterize the boundary behaviour of biinvariant holomorphic positive definite kernels on a bi-invariant domain $D = G \operatorname{Exp}(iD_h)$ by means of the boundary behaviour on the abelian submanifold $D_T := T \operatorname{Exp}(i(D_h \cap \mathfrak{t}))$. If the convex invariant set $D_h \subseteq \mathfrak{g}$ is a pointed cone, we show that $\lim_{z \rightarrow \partial D} K(z, z) = \infty$ if and only if $\lim_{z \rightarrow \partial D_T} K(z, z) = \infty$. As abelian domains are comparable easily to deal with contrary to the highly non-commutative bi-invariant domains D , this result allows us in the sequel to make quite explicit computations.

Definition I.1. Let V be a finite dimensional real vector space and V^* its dual.

(a) For each subset $E \subseteq V$ we define its *dual cone* by $E^* := \{\alpha \in V^* : (\forall x \in E) \alpha(x) \geq 0\}$. We note that E^* is a convex closed subcone of V^* .

(b) For a convex subset $E \subseteq V$ we set

$$H(E) := \{x \in V : x + E = E\}, \quad \text{and} \quad \lim E := \{x \in V : x + E \subseteq E\}.$$

We call $H(E)$ the *edge* and $\lim E$ the *limit cone* of E . Note that $H(E)$ is a vector space, $H(E) = H(\overline{E})$ if E is open and that $\lim E$ is a convex cone in V .

(c) A convex set E is called *pointed* if it contains no affine lines. Note that if E is open or closed then E is pointed if and only if its edge is zero. \square

Definition I.2. Let \mathfrak{g} be a finite dimensional Lie algebra over \mathbb{R} .

(a) An element $X \in \mathfrak{g}$ is called *elliptic* if $\text{ad } X$ operates semisimply with purely imaginary spectrum. A convex cone $W \subseteq \mathfrak{g}$ is said to be *elliptic* if $W^0 \neq \emptyset$ and all $X \in W^0$ are elliptic.

(b) For a subalgebra $\mathfrak{a} \subseteq \mathfrak{g}$ we write $\text{Inn}(\mathfrak{a}) := \langle e^{\text{ad } \mathfrak{a}} \rangle \subseteq \text{Aut}(\mathfrak{g})$ for the corresponding group of inner automorphisms. A subalgebra $\mathfrak{a} \subseteq \mathfrak{g}$ is said to be *compactly embedded* if $\text{Inn}(\mathfrak{a})$ is relatively compact in $\text{Aut}(\mathfrak{g})$.

(c) Let $\mathfrak{t} \subseteq \mathfrak{g}$ be a compactly embedded Cartan subalgebra and recall that there exists a unique maximal compactly embedded subalgebra \mathfrak{k} containing \mathfrak{t} (cf. [HHL89, A.2.40]).

(d) Associated to the Cartan subalgebra $\mathfrak{t}_{\mathbb{C}}$ in the complexification $\mathfrak{g}_{\mathbb{C}}$ is a root decomposition as follows. For a linear functional $\alpha \in \mathfrak{t}_{\mathbb{C}}^*$ we set

$$\mathfrak{g}_{\mathbb{C}}^{\alpha} := \{X \in \mathfrak{g}_{\mathbb{C}} : (\forall Y \in \mathfrak{t}_{\mathbb{C}}) [Y, X] = \alpha(Y)X\}$$

and write $\Delta := \{\alpha \in \mathfrak{t}_{\mathbb{C}}^* \setminus \{0\} : \mathfrak{g}_{\mathbb{C}}^{\alpha} \neq \{0\}\}$ for the set of roots. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\mathbb{C}}^{\alpha}$, $\alpha(\mathfrak{t}) \subseteq i\mathbb{R}$ for all $\alpha \in \Delta$ and $\mathfrak{g}_{\mathbb{C}}^{\alpha} = \mathfrak{g}_{\mathbb{C}}^{-\alpha}$, where $X \rightarrow \bar{X}$ denotes complex conjugation on $\mathfrak{g}_{\mathbb{C}}$ with respect to \mathfrak{g} .

(e) A root α is said to be *compact* if $\mathfrak{g}_{\mathbb{C}}^{\alpha} \subseteq \mathfrak{k}_{\mathbb{C}}$ and *non-compact* otherwise. We write Δ_k for the set of compact roots and Δ_n for the non-compact ones. If $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ is a \mathfrak{k} -invariant Levi decomposition, then we set

$$\Delta_r := \{\alpha \in \Delta : \mathfrak{g}_{\mathbb{C}}^{\alpha} \subseteq \mathfrak{r}_{\mathbb{C}}\} \quad \text{and} \quad \Delta_s := \{\alpha \in \Delta : \mathfrak{g}_{\mathbb{C}}^{\alpha} \subseteq \mathfrak{s}_{\mathbb{C}}\}$$

and recall that $\Delta = \Delta_r \dot{\cup} \Delta_s$ (cf. [Ne99, Ch. V]).

(f) A positive system Δ^+ of roots is a subset of Δ for which there exists a regular element $X_0 \in i\mathfrak{t}^*$ with $\Delta^+ := \{\alpha \in \Delta : \alpha(X_0) > 0\}$. A positive system is said to be *\mathfrak{k} -adapted* if the set $\Delta_n^+ := \Delta_n \cap \Delta^+$ is invariant under the *Weyl group* $\mathcal{W}_{\mathfrak{t}} := N_{\text{Inn}(\mathfrak{t})}(\mathfrak{t})/Z_{\text{Inn}(\mathfrak{t})}(\mathfrak{t})$ acting on \mathfrak{t} . We recall from [Ne99, Ch. V] that there exists a \mathfrak{k} -adapted positive system if and only if $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{z}(\mathfrak{k})) = \mathfrak{k}$. In this case we call \mathfrak{g} *quasihermitian*. In this case it is easy to see that \mathfrak{s} is quasihermitian too, and so all simple ideals of \mathfrak{s} are either compact or hermitian.

(g) We associate to a positive system Δ^+ the convex cones

$$C_{\min} := \text{cone}\{i[\bar{X}_{\alpha}, X_{\alpha}] : X_{\alpha} \in \mathfrak{g}_{\mathbb{C}}^{\alpha}, \alpha \in \Delta_n^+\},$$

and $C_{\max} := (i\Delta_n^+)^* = \{X \in \mathfrak{t} : (\forall \alpha \in \Delta_n^+) i\alpha(X) \geq 0\}$. Note that both C_{\min} and C_{\max} are closed convex cones in \mathfrak{t} .

(h) Write $p_{\mathfrak{t}}: \mathfrak{g} \rightarrow \mathfrak{t}$ for the orthogonal projection along $[\mathfrak{t}, \mathfrak{g}]$ and set $\mathcal{O}_X := \text{Inn}(\mathfrak{g}).X$ for the adjoint orbit through $X \in \mathfrak{g}$. We define the *minimal* and *maximal cone* associated to Δ^+ by

$$\begin{aligned} W_{\min} &:= \{X \in \mathfrak{g} : p_{\mathfrak{t}}(\mathcal{O}_X) \subseteq C_{\min}\} & \text{and} \\ W_{\max} &:= \{X \in \mathfrak{g} : p_{\mathfrak{t}}(\mathcal{O}_X) \subseteq C_{\max}\} \end{aligned}$$

and note that both cones are convex closed and $\text{Inn}(\mathfrak{g})$ -invariant. \square

From now on we assume that \mathfrak{g} contains a compactly embedded Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$ and that there exists an elliptic cone $W \subseteq \mathfrak{g}$. Then there exists a \mathfrak{k} -adapted positive system Δ^+ such that

$$C_{\min} \subseteq W \cap \mathfrak{t} \subseteq C_{\max}$$

holds and W_{\max} is an elliptic cone (cf. [Ne96b, Th. II.11]). Moreover, we have $W_{\min} \cap \mathfrak{t} = C_{\min}$ and $W_{\max} \cap \mathfrak{t} = C_{\max}$ (cf. [Ne97, Lemma I.1]).

