JOININGS OF GAUSSIAN SYSTEMS

Gaussian systems

Suppose $(Y, \mathcal{C}, \nu)$ is a probability space. A (real) *Gaussian subspace* of $L^2_{\mathbb{R}}(\nu)$ is a closed subspace $\mathcal{H}$ such that each $f \in \mathcal{H}$ has a centered Gaussian distribution. Say $\mathcal{H}$ is *generating* if the generated sigma-algebra $\Sigma(\mathcal{H}) = C$.

Example. If γ is the standard Gaussian measure on $\mathbb{R}$, $d\gamma = \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$ and $\gamma^{\mathbb{N}}$ is the product measure on $\mathbb{R}^{\mathbb{N}}$ then the co-ordinate projections are Gaussian random variables and the subspace they generate is a generating Gaussian subspace of $L^2(\gamma^{\mathbb{N}})$.

Every separable non-atomic probability space has a generating Gaussian subspace.

Facts:

1. If $\{E_i\}$ is a family of mutually orthogonal subspaces of $\mathcal{H}$ then the family $\{\Sigma(E_i)\}$ of sigma-algebras is (jointly) independent.
2. If $\mathcal{H}$ is a generating Gaussian space and U is any unitary operator on $\mathcal{H}$ then there is a unique automorphism
$$T = T_U : (Y, \mathcal{C}, \nu) \to (Y, \mathcal{C}, \nu)$$
such that $Tf = Uf \quad \forall f \in \mathcal{H}$.
3. If E_1 and E_2 are closed subspaces of $\mathcal{H}$ then
$$\Sigma(E_1 \cap E_2) = \Sigma(E_1) \cap \Sigma(E_2)$$
4. If $E_1 \supseteq E_2 \supseteq \dots$ are subspaces of $\mathcal{H}$ then
$$\bigcap_n \Sigma(E_n) = \Sigma(\bigcap E_n) .$$

Remark: 3 and 4 are non-trivial theorems proved by Thouvenot. (Earlier references?).

Definition: A stationary Gaussian process is a stationary process $\{f_i\} \subset L^2_{\mathbb{R}}(\nu)$ whose closed linear span in $L^2_{\mathbb{R}}(\nu)$ is a Gaussian space.

A measure σ on the unit circle $\mathbb{T}$ is called **symmetric** if it is invariant under $z \mapsto \bar{z}$. Given any stationary $\mathbb{R}$- valued process there is a symmetric measure $\sigma = \sigma(\{f_i\})$ on $\mathbb{T}$ such that $\hat{\sigma}(i) = \langle f_0, f_i \rangle = \langle f_j, f_{j+i} \rangle$. Given any symmetric measure σ on $\mathbb{T}$ there is a unique stationary Gaussian process $\{f_i\}$ such that $\langle f_0, f_i \rangle = \hat{\sigma}(i)$. The distribution of a Gaussian process $\{f_i\}$ is completely determined by the covariance sequence $a_i = \langle f_0, f_i \rangle$

Traditionally, a Gaussian automorphism has been defined as the shift on a stationary Gaussian process. We will take a slightly different point of view.

Definition: $\mathbf{Y} = (Y, \mathcal{C}, \nu, S)$ is called a *Gaussian system* if there is an S-invariant generating Gaussian subspace $\mathcal{H}$ of $L^2_{\mathbb{R}}(\nu)$. (i.e. $S = T_U$ where $U = T\big|_{\mathcal{H}}$). $\mathcal{H}$ is *not* unique in general. We refer to $(U, \mathcal{H})$ as the underlying unitary of Y.

It is easy to see that the above definition is equivalent to being a direct product of Gaussians in the traditional sense.

The maximal spectral type $\sigma(U)$ of a real unitary U is a symmetric measure σ on the circle $\mathbb{T}$.

Theorem. *T_U is weakly mixing $\Leftrightarrow$ T_U is ergodic $\Leftrightarrow$ $\sigma(U)$ is continuous. Henceforth we deal only with weakly mixing Gaussian U.*

Gaussian centralizer of a Gaussian system

We let $U(\mathcal{H})$ denote the unitary group of $\mathcal{H}$ equipped with the weak operator topology. $U_n \to U$ in $U(\mathcal{H})$ iff $T_{U_n} \to T_U$ in the weak topology. If U is a unitary on $\mathcal{H}$ we let $C(U)$ denote the unitary centralizer of U

$$C(U) = \{V \in U(\mathcal{H}) : UV = VU\} \ ,$$

a closed subgroup of $U(\mathcal{H})$.

Example. Call $f : \mathbb{T} \to \mathbb{C}$ (hermitian) *symmetric* if $f(\overline{z}) = \overline{f(z)}$. Suppose σ is a symmetric measure on the circle and let $L^2_h(\sigma) = \{f \in L^2_{\mathbb{C}} : f \text{ symmetric }\}$. This is a real subspace of $L^2_{\mathbb{C}}$. If $f : \mathbb{T} \to \mathbb{T}$ is a measurable symmetric function

$$M_f(g) = fg$$

defines a unitary operator on $L^2_h(\sigma)$. Any real unitary operator with a cyclic vector can be represented as M_z for some σ. Then $C(M_z)$ consists of all M_f as above. This is a large group!

If $\mathbf{Y}$ is a Gaussian system with underlying unitary $(U, \mathcal{H})$ and $V \in C(U)$ then $T_V \in C(T_U) = C(\mathbf{Y})$ so $C(U)$ is embedded in $C(T_U)$. We will see that in some cases $C(U) = C(\mathbf{Y})$. In general we refer to $C(U) \subset C(\mathbf{Y})$ as the Gaussian centralizer of T.

Splittings

A *splitting* of a system $\mathbf{Y}$ is a pair $(\mathcal{A}_1, \mathcal{A}_2)$ of independent factor algebras such that $\mathcal{A}_1 \vee \mathcal{A}_2 = \mathcal{C}$. If $\mathbf{Y}$ is Gaussian and E, F are complementary orthogonal S-invariant subspaces of $\mathcal{H}$ then $(\Sigma(E), \Sigma(F))$ is a splitting of $\mathbf{Y}$. $\Sigma(E)$ and $\Sigma(F)$ are again Gaