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SMALL EXTENSIONS OF WITT RINGS

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We consider certain Witt ring extensions S of a noetherian Witt ring R obtained by adding one new generator. The conditions on the new generator are those known to hold when R is the Witt ring of a field F, S is the Witt ring of a field K and K/F is an odd degree extension. We show that if R is of elementary type then so is S.

The elementary type conjecture is a proposed classification of noetherian Witt rings. A potential source of counter-examples is as follows: Start with a field F where WF is known (necessarily of elementary type) then look at noetherian WK for extension fields K of F. Jacob and Ware [3] have shown that WK is again of elementary type when $[K:F]=2$. Here we look at the simplest case of odd degree extensions, again showing WK is of elementary type. We note that WF is noetherian iff $G(F) \equiv F^{\cdot}/F^{\cdot 2}$ is finite. Also when K/F has odd degree then $G(F) \cong F^{\cdot}K^{\cdot 2}/K^{\cdot 2}$ embeds into $G(K)$.

We will in fact work with abstract Witt rings R (as defined by Marshall [4]) with associated group of one dimensional forms $G(R)$. The small extensions considered here are as follows. Let H be a subgroup of $G(R)$. We say a Witt ring S is an H-extension of R if there exists an $\alpha \in G(S)$ such that:

(1) $G(S) = \{1,\alpha\}G(R)$, and
(2) For all $x \in G(R)$ we have:

$$D_S\langle 1,-x\rangle = \begin{cases} D_R\langle 1,-x\rangle, & \text{if } x \notin H \\ \{1,\alpha\}D_R\langle 1,-x\rangle & \text{if } x \in H \end{cases}$$

$$D_S\langle 1,-\alpha x\rangle = \{1,-\alpha x\}\left(D_R\langle 1,-x\rangle \cap H\right).$$

These conditions hold for $R = WF, S = WK$ when K/F is an odd degree extension and $[G(K):G(F)] = 2$ by [2, 4.7] (we note that [2, 4.7] should include the condition that $N_{K/F}(a) = 1$). No such field extensions are known. However, there are many examples of H-extensions of abstract Witt rings, which we determine inductively. This can be viewed as a first step in classifying extensions of noetherian Witt rings. It also helps the search for odd degree extensions K/F with $[G(K):G(F)] = 2$, while lessening the motivation for such a search.

The proof of the main result is a long series of technical lemmas, only one of which (3.1) has independent interest. However, most of the results mimic the expected behavior of the field extension case, evidence the elementary type conjecture holds. This is speculative since there may be no field extensions yielding H-extensions. However, we indulge in this suggestive speculation once, after (1.5).

For any group H, $H^{\cdot}$ denotes $H \setminus \{1\}$. The quaternionic mapping associated to R will be denoted by q. For $x \in G(R)$, $Q(x) = \{q(x,y) : y \in G(R)\}$ and for a subgroup H, $Q(H) = \{q(h,y) : h \in H, y \in G(R)\}$. The value set of $\langle 1, -x\rangle$ is $D\langle 1, -x\rangle = \{y \in G(R) : q(x,y) = 1\}$. We will often work with several Witt rings at once and write $q_R, Q_R(x)$ and $D_R\langle 1, -x\rangle$ to indicate these objects for R.

R is of local type if $|q(G(R), G(R))| = 2$. We let E_n denote the elementary 2-group of order n. The group ring $R[E_n]$ is again a Witt ring. An element $t \in G(R)$ is rigid if $D\langle 1, t\rangle = \{1, t\}$ and t is birigid if both t and $-t$ are rigid. The basic part of R, $B(R)$, consists of ± 1 and all $x \in G(R)$ with either x or $-x$ not rigid. $B(R)$ is a subgroup of $G(R)$ and $R = R_0[G(R)/B(R)]$, where R_0 is the Witt ring generated by $B(R)$. We express this last statement by writing $R_0 = W(B(R))$.

The product in the category of Witt rings is:

$$R_1 \sqcap R_2 = \{(r_1, r_2) : r_i \in R_i \quad \text{and} \quad \dim r_1 \equiv \dim r_2 \pmod 2\}.$$

If $R = R_1 \sqcap R_2$ then $G(R) = G(R_1) \times G(R_2)$ and:

$$D_R\langle (1,1), (x,y)\rangle = D_{R_1}\langle 1, x\rangle \times D_{R_2}\langle 1, y\rangle.$$

The radical of R is $\text{rad}(R) = \{x \in G(R) : D\langle 1, -x\rangle = G(R)\}$. We say R is degenerate if $\text{rad}(R) \neq 1$ and totally degenerate if $\text{rad}(R) = G(R)$. D_n denotes a totally degenerate Witt ring with square class group of order 2^n. There are two possibilities for D_n depending on whether -1 is a square or not. Specifically, D_n is either a product of n copies of $(\mathbb{Z}/2\mathbb{Z})[E_1]$ or n copies of $\mathbb{Z}/4\mathbb{Z}$. If R is degenerate then there exist uniquely determined n and non-degenerate Witt ring R_0 such that $R = D_n \sqcap R_0$. R_0 is the non-degenerate part of R.

R is of elementary type if it can be built from $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}$ and Witt rings of local type by a succession of group ring extensions (for some E_n) and products. The elementary type conjecture is that every noetherian Witt ring is of elementary type.

1. Group ring extensions.

To help the reader navigate the following six sections of lemmas, we point out here the highlights. We will prove S is of elementary type by induction on $|G(R)|$. If R is of local type then generally $H = 1$ or $G(R)$ and S is determined (2.3), (1.1). The exceptions occur when $R = L_{2,0}$ (there are three possible H's) and when $R = L_{2,1}$ (four possible H's). The resulting extensions are determined in (1.5).

If $R = R_0[E_n]$ is a group ring either H is one of three special subgroups or $S = S_0[E_n]$ with S_0 an H-extension of R_0 (1.5). If $R = R_1 \sqcap R_2$ is a product then S is generally a product of one factor with an H-extension of the other factor (5.7), (6.1). There are exceptional cases when $R_1 = \mathbb{Z}$ (5.9) or R_1 is totally degenerate (4.1).

Lemma 1.1. *Let S be an H-extension of R.*

(a) *If $H = 1$ then $S = R[E_1]$, with E_1 generated by α.*
(b) *If $H = G(R)$ then $S = D_1 \sqcap R$, with D_1 generated by α.*

Proof. Suppose first that $H = 1$. Then for all $g \in G(R)$ we have from the definition of an H-extension that $D_S\langle 1, -\alpha g\rangle = \{1, -\alpha g\}(D_R\langle 1, -g\rangle \cap H)$ and so αg is birigid. Thus $B(S) \subset G(R)$ and $G(S) = \{1, \alpha\}G(R)$. So by [4, 5.19] $S = R[E_1]$, where E_1 is generated by α.

Next suppose that $H = G(R)$. Then $D_S\langle 1, -\alpha\rangle = \{1, -\alpha\}(D_R\langle 1, -1\rangle \cap H) = G(S)$. Hence $\alpha \in \mathrm{rad}(S)$. Then by [4, pp. 105-106] $S = D_1 \sqcap R$, where D_1 is generated by α. □

When $H \neq 1$, which we will often assume in light of (1.1), we use the following notation (recall that $B(R)$ is the basic part of R):

$$\begin{aligned} T &= \bigcup_{h \in H^{\cdot}} D_R\langle 1, -h\rangle \\ T_0 &= \bigcup_{h \in H^{\cdot}} (D_R\langle 1, -h\rangle \setminus \{-h\}) \\ B(H) &= H \cap B(R) \\ BT &= \bigcup_{h \in B(H)^{\cdot}} D_R\langle 1, -h\rangle \\ BT_0 &= \bigcup_{h \in B(H)^{\cdot}} (D_R\langle 1, -h\rangle \setminus \{-h\}). \end{aligned}$$

Lemma 1.2. *Let S be an H-extension of R with $|H| > 1$. Then:*

(a) $\pm T_0 \subset B(R) \subset \pm T$,
(b) $B(S) = \pm\{1, \alpha\}T$.

Proof. First note that $D_S\langle 1, -\alpha\rangle = \{1, -\alpha\}H$, and $|H| > 1$ imply $\alpha \in B(S)$. If $x \in G(R) \setminus \pm T$ then $D_R\langle 1, \pm x\rangle \cap H = \{1\}$ so by the definition of H-extensions, $D_S\langle 1, \pm\alpha x\rangle = \{1, \pm\alpha x\}$. Hence $\alpha x \notin B(S)$, and as $\alpha \in B(S)$, $x \notin B(S)$. That is, x is birigid in S, and so also in R. Thus $x \notin B(R)$ and we have:

$$\begin{aligned} B(R) &\subset \pm T, \\ B(S) &\subset \pm\{1, \alpha\}T. \end{aligned}$$

Let $x \in T_0, x \neq -1$ so that for some $h \in H^{\cdot}, x \in D_R\langle 1, -h\rangle$ and $x \neq -h$. Then $D_R\langle 1, -x\rangle$ contains $-x, h$ which are distinct and not equal to 1. So $-x \in B(R)$ and also $x \in B(R)$. If $x = -1$ then again $x \in B(R)$. Thus $T_0 \subset B(R)$, and so $\pm T_0 \subset B(R)$ completing the proof of (a).

If $x \in T$ with $x \in D_R\langle 1, -h\rangle, h \in H^{\cdot}$ then

$$D_S\langle 1, -x\rangle = \{1, -\alpha x\}\,(D_R\langle 1, -x\rangle \cap H)$$

contains $\{1, -\alpha x, h, -\alpha x h\}$. Thus $-\alpha x \in B(S)$. Again $\alpha \in B(S)$ so $x \in B(S)$. This shows $\pm\{1, \alpha\}T \subset B(S)$, completing the proof of (b). □

Lemma 1.3. *If S is an H-extension of R with $|H| > 1$ then $B(R) = \pm BT$ and either:*

(a) *$B(R) = \pm B(H)$ and $B(S) = \pm\{1, \alpha\}H$, or*
(b) *$B(R) = \pm T$ and $B(S) = \{1, \alpha\}B(R)$.*

Proof. If $h \in H \setminus B(H)$ then $D_R\langle 1, -h\rangle = \{1, -h\}$. So:

$$\pm T = \pm BT \cup \pm(H \setminus B(H)). \tag{1.4}$$

Now $B(R) \subset \pm T$ by (1.2)(a) and $B(R) \cap \pm(H \setminus B(H)) = \emptyset$ so $B(R) \subset \pm BT$. Conversely,

$$\pm BT = \pm(BT \cup B(H)) \subset \pm T_0 \cup \pm B(H) \subset B(R),$$

which proves the first statement.

Now (1.4) gives:

$$\pm T = \pm BT \cup \pm H = B(R) \cup \pm H,$$

and (1.2)(b) gives:

$$B(S) = \pm\{1, \alpha\}T = \{1, \alpha\}B(R) \cup \pm\{1, \alpha\}H.$$

This expresses the group $B(S)$ as the union of two subgroups, hence either:

(i) $\{1, \alpha\}B(R) \subset \pm\{1, \alpha\}H$, or
(ii) $\pm\{1, \alpha\}H \subset \{1, \alpha\}B(R)$.