Definition I.3. (a) Let $W \subseteq \mathfrak{g}$ be a closed elliptic cone. Let \tilde{G} , resp. $\tilde{G}_{\mathbb{C}}$, be the simply connected Lie groups associated to \mathfrak{g} , resp. $\mathfrak{g}_{\mathbb{C}}$, and set $G_1 := \langle \exp \mathfrak{g} \rangle \subseteq \tilde{G}_{\mathbb{C}}$. Then Lawson's Theorem (cf. [HiNe93, Th. 7.34, 35]) says that the subset $\Gamma_{G_1}(W) := G_1 \exp(iW)$ is a closed subsemigroup of $G_{\mathbb{C}}$ and the polar map

$$G_1 \times W \rightarrow \Gamma_{G_1}(W), \quad (g, X) \mapsto g \exp(iX)$$

is a homeomorphism.

Now the universal covering semigroup $\Gamma_{\tilde{G}}(W) := \tilde{\Gamma}_{G_1}(W)$ has a similar structure. We can lift the exponential function $\exp: \mathfrak{g} + iW \rightarrow \Gamma_{G_1}(W)$ to an exponential mapping $\text{Exp}: \mathfrak{g} + iW \rightarrow \Gamma_{\tilde{G}}(W)$ with $\text{Exp}(0) = \mathbf{1}$ and thus obtain a polar map

$$\tilde{G} \times W \rightarrow \Gamma_{\tilde{G}}(W), \quad (g, X) \mapsto g \text{Exp}(iX)$$

which is a homeomorphism.

If G is a connected Lie group associated to \mathfrak{g} , then $\pi_1(G)$ is a discrete central subgroup of $\Gamma_{\tilde{G}}(W)$ and we obtain a covering homomorphism $\Gamma_{\tilde{G}}(W) \rightarrow \Gamma_G(W) := \Gamma_{\tilde{G}}(W)/\pi_1(G)$ (cf. [HiNe93, Ch. 3]). It is easy to see that there is also a polar map $G \times W \rightarrow \Gamma_G(W)$, $(g, X) \mapsto g \text{Exp}(iX)$ which is a homeomorphism. The semigroups of the type $\Gamma_G(W)$ are called *complex Ol'shanskiĭ semigroups*.

The subset $\Gamma_G(W^0) \subseteq \Gamma_G(W)$ is an open semigroup carrying a complex manifold structure such that semigroup multiplication is holomorphic. Moreover there is an involution on $\Gamma_G(W)$ given by

$$*: \Gamma_G(W) \rightarrow \Gamma_G(W), \quad s = g \text{Exp}(iX) \mapsto s^* = \text{Exp}(iX)g^{-1}$$

which is antiholomorphic on $\Gamma_G(W^0)$ (cf. [HiNe93, Th. 9.15] for a proof of all that). Thus $\Gamma_G(W)$ is an involutive semigroup.

(b) A *bi-invariant domain* $D \subseteq \Gamma_G(W_{\max}^0)$ is an open connected $G \times G$ bi-invariant subset of $\Gamma_G(W_{\max}^0)$. Note that

$$D = G \text{Exp}(iD_h) = G \text{Exp}(i\mathcal{D})G,$$

where $D_h \subseteq W_{\max}^0$ and $\mathcal{D} = D_h \cap \mathfrak{t}$. Recall that D is a Stein manifold if and only if D_h is convex (cf. [Ne98, Th. 6.1]). In this case D is called a *bi-invariant Stein domain*. We call D *pointed* if D_h is pointed in \mathfrak{g} . The

boundary of a left G -invariant subset $E = G \operatorname{Exp}(iE_h) \subseteq \Gamma_G(W_{\max})$ is defined as $\partial E := G \operatorname{Exp}(i\partial E_h)$. Note that $\partial D = \overline{D} \setminus D$ for every bi-invariant domain D , where the closure \overline{D} is taken in $\Gamma_G(W_{\max})$. \square

Lemma I.4. *Let $W \subseteq \mathfrak{g}$ be an invariant elliptic pointed convex cone and set $C := W \cap \mathfrak{t}$. Let $(X_n)_{n \in \mathbb{N}}$ be a sequence in W^0 converging to $X \in \partial W$. Then there exists a subsequence $(X_{n_k})_{k \in \mathbb{N}}$ and a sequence $(Y_{n_k})_{k \in \mathbb{N}}$ in C^0 with $Y_{n_k} \in \operatorname{Inn}(\mathfrak{g}) \cdot X_{n_k}$ and $Y_{n_k} \rightarrow Y \in \partial C$.*

Proof. W.l.o.g. we may assume that W is closed. According to [HiNe93, Th. 7.27], we can reconstruct W^0 from C^0 , i.e., we have $W^0 = \operatorname{Inn}(\mathfrak{g}) \cdot C^0$. In particular, we find a sequence $(Y_n)_{n \in \mathbb{N}}$ in C^0 and a sequence $(g_n)_{n \in \mathbb{N}}$ in $\operatorname{Inn}(\mathfrak{g})$ such that $g_n \cdot X_n = Y_n$. We claim that $(Y_n)_{n \in \mathbb{N}}$ is bounded.

The Convexity Theorem for Adjoint Orbits (cf. [KrNe96, Th. VIII.9]) implies that

$$(1.1) \quad p_{\mathfrak{t}}(X_n) \in \operatorname{conv}(\mathcal{W}_{\mathfrak{t}} \cdot Y_n) + C_{\min} \subseteq C$$

for all $n \in \mathbb{N}$.

As C is pointed, a sequence $(Z_n)_{n \in \mathbb{N}}$ in C is unbounded if and only if $\overline{\lim}_{n \rightarrow \infty} \alpha(Z_n) = \infty$ holds for one $\alpha \in \operatorname{int} C^*$. Thus if $(Y_n)_{n \in \mathbb{N}}$ is unbounded, then (1.1) together with the invariance of C under $\mathcal{W}_{\mathfrak{t}}$ implies that $(p_{\mathfrak{t}}(X_n))_{n \in \mathbb{N}}$ is unbounded. But this contradicts the fact that $(p_{\mathfrak{t}}(X_n))_{n \in \mathbb{N}}$ being a continuous image of a Cauchy sequence is bounded, proving the claim.

Let now $(Y_{n_k})_{k \in \mathbb{N}}$ be a convergent subsequence of $(Y_n)_{n \in \mathbb{N}}$ and $Y = \lim_{k \rightarrow \infty} Y_{n_k}$ the corresponding limit in C . It remains to show that $Y \in \partial C$. To obtain a contradiction we assume that $Y \in C^0$.

We write $\operatorname{Sl}(W)$ for the special automorphism group of the cone W and note that $\operatorname{Inn}(\mathfrak{g}) \subseteq \operatorname{Sl}(W)$ (cf. [HiNe93, Prop. 7.3(v)]). Then [HiNe93, Prop. 1.11] implies that there exists a convergent subsequence of $(g_{n_k})_{k \in \mathbb{N}}$ in $\operatorname{Sl}(W)$ which we also denote by $(g_{n_k})_{k \in \mathbb{N}}$. Write g for the corresponding limit. Then

$$X = \lim_{k \rightarrow \infty} g_{n_k}^{-1} \cdot Y_{n_k} = g^{-1} \cdot Y.$$

Since $\operatorname{Sl}(W) \cdot W^0 = W^0$ and $Y \in W^0$, this implies that $X \in W^0$; a contradiction, concluding the proof of the lemma. \square

Definition I.5. Let M be a complex manifold and $\operatorname{Hol}(M)$ denote the space of holomorphic functions on M . We write \overline{M} for M equipped with the opposite complex structure.

(a) A function $K \in \operatorname{Hol}(M \times \overline{M})$ is called a *holomorphic positive definite kernel* if for every sequence z_1, \dots, z_n in M the matrix $(K(z_i, z_j))_{i,j}$ is positive semi-definite. We write $\mathcal{P}(M^2)$ for the convex cone of all holomorphic positive definite kernels on M . Note that every $K \in \mathcal{P}(M^2)$ satisfies the

inequality

$$(1.2) \quad (\forall z, w \in M) \quad |K(z, w)| \leq \sqrt{K(z, z)}\sqrt{K(w, w)}.$$

Recall that $K \in \mathcal{P}(M^2)$ if and only if there exists a Hilbert space $\mathcal{H} \subseteq \text{Hol}(M)$ with continuous point evaluations $K_z: \mathcal{H} \rightarrow \mathbb{C}$, $f \mapsto f(z)$ such that $K(z, w) = \langle K_w, K_z \rangle$ holds for all $(z, w) \in M \times M$ (cf. [Ne99, Ch. II]). In this case we also write \mathcal{H}_K instead of \mathcal{H} and refer to \mathcal{H}_K as the *reproducing kernel Hilbert space* corresponding to K .