In case (i) $B(S) = \pm\{1, \alpha\}H$ and $B(R) \subset \pm\{1, \alpha\}H \cap G(R) = \pm H$. Hence $B(R) = \pm B(H)$. In case (ii) $B(S) = \{1, \alpha\}B(R)$ and $H \subset B(R)$. Then $H = B(H), BT = T$ and by the first statement $B(R) = \pm T$. □

Recall that any Witt ring R can be written as $R_0[G(R)/B(R)]$, where $R_0 = W(B(R))$, the Witt ring generated by $B(R)$. See [**4**, Chapter 5, Section 7] for details.

Proposition 1.5. *Let $R = R_0[E_n]$, with R_0 basic. Let S be an H-extension of R. Then:*

(a) *If $H = 1$ then $S = R_0[E_{n+1}]$.*
(b) *If $|H| > 1$ and $H \subset G(R_0)$ then $S = S_0[E_n]$, for some Witt ring S_0 that is an H-extension of R_0 (with the same α).*

(c) *If* $H \not\subset G(R_0)$ *and* $-1 \in H$ *then* $S = (D_1 \sqcap R_0[H/B(H)])\,[G(R)/H]$, *and* $G(R_0) \subset H$.
(d) *If* $H \not\subset G(R_0)$ *and* $-1 \notin H$ *then* $S = (\mathbb{Z} \sqcap R_0[H/B(H)])\,[G(R)/\pm H]$ *and* $G(R_0) \subset \pm H$.

Proof. If $H = 1$ then $S = R[E_1]$ by (1.1), which gives (a). So assume $|H| > 1$. Further suppose that $H \subset G(R_0)$ so that $B(H) = H \cap B(R) = H \cap G(R_0) = H$. Then $-H^{\cdot} \subset T \subset G(R_0)$. Hence $\pm H \subset \pm T \subset G(R_0)$. Thus if (1.3)(a) holds, so that $G(R_0) = B(R) = \pm B(H) = \pm H$, then $B(R) = \pm T$ also. So we are always in case (b) of (1.3). Then, since $G(S)/B(S) \cong G(R)/B(R)$, we have $S = S_0[E_n]$, where $S_0 = W(B(S))$. From $H \subset B(R)$ we have that S_0 is an H-extension of R_0.

Next suppose that $H \not\subset G(R_0)$. We still have that $G(R_0) = B(R)$. If $B(R) = \pm T$ then $H \subset B(R)$, contrary to our assumption. Thus we are in Case (a) of (1.3). First say that $-1 \in H$. Note that $-1 \in H \cap B(R) = B(H)$ also. Then $B(R) = B(H)$ and $B(S) = \{1, \alpha\}H$. Thus $S = S_0[G(R)/H]$, for $S_0 = W(\{1, \alpha\}H)$, since $G(S)/B(S) \cong G(R)/H$.

Now $D_{S_0}\langle 1, -\alpha\rangle = D_S\langle 1, -a\rangle \cap \{1, \alpha\}H = \{1, \alpha\}H$. Hence $\alpha \in \operatorname{rad}(S_0)$. Write $S_0 = D_1 \sqcap S_1$, for some Witt ring S_1 and with D_1, generated by α, being $\mathbb{Z}_2[E_1]$ or $\mathbb{Z}_4$ using [**4**, p. 104]. Note that $S_1 = W(H)$.

If $h \in H \setminus B(H)$ then $D_R\langle 1, -h\rangle = \{1, -h\}$ and $D_S\langle 1, -h\rangle = \{1, \alpha, -h, -\alpha h\}$ so that $D_{S_1}\langle 1, -h\rangle = D_S\langle 1, -h\rangle \cap H = \{1, -h\}$. Similarly, $D_{S_1}\langle h\rangle = \{1, h\}$. And if $h \in B(H)$ then $D_{S_1}\langle 1, -h\rangle = \{1, \alpha\}D_R\langle 1, -h\rangle \cap H = D_R\langle 1, -h\rangle \cap H = D_R\langle 1, -h\rangle \cap B(H) = D_{R_0}\langle 1, -h\rangle$. Thus $S_1 = R_0[H/B(H)]$.

We still suppose $H \not\subset G(R_0)$, so that we are in Case (a) of (1.3), and now say that $-1 \notin H$. Then $S = S_0[G(R)/\pm H]$ as $G(S)/B(S) = \{1, \alpha\}G(R)/\pm\{1, \alpha\}H \cong G(R)/\pm H$. Here $S_0 = W(\pm\{1, \alpha\}H)$.

Now $D_{S_0}\langle 1, -\alpha\rangle = \{1, \alpha\}H$ has index two in $G(S_0) = \pm\{1, \alpha\}H$. Further $\alpha \notin D_S\langle 1, -\alpha\rangle$ else $-1 \in \{1, -\alpha\}H$ and $-1 \in H$. Thus we have an orthogonal decomposition in the sense of [**1**]:

$$G(S_0) = \{1, \alpha\} \perp D_S\langle 1, -\alpha\rangle. \tag{1.6}$$

Set $S_1 = W(\{1, \alpha\})$ and $S_2 = W(D_S\langle 1, -\alpha\rangle)$. Note that $S_1 = \mathbb{Z}$ as $-1 \notin D_S\langle 1, -\alpha\rangle$. If $h \in \pm H \setminus \pm B(H)$ then $D_R\langle 1, \pm h\rangle = \{1, \pm h\}, D_S\langle 1, -h\rangle = \{1, \alpha, -h, -\alpha h\}$ and $D_S\langle 1, h\rangle = \{1, h\}$. Thus we have $D_{S_1}\langle 1, \pm h\rangle = D_S\langle 1, \pm h\rangle \cap \pm H = \{1, \pm h\}$. So $S_2 = S_3[H/B(H)]$, for some Witt ring S_3. S_2 is indeed a group ring as $H \not\subset G(R_0)$ implies $H \neq B(H)$.

We wish to apply [**1**, 3.4] and deduce that the decomposition (1.6) yields a product of Witt rings. First we need to handle the case where S_2 is decomposable, that is, $S_2 = \mathbb{Z}[E_1]$. In this case $|G(S_2)| = |D_S\langle 1, -\alpha\rangle| = 4$ so that $H = \{1, t\}$, for some $t \notin G(R_0)$ and $G(R_0) = \{\pm 1\}$ as $G(R_0) \subset \pm H$. Now $D_S\langle 1, -t\rangle = \{1, \alpha, -t, -\alpha t\}$ so we consider instead the orthogonal decomposition:

$$\{1, t\} \perp D_S\langle 1, -t\rangle.$$

Now $Q_S(\{1,t\})=\{1,q(t,-1)\}$ and $Q_S(D_S\langle 1,-t\rangle)=\{1,q(\alpha,-1),q(-t,-1),q(-\alpha t,-1)\}$, using $q(\alpha,-t)=q(\alpha,-1)$ and $q(t,-\alpha t)=1$. Then $Q_S(\{1,t\})\cap Q_S(D_S\langle 1,-t\rangle)=1$. Thus:

$$S_0=W(\{1,t\})\sqcap W(D_S\langle 1,-t\rangle).$$

Now $W(\{1,t\})=\mathbb{Z}$ as $-1\notin D_S\langle 1,-t\rangle$ and $W(D_S\langle 1,-t\rangle)=R_0[\{1,\alpha\}]$ since α is birigid in $qfst$ and $G(R_0)=\{\pm 1\}$. This gives the desired result of (d).

We now apply [**1**, 3.4] and obtain that either (1.6) yoilds a product or $Q_S(\{1,\alpha\})=Q_S(D_S\langle 1,-\alpha\rangle)$. But if $t\in H\setminus G(R_0)$ then $q(t,t)=q(t,-1)$ and $q(\alpha,\alpha)=q(\alpha,-1)$ are distinct, since $-1\notin D_S\langle 1,-\alpha t\rangle$. Hence the decomposition (1.6) in fact yields the product $S_0=S_1\sqcap S_2$. We have already seen that $S_1=\mathbb{Z}$ and that $S_2=S_3[H/B(H)]$. If $h\in\pm B(H)$ then $D_{S_3}\langle 1,-h\rangle=D_{S_1}\langle 1,-h\rangle=D_R\langle 1,-h\rangle=D_{R_0}\langle 1,-h\rangle$. So $S_3=R_0$. □

H-extensions are motivated by the behavior of odd degree field extensions. (1.5) and other lemmas do mimic the results expected in the field case, at least for valued fields. Pointing out these parallels is speculation (there may be no field extensions yielding an H-extension) but it is instructive.

Suppose then that K/F is an odd degree extension with WK an H-extension of WF. Further suppose that K has a 2-henselian valuation B with a basic residue field k_B. If α is not a unit, modulo squares, then α is birigid and we have case (a) of (1.5). Otherwise, α pushes down to $\overline{\alpha}\in k_B$. Set $A=B\cap F$. If A has a basic residue field k_A then $Wk_A\subset Wk_B$, $\overline{H}=D\langle 1,-\overline{\alpha}\rangle\cap k_B$ and Wk_B is an $\overline{H}$-extension of Wk_A. This is (1.5)(b). If k_A is not basic then H contains birigid elements of F and so $B(k_A)\subset\pm H$. This yields cases (c) and (d) of (1.5).

2. Local type rings.

Notation. For a subset $A\subset G(R)$ set:

$$C_R(A)=\bigcap_{a\in A}D_R\langle 1,-a\rangle.$$

Lemma 2.1. *Suppose that S is an H-extension of R. Suppose $k\in C_R(H)\setminus H$. Then $Q_R(H)\cap Q_R(k)=1$.*

Proof. Let $\rho\in Q_R(H)\cap Q_R(k)$ so that $\rho=q(h,x)=q(k,y)$ with $h\in H$, and $x,y\in G(R)$. Since $H\subset D_S\langle 1,-\alpha\rangle$ we have that $q(k,y)=q(\alpha x,h)$. By linkage there exists a $t\in G(S)$ such that:

$$q(k,y)=q(k,t)=q(\alpha x,t)=q(\alpha x,h).$$

The first equality gives $ty\in D_S\langle 1,-k\rangle=D_R\langle 1,-k\rangle\subset G(R)$, since $k\notin H$. Hence $t\in G(R)$. The second equality gives:

$$t\in D_S\langle 1,-\alpha xk\rangle\cap G(R)=D_R\langle 1,-xk\rangle\cap H.$$

This implies $t \in D_R\langle 1, -x\rangle$ since $H \subset D_R\langle 1, -k\rangle$. The third equality gives:

$$ht \in D_S\langle 1, -\alpha x\rangle \cap G(R) = D_R\langle 1, -x\rangle \cap H.$$

Hence $h \in D_R\langle 1, -x\rangle$ and $\rho = q(x, h) = 1$. □

The small Witt rings of local type will often be treated separately. The only local type Witt ring with two generators is $\mathbb{Z}$. There are two Witt rings of local type on four generators and both are group rings. If L is local type and $|G(L)| \geq 8$ then L is not a group ring. See [4, Chapter 5, Section 3] for details.