(b) An *involutive semigroup* is a semigroup S together with an involutive antiautomorphism $*$: $S \rightarrow S$, i.e., $(s^*)^* = s$ and $(st)^* = t^*s^*$ holds for all $s, t \in S$.

A mapping $\alpha: S \rightarrow \mathbb{R}^+$ is called an *absolute value* if $\alpha(s^*) = \alpha(s)$ and $\alpha(st) \leq \alpha(s)\alpha(t)$ hold for all $s, t \in S$. We denote by $\mathcal{A}(S)$ the collection of all absolute values on S .

(c) Let S be an involutive semigroup acting on M from the left by holomorphic mappings. A positive definite kernel K is said to be *S -invariant* if $K(s.z, w) = K(z, s^*.w)$ holds for all $s \in S$, $z, w \in M$. We write $\mathcal{P}_S(M^2)$ for the subcone of $\mathcal{P}(M^2)$ of all S -invariant elements.

(d) An S -invariant positive definite kernel $K \in \mathcal{P}_S(M^2)$ is called *α -bounded* for some $\alpha \in \mathcal{A}(S)$ if

$$K(s.z, s.z) \leq \alpha(s)K(z, z)$$

holds for all $z \in M$, $s \in S$. The set of all α -bounded positive definite kernels is denoted by $\mathcal{P}_S(M^2, \alpha)$. Note that each $K \in \mathcal{P}_S(M^2, \alpha)$ gives rise to an *involutive representation* of S given by

$$\pi_K: S \rightarrow B(\mathcal{H}_K), \quad (\pi_K(s).f)(z) = f(s^*.z),$$

i.e., (π_K, \mathcal{H}_K) is a representation of S satisfying $\pi_K(s^*) = \pi_K(s)^*$ for all $s \in S$ (cf. [Ne99, Ch. II]). \square

We equip $G \times G$ with the involution $(g_1, g_2)^* = (g_1^{-1}, g_2^{-1})$ for $g_1, g_2 \in G$. Note that every $K \in \mathcal{P}_{G \times G}$ is trivially α -bounded with $\alpha = \mathbf{1}$, and that

$$\pi_K: G \times G \rightarrow U(\mathcal{H}_K), \quad (\pi_K(g_1, g_2).f)(z) = f(g_1^{-1}zg_2)$$

is a unitary representation of $G \times G$ (cf. [Kr97, Lemma III.6]).

Lemma I.6. *Let V be a finite dimensional real vector space, $V^\sharp := V \oplus \mathbb{R}$ and $E \subseteq V$ a convex subset. Set $E^\sharp := \mathbb{R}^+(E \times \{1\})$. Then E^\sharp is a convex subcone of V^\sharp and the following assertions hold:*

- (i) E is closed in E^\sharp .
- (ii) E is pointed if and only if E^\sharp is pointed.
- (iii) If E is open or closed, then $\partial E^\sharp =]0, \infty[.(\partial E \times \{1\}) \cup (\lim E \times \{0\})$.

Proof. (i) This is clear.

(ii) If E^\sharp is not pointed, then there exists a non-zero element $y = (x, r) \in V^\sharp$ such that $\mathbb{R}y \subseteq E^\sharp$. In view of $E^\sharp = \mathbb{R}^+(E \times \{1\})$, we must have $r = 0$. Now $\mathbb{R}y + E^\sharp = E^\sharp$ implies that $\mathbb{R}x + E = E$, i.e., E is not pointed.

Conversely, if E is not pointed, then there exists a non-zero element $x \in V$ such that $\mathbb{R}x \subseteq H(E)$. Then $\mathbb{R}(x, 0) \subseteq H(E^\sharp)$, i.e., C is not pointed.

(iii) Note that $]0, \infty[(E^0 \times \{1\})$ is open and that $]0, \infty[(\overline{E} \times \{1\})$ is closed in $V^\sharp \setminus (V \times \{0\})$. Hence $\partial E^\sharp \cap V^\sharp \setminus (V \times \{0\}) =]0, \infty[(\partial E \times \{1\})$. By the definition of E^\sharp we have $(x, 0) \in \partial E^\sharp$ if and only if there exists a sequence of positive real numbers $(\lambda_n)_{n \in \mathbb{N}}$ and a sequence $(x_n)_{n \in \mathbb{N}}$ such that $\lim_{n \rightarrow \infty} \lambda_n = 0$ and $\lim_{n \rightarrow \infty} \lambda_n x_n = x$. In view of [Ne99, Prop. III.1.5(iii)], this means that $x \in \lim E$, concluding the proof of (iii). \square

Lemma I.7. *Let V be a finite dimensional real vector space and $E \subseteq V$ a convex set. Further let $V_1 := V/H(E)$, denote $q: V \rightarrow V_1$ the corresponding quotient homomorphism and set $E_1 := q(E)$. Then we have $q(\partial E) = \partial E_1$.*

Proof. As $E + H(E) = E$ it follows that $E^0 + H(E) = E^0$ and $\overline{E} + H(E) = H(E)$. Thus $q(\overline{E}) = \overline{E}_1$, and $q(E^0) = E_1^0$ since q is an open mapping. This proves the lemma. \square

Proposition I.8. *Let $\mathfrak{g}^\sharp = \mathfrak{g} \oplus \mathbb{R}$, $G^\sharp = G \times \mathbb{R}$, and*

$$D^\sharp := \Gamma_{G^\sharp} \left(D_h^\sharp \right) \subseteq \Gamma_{G^\sharp} (W_{\max}^0 \oplus \mathbb{R}) \cong \Gamma_G (W_{\max}^0) \oplus \mathbb{C}.$$

(i) *The map $j: D \rightarrow D^\sharp$, $s \mapsto (s, i)$ is a $G \times G$ -equivariant holomorphic closed embedding inducing a map*

$$\mathcal{P}_{G^\sharp \times G^\sharp} \left(D^{\sharp 2} \right) \rightarrow \mathcal{P}_{G \times G} (D^2), \quad K^\sharp \mapsto K := K^\sharp \circ j.$$

(ii) *Let $K \in \mathcal{P}_{G \times G} (D^2)$. Then the following statements are equivalent:*

- (a) $\lim_{z \rightarrow \partial D} K(z, z) = \infty$.
- (b) $\lim_{X \rightarrow \partial D_h} K(\text{Exp}(iX), \text{Exp}(iX)) = \infty$. Moreover, if $K = K^\sharp \circ j$ with some $K^\sharp \in \mathcal{P}_{G^\sharp \times G^\sharp} (D^{\sharp 2})$, then (a)-(b) are implied by
- (c) $\lim_{X \rightarrow \partial D_h^\sharp} K^\sharp(\text{Exp}(iX), \text{Exp}(iX)) = \infty$.

(iii) *Let $\mathfrak{a} := H(D_h)$ be the edge of D_h , $\mathfrak{g}_1 := \mathfrak{g}/\mathfrak{a}$ and $q: \mathfrak{g} \rightarrow \mathfrak{g}_1$ the corresponding quotient morphism. Further let $A := \langle \exp(\mathfrak{a}) \rangle$, $G_1 := G/A$ and $D_1 := G_1 \text{Exp}(iD_{h,1})$. Then the quotient morphism $\tilde{q}: D \rightarrow D_1$ induced by q gives rise to an injection*

$$\tilde{q}^*: \mathcal{P}_{G_1 \times G_1} (D_1^2) \rightarrow \mathcal{P}_{G \times G} (D^2), \quad K_1 \mapsto K := K_1 \circ \tilde{q}$$

and the following statements are equivalent:

- (a) $\lim_{X \rightarrow \partial D_h} K(\text{Exp}(iX), \text{Exp}(iX)) = \infty$.
- (b) $\lim_{X \rightarrow \partial D_{h,1}} K_1(\text{Exp}(iX), \text{Exp}(iX)) = \infty$.
- (c) $\lim_{X \rightarrow \partial D_1} K_1(\text{Exp}(iX), \text{Exp}(iX)) = \infty$.

Proof. (i) Obviously, j is holomorphic and $G \times G$ -equivariant. Further the closedness of j follows from Lemma I.6(i) together with the Polar Decomposition of a complex Ol'shanskiĭ semigroup. The second assertion is clear.

(ii) (a) \Rightarrow (b) is obvious.

(b) \Rightarrow (a): This follows from the biinvariance of K .

The remaining statement follows from Lemma I.6(iii).

(iii) (a) \iff (b): In view of $q(\partial D_h) = \partial D_{h,1}$ (cf. Lemma I.7), this follows from the equivalence of (a) and (b) in (ii).

(b) \Rightarrow (c) is clear.

(c) \Rightarrow (b) follows from the $\text{Inn}(\mathfrak{g})$ -invariance of the map

$$D_{h,1} \rightarrow \mathbb{R}^+, \quad X \mapsto K_1(\text{Exp}(iX), \text{Exp}(iX))$$

together with Lemma I.4. □

II. A kernel tending to infinity at the boundary.

In this section we construct a kernel $K \in \mathcal{P}_{G \times G}(D^2)$ tending to infinity at the boundary. In view of Proposition I.8, this reduces to the case where $D = \Gamma_G(W)$ is a pointed complex Ol'shanskiĭ semigroup. But Proposition I.8(iii) tells us even more: We only have to check that $\lim_{X \rightarrow \partial C} K(\text{Exp}(iX), \text{Exp}(iX)) = \infty$. To start out we need some notation concerning highest weight representations and their associated characters.

Definition II.1. Let Δ^+ be a positive system.

(a) For a $\mathfrak{g}_{\mathbb{C}}$ -module V and $\beta \in (\mathfrak{t}_{\mathbb{C}})^*$ we write $V^\beta := \{v \in V : (\forall X \in \mathfrak{t}_{\mathbb{C}}) X.v = \beta(X)v\}$ for the *weight space of weight β* and $\mathcal{P}_V = \{\beta : V^\beta \neq \{0\}\}$ for the set of weights of V .

(b) Let V be a $\mathfrak{g}_{\mathbb{C}}$ -module and $v \in V^\lambda$ a $\mathfrak{t}_{\mathbb{C}}$ -weight vector. We say that v is a *primitive element of V* (with respect to Δ^+) if $\mathfrak{g}_{\mathbb{C}}^\alpha.v = \{0\}$ holds for all $\alpha \in \Delta^+$.

(c) A $\mathfrak{g}_{\mathbb{C}}$ -module V is called a *highest weight module* with highest weight λ (with respect to Δ^+) if it is generated by a primitive element of weight λ .

(d) Let $\lambda \in \mathfrak{t}^*$ be dominant integral w.r.t. Δ_k^+ and $F(\lambda)$ the corresponding highest weight module for $\mathfrak{k}_{\mathbb{C}}$. Assume that Δ^+ is \mathfrak{k} -adapted and set $\mathfrak{p}^\pm = \bigoplus_{\alpha \in \Delta_n^\pm} \mathfrak{g}_{\mathbb{C}}^\alpha$. We define the *generalized Verma module* by

$$N(\lambda) := \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \otimes_{\mathcal{U}(\mathfrak{k}_{\mathbb{C}} + \mathfrak{p}^+)} F(\lambda).$$

Note that $N(\lambda)$ is a highest weight module for $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ with highest weight λ . We denote by $L(\lambda)$ the unique irreducible quotient of $N(\lambda)$.

(e) Let G be a connected Lie group with Lie algebra \mathfrak{g} . We write K for the analytic subgroup of G corresponding to \mathfrak{k} . Let (π, \mathcal{H}) be a unitary representation of G . A vector $v \in \mathcal{H}$ is called *K -finite* if it is contained in

a finite dimensional K -invariant subspace. We write $\mathcal{H}^{K,\omega}$ for the space of analytic K -finite vectors.

(f) An irreducible unitary representation (π, \mathcal{H}) of G is called a *highest weight representation* w.r.t. Δ^+ with highest weight $\lambda \in \mathfrak{it}^*$ if $\mathcal{H}^{K,\omega}$ is a highest weight module for $\mathfrak{g}_{\mathbb{C}}$ w.r.t. Δ^+ and highest weight λ . We say that the irreducible highest weight module $L(\lambda)$ is *unitarizable* if there exists a unitary highest weight representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of \tilde{G} with $\mathcal{H}_\lambda^{K,\omega} \cong L(\lambda)$ as $\mathfrak{g}_{\mathbb{C}}$ -modules. We write $HW(G, \Delta^+) \subset \mathfrak{it}^*$ for the set of highest weights corresponding to unitary highest weight representations of G w.r.t. Δ^+ and write $HW(\Delta^+) := HW(\tilde{G}, \Delta^+)$ for the set of all highest weights w.r.t. Δ^+ which correspond to a unitarizable $L(\lambda)$.

(g) Let $\lambda \in HW(\Delta^+)$. We call λ *singular* if the natural map $N(\lambda) \rightarrow L(\lambda)$ has a non-trivial kernel and *non-singular* otherwise. \square

For each unitary representation (π, \mathcal{H}) of G we write (π^*, \mathcal{H}^*) for the corresponding dual representation. Let $B_2(\mathcal{H})$ be the space of Hilbert Schmidt operators on \mathcal{H} . We define a representation of $G \times G$ on $B_2(\mathcal{H})$ by

$$\pi^c: G \times G \rightarrow U(B_2(\mathcal{H})), \quad \pi^c(g_1, g_2).A := \pi(g_2)A\pi(g_1)^*.$$

Note that there is a canonical isomorphism between $(\pi^* \otimes \pi, \widehat{\mathcal{H}})$ and $(\pi^c, B_2(\mathcal{H}))$.

Now we fix a positive system Δ^+ associated to W_{\max} .

Recall from [HiNe96, Th. 3.6, Th. B] that each highest weight representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of G extends to an holomorphic representation of $\Gamma_G(W_{\max})$ denoted by the same symbol. Moreover all operators $\pi_\lambda(s), s \in \Gamma_G(W_{\max}^0)$, are of trace class (cf. [Ne94, Th. III.8]), so that $\Theta_\lambda(s) := \text{tr } \pi_\lambda(s)$ makes sense for all $s \in \Gamma_G(W_{\max}^0)$. We call Θ_λ the *character* of $(\pi_\lambda, \mathcal{H}_\lambda)$ and note that Θ_λ is holomorphic on $\Gamma_G(W_{\max}^0)$ (cf. [Ne94, Th. IV.11]).

Associated to a \mathfrak{k} -adapted positive system, we define the function

$$\phi: iC_{\max}^0 \rightarrow \mathbb{R}^+, \quad X \mapsto \frac{1}{\prod_{\alpha \in \Delta_n^+} (1 - e^{-\alpha(X)})^{m_\alpha}},$$

where $m_\alpha := \dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}}^\alpha$ for all $\alpha \in \Delta$.

Lemma II.2. *Let $\lambda \in HW(G, \Delta^+)$ be non-singular and $(\pi_\lambda, \mathcal{H}_\lambda)$ an associated highest weight representation of G . If Θ_λ^K denotes the character of $F(\lambda)$, then*

$$\Theta_\lambda(\text{Exp } X) = \phi(X)\Theta_\lambda^K(\text{Exp}(X))$$

for all $X \in iC_{\max}^0$.

Proof. [Kr97, Lemma IV.8(i)]. \square

Proposition II.3. *Let K be a connected Lie group with compact Lie algebra \mathfrak{k} . Let \mathfrak{t} be a Cartan subalgebra of \mathfrak{k} , Δ_k^+ be a positive system of roots and $\lambda \in \mathfrak{it}^*$ be a dominant analytically integral element. Further let Θ_λ^K denote*

the character of the holomorphic representation $(\pi_\lambda^K, F(\lambda))$ of $K_{\mathbb{C}}$ and set $d_\lambda := \dim F(\lambda)$.