Lemma 2.2. *Suppose $R = L \sqcap R_2$, with L a Witt ring of local type and $|G(L)| \geq 8$. Let π_1 be the projection map of $G(R)$ onto $G(L)$. Let S be an H-extension of R. Then $\pi_1(H) = 1$ or $G(L)$.*

Proof. Set $B = \pi_1(H)$ and write $Q(L) = \{1, \rho\}$. Suppose that $B \neq 1$. If $(u, v) \in H$ with $u \neq 1$ then pick $r \in G(L) \setminus D_L\langle 1, -u\rangle$. We get $q((u, v), (r, 1)) = (\rho, 1)$ and so $(\rho, 1) \in Q(H)$.

Now $H \subset B \times G(R_2)$ so that $C_L(B) \times 1 = C_R(B \times G(R_2)) \subset C_R(H)$. If $C_L(B) = 1$ then $B = G(L)$ and we are done. So suppose there exists $1 \neq z \in C_L(B)$. Set $h = (z, 1) \in C_R(H)$. Then $Q(h) = \{1, (\rho, 1)\} \subset Q(H)$. By (2.1) we must have $h \in H$ and so $C_L(B) \times 1 \subset H$. In particular, $C_L(B) \subset B$.

Continue to let $h = (z, 1)$ where $1 \neq z \in C_L(B)$. We claim there exist an $h_1 \in H$ and an $x \in D_R\langle 1, -h\rangle$ such that $q(x, h_1) = (\rho, 1)$. Suppose not. We consider any $x = (a, 1)$ with $a \in D_L\langle 1, -z\rangle$. Then $q(x, (u, v)) = (\rho, 1)$ unless $a \in D_L\langle 1, -u\rangle$. Thus $D_L\langle 1, -z\rangle \subset D_L\langle 1, -u\rangle$ for all $(u, v) \in H$, that is, $D_L\langle 1, -z\rangle \subset C_L(B)$. Then $C_L(B) = D_L\langle 1, -z\rangle$ and so $B = \{1, z\}$. But $C_L(B) \subset B$ so that $D_L\langle 1, -z\rangle \subset \{1, z\}$ and $|G(L)| = 4$, a case we are excluding.

Thus there does exist an $h_1 \in H$ and an $x \in D_R\langle 1, -h\rangle$ such that $q(x, h_1) = (\rho, 1)$. Then in S we have $q(\alpha x, h_1) = (\rho, 1)$ and so $Q_S(h) \subset Q_S(\alpha x)$. We obtain:

$$|Q_S(\alpha x) \cap Q_S(h)| = 2.$$

On the other hand:

$$D_S\langle 1, -\alpha x\rangle \cap D_S\langle 1, -h\rangle = \{1, -\alpha x\}\,(D_R\langle 1, -x\rangle \cap H) \cap \{1, \alpha\} D_R\langle 1, -h\rangle.$$

Here $D_R\langle 1, -x\rangle \cap H \subset D_R\langle 1, -h\rangle$ as $h \in C_R(H)$. Also, by construction, x is an element of $D_R\langle 1, -h\rangle$ and $h \in C_R(H) \subset D_R\langle 1, -h\rangle$. So $-1, x \in D_R\langle 1, -h\rangle$ and $-\alpha x \in \alpha D_R\langle 1, -h\rangle$. Thus $D_S\langle 1, -\alpha x\rangle \subset D_S\langle 1, -h\rangle$. By [4, 5.2]:

$$|Q_S(\alpha x) \cap Q_S(h)| = \frac{|D_S\langle 1, -\alpha x h\rangle|}{|D_S\langle 1, -\alpha x\rangle \cap D_S\langle 1, -h\rangle|} = \frac{2|D_R\langle 1, -xh\rangle \cap H|}{2|D_R\langle 1, -x\rangle \cap H|},$$

and $D_R\langle 1, -xh\rangle \cap H = D_R\langle 1, -x\rangle \cap H$ as $h \in C_R(H)$ implies $H \subset D_R\langle 1, -h\rangle$. So $|Q_S(\alpha x) \cap Q_S(h)| = 1$, a contradiction. Hence $\pi_1(H) = G(L)$. □

Corollary 2.3. *Suppose R is of local type with $|G(R)| \neq 4$. If S is an H-extension of R then either:*

(a) *$H = 1$ and $S = R[E_1]$, with E_1 generated by α,* *or*
(b) *$H = G(R)$ and $S = D_1 \sqcap R$, with D_1 generated by α.*

Proof. If $|G(R)| = 2$ then it is clear that $H = 1$ or $H = G(R)$. If $|G(R)| \geq 8$ the take $R_2 = 1$ in (2.2) to get $H = 1$ or $G(R)$. Now apply (1.1). □

3. Products: General Lemmas.

We start with a lemma that may be of general interest.

Lemma 3.1. *Let R be a Witt ring of elementary type. Let K be a proper subgroup of $G(R)$ and let $y \in G(R)$. If*

$$G(R) = \bigcup_{k \in K} D\langle 1, -yk\rangle,$$

then $y \in \mathrm{rad}(R) \cdot K$.

Proof. We first prove the result for non-degenerate R where we must show $y \in K$. It suffices to prove this for subgroups K of index two. Namely, if K_0 is any subgroup satisfying the hypothesis let $\mathcal{A}$ denote the set of subgroups K of index two that contain K_0. Then for any $K \in \mathcal{A}$:

$$G(R) = \bigcup_{k \in K_0} D\langle 1, -yk\rangle \subset \bigcup_{k \in K} D\langle 1, -yk\rangle.$$

Assuming the result holds for subgroups of index two, we obtain $y \in K$. Then $y \in \bigcap_{K \in \mathcal{A}} K = K_0$, as desired.

So suppose $[G(R) : K] = 2$. We work by induction on $|G(R)|$. We need to prove that $y \in K$ when R is of local type, a group ring or a product. First suppose R is of local type. Then $K = D\langle 1, -a\rangle$, for some $a \in G(R)$. We have:

$$G(R) = \bigcup_{k \in D\langle 1,-a\rangle} D\langle 1, -yk\rangle = D\langle 1, -y, ay\rangle.$$

Multiplying by $-a$ gives $G(R) = D\langle\langle -a, -y\rangle\rangle'$, the pure part of the Pfister form $\langle\langle -a, -y\rangle\rangle$. In particular, $-1 \in D\langle\langle -a, -y\rangle\rangle'$ so $\langle\langle -a, -y\rangle\rangle = 0$ and $y \in D\langle 1, -a\rangle = K$.

Next let $R = R_0[E_1]$, with $E_1 = \{1, t\}$ and $|G(R_0)| \geq 2$ (if $G(R_0) = 1$ then R is degenerate). Suppose $y \notin K$. We claim $G(R_0) \subset K$. Choose any $g \in G(R_0)$. Then there exist $k_1, k_2 \in K$ with $-gt \in D\langle 1, -yk_1\rangle$ and $-t \in D\langle 1, -yk_2\rangle$. We get $gt = yk_1$ and $t = yk_2$ since $y \notin K$. Hence $g = k_1k_2 \in K$. This proves the claim. Both $G(R_0)$ and K have index two so $K = G(R_0)$. From $t = yk_2$ we have $y \in tG(R_0)$. Pick $g \in G(R_0)^{\cdot}$. Then $g \in D\langle 1, -yk_3\rangle$ for some $k_3 \in K$. But $yk_3 \in yK = tG(R_0)$, so this is impossible. The contradiction implies $y \in K$.

Lastly, say $R = R_1 \sqcap R_2$. Write $y = (y_1, y_2)$. Now $K \cap (G(R_1) \times 1)$ is a subgroup of index at most two in $G(R_1) \times 1$. Let K_1 be its projection into $G(R_1)$. Similarly, let K_2 be the projection of $K \cap (1 \times G(R_2))$ into $G(R_2)$. Then $[G(R_i) : K_i] \leq 2$, for $i = 1, 2$. If $K_2 = G(R_2)$ then:

$$\bigcup_{k \in K} D_R\langle 1, -yk\rangle = \bigcup_{k_1 \in K_1} D\langle 1, -y_1 k_1\rangle \times G(R_2)$$

so that $G(R_1) = \bigcup_{K_1} D\langle 1, -y_1 k_1\rangle$. By induction then $y_1 \in K_1$ and hence $y = (y_1, y_2) \in K_1 \times G(R_2) = K$. In the same way, if $K_1 = G(R_1)$ then $y \in K$ as desired. So we may assume that $[G(R_i) : K_i] = 2$ for $i = 1, 2$. Write $K = \{1, \gamma\} K_1 \times K_2$. We have:

$$G(R) = \bigcup_{k_1 \in K_1, k_2 \in K_2} \big(D\langle 1, -y_1 k_1\rangle \times D\langle 1, -y_2 k_2\rangle \\ \cup D\langle 1, -y_1 \gamma_1 k_1\rangle \times D\langle 1, -y_2 \gamma_2 k_2\rangle\big),$$

where $\gamma = (\gamma_1, \gamma_2)$.

Suppose $y_1 \in K_1$. If $G(R_1) = \cup D\langle 1, -y_1\gamma_1 k_1\rangle$ then by induction we have $y_1\gamma_1 \in K_1$ and so $\gamma_1 \in K_1$. Then $K = K_1 \times G(R_2)$ and $K_2 = G(R_2)$ a case we have already covered. We may thus assume there exists a $g_1 \in G(R_1) \setminus \cup D\langle 1, -y_1\gamma_1 k_1\rangle$. Then $g_1 \times G(R_2) \subset \cup(D\langle 1, -y_1 k_1\rangle \times D\langle 1, -y_2 k_2\rangle)$ and so $G(R_2) = \cup D\langle 1, -y_2 k_2\rangle$. By induction $y_2 \in K_2$ and $y = (y_1, y_2) \in K_1 \times K_2 \subset K$, and we are done.

We may thus assume $y_1 \notin K_1$. Similarly, $y_2 \notin K_2$. Pick, for $i = 1, 2$, a $g_i \in G(R_i) \setminus \cup D\langle 1, -y_i k_i\rangle$, which is possible by induction. Then $g_1 \times G(R_2) \subset \cup(D\langle 1, -y_1\gamma_1 k_1\rangle \times D\langle 1, -y_2\gamma_2 k_2\rangle)$ and so $G(R_2) = \cup D\langle 1, -y_2\gamma_2 k_2\rangle$. By induction once more, we have $y_2\gamma_2 \in K_2$. Similarly, $y_1\gamma_1 \in K_1$. Then $y \in (\gamma_1, \gamma_2)(K_1 \times K_2) \subset K$, as desired. This proves the result for non-degenerate R.

Now suppose R is degenerate. Write $R = D \sqcap R_2$, with $\operatorname{rad}(R) = G(D) \times 1$ and R_2 non-degenerate. Let π_2 be the projection of $G(R)$ onto $G(R_2)$. Set $K_2 = \pi_2(K)$ and write $y = (y_1, y_2)$, with $y_1 \in G(D)$ and $y_2 \in G(R_2)$. Our assumption is:

$$G(R) = G(D) \times G(R_2) = \bigcup_{(k_1, k_2) \in K} D\langle (1,1), -(y_1 k_1, y_2 k_2)\rangle \\ = G(D) \times \left(\bigcup_{k_2 \in K_2} D_{R_2}\langle 1, -y_2 k_2\rangle \right).$$

From the non-degenerate case we get $y_2 \in K_2 = \pi_2(K)$. Thus there exists a $d \in G(D)$ such that $(d, y_2) \in K$. Hence $y = (y_1, y_2) = (dy_1, 1)(d, y_2) \in \operatorname{rad}(R) \cdot K$. □

Our key reduction lemma follows.

Lemma 3.2. *Let $R = R_1 \sqcap R_2$ and suppose S is an H-extension of R. If $H = H_1 \times G(R_2)$ then there exists a Witt ring T that is an H_1-extension of R_1 such that $S \cong T \sqcap R_2$.*

Proof. We first construct T. Let $G(T)$ be a group containing $G(R_1)$ as a subgroup of index 2; write $G(T) = \{1, \beta\}G(R_1)$. Let $\varphi : G(T) \to \{1, \alpha\}(G(R_1) \times 1)$ be the isomorphism sending $g_1 \mapsto (g_1, 1)$ and $\beta g_1 \mapsto \alpha(g_1, 1)$, where $g_1 \in G(R_1)$. For $z \in G(T)$ define:

$$D_T\langle 1, -z\rangle = \varphi^{-1}(D_S\langle 1, -\varphi(z)\rangle \cap \{1, \alpha\}(G(R_1) \times 1)).$$

We check that T is an H_1-extension of R_1. If $z \in G(R_1) \setminus H_1$ then:

$$\begin{aligned} D_T\langle 1, -z\rangle &= \varphi^{-1}((D_{R_1}\langle 1, -z\rangle \times G(R_2)) \cap \{1, \alpha\}(G(R_1) \times 1)) \\ &= \varphi^{-1}(D_{R_1}\langle 1, -z\rangle \times 1) \\ &= D_{R_1}\langle 1, -z\rangle. \end{aligned}$$

If $z \in H_1$ then:

$$\begin{aligned} D_T\langle 1, -z\rangle &= \varphi^{-1}(\{1, \alpha\}(D_{R_1}\langle 1, -z\rangle \times G(R_2)) \cap \{1, \alpha\}(G(R_1) \times 1)) \\ &= \varphi^{-1}(\{1, \alpha\}(D_{R_1}\langle 1, -z\rangle \times 1)) \\ &= \{1, \beta\}D_{R_1}\langle 1, -z\rangle. \end{aligned}$$

Lastly, if $z \in G(R_1)$ then:

$$D_T\langle 1, -\beta z\rangle = \varphi^{-1}(D_S\langle 1, -\alpha(z, 1)\rangle \cap \{1, \alpha\}(G(R_1) \times 1)).$$

Now:

$$\begin{aligned} D_S\langle 1, -\alpha(z, 1)\rangle &= \{1, -\alpha(z, 1)\}\left((D_{R_1}\langle 1, -z\rangle \times G(R_2)) \cap (H_1 \times G(R_2))\right) \\ &= \{1, -\alpha(z, 1)\}\left((D_{R_1}\langle 1, -z\rangle \cap H_1) \times G(R_2)\right). \end{aligned}$$

Since $(1, -1) \in D_{R_1}\langle 1, -z\rangle \cap H_1 \times G(R_2)$ we have:

$$D_S\langle 1, -\alpha(z, 1)\rangle = \{1, \alpha(-z, 1)\}\left((D_{R_1}\langle 1, -z\rangle \cap H_1) \times G(R_2)\right).$$

Hence:

$$\begin{aligned} D_T\langle 1, -\beta z\rangle &= \varphi^{-1}(\{1, \alpha(-z, 1)\}(D_{R_1}\langle 1, -z\rangle \cap H_1) \times G(R_2)) \\ &\qquad \cap \{1, \alpha(-z, 1)\}(G(R_1) \times 1) \\ &= \varphi^{-1}(\{1, \alpha(-z, 1)\}(D_{R_1}\langle 1, -z\rangle \cap H_1) \times 1) \\ &= \{1, -\beta z\}(D_{R_1}\langle 1, -z\rangle \cap H_1). \end{aligned}$$

We begin the verification that $(G(T), D_T)$ is linked, so that T is indeed a Witt ring. Let $t = \alpha^{\epsilon_1}(u, v) \in G(S)$ and let $\beta^{\epsilon_2}x, \beta^{\epsilon_3}y \in G(T)$ with each $\epsilon_i = 0$ or 1.

Claim. If $t \in \varphi(\beta^{\epsilon_3}y)D_S\langle 1, -\varphi(\beta^{\epsilon_2}x)\rangle$ then $\beta^{\epsilon_1}u \in \beta^{\epsilon_3}yD_T\langle 1, -\beta^{\epsilon_2}x\rangle$.

We first assume that $\epsilon_1 = 0$. We have four cases:

Case 1. $\epsilon_2 = 0, \epsilon_3 = 0$. Here $(uy, v) \in D_S\langle 1, -(x, 1)\rangle$, hence $uy \in D_{R_1}\langle 1, -x\rangle \subset D_T\langle 1, -x\rangle$.

Case 2. $\epsilon_2 = 0, \epsilon_3 = 1$. Here $\alpha(uy, v) \in D_S\langle 1, -(x, 1)\rangle$. We must have that $x \in H_1$ so $\alpha(uy, v) \in \{1, \alpha\}(D_{R_1}\langle 1, -x\rangle \times G(R_2))$. Then $uy \in D_{R_1}\langle 1, -x\rangle$. Thus $\beta uy \in D_T\langle 1, -x\rangle = \{1, \beta\}D_{R_1}\langle 1, -x\rangle$.

Case 3. $\epsilon_2 = 1, \epsilon_3 = 0$. Here:

$$\begin{aligned}(uy, v) &\in \{1, -\alpha(x, 1)\}((D_{R_1}\langle 1, -x\rangle \times G(R_2)) \cap H) \\ &= \{1, -\alpha(x, 1)\}((D_{R_1}\langle 1, -x\rangle \cap H_1) \times G(R_2)).\end{aligned}$$

Thus $uy \in D_{R_1}\langle 1, -x\rangle \cap H_1$. We obtain

$$uy \in D_T\langle 1, -\beta x\rangle = \{1, -\beta x\}(D_{R_1}\langle 1, -x\rangle \cap H_1).$$

Case 4. $\epsilon_2 = 1, \epsilon_3 = 1$. Here $\alpha(uy, v) \in \{1, -\alpha(x, 1)\}((D_{R_1}\langle 1, -x\rangle \cap H_1) \times G(R_2))$ so that $(-xuy, v) \in (D_{R_1}\langle 1, -x\rangle \cap H_1) \times G(R_2)$. Thus $-xuy \in D_{R_1}\langle 1, -x\rangle \cap H_1$ and $\beta uy \in D_T\langle 1, -\beta x\rangle = \{1, -\beta x\}(D_{R_1}\langle 1, -x\rangle \cap H_1)$.

The four cases with $\epsilon_1 = 1$ are identical to the four above cases. For example, if $\epsilon_1 = 1, \epsilon_2 = 0, \epsilon_3 = 0$ then we have $\alpha(uy, v) \in D_S\langle 1, -(x, 1)\rangle$, which is *Case* 2 above. Thus the Claim is proven.

We can now check linkage in T. Let $x, y, z, w \in G(T)$ and suppose:

$$xD_T\langle 1, -y\rangle \cap D_T\langle 1, -yz\rangle \cap wD_T\langle 1, -z\rangle \neq \emptyset.$$

Apply φ to get:

$$\varphi(x)D_S\langle 1, -\varphi(y)\rangle \cap D_S\langle 1, -\varphi(yz)\rangle \cap \varphi(w)D_S\langle 1, -\varphi(z)\rangle \neq \emptyset.$$

By linkage on S, there exists a $t \in G(S)$ in:

$$\varphi(y)D_S\langle 1, -\varphi(x)\rangle \cap D_S\langle 1, -\varphi(xw)\rangle \cap \varphi(z)D_S\langle 1, -\varphi(w)\rangle.$$

Now apply the

Claim.

$$yD_T\langle 1, -x\rangle \cap D_T\langle 1, -xw\rangle \cap zD_T\langle 1, -w\rangle \neq \emptyset,$$

as desired.

Lastly, set $W = T \sqcap R_2$. Then $G(W) = (\{1, \beta\}G(R_1)) \times G(R_2)$. Set $\gamma = (\beta, 1)$ so that $G(W) = \{1, \gamma\}(G(R_1) \times G(R_2)) = \{1, \gamma\}G(R)$. We will show W is an H-extension of R, via γ, and hence that $S \cong W$.

First let $h = (h_1, g_2) \in H$, where $h_1 \in H_1 \subset G(R_1)$ and $g_2 \in G(R_2)$. Then:

$$\begin{aligned}D_W\langle 1, -h\rangle &= D_T\langle 1, -h_1\rangle \times D_{R_2}\langle 1, -g_2\rangle \\ &= (\{1, \beta\}D_{R_1}\langle 1, -h_1\rangle) \times D_{R_2}\langle 1, -g_2\rangle \\ &= \{1, \gamma\}D_R\langle 1, -h\rangle.\end{aligned}$$

Next let $g = (g_1, g_2) \in G(R) \setminus H$, with $g_1 \in G(R_1) \setminus H_1$ and $g_2 \in G(R_2)$. Then:

$$\begin{aligned} D_W\langle 1, -g\rangle &= D_T\langle 1, -g_1\rangle \times D_{R_2}\langle 1, -g_2\rangle \\ &= D_{R_1}\langle 1, -g_1\rangle \times D_{R_2}\langle 1, -g_2\rangle = D_R\langle 1, -g\rangle. \end{aligned}$$

Lastly, let $g = (g_1, g_2) \in G(R)$, with $g_1 \in G(R_1)$ and $g_2 \in G(R_2)$. Then:

$$\begin{aligned} D_W\langle 1, -\gamma g\rangle &= D_W\langle 1, -(\beta g_1, g_2)\rangle \\ &= D_{R_1}\langle 1, -\beta g_1\rangle \times D_{R_2}\langle 1, -g_2\rangle \\ &= \big(\{1, -\beta g_1\}(D_{R_1}\langle 1, -g_1\rangle \cap H_1)\big) \times D_{R_2}\langle 1, -g_2\rangle \\ &= (D_{R_1}\langle 1, -g_1\rangle \cap H_1) \times D_{R_2}\langle 1, -g_2\rangle \\ &\quad \cup \gamma\big(-g_1(D_{R_1}\langle 1, -g_1\rangle \cap H_1) \times -g_2 D_{R_2}\langle 1, -g_2\rangle\big) \\ &= \{1, -\gamma(g_1, g_2)\}(D_R\langle 1, -(g_1, g_2)\rangle \cap H) \\ &= \{1, -\gamma g\}(D_R\langle 1, -g\rangle \cap H). \end{aligned}$$

Thus $W = T \sqcap R_2$ is an H-extension of R and so is isomorphic to S. □

Our last general lemma is the most technical, but it also does most of the work.