- (i) $(\forall X \in \mathfrak{it}) e^{\lambda(X)} \leq \Theta_\lambda^K(\exp_{K_{\mathbb{C}}}(X)) \leq d_\lambda \sup_{w \in \mathcal{W}_k} e^{\lambda(w.X)}$.
- (ii) If $\|\cdot\|$ denotes a norm on \mathfrak{it}^* , then there exists a constant $c > 0$ and an element $n \in \mathbb{N}$ such that

$$d_\lambda \leq c\|\lambda\|^n + c$$

holds for all integral elements λ .

Proof. (i) Let $\{v_j : 1 \leq j \leq d_\lambda\}$ be an orthonormal basis of weight vectors of $F(\lambda)$. For each j let α_j be the weight corresponding to v_j . Then we have for all $X \in \mathfrak{t}_{\mathbb{C}}$ that

$$(2.1) \quad \Theta_\lambda^K(\exp_{K_{\mathbb{C}}}(X)) = \sum_{j=1}^{d_\lambda} \langle \pi_\lambda^K(\exp_{K_{\mathbb{C}}}(X)).v_j, v_j \rangle = \sum_{j=1}^{d_\lambda} e^{\alpha_j(X)}.$$

Since $\lambda = \alpha_j$ for some $1 \leq j \leq d_\lambda$, this proves the first inequality.

To prove the second inequality, we first observe that both $\Theta_\lambda^K(\exp_{K_{\mathbb{C}}}(X))$ and $d_\lambda \sup_{w \in \mathcal{W}_k} e^{\lambda(w.X)}$ considered as functions of $X \in \mathfrak{it}^*$ are invariant under the Weyl group \mathcal{W}_k . Thus we may assume that $X \in -(\Delta_k^+)^* := \{Y \in \mathfrak{it}^* : (\forall \alpha \in \Delta_k^+) \alpha(X) \leq 0\}$. Since $\lambda(X) = \sup_{1 \leq j \leq d_\lambda} \alpha_j(X)$ whenever $X \in -(\Delta_k^+)^*$, (2.1) implies that $\Theta_\lambda^K(\exp_{K_{\mathbb{C}}}(X)) \leq d_\lambda e^{\lambda(X)}$, concluding the proof of (i).

- (ii) This is a direct consequence of the Weyl Dimension Formula. \square

Corollary II.4. *If $\lambda \in HW(G, \Delta^+)$ is non-singular, then there exist a constant $c > 0$ and $n \in \mathbb{N}$ such that*

$$e^{\lambda(X)} \leq \Theta_\lambda(\text{Exp}(X)) \leq (c\|\lambda\|^n + c)\phi(X) \sup_{w \in \mathcal{W}_k} e^{\lambda(w.X)}$$

holds for all $X \in iC_{\max}^0$.

Proof. As $1 = \inf_{X \in iC_{\max}^0} \phi(X)$, the corollary follows from Lemma II.2 and Proposition II.3. \square

Lemma II.5. *Let V be a finite dimensional real vector space, $C \subseteq V$ a convex cone with non-empty interior, $\Gamma \subseteq V$ a lattice and $Q \subseteq V$ a compact subset. Then*

$$\Gamma(C, Q) := \{\gamma \in \Gamma : \gamma \in C, \gamma + Q \subseteq C\}$$

is an additive subsemigroup of Γ , $\overline{\mathbb{R}^+\Gamma(C, Q)}$ is a closed convex cone and the following equalities hold:

- (i) $\overline{\mathbb{R}^+\Gamma(C, Q)} = \overline{C}$.
- (ii) $\Gamma(C, Q)^* = C^*$.

Proof. First we show that $\Gamma(C, Q)$ is an additive semigroup. Let $\gamma_1, \gamma_2 \in \Gamma(C, Q)$. Then $\gamma_1 + \gamma_2 \in C$ since C is a convex cone and further we have for all $x \in Q$

$$\gamma_1 + \gamma_2 + x = \gamma_1 + \underbrace{(\gamma_2 + x)}_{\in C} \in C + C = C,$$

proving that $\Gamma(C, Q)$ is an additive semigroup.

It follows in particular that $\mathbb{Q}^+\Gamma(C, Q)$ is an additive semigroup and hence the same holds for $\overline{\mathbb{R}^+\Gamma(C, Q)} = \overline{\mathbb{Q}^+\Gamma(C, Q)}$. This proves that $\overline{\mathbb{R}^+\Gamma(C, Q)}$ is a closed convex cone.

(i) Since $\Gamma(C, Q) \subseteq C$ we obtain in particular that $\overline{\mathbb{R}^+\Gamma(C, Q)} \subseteq \overline{C}$.

To prove the converse inclusion, we assume that $\overline{\mathbb{R}^+\Gamma(C, Q)} \neq \overline{C}$. Then we find an open ball $B \subseteq V$ such that $B \subseteq C^0 \setminus \overline{\mathbb{R}^+\Gamma(C, Q)}$. Since $C^0 \setminus \overline{\mathbb{R}^+\Gamma(C, Q)}$ is an open cone, this implies in particular that

$$(2.2) \quad (\forall \lambda > 0) \quad \lambda B \cap \Gamma(C, Q) = \emptyset.$$

If λ is sufficiently large, then we have $\lambda B \subseteq C - x$ for all $x \in Q$, because $B + \frac{1}{\lambda}x$ is contained in C for sufficiently large λ and Q is compact. Let $\lambda_0 > 0$ such that $\lambda B \subseteq C - x$ for all $x \in Q$ and $\lambda > \lambda_0$.

In view of this and $\Gamma(C, Q) = \Gamma \cap C \cap \bigcap_{x \in Q} (C - x)$, (2.2) implies in particular that

$$(\forall \lambda > \lambda_0) \quad \Gamma \cap C \cap \lambda B = \Gamma(C, Q) \cap \lambda B = \emptyset;$$

a contradiction, concluding the proof of (i).

(ii) This follows from (i) and $\overline{\mathbb{R}^+\Gamma(C, Q)}^* = \Gamma(C, Q)^*$. \square

Let $\mathfrak{t}_e := \{X \in \mathfrak{t} : \exp(X) = \mathbf{1}\}$ and note that \mathfrak{t}_e is a lattice in $\text{span}\{\mathfrak{t}_e\}$. Hence we find a lattice $\Gamma \in i\mathfrak{t}^*$ which is contained in the set $\{\alpha \in i\mathfrak{t}^* : (\forall X \in \mathfrak{t}_e) \alpha(X) \in 2\pi i\mathbb{Z}\}$. From now on we fix a lattice $\Gamma \subseteq i\mathfrak{t}^*$ having this property.

Lemma II.6. *Let $W \subseteq \mathfrak{g}$ be a pointed closed $\text{Inn}(\mathfrak{g})$ -invariant cone with non-empty interior, Δ^+ a positive system satisfying $C_{\min} \subseteq C = W \cap \mathfrak{t} \subseteq C_{\max}$ and $\rho := \frac{1}{2} \sum_{\alpha \in \Delta^+} m_\alpha \alpha$. Then*

$$\Gamma(i \text{int } C^*, 2\rho) := \{\lambda \in \Gamma : \lambda \in i \text{int } C^*, \lambda + 2\rho \in i \text{int } C^*\}$$

consists of non-singular G -analytically integral elements.

Proof. First note that Γ consists of G -analytically integral elements, hence the same holds for the subset $\Gamma(i \text{int } C^*, 2\rho)$. We claim that $\Gamma(i \text{int } C^*, 2\rho) \subseteq \Gamma(i \text{int } C^*, \rho)$. In fact, Lemma II.5 implies that $\Gamma(i \text{int } C^*, 2\rho)$ is an additive semigroup, so that $\lambda \in \Gamma(i \text{int } C^*, 2\rho)$ implies that $2\lambda \in \Gamma(i \text{int } C^*, 2\rho)$ which means that $2\lambda + 2\rho \in i \text{int } C^*$ or equivalently $\lambda + \rho \in \Gamma(i \text{int } C^*, \rho)$. This proves the claim.