Lemma 3.3. *Let $u \in G(R)$ and $h \in H$. Then:*

$$D_R\langle 1, uh, -h\rangle \setminus uH \subset \bigcup_{t \in h(D\langle 1, -u\rangle \cap H)} D_R\langle 1, -t\rangle.$$

Proof. Let $w \in D_R\langle 1, uh, -h\rangle \setminus uH$. Then $w \in D_R\langle 1, -hv\rangle$ for some $v \in D_R\langle 1, -u\rangle$ and $uw \notin H$. We have:

$$q(\alpha u, h) = q(u, h) = q(u, vh) = q(uw, vh).$$

Thus, by linkage, there exists a $t \in G(S)$ such that:

$$q(\alpha u, h) = q(\alpha u, t) = q(uw, t) = q(uw, vh).$$

The third equality gives $vht \in D_S\langle 1, -uw\rangle$. Since $uw \notin H$ this implies $t \in G(R)$. Then the first two equalities give:

$$\begin{aligned} ht &\in D_S\langle 1, -\alpha u\rangle \cap G(R) = D_R\langle 1, -u\rangle \cap H, \\ t &\in D_S\langle 1, -\alpha w\rangle \cap G(R) = D_R\langle 1, -w\rangle \cap H. \end{aligned}$$

Hence $w \in D_R\langle 1, -t\rangle$ where $t \in h(D_R\langle 1, -u\rangle \cap H)$. □

4. Products: Degenerate Witt rings.

If R is a degenerate Witt ring then $R = D \sqcap R_2$, for some Witt ring R_2 and where $G(D) = \{1, d\}$, with $D_D\langle 1, 1\rangle = D_D\langle 1, d\rangle = \{1, d\}$. We will often use the fact that if $(u, v) \in G(R)$ then $D_R\langle 1, -(u, v)\rangle = D_R\langle 1, -(du, v)\rangle$.

Lemma 4.1. *Suppose $R = D \sqcap R_2$ is degenerate. Let π_1 be the projection of $G(R)$ onto $G(D)$. Let S be an H-extension of R. Then either $G(D) \times 1 \subset H$ or S is isomorphic to an H_0-extension of R, for some subgroup $H_0 \subset G(R)$ with $\pi_1(H_0) = 1$.*

Proof. Suppose $(d, 1) \notin H$ and that $\pi_1(H) \neq 1$, so that $(d, y) \in H$ for some $y \in G(R_2)$. Let G_2 be the subgroup of $G(R_2)$ such that $1 \times G_2 = H \cap (1 \times G(R_2))$. Then $H = (1 \times G_2) \cup (d \times yG_2)$. Set $H_0 = 1 \times \{1, y\}G_2$, and note that $\pi_1(H_0) = 1$.

Let $\beta^2 = 1$ and set $G(S_0) = \{1, \beta\}G(R_2)$. Define S_0-value set s so that S_0 is an H_0-extension of R. We wish to show $S \cong S_0$. Extend G_2 to a subgroup K of index two in $G(R_2)$ that does not contain y. Define $\varphi : G(S) \to G(S_0)$ by $\alpha \mapsto \beta$ and for $(u, v) \in G(R)$:

$$\varphi(u, v) = \begin{cases} (u, v), & \text{if } v \in K \\ (du, v), & \text{if } v \notin K. \end{cases}$$

It is quickly checked that φ is an isomorphism. We will show:

$$\varphi(D_S\langle 1, -s\rangle) = D_{S_0}\langle 1, -\varphi(s)\rangle, \tag{4.2}$$

for all $s \in G(S)$. This shows both that S_0 is a Witt ring and that $S \cong S_0$.

Claim. If $(u, v) \in G(R)$ then $\varphi(D_R\langle 1, -(u, v)\rangle) = D_R\langle 1, -(u, v)\rangle$.

$$\begin{aligned} D_R\langle 1, -(u, v)\rangle &= \{1, d\} \times D_{R_2}\langle 1, -v\rangle \\ &= 1 \times (D_{R_2}\langle 1, -v\rangle \cap K) \cup d \times (D_{R_2}\langle 1, -v\rangle \cap K) \\ &\quad \cup 1 \times (D_{R_2}\langle 1, -v\rangle \cap yK) \cup d \times (D_{R_2}\langle 1, -v\rangle \cap yK). \end{aligned}$$

Thus:

$$\begin{aligned} \varphi(D_R\langle 1, -(u, v)\rangle) &= 1 \times (D_{R_2}\langle 1, -v\rangle \cap K) \cup d \times (D_{R_2}\langle 1, -v\rangle \cap K) \\ &\quad \cup d \times (D_{R_2}\langle 1, -v\rangle \cap yK) \cup 1 \times (D_{R_2}\langle 1, -v\rangle \cap yK) \\ &= D_R\langle 1, -(u, v)\rangle, \end{aligned}$$

proving the Claim.

We now check (4.2) in various cases. First suppose $s = (u, v) \in G(R)$, with $v \in K$. Then $s \in H$ iff $u = 1$ and $v \in G_2$. We have $\varphi(s) = s$ and $\varphi(s) \in H_0$ iff $u = 1$ and $v \in G_2$ iff $s \in H$. $D_S\langle 1, -s\rangle = \{1, \alpha\}D_R\langle 1, -s\rangle$ or $D_R\langle 1, -s\rangle$ depending on whether or not $s \in H$. So by the Claim, $\varphi(D_S\langle 1, -s\rangle) = \{1, \beta\}D_R\langle 1, -s\rangle$ or $D_R\langle 1, -s\rangle$ depending on whether or not $\varphi(s) \in H_0$. Thus $\varphi(D_S\langle 1, -s\rangle) = D_{S_0}\langle 1, -\varphi(s)\rangle$.

Next suppose that $s = (u, v) \in G(R)$ with $v \in yK$. Then $s \in H$ iff $u = d$ and $v \in yG_2$. We have $\varphi(s) = (du, v)$ so that $\varphi(s) \in H_0$ iff $u = d$ and $v \in yG_2$ iff $s \in H$. Again using the Claim:

$$D_S\langle 1, -s\rangle = \begin{cases} \{1, \alpha\}D_R\langle 1, -s\rangle, & \text{if } s \in H \\ D_R\langle 1, -s\rangle, & \text{if } s \notin H. \end{cases}$$

Thus:

$$\varphi(D_S\langle 1,-s\rangle) = \begin{cases} \{1,\beta\}D_R\langle 1,-s\rangle, & \text{if } \varphi(s)\in H_0 \\ D_R\langle 1,-s\rangle & \text{if } \varphi(s)\notin H_0. \end{cases}$$

Now $D_R\langle 1,-s\rangle = D_R\langle 1,-(u,v)\rangle = D_R\langle 1,-(du,v)\rangle = D_R\langle 1,-\varphi(s)\rangle$. Hence we have as desired that $\varphi(D_S\langle 1,-s\rangle) = D_{S_0}\langle 1,-\varphi(s)\rangle$.

Now suppose $s = \alpha(u,v)\in\alpha G(R)$. Then:

$$\begin{aligned} D_S\langle 1,-s\rangle &= \{1,-\alpha(u,v)\}(D_R\langle 1,-(u,v)\rangle\cap H) \\ &= \{1,-\alpha(u,v)\}[(\{1,d\}\times D_{R_2}\langle 1,-v\rangle)\cap(1\times G_2\cup d\times yG_2)] \\ &= \{1,-\alpha(u,v)\}[1\times(D_{R_2}\langle 1,-v\rangle\cap G_2)\cup d\times(D_{R_2}\langle 1,-v\rangle\cap yG_2)]. \end{aligned}$$

Now:

$$\begin{aligned} &\varphi((1\times(D_{R_2}\langle 1,-v\rangle\cap G_2))\cup(d\times(D_{R_2}\langle 1,-v\rangle\cap yG_2))) \\ &= 1\times(D_{R_2}\langle 1,-v\rangle\cap G_2)\cup 1\times(D_{R_2}\langle 1,-v\rangle\cap yG_2) \\ &= 1\times(D_{R_2}\langle 1,-v\rangle\cap\{1,y\}G_2) \\ &= D_R\langle 1,-(u,v)\rangle\cap H_0. \end{aligned}$$

Thus if $v\in K$ then:

$$\begin{aligned} \varphi(D_S\langle 1,-s\rangle) &= \varphi(D_S\langle 1,-\alpha(u,v)\rangle) \\ &= \{1,-\beta(u,v)\}(D_R\langle 1,-(u,v)\rangle\cap H_0) = D_{S_0}\langle 1,-\varphi(s)\rangle, \end{aligned}$$

verifying (4.2) in this case.

Lastly, if $v\in yK$ then:

$$\begin{aligned} \varphi(D_S\langle 1,-s\rangle) &= \varphi(D_S\langle 1,-\alpha(u,v)\rangle) \\ &= \{1,-\beta(du,v)\}(D_R\langle 1,-(u,v)\rangle\cap H_0) \\ &= \{1,-\beta(du,v)\}(D_R\langle 1,-(du,v)\rangle\cap H_0) \\ &= D_{S_0}\langle 1,-\varphi(s)\rangle. \end{aligned}$$

Thus (4.2) holds in all cases. □

5. Products: Local type factors.

Lemma (3.3) looks simpler when one factor has local type.

Lemma 5.1. *Suppose $R = L\sqcap R_2$, with L of local type. Suppose S is an H-extension of R. Let $h=(h_1,h_2)\in H$ and $u=(u_1,u_2)\in G(R)$ such that $u_1\notin D_L\langle 1,-h_1\rangle$ while $u_2\in D_{R_2}\langle 1,-h_2\rangle$. Then:*

$$G(R) = uH\cup(u_1\times G(R_2))\cup\bigcup_{t\in h(D_R\langle 1,-u\rangle\cap H)} D_R\langle 1,-t\rangle.$$

Proof. Write $Q(L) = \{1, \rho\}$. Then $\langle\langle -u, -h\rangle\rangle = (\rho, 1)$. Hence $-D_R\langle\langle -u, -h\rangle\rangle' = \{(x, y) \in G(R) : x \neq 1\}$. Now $-u \cdot \langle -u, -h, uh\rangle \simeq \langle 1, uh, -h\rangle$. Thus $D_R\langle 1, uh, -h\rangle = \{(x, y) \in G(R) : x \neq u_1\}$. Apply (3.3). □

Lemma 5.2. *Suppose $R = L \sqcap R_2$, with L of local type. Let π_1 be the projection of $G(R)$ onto $G(L)$. Let S be an H-extension of R and suppose that $\pi_1(H) = G(L)$. Let $u = (u_1, u_2) \in G(R)$.*

(a) *If $|G(L)| \geq 4$ then $\pi_1(D_R\langle 1, -u\rangle \cap H) = D_L\langle 1, -u_1\rangle$.*
(b) *If $L = \mathbb{Z}$ and $\pi_1(D_R\langle 1, -u\rangle \cap H) \neq D_L\langle 1, -u_1\rangle$ for some u, then $H = 1 \times H_2 \cup -1 \times -H_2$, where $1 \times H_2 = H \cap (1 \times G(R_2))$ and H_2 is an ordering on R_2.*

Proof. Suppose that $\pi_1(D_R\langle 1, -u\rangle \cap H) = K < D_L\langle 1, -u_1\rangle$.

Claim. If $v \in G(L) \setminus K$ then $(v, -1) \in H$.

Since $\pi_1(H) = G(L)$ there exists a $w \in G(R_2)$ such that $(v, w) = h \in H$. Now:

$$\begin{aligned} D_R\langle 1, uh, -h\rangle &= -u(D_L\langle\langle -u, -v\rangle\rangle' \times D_{R_2}\langle\langle -u_2, -w\rangle\rangle') \\ &\supset u_1 G(L)^{\cdot} \times u_2 T, \end{aligned}$$

where $T = -D_{R_2}\langle\langle -u_2, -w\rangle\rangle'$. Also:

$$\bigcup_{t \in h(D_R\langle 1, -u\rangle \cap H)} D_R\langle 1, -t\rangle \subset \left(\bigcup_{k \in K} D_L\langle 1, -vk\rangle\right) \times G(R_2).$$

Now $v \notin K$ implies $G(L) \neq \cup D_L\langle 1, -vk\rangle$ by (3.1). Choose a $g \in G(L) \setminus \cup D_L\langle 1, -vk\rangle$.

We check that we may assume $g \neq u_1$. If $v \in D_L\langle 1, -u_1\rangle$ then we have $u_1 D_L\langle 1, -v\rangle = D_L\langle 1, -v\rangle \subset \cup D_L\langle 1, -vk\rangle$ and so no $g \in G(L) \setminus \cup D_L\langle 1, -vk\rangle$ is equal to u_1. If instead $v \notin D_L\langle 1, -u_1\rangle$ then, as $|K| < |D_L\langle 1, -u_1\rangle|$, there exists a $w \neq u_1$ such that $K \subset D_L\langle 1, -u_1\rangle \cap D_L\langle 1, -w\rangle$. If $v \notin D_L\langle 1, -w\rangle$ then w is not in any $D_L\langle 1, -vk\rangle$, for $k \in K$, and we may take $g = w$. If $v \in D_L\langle 1, -w\rangle$ then $v \notin D_L\langle 1, -u_1 w\rangle$ and we may take $g = u_1 w$.

We thus have $g \in u_1 G(L)^{\cdot} \setminus \cup D_L\langle 1, -vk\rangle$. So $g \times u_2 T \subset uH$ by (3.3). Hence $u_1 g \times T \subset H$. Then $(u_1 g, u_2), (u_1 g, -u_2 w) \in H$ and so $(1, -w) \in H$. We obtain that $(v, -1) = (v, w)(1, -w) \in H$ and the Claim is proven.

Now suppose $u_1 \neq 1$. Let $x \in G(L)^{\cdot}$. Since $|K| < |D_L\langle 1, -u_1\rangle|$ we have $|K| \leq \frac{1}{4}|G(L)|$. So we can choose $v \in G(L) \setminus \{1, x\}K$. Then $(v, -1)$ and $(vx, -1)$ are in H by the Claim. Hence $(x, 1) \in H$. This shows that $G(L) \times 1 \subset H$. But then $D_L\langle 1, -u_1\rangle \times 1 \subset D_R\langle 1, -u\rangle \cap H$ and $\pi_1(D_R\langle 1, -u\rangle \cap H) = D_L\langle 1, -u_1\rangle$, as desired.

Next suppose $u_1 = 1$ and $|G(L)| \geq 4$. We show $\pi_1(D_R\langle 1, -u\rangle \cap H) = G(L)$. Pick any $a \in G(L)$ and pick a $b \in G(L)^{\cdot}$ such that $a \in D_L\langle 1, -b\rangle$. This is possible since $|G(L)| \geq 4$ implies there are at least two b's such that $a \in$

$D_L\langle 1,-b\rangle$. So there is such a b not equal to 1. Then, by the above paragraph, $\pi_1(D_R\langle 1,-(b,u_2)\rangle \cap H) = D_L\langle 1,-b\rangle$ contains a. Thus there exists a $k \in D_{R_2}\langle 1,-u_2\rangle$ such that $(a,k) \in H$. Thus $(a,k) \in D_R\langle 1,-(u_1,u_2)\rangle$, as $u_1 = 1$, and so $a \in \pi_1(D_R\langle 1,-u\rangle \cap H)$.

Lastly, suppose $u_1 = 1$ and $L = \mathbb{Z}$. Here $K = \{1\}$ and $v = -1$. The Claim shows that $(-1,-1) \in H$. Now $H_2 = H \cap (1 \times G(R_2))$ has index 2 in H since $1 \times G(R_2)$ has index 2 in $G(R)$ and $H \not\subset 1 \times G(R_2)$. Hence $H = 1 \times H_2 \cup -1 \times -H_2$. The last paragraph of the proof of the Claim gives $u_1 g \times T \subset H$, where $g \neq u_1$. Thus $g = -1$ and after multiplying by $-1 \in H$ we get:

$$1 \times D_{R_2}\langle\langle -u_2, -w\rangle\rangle' \subset H. \tag{5.3}$$

This holds for all $w \in G(R_2)$ such that $(-1,w) \in H$, that is, for all $w \in -H_2$. Thus for any $h_2 \in H_2$:

$$\begin{aligned} -u_2 D_{R_2}\langle 1,h_2\rangle &\subset D_{R_2}\langle -u_2, h_2, -u_2h_2\rangle \subset H_2 \\ D_{R_2}\langle 1,h_2\rangle &\subset H_2. \end{aligned}$$

Thus H_2 is a preordering. Also (5.3) holds for any $u_2 \in G(R_2)$ with $\pi_1(D_R\langle 1,-(1,u_2)\rangle) = 1$. That is, $D_R\langle 1,-(1,u_2)\rangle \cap (-1 \times -H_2) = \emptyset$ or equivalently, u_2 is not in $D_{R_2}\langle 1,h_2\rangle$ for any $h_2 \in H_2$. For such a u_2, (5.3) implies $-u_2 \in H_2$. Hence:

$$G(R_2) = -H_2 \cup \bigcup_{h_2 \in H_2} D_{R_2}\langle 1,h_2\rangle.$$

But H_2 is a preordering so that $\cup D_{R_2}\langle 1,h_2\rangle \subset H_2$. Thus $G(R_2) = -H_2 \cup H_2$, H_2 has index 2 in $G(R_2)$ and so H_2 is an ordering. □

Notation. Suppose $R = R_1 \sqcap R_2$ and that H is a subgroup of $G(R)$. For $x \in G(R_1)$ set $F(x) = \{y \in G(R_2) : (x,y) \in H\}$.

Lemma 5.4. *Let $R = R_1 \sqcap R_2$ and let S be an H-extension of R. Let π_1 be the projection of $G(R)$ onto $G(R_1)$ and suppose $\pi_1(H) = G(R_1)$. Then for all $x \in G(R_1)$:*

(a) *$F(x)$ is non-empty.*
(b) *$F(1)$ is a subgroup of $G(R_1)$.*
(c) *$F(x)$ is a coset of $F(1)$.*

Proof. No $F(x)$ is empty since $\pi_1(H) = G(R_1)$. Clearly $F(1)$ is a subgroup. Fix $y_0 \in F(x)$. If $y \in F(1)$ then $(1,y),(x,y_0) \in H$ implies $(x,yy_0) \in H$ and so $yy_0 \in F(x)$. This says $y_0F(1) \subset F(x)$.

Now let $y \in F(x)$. Then $(x,y_0),(x,y) \in H$ so that $(1,yy_0) \in H$. Hence $yy_0 \in F(1)$ and we have the reverse inclusion $F(x) \subset y_0F(1)$. □

Lemma 5.5. *Let $R = R_1 \sqcap R_2$ be of elementary type. Let S be an H-extension of R. Let π_1 be the projection of $G(R)$ onto $G(R_1)$. Suppose the following:*

(1) $\pi_1(H) = G(R_1)$.
(2) $F(a) \cap \mathrm{rad}(R_2) \subset \{1\}$, *for all* $a \in G(R_1)$.
(3) *For all* $u = (u_1, u_2) \in G(R)$ *we have* $\pi_1(D_R\langle 1, -u\rangle \cap H) = D_{R_1}\langle 1, -u_1\rangle$.

Then $H = G(R_1) \times H_2$, for some subgroup $H_2 \subset G(R_2)$.

Proof. Let $a \in G(R_1)$. We will first show that:

$$G(R_2) = \bigcup_{k \in F(a)} D_{R_2}\langle 1, -k\rangle. \tag{5.6}$$

Pick any $b \in D_{R_1}\langle 1, -a\rangle$ and any $g \in G(R_2)$. Then $a \in D_{R_1}\langle 1, -b\rangle = \pi_1(D_R\langle 1, -(b, g)\rangle \cap H)$ by assumption (3). Hence there exists a $k \in D_{R_2}\langle 1, -g\rangle$ with $(a, k) \in H$. That is, $g \in D_{R_2}\langle 1, -k\rangle$ for some $k \in F(a)$, proving (5.6).

Write $F(a) = yF(1)$ as in (5.4). Then (5.6) becomes:

$$G(R_2) = \bigcup_{k \in F(1)} D_{R_2}\langle 1, -yk\rangle.$$

Thus $y \in \mathrm{rad}(R_2) \cdot F(1)$ by (3.1). That is, there exists a $d \in \mathrm{rad}(R_2)$ such that $d \in yF(1) = F(a)$. By assumption (2) then $d = 1$. Hence $y \in F(1)$ and so $F(a) = F(1)$. By assumption (1) we have $(a, m) \in H$ for some $m \in G(R_2)$. Then $m \in F(a) = F(1)$ so that $(1, m) \in H$ also. So $(a, 1) = (a, m)(1, m) \in H$. Hence $G(R_1) \times 1 \subset H$ and $H = G(R_1) \times F(1)$. □

We first complete the case of a local factor L with $|G(L)| \geq 8$.

Corollary 5.7. *Let $R = L \sqcap R_2$, with R_2 of elementary type, L of local type and $|G(L)| \geq 8$. Let S be an H-extension of R. Suppose $F(a) \cap \mathrm{rad}(R_2) \subset \{1\}$ for all $a \in G(L)$. Then either $H = 1 \times H_2$ or $H = G(L) \times H_2$ for some subgroup $H_2 \subset G(R_2)$.*

Proof. Again let π_1 denote the projection of $G(R)$ onto $G(L)$. We know that $\pi_1(H) = 1$ or $G(L)$, by (2.2). If $\pi_1(H) = 1$ then clearly $H = 1 \times H_2$ for some subgroup H_2. So suppose that $\pi_1(H) = G(L)$, the first hypothesis of (5.5). We are assuming the second hypothesis as well. And (5.2) shows the third hypothesis of (5.5) holds. Hence $H = G(L) \times H_2$, for some subgroup H_2. □

The argument for $R = \mathbb{Z} \sqcap R_2$ is different.

Lemma 5.8. *Let R be a real Witt ring of elementary type. Let $P \subset G(R)$ be an ordering. Suppose that for all $x \in P$ that:*

$$P = \bigcup_{k \in D\langle 1, -x\rangle \cap P} D\langle 1, xk\rangle.$$

Then $R = \mathbb{Z} \sqcap R_2$, for some Witt ring R_2.