Further, $C_{\min} \subseteq C$ implies that $i \text{int } C^* \subseteq i \text{int } C_{\min}^*$ and we therefore get

$$\Gamma(i \text{int } C^*, 2\rho) \subseteq \{\lambda \in \Gamma : \lambda + \rho \in i \text{int } C_{\min}^*\}.$$

In view of [Ne99, Ch. IX], this implies that all elements of $\Gamma(\text{int } C^*, 2\rho)$ are non-singular. \square

Lemma II.7. *Let V be a finite dimensional real vector space, $C \subseteq V$ a convex pointed cone with non-empty interior, $\Gamma \subseteq V^*$ a lattice and $Q \subseteq V^*$ a compact subset. We fix a norm $\|\cdot\|$ on V^* with $\|\gamma\| \geq 1$ for all $\gamma \in \Gamma \setminus \{0\}$ and consider for each $N \in \mathbb{N}_0$ the mapping*

$$F^N: V \rightarrow \mathbb{R}^+ \cup \{\infty\}, \quad x \mapsto \sum_{\gamma \in \Gamma(\text{int } C^*, Q)} \|\gamma\|^N e^{-\gamma(x)}.$$

- (i) *For all $N \in \mathbb{N}_0$ the series defining F^N converges compactly on C^0 . In particular, $F^N|_{C^0}$ is continuous.*
- (ii) *For all $N \geq 1$ we have $\lim_{\substack{x \rightarrow \partial C \\ x \in C^0}} F^N(x) = \infty$.*

Proof. (i) Since C^* is pointed we find for every $x \in C^0$ a constant $C_x > 0$ such that $\|\alpha\| \leq C_x \alpha(x)$ holds for every $\alpha \in C^*$. Thus we find for every compact subset $K \subseteq C^0$ a constant $C_K > 0$ such that

$$(\forall x \in K)(\forall \alpha \in C^*) \|\alpha\| \leq C_K \alpha(x).$$

This in turn implies that

$$\sup_{x \in K} \sum_{\gamma \in \Gamma(\text{int } C^*, Q)} \|\gamma\|^N e^{-\gamma(x)} \leq \sum_{\gamma \in \Gamma(\text{int } C^*, Q)} \|\gamma\|^N e^{-\frac{1}{C_K} \|\gamma\|} < \infty,$$

proving (i).

(ii) As $\|\gamma\| \geq 1$ for all $\gamma \in \Gamma \setminus \{0\}$, we have $F^1 \leq F^N$ for all $N \in \mathbb{N}$ and hence we only have to prove the assertion for $N = 1$. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in C^0 converging to $x \in \partial C$. Then there exists an element $0 \neq \alpha \in \partial C^*$ such that $\alpha(x) = 0$. Choose $\alpha_n \in \mathbb{R}^+ \alpha$ with $\alpha_n(x_n) = 1$. We claim that $\lim_{n \rightarrow \infty} \|\alpha_n\| = \infty$. Indeed, otherwise we find a number $L > 0$ and a subsequence $(\alpha_{n_k})_{k \in \mathbb{N}}$ such that $\alpha_{n_k} \in [0, L]\alpha$. But then

$$1 = \alpha_{n_k}(x_{n_k}) \leq L \alpha(x_{n_k}) \rightarrow 0$$

yields a contradiction and proves the claim.

Next we claim that there exists a constant $c > 0$ and elements $\gamma_n \in \Gamma(\text{int } C^*, Q)$ such that $\|\gamma_n - \alpha_n\| < c$ for all $n \in \mathbb{N}$. Let $r > 0$ such that $B(\beta, r) := \{\mu \in V^* : \|\beta - \mu\| \leq r\}$ intersects Γ for all $\beta \in V^*$. Now choose $\mu \in \text{int } C^*$ such that $B(\mu + t\alpha, r) + Q \subseteq \text{int } C^*$ holds for all $t \geq 0$. Then we find elements $\gamma_n \in \Gamma(\text{int } C^*, Q) \cap B(\mu + \alpha_n, r)$. These elements γ_n satisfy $\|\gamma_n - \alpha_n\| \leq c$ with $c = r + \|\mu\|$, proving our second claim.

Now we get

$$\begin{aligned} F_1(x_n) &= \sum_{\gamma \in \Gamma(\text{int } C^*, \mathcal{Q})} \|\gamma\| e^{-\gamma(x_n)} \geq \|\gamma_n\| e^{-(\gamma_n(x_n) - \alpha_n(x_n))} e^{-\alpha_n(x_n)} \\ &\geq \frac{\|\gamma_n\|}{e} e^{-\|\gamma_n - \alpha_n\| \cdot \|x_n\|} \geq \frac{1}{e} e^{-c\|x_n\|} (\|\alpha_n\| - c), \end{aligned}$$

and so

$$\lim_{n \rightarrow \infty} F_1(x_n) \geq \lim_{n \rightarrow \infty} \frac{1}{e} e^{-c\|x_n\|} (\|\alpha_n\| - c) = \infty.$$

This proves (ii). \square

Theorem II.8. *Let $S = \Gamma_G(W)$ be a pointed complex Ol'shanskii semi-group. Then for all $N \in \mathbb{N}$ the prescription*

$$K^N : S^0 \times S^0 \rightarrow \mathbb{C}, \quad (z, w) \mapsto \sum_{\lambda \in \Gamma(i \text{int } C^*, 2\rho)} \|\lambda\|^N \Theta_\lambda(zw^*)$$

defines an element of $\mathcal{P}_{G \times G}(S^{0^2})$ satisfying

$$\lim_{z \rightarrow \partial S^0} K^N(z, z) = \infty.$$

Proof. First we show that $K^N \in \mathcal{P}_{G \times G}(S^{0^2})$ for all $N \in \mathbb{N}$. Since all kernels

$$K^\lambda : S^0 \times S^0 \rightarrow \mathbb{C}, \quad (z, w) \mapsto \Theta_\lambda(zw^*)$$

belong to $\mathcal{P}_{G \times G}(S^{0^2})$ (cf. [Ne94, Th. IV.11]), we only have to show that the series defining K^N converges uniformly on compact subsets. In view of (1.2), a series of holomorphic positive definite kernels on a complex manifold converges compactly if and only if it converges uniformly on compact subsets on the diagonal. Therefore the bi-invariance of the kernels K^λ together with the Polar Decomposition of S^0 imply that it suffices to prove the compact convergence of the series defining the function

$$\Theta^N : iW^0 \rightarrow \mathbb{R}^+, \quad X \mapsto \sum_{\lambda \in \Gamma(i \text{int } C^*, 2\rho)} \|\lambda\|^N \Theta_\lambda(\text{Exp}(X)).$$

If $C(W^0)^G$ denotes the $\text{Ad}(G)$ -invariant continuous functions on W^0 endowed with the topology of uniform convergence on compact subsets, then [Ne96b, Prop. III.6] entails that the restriction mapping $C(W^0)^G \rightarrow C(C^0)^{\mathcal{W}_\mathfrak{k}}$ is an isomorphism of Fréchet spaces. Thus we have reduced the problem to showing the compact convergence of the series defining $\Theta^N|_{C^0}$. In view of Corollary II.4, we therefore have to prove that the series defined by

$$F^N|_{iC^0} : iC^0 \rightarrow \mathbb{R}^+, \quad X \mapsto \sum_{\lambda \in \Gamma(i \text{int } C^*, 2\rho)} \|\lambda\|^N e^{\lambda(X)}$$

converges compactly. But this is exactly the contents of Lemma II.7(i).

In order to prove the second assertion, Proposition I.8(iii) implies that we only have to check that $\lim_{\substack{X \rightarrow \partial C \\ X \in C^0}} K^N(\text{Exp}(iX), \text{Exp}(iX)) = \infty$. According to Corollary II.4, this follows from $\lim_{\substack{X \rightarrow \partial C \\ X \in C^0}} F^N(X) = \infty$ which is the contents of Lemma II.7(ii). This proves the theorem. \square

Corollary II.9. *Let $D \subseteq \Gamma_G(W_{\max}^0)$ be a bi-invariant Stein domain. Then there exists a kernel $K \in \mathcal{P}_{G \times G}(D^2)$ such that*

$$\lim_{z \rightarrow \partial D} K(z, z) = \infty.$$

Proof. In view of Proposition I.8(ii), we may assume that $D_h = D_h^\sharp$ is a cone, and hence Proposition I.8(iii) implies that we even may assume that $D_h = D_{h,1}^\sharp$ is a pointed cone. Now the corollary follows from Theorem II.8. \square

Remark II.10. One can modify Γ a little bit without affecting the contents Theorem II.8 as follows. If one takes $\Gamma' = \Gamma \cup F$, where $F \subseteq HW(G, \Delta^+)$ is a finite set of highest weights, then Theorem II.8 remains true with Γ replaced by Γ' .