Proof. Suppose $\mathbb{Z}$ is not a factor of R. Then R has a group ring factor that is real. Thus $R = R_0[E_1] \sqcap R_2$, for some Witt rings R_0, R_2, and we may assume $P = P_0\{1,t\} \times G(R_2)$, where $P_0 \subset G(R_0)$ is an ordering on R_0 and $E_1 = \{1,t\}$. Then take $x = (t,1)$. We have that $D\langle 1,-x\rangle = \{1,-t\} \times G(R_2)$ and $D\langle 1,-x\rangle \cap P = 1 \times G(R_2)$. Thus:

$$P = \bigcup_{k \in D\langle 1,-x\rangle \cap P} D\langle 1, xk\rangle = \bigcup_{g_2 \in G(R_2)} D\langle (1,1),(t,g_2)\rangle$$
$$= \{1,t\} \times G(R_2).$$

Hence $P_0 = 1$ and $R_0 = \mathbb{Z}$, giving a contradiction. □

Lemma 5.9. *Let $R = \mathbb{Z} \sqcap R_2$ and suppose S is an H-extension of R. Then one of the following occurs.*

(a) *$H = 1 \times H_2$ for some subgroup $H_2 \subset G(R_2)$.*
(b) *$R = \mathbb{Z} \sqcap R_3$, for some Witt ring R_3, and $\{\pm 1\} \times 1 \subset H$.*
(c) *$R = \mathbb{Z} \sqcap \mathbb{Z} \sqcap R_3$, for some Witt ring R_3, and $(1,1) \times G(R_3) \subset H$.*

Proof. Again let π_1 be the projection of $G(R)$ onto $G(\mathbb{Z}) = \{\pm 1\}$. If $\pi_1(H) = 1$ the we are in case (a). Thus we may assume that $\pi_1(H) = G(\mathbb{Z})$. If for every $u \in G(R)$ we have that $\pi_1(D_R\langle 1,-u\rangle \cap H) = D\langle 1, -\pi_1(u)\rangle$ then (5.5) implies we are in case (b). So suppose this fails for some $u \in G(R)$. Then by (5.2) $H = 1 \times H_2 \cup (-1 \times -H_2)$, for some ordering H_2 of $G(R_2)$. We will first show that for every $h_2 \in H_2$ that:

$$H_2 = \bigcup_{k \in D_{R_2}\langle 1,-h_2\rangle \cap H_2} D_{R_2}\langle 1, h_2 k\rangle.$$

Consider $\varphi = \langle (1,1),(1,h_2),(1,-1)\rangle \in S$. We compute its value set two ways. First:

$$D_S\langle (1,1),(1,h_2),(1,-1)\rangle = \bigcup_{\beta \in D_S\langle (1,1),(1,-h_2)\rangle} D_S\langle (1,1), \beta(1,h_2)\rangle.$$

Now $(-1,h_2) \notin H$ so $D_S\langle (1,1),(1,-h_2)\rangle = 1 \times D_{R_2}\langle 1,-h_2\rangle$. For φ to represent an element of $\alpha(1 \times G(R_2))$ we must have $\beta \in -H = H$. That is, $\beta = (1,\beta_2)$ with $\beta_2 \in H_2$. Thus:

$$D_S(\varphi) \cap \alpha(1 \times G(R_2)) = \alpha \cdot \bigcup_{\beta_2 \in D_{R_2}\langle 1,-h_2\rangle \cap H_2} (1 \times D_{R_2}\langle 1, \beta_2 h_2\rangle).$$

Next:

$$D_S(\varphi) = (1,h_2) \cdot \bigcup_{\gamma \in D_S\langle (1,1),(1,-1)\rangle} D_S\langle (1,1), \gamma(1,h_2)\rangle.$$

For any $x \in H_2$ take $\gamma = (1, x) \in D_S\langle(1,1),(1,-1)\rangle$. Then since $(-1,-xh_2) \in H$:

$$\begin{aligned}\alpha(1,x) &\in (1,h_2)D_S\langle(1,1),-(-1,-xh_2)\rangle \\ &= (1,h_2)\cdot\{1,\alpha\}(1\times D_{R_2}\langle 1,xh_2\rangle).\end{aligned}$$

Thus $D_S(\varphi)\cap\alpha(1\times G(R_2)) = \alpha(1\times H_2)$. The two computations of $D_S(\varphi)$ thus yield $H_2 = \cup D_{R_2}\langle 1,\beta h_2\rangle$, over $\beta \in D_{R_2}\langle 1,-h_2\rangle \cap H_2$.

We may now apply (5.8) to obtain $R_2 = \mathbb{Z} \sqcap R_3$, for some Witt ring R_3. Let $H_3 \subset G(R_3)$ be the subgroup such that $H_2 \cap (1\times G(R_3)) = 1\times H_3$. We note that both H_2 and $1\times G(R_3)$ have index two in $G(R_2)$.

If $H_2 = 1\times G(R_3)$ then $(1,1)\times G(R_3) \subset H$ and we are in case (c). So suppose $H_2 \neq 1\times G(R_3)$. Then $1\times H_3$ has index two in H_2 and H_3 has index two in $G(R_3)$. Write $H_2 = 1\times H_3 \cup (-1\times zH_3)$, for some $z \in G(R_3)$. Then:

(5.10)

$$H = [(1,1)\times H_3]\cup[(1,-1)\times zH_3]\cup[(-1,-1)\times -H_3]\cup[(-1,1)\times -zH_3].$$

Now $[G(R_3) : H_3] = 2$ implies at least one of the cosets $zH_3, -H_3, -zH_3$ equals H_3. Say $zH_3 = H_3$. Then the second term of (5.10) shows $(1,1,1)$, $(1,-1,1) \in H$. Set R_4 equal to the product of the first copy of $\mathbb{Z}$ and R_3. Then $R = \mathbb{Z}\sqcap R_4$ and $\{\pm 1\}\times 1 \subset H$. We are thus in case (b). Next say $-H_3 = H_3$. Then $(1,-1) \in 1\times H_3 \subset H_2$. Since H_2 is an ordering we have $D_{R_2}\langle(1,1),(1,-1)\rangle \subset H_2$. But the $1\times G(R_3) \subset H_2$, a case we have already considered. Lastly, suppose $-zH_3 = H_3$. Then the fourth term of (5.10) shows $(1,1,1),(-1,1,1) \in H$. This is case (b) again. □

6. Products: Group ring factors.

Lemma 6.1. *Let $R = R_1 \sqcap R_2$, with $R_1 = R_0[E_1]$ and E_1 generated by t. Let S be an H-extension of R. Let π_1 be the projection of $G(R)$ onto $G(R_1)$ and suppose $\pi_1(H) \not\subset G(R_0)$. Then either $\pi_1(H) = G(R_1)$ or $1\times G(R_2) \subset H$.*

Proof. From $\pi_1(H) \not\subset G(R_0)$ we may assume $h = (t, g_2) \in H$, for some $g_2 \in G(R_2)$. Suppose $\pi_1(H) \neq G(R_1)$, Choose $-g_1 \in G(R_1)\setminus\pi_1(H)$. Then $-g_1 t \notin \pi_1(H)$. Set $u = (g_1 t, 1)$ and note that $\pi_1(D_R\langle 1,-u\rangle \cap H) = 1$. Now:

$$\begin{aligned}D_R\langle 1,uh,-h\rangle &= D_{R_1}\langle 1,g_1,-t\rangle \times D_{R_2}\langle 1,g_2,-g_2\rangle \\ &\supset g_1\times G(R_2).\end{aligned}$$

Also:

$$\bigcup_{k\in h(D_R\langle 1,-u\rangle\cap H)} D_R\langle 1,-k\rangle \subset D_{R_1}\langle 1,-t\rangle \times G(R_2) = \{1,-t\}\times G(R_2).$$

Hence by (3.3), $g_1\times G(R_2) \subset uH$. Multiplying by u gives $t\times G(R_2) \subset H$. Thus $1\times G(R_2) \subset H$. □

Lemma 6.2. *Let $R = R_1 \sqcap R_2$, with $R_1 = R_0[E_1]$ and E_1 generated by t. Let S be an H-extension of R. Let π_1 be the projection of $G(R)$ onto $G(R_1)$ and suppose $\pi_1(H) = G(R_1)$. If $u_1 \in G(R_0)^{\cdot}$ and $u = (u_1, u_2)$ then $\pi_1(D_R\langle 1, -u\rangle \cap H) = D_{R_1}\langle 1, -u_1\rangle$.*

Proof. Set $K = \pi_1(D_R\langle 1, -u\rangle \cap H)$ and suppose $K < D_{R_1}\langle 1, -u_1\rangle$. Let $g \in G(R_0)$. Then $(gt, g_2) \in H$ for some $g_2 \in G(R_2)$, since $\pi_1(H) = G(R_1)$. Now $D_R\langle 1, uh, -h\rangle$ contains $-gtD_{R_1}\langle 1, -u_1\rangle \times u_2T$, where $T = -D_{R_2}\langle\langle -u_2, -g_2\rangle\rangle'$. Also:

$$\bigcup_{w\in h(D_R\langle 1,-u\rangle\cap H)} D_R\langle 1, -w\rangle \subset \bigcup_{k\in K} D_{R_1}\langle 1, -kgt\rangle \times G(R_2)$$
$$= (\{1\} \cup -gtK) \times G(R_2).$$

Hence by (3.3), if $y \in D_{R_1}\langle 1, -u_1\rangle \setminus K$ then:

$$-gty \times u_2T \subset uH$$
$$-gtu_1y \times T \subset H.$$

Now u_2 and $-u_2g_2$ are in T so $(-gtu_1y, u_2)$ and $(-gtu_1y, -u_2g_2)$ are in H. Thus $(1, -g_2) \in H$ and as result $(gt, -1) \in H$.

This holds for all $g \in G(R_0)$ so we have that $tG(R_0) \times -1 \subset H$. Thus $G(R_0)\times 1 \subset H$. But the $D_{R_1}\langle 1, -u_1\rangle\times 1 \subset D_R\langle 1, -u\rangle\cap H$ and $\pi_1(D_R\langle 1, -u\rangle\cap H) = D_{R_1}\langle 1, -u_1\rangle$, a contradiction. □

Lemma 6.3. *Let $R = R_1 \sqcap R_2$, with $R_1 = R_0[E_1]$ non-degenerate and E_1 generated by t. Let S be an H-extension of R. Let π_1 be the projection of $G(R)$ onto $G(R_1)$ and suppose $\pi_1(H) = G(R_1)$. Then $G(R_1) \times 1 \subset H$.*

Proof. We will first show $F(g) = F(1)$ for all $g \in G(R_0)$. Let $g \in G(R_0)^{\cdot}$. Pick $u_1 \in G(R_0)^{\cdot}$ such that $g \in D_{R_1}\langle 1, -u_1\rangle$. For all $u_2 \in G(R_2)$, since $g \in D_{R_1}\langle 1, -u_1\rangle = \pi_1(D_R\langle 1, -(u_1, u_2)\rangle \cap H)$, there exists a $k \in G(R_2)$ with $(g, k) \in H$ and $k \in D_{R_2}\langle 1, -u_2\rangle$. That is, $G(R_2) = \cup_{k\in F(g)} D_{R_2}\langle 1, -k\rangle$. By (3.1) and (5.4), $F(g) = F(1)$.

We next show $G(R_0) \times 1 \subset H$. Continue to let $g \in G(R_0)^{\cdot}$. Now, as $g \in \pi_1(H) = G(R_1)$, we have $(g, m) \in H$ for some $m \in G(R_2)$. Then $m \in F(g) = F(1)$ so $(1, m) \in H$ and hence $(g, 1) = (g, m)(1, m) \in H$. This shows $G(R_0) \subset H$.

We will be done if we show $F(t) = F(1)$. Then, if $(t, k) \in H$ we get $(1, k)$ and hence $(t, 1)$ are in H. Apply the previous paragraph to get $G(R_1) \times 1 = \{1, t\}G(R_0) \times 1 \subset H$.

So suppose $F(t) \neq F(1)$. We have by (3.1):

$$G(R_2) \neq \bigcup_{k\in F(t)} D_{R_2}\langle 1, -k\rangle.$$

Pick $u_2 \in G(R_2) \setminus \cup D_{R_2}\langle 1, -k\rangle$. Set $u = (-t, u_2)$. Then, as there is no k with $(t, k) \in H$ and $k \in D_{R_2}\langle 1, -k\rangle$, we have $\pi_1(D_R\langle 1, -u\rangle \cap H) = 1$. Pick any $g \in G(R_0)^{\cdot}$ (we note that $|G(R_0)| > 1$, else R_1 is degenerate). Pick any $g_2 \in F(g)$ and set $h = (g, g_2) \in H$. Now:

$$\begin{aligned} D_R\langle 1, uh, -h\rangle &= D_{R_1}\langle 1, -gt, -g\rangle \times D_{R_2}\langle 1, u_2g_2, -g_2\rangle \\ &\supset -gt \times D_{R_2}\langle 1, u_2g_2, -g_2\rangle. \end{aligned}$$

Also:

$$\bigcup_{w \in h(D_R\langle 1,-u\rangle \cap H)} D_R\langle 1, -w\rangle \subset D_{R_1}\langle 1, -g\rangle \times G(R_2).$$

Hence, by (3.3),

$$\begin{aligned} -gt \times D_{R_2}\langle 1, u_2g_2, -g_2\rangle &\subset uH \\ -t \times D_{R_2}\langle g_2, u_2, -1\rangle &\subset H. \end{aligned}$$

In particular, $(-t, g_2) \in H$. Since $(-1, 1) \in G(R_0) \times 1 \subset H$, we get $(t, g_2) \in H$. But then $g_2 \in F(t) \cap F(g)$, which equals $F(t) \cap F(1)$ by previous work. $F(t)$ is a coset of $F(1)$, by (5.4), so in fact $F(t) = F(1)$ as desired. □

7. The Main Theorem.

Theorem 7.1. *Let R be a Witt ring of elementary type. If S is an H-extension of R, for some subgroup $H \subset G(R)$, then S is also of elementary type.*

Proof. We argue by induction on $|G(R)|$. If $|G(R)| \le 2$ then either $H = 1$ or $H = G(R)$ and we are done by (1.1). Suppose $|G(R)| > 2$. If $R = A \sqcap B$ then π_A will denote the projection of $G(R)$ onto $G(A)$.

Now suppose R is degenerate. Write $R = D_k \sqcap R_2$, where R_2 is non-degenerate and $G(D_k) \times 1 = \mathrm{rad}(R)$. If some $d \in \mathrm{rad}(R) \cap H$ then write $R = D_1 \sqcap R_3$, where d generates D_1. We have $G(D_1) \times 1 \subset H$ and so $H = G(D_1) \times H_3$, for some subgroup $H_3 \subset G(R_3)$. Then (3.2) implies $S = D_1 \sqcap S_3$, where S_3 is an H_3-extension of R_3. By induction, S_3 is of elementary type and so S is also. We may thus assume $\mathrm{rad}(R) \cap H = 1$. If $\pi_{D_k}(H) \cap G(D_k) \ne 1$ then by (4.1) we may replace H, without affecting S, by another subgroup H_0 such that $\pi_{D_k}(H_0) \cap G(D_k) = 1$. We assume this has already been done so that $\pi_{D_k}(H) \cap G(D_k) = 1$. We note this also holds trivially if R is non-degenerate and $k = 0$.

Next suppose R has a local type factor L with $|G(L)| \ge 8$. Write $R = L \sqcap R_4$. D_k is a factor of R_4. We check the hypothesis of (5.7). Let $a \in G(L)$ and suppose $x \in F(a) \cap \mathrm{rad}(R_4)$. This means $(a, x) \in H$ and so $x \in \pi_{D_k}(H) = 1$. Thus $F(a) \cap \mathrm{rad}(R_4) \subset \{1\}$, as desired. Apply (5.6) to get that either $G(L) \times 1 \subset H$ or $\pi_L(H) = 1$. In the first case, $H = G(L) \times H_4$, for some subgroup H_4 of $G(R_4)$. Then (3.2) implies $S = L \sqcap S_4$, where S_4 is an H_4-extension of R_4. By induction, S_4 is of elementary type and so S is also. We may thus assume we are in the second case: $\pi_L(H) = 1$.

Now suppose the local factor is $L_1 = \mathbb{Z}$. There are three cases according to (5.9). In case (b) we can write $R = \mathbb{Z} \sqcap R_5$, with $\{\pm 1\} \times 1 \subset H$. Then $H = G(\mathbb{Z}) \times H_5$, for some subgroup $H_5 \subset G(R_5)$. Applying (3.2) again gives $S = \mathbb{Z} \sqcap S_5$, where S_5 is an H_5-extension of R_5. Induction again shows S is of elementary type. In case (c) we can write $R = \mathbb{Z} \sqcap \mathbb{Z} \sqcap R_6$, with $(1,1) \times G(R_6) \subset H$. Once again (3.2) yields $S = S_0 \sqcap R_6$, where S_0 is an H_0-extension of $\mathbb{Z} \sqcap \mathbb{Z}$, for some subgroup H_0. Since $\mathbb{Z} \sqcap \mathbb{Z} \cong \mathbb{Z}[E_1]$, (1.5) shows S_0 is of elementary type (in (1.5)(b) we have $R_0 = \mathbb{Z}$ so that its extension is of elementary type as $|G(R_0)| = 2$). Thus S is also of elementary type. We may thus assume we are in case (a) of (5.9), namely, that $\pi_{L_1}(H) = 1$. This is the same conclusion as when the local factor has at least 8 square classes.

The only local type factors we have omitted are those with 4 square classes and these Witt rings are group rings. We are thus in the following position: $R = Y \sqcap W_1 \sqcap \ldots \sqcap W_n$, where Y is a product of D_k with $k \geq 0$, and local type rings L with $|G(L)| \neq 4$ and each W_i is a non-degenerate group ring. (It is possible that $Y = 1$.) We also have $\pi_Y(H) = 1$, so that if $n = 0$ then $H = 1$ and we are done by (1.1). So suppose $n \geq 1$. Write $W_i = V_i[E_1]$. We first suppose that $\pi_{W_i}(H) \not\subset G(V_i)$ for some i. Write $R = W_i \sqcap R_7$. There are two possibilities according to (6.1).

The first possibility is that $1 \times G(R_7) \subset H$. Write $H = H_0 \times G(R_7)$, for some subgroup H_0 of $G(W_i)$. Then (3.2) gives that $S = S_0 \sqcap R_7$, for S_0, some H_0-extension of W_i. If $G(R_7) \neq 1$ then we are done by induction. We drop the subscript i and suppose then that $R = W = W_0[E_n]$, where W_0 is basic and $n \geq 1$. In cases (a),(c),(d) we have S is of elementary type. In case (b) $S = S_0[E_n]$, where S_0 is an H-extension of W_0, and so again S is of elementary type by induction.

The second possibility in (6.1) is that $\pi_{W_i}(H) = G(W_i)$. Then by (6.3) we have $H = G(W_i) \times H_7$, where H_7 is a subgroup of R_7. Apply (3.2) once again to get that $S = W_i \sqcap S_7$, where S_7 is an H_7-extension of R_7. Induction gives that S is of elementary type. This completes the result when (6.1) applies, that is, when $\pi_{W_i}(H) \not\subset G(V_i)$, for some i.

We may thus assume we have $R = Y \sqcap W_1 \sqcap \ldots \sqcap W_n$, with $\pi_Y(H) = 1$ and every $\pi_{W_i}(H) \subset V_i$. Choose $t_i \notin V_i$, for each i and set $g = (1, t_1, \ldots, t_n) \in G(R)$. Then:

$$g \notin \pm \bigcup_{h \in H^{\cdot}} D_R\langle 1, -h \rangle,$$

as for any $h \in H^{\cdot}$ has a coordinate in $G(V_i)^{\cdot}$. By (1.2), $g \notin B(R)$. Thus R is itself a group ring, a case we covered two paragraphs ago. □

The previous sections can be used to determine the possible H-extensions of a given ring R. As an example, consider $R = (D_1 \sqcap L_3)[E_2]$. Let d generate D_1, $-1, a, b$ generate L_3 and s, t generate E_2. Thus $G(R) = \text{gp}(d, -1, a, b, s, t)$, where $\text{gp}(A)$ denotes the group generated by A. $G(R)$ has 2825 subgroups, 47 of which will yield H-extensions. Up to isomorphism, there are exactly 8 H-extensions of R. Below we list the 8 extensions S along with one choice for the corresponding subgroup H.

1. $(D_1 \sqcap L_3)[E_3]$	1
2. $(D_1 \sqcap (D_1 \sqcap L_3)[E_1])[E_1]$	$\mathrm{gp}(d, -1, a, b, s)$
3. $D_1 \sqcap (D_1 \sqcap L_3)[E_2]$	$G(R)$
4. $(\mathbb{Z} \sqcap (D_1 \sqcap L_3)[E_1])[E_1]$	$\mathrm{gp}(d, a, b, s)$
5. $\mathbb{Z} \sqcap (D_1 \sqcap L_3)[E_2]$	$\mathrm{gp}(d, a, b, s, t)$
6. $(D_1 \sqcap L_3[E_1])[E_2]$	$\mathrm{gp}(d)$
7. $(D_2 \sqcap L_3)[E_2]$	$\mathrm{gp}(d, -1, a, b)$
8. $(D_1[E_1] \sqcap L_3)[E_2]$	$\mathrm{gp}(-1, a, b)$

We give a brief sketch of how this list was derived. Begin by running through the cases of (1.5). In (a) $H = 1$ and S is (1) by (1.1). In (c), $G(R_0)$ is a proper subgroup of H, so S is (2) or (3), depending on whether or not $H = G(R)$. In (d) $G(R_0) \subset \pm H, -1 \notin H$ and $H \not\subset G(R_0)$. Thus H looks like a subgroup K of index 2 in $G(R_0)$ that does not contain -1, together with one or more elements from $\{t, s, ts\}$. S is (4) if $|H \cap E_2| = 2$ and (5) if $|H \cap E_2| = 4$. In (1.5)(b) $S = S_0[E_2]$, where S_0 is an H-extension of R_0. Now $R_0 = D_1 \sqcap L_3$. By (4.1) we can assume that either $H = G(D_1) \times H_2$, or that $H = 1 \times H_2$, for some $H_2 \subset G(L_3)$. Now $H_2 = 1$ or $H_2 = G(L_3)$ by (2.3) and (5.7). Since we have already done the case $H = 1$ this gives three choices: $G(D_1) \times 1, G(D_1) \times G(L_3)$ and $1 \times G(L_3)$. The corresponding S is (6), (7) and (8), respectively.

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