III. The equivariant embedding theorem.

In this section we apply the results of Section II to construct a kernel $K \in \mathcal{P}_{G \times G}(D^2)$ such that the map

$$e_K: D \mapsto \mathcal{H}_K^* \setminus \{0\}, \quad z \mapsto K_z$$

defines a $G \times G$ -equivariant holomorphic embedding with closed image.

Proposition III.1. *Let M be a complex manifold, S an involutive semi-group acting on M by holomorphic maps and $K \in \mathcal{P}_S(M^2)$. Then*

$$e_K: M \rightarrow \mathcal{H}_K^*, \quad z \mapsto K_z$$

is an S -equivariant holomorphic map.

Proof. From the S -invariance of K it follows that

$$e_K(s.z) = K_{s.z} = \pi_K(s).K_z = \pi_K(s).e_K(z)$$

for all $s \in S$ and $z \in M$, proving the S -equivariance of the map e_K . It remains to show that e_K is holomorphic. As \mathcal{H}_K is a Hilbert space, e_K is holomorphic if and only if the following conditions are satisfied:

- (1) e_K is locally bounded.
- (2) There exists a total subset $T \subseteq \mathcal{H}_K$ such that the mappings $M \rightarrow \mathbb{C}$, $z \mapsto K_z(f) = \langle f, K_z \rangle$ are holomorphic for all $f \in T$.

If $Q \subseteq D$ is a compact subset, then we have

$$\sup_{z \in Q} \|K_z\|^2 = \sup_{z \in Q} K(z, z) < \infty,$$

proving (1). Finally $T = \{K_w : w \in M\}$ is a total subset in \mathcal{H}_K and $z \mapsto K_z(K_w) = \langle K_w, K_z \rangle = K(z, w)$ is holomorphic because $K \in \text{Hol}(M \times \overline{M})$. This proves (2) and concludes the proof of the lemma. \square

Corollary III.2. *If $K \in \mathcal{P}_{G \times G}(D^2)$ is non-zero, then $K_z \neq 0$ for all $z \in D$ and the mapping*

$$e_K : D \rightarrow \mathcal{H}_K^* \setminus \{0\}, \quad z \mapsto K_z$$

is $G \times G$ -equivariant and holomorphic.

Proof. It follows from [Ne97, Lemma III.6] that $K_z \neq 0$ for all $z \in D$ and thus the map e_K is well defined. Now the assertions follow from Proposition III.1. \square

Definition III.3. A connected Lie group is called a (CA)-Lie group if $\text{Ad}(G) \subseteq \text{Aut}(\mathfrak{g})$ is closed. Note that all connected reductive and nilpotent Lie groups are (CA)-Lie groups. \square

The next lemma is our key observation. The proof depends heavily on the special choice of the lattice $\Gamma(i \text{int } C^*, 2\rho)$ and is a little bit tricky.

Lemma III.4. *Let $S = \Gamma_G(W)$ be a pointed complex Ol'shanskiĭ semi-group and suppose that G is a (CA)-Lie group. Let K^N , $N \in \mathbb{N}$, be as in Theorem II.8 and $(K_{s_n}^N)_{n \in \mathbb{N}}$ a convergent sequence in \mathcal{H}_{K^N} with limit different from zero. Further let Z denote the center of G .*

- (i) *The set $\{s_n Z : n \in \mathbb{N}\}$ is relatively compact in S^0/Z .*
- (ii) *If, in addition, Z is compact, then $\{s_n : n \in \mathbb{N}\}$ is relatively compact in S^0 .*

Proof. (i) For simplicity, we write K instead of K^N . Since $K = \sum_{\lambda} \|\lambda\|^N K^{\lambda}$ is a direct sum of positive definite kernels corresponding to inequivalent irreducible unitary representations of $G \times G$, it follows in particular that $\mathcal{H}_K = \widehat{\bigoplus}_{\lambda} \mathcal{H}_{K^{\lambda}}$ (cf. [Ne99, Th. I.11, Rem. I.12(a)]). Thus $(K_{s_n})_{n \in \mathbb{N}}$ being a convergent sequence with non-zero limit implies in particular that all sequences $(K_{s_n}^{\lambda})_{n \in \mathbb{N}}$ are convergent and at least one limit $f^{\lambda} = \lim_{n \rightarrow \infty} K_{s_n}^{\lambda}$ is different from zero.

Step 1: The set $\{s_n Z : n \in \mathbb{N}\}$ is relatively compact in S/Z .

Let $\lambda \in \Gamma(i \text{int } C^*, 2\rho)$ be such that $(K_{s_n}^{\lambda})_{n \in \mathbb{N}}$ converges with limit different from zero. Let

$$\chi_{\lambda} : Z \rightarrow S^1, \quad z = \exp(X) \mapsto e^{-\lambda(X)}$$

and note that χ_{λ} is an element of \widehat{Z} .

The *Bergman space* corresponding to the character χ_λ is defined as

$$\mathcal{B}^2(S/Z, \chi_\lambda) = \left\{ f \in \text{Hol}(S^0) : (\forall z \in Z, s \in S^0) f(sz) = \chi_\lambda(z)^{-1} f(s), \right. \\ \left. \|f\|_2^2 := \int_{S^0/Z} |f(s)|^2 d\mu_{S^0/Z}(sZ) < \infty \right\},$$

where $\mu_{S^0/Z}$ denotes the canonical left S -invariant measure on S^0/Z (cf. [Kr98, Sect. II]). Recall from [Kr98, Prop. II.4, Th. IV.5] that $\mathcal{B}^2(S/Z, \chi_\lambda)$ is a closed subspace of the Hilbert space $L^2(S/Z, \chi_\lambda)$ and that there exists a positive constant $c > 0$ such that the prescription

$$\mathcal{H}_{K^\lambda} \rightarrow \mathcal{B}^2(S/Z, \chi_\lambda), \quad K_z^\lambda \mapsto cK_z^\lambda$$

defines an $S \times S$ -equivariant isometric embedding.

We obtain in particular that $(K_{s_n}^\lambda)_{n \in \mathbb{N}}$ is a convergent sequence in $\mathcal{B}^2(S/Z, \chi_\lambda)$ with limit $f^\lambda \neq 0$. To obtain a contradiction, we now assume that there exists a subsequence of $(s_n Z)_{n \in \mathbb{N}}$ leaving every compact subset of S/Z . To avoid further notation we denote this subsequence again by $(s_n Z)_{n \in \mathbb{N}}$. Note that $(s_n^* Z)_{n \in \mathbb{N}}$ also leaves every compact subset of S/Z , since the involution on S induces an involution $*$: $S/Z \rightarrow S/Z$. We write

$$\rho: S \rightarrow B(\mathcal{B}^2(S/Z, \chi_\lambda)), \quad (\rho(s).f)(z) = f(zs)$$

for the right regular representation of S on $\mathcal{B}^2(S/Z, \chi_\lambda)$ and note that $(\rho, \mathcal{B}^2(S/Z, \chi_\lambda))$ is a holomorphic contraction representation of S (cf. [Kr98, Prop. II.4]). Let $s_0 \in S^0$. Then $\rho(s_0).K_{s_n}^\lambda \rightarrow \rho(s_0).f^\lambda$ and $\rho(s_0).f^\lambda \neq 0$ since $\rho(s_0)$ is an injective operator. It follows in particular that there exists a convergent subsequence of $(\rho(s_0).K_{s_n}^\lambda)_{n \in \mathbb{N}}$ converging to $\rho(s_0).f^\lambda$ pointwise. Note that $|K_s^\lambda| \in C_0(S/Z) \upharpoonright_{S^0/Z}$ for all $s \in S^0$ (cf. [Kr98, Prop. II.4]). Thus we obtain from

$$\begin{aligned} \rho(s_0).K_{s_n}^\lambda(z) &= K_{s_n}^\lambda(zs_0) = K^\lambda(zs_0, s_n) = K^\lambda(z, s_n s_0^*) \\ &= K^\lambda(s_n^* z, s_0^*) = K_{s_0^*}^\lambda(s_n^* z) \end{aligned}$$

for all $z \in S^0$ that $\rho(s_0).K_{s_n}^\lambda \rightarrow 0$ pointwise. This is a contradiction to $\rho(s_0).f^\lambda \neq 0$ and proves our **first** step.

Step 2: Every cluster point of $(s_n Z)_{n \in \mathbb{N}}$ lies in S^0/Z .

As the Polar Decomposition of S is inherited by S/Z , i.e., the mapping

$$G/Z \times W \rightarrow S/Z, \quad (gZ, X) \mapsto g \text{Exp}(iX)Z$$

is a homeomorphism, we can write $s_n = g_n \text{Exp}(X_n)$, where $g_n \in G$ and $X_n \in iW^0$. According to Step 1, we now may assume that both $(g_n Z)_{n \in \mathbb{N}}$ and $(X_n)_{n \in \mathbb{N}}$ converge. Let $X \in iW$ be the limit of $(X_n)_{n \in \mathbb{N}}$. Note that it suffices to show that $X \in iW^0$. As $(K_{s_n}^\lambda)_{n \in \mathbb{N}}$ is convergent with non-zero

limit, $(\|K_{s_n}\|^2)_{n \in \mathbb{N}}$ is a convergent sequence in \mathbb{R}^+ with positive limit. The bi-invariance of K further implies that

$$\begin{aligned} \|K_{s_n}\|^2 &= K(s_n, s_n) = K(g_n \operatorname{Exp}(X_n), g_n \operatorname{Exp}(X_n)) \\ &= K(\operatorname{Exp}(X_n), \operatorname{Exp}(X_n)), \end{aligned}$$

so that Step 2 follows from Theorem II.8.

(ii) This is a direct consequence of (i) and the compactness of Z . \square

Lemma III.5. *If $S = \Gamma_G(W)$ is a pointed complex Ol'shanskii semigroup, then there exists a finite set $F \subseteq HW(G, \Delta^+)$ such that all representations $\pi_{K^N}: S \rightarrow B(\mathcal{H}_{K^N})$, $N \in \mathbb{N}$, associated to $\Gamma' = \Gamma \cup F$ (cf. Remark II.10) are injective.*

Proof. Since S is pointed, it follows from [Kr98, Prop. V.7] that π_{K^N} is injective if and only if $\pi_{K^N}|_T$ is injective, where $T = \exp \mathfrak{t}$. Evaluation of the operators $\pi_{K^N}(t)$, $t \in T$, on highest weight vectors v_λ , $\lambda \in \Gamma(i \operatorname{int} C^*, 2\rho)$, now easily shows how one can choose F to obtain injective representations. For more details we refer to [Ne96a, Sect. V]. \square

Theorem III.6. *Let $D \subseteq \Gamma_G(W_{\max}^0)$ be a pointed bi-invariant Stein domain and suppose that G is a (CA)-Lie group and Z is compact. Then there exists a kernel $K \in \mathcal{P}_{G \times G}(D^2)$ such that the mapping*

$$e_K: D \rightarrow \mathcal{H}_K^* \setminus \{0\}, \quad z \mapsto K_z$$

defines a $G \times G$ -equivariant holomorphic embedding with closed range.

Proof. In view of Proposition I.8(i) and Lemma I.6(ii), we may assume that $D = S^0 = \Gamma_G(W^0)$ is a pointed open complex Ol'shanskii semigroup. Now let Γ' as in Lemma III.5 and $K = K^N$ for some $N \in \mathbb{N}$. As $K \neq 0$, the map e_K is a well defined $G \times G$ -equivariant holomorphic map (cf. Corollary III.2). Lemma III.5 implies that the representation $\pi_K: S \rightarrow B(\mathcal{H}_K)$ is injective so that \mathcal{H}_K separates points by [Kr98, Prop. V.10], which in turn means that e_K is injective.

Next we show that $\operatorname{ime} e_K$ is closed. In fact, if $K_{s_n} \rightarrow f \neq 0$, then Lemma III.4 implies that $(s_n)_{n \in \mathbb{N}}$ is a bounded sequence in S^0 with all accumulation points in S^0 . Thus we find a convergent subsequence $(s_{n_k})_{k \in \mathbb{N}}$ with limit $s \in S^0$. Now we get

$$f = \lim_{n \rightarrow \infty} K_{s_n} = \lim_{k \rightarrow \infty} K_{s_{n_k}} = \lim_{k \rightarrow \infty} e_K(s_{n_k}) = e_K(s) = K_s,$$

proving the closedness of $\operatorname{ime} e_K$.

Finally another easy application of Lemma III.4 shows that e_K is homeomorphic onto its image, concluding the proof of the theorem. \square

Example III.7 (The Bergman kernel associated to $\mathrm{Sl}(2, \mathbb{R})$). Let $G = \mathrm{Sl}(2, \mathbb{R})$ and $\mathfrak{g} := \mathfrak{sl}(2, \mathbb{R})$. We choose

$$U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

as a basis for \mathfrak{g} .

Then $\mathfrak{t} := \mathbb{R}U$ is a compactly embedded Cartan subalgebra. Let $\alpha \in i\mathfrak{t}^*$ be defined by $\alpha(U) = -2i$. The root system of \mathfrak{g} is given by $\Delta = \{\pm\alpha\}$ with root spaces $\mathfrak{g}_{\mathbb{C}}^{\alpha} = \mathbb{C}(T + iH)$ and $\mathfrak{g}_{\mathbb{C}}^{-\alpha} = \mathbb{C}(T - iH)$. We define a positive system by $\Delta^+ := \{\alpha\}$ and write κ for the Cartan-Killing form on \mathfrak{g} . Then the upper light cone

$$\begin{aligned} W &:= \{X = uU + tT + hH : u \geq 0, \kappa(X, X) \leq 0\} \\ &= \{X = uU + tT + hH : u \geq 0, h^2 + t^2 - u^2 \leq 0\} \end{aligned}$$

is an invariant pointed cone in \mathfrak{g} . Moreover, W is up to sign the unique invariant elliptic cone in \mathfrak{g} (cf. [HiNe93, Th. 7.25]). Thus up to isomorphism $S := \Gamma_G(W)$ is the unique complex Ol'shanskiĭ semigroup corresponding to G .

In the following we identify $\mathfrak{t}_{\mathbb{C}}$ with \mathbb{C} via the isomorphism $\mathfrak{t}_{\mathbb{C}} \rightarrow \mathbb{C}$, $\lambda \mapsto \lambda(iU)$. Then $HW(G, \Delta^+) = HW(G, W) = \{\lambda \in \mathbb{Z} : \lambda \leq 0\}$, $\lambda + 2\rho \in i \operatorname{int} C^*$ if and only if $\lambda \leq -3$, and thus $\Gamma(i \operatorname{int} C^*, 2\rho) = \{\lambda \in \mathbb{Z} : \lambda \leq -3\}$. The main point is that $\Gamma(i \operatorname{int} C^*, 2\rho)$ coincides with the weights in the decomposition of the Bergman kernel B of the *Bergman space*

$$\mathcal{B}^2(S) := \left\{ f \in \operatorname{Hol}(S^0) : \|f\|_2^2 := \int_{S^0} |f(s)|^2 d\mu_{G_{\mathbb{C}}}(s) < \infty \right\}$$

(cf. [Kr98, Th. IV.7]). Further one knows that

$$B = \sum_{\lambda \leq -3} \lambda(1 + \lambda)^2(4 - \lambda^2)K^{\lambda}$$

(cf. [Kr98, Th. IV.7, Ex. IV.8]) so that Theorem II.8 and Theorem III.6 imply that

$$\lim_{z \rightarrow \partial S} B(z, z) = \infty$$

and that the map

$$e_B : S^0 \rightarrow \mathcal{B}^2(S)^* \setminus \{0\}, \quad s \mapsto B_s$$

is a $G \times G$ -equivariant holomorphic embedding with closed range. \square

Problems III.8. (a) What is the reason for that one has to exclude zero in \mathcal{H}_K^* to obtain the closedness of the map e_K ?

(b) Given a pointed biinvariant Stein domain D , does there exist an equivariant closed embedding of D into a complex topological vector space E endowed with a continuous $G \times G$ -action?

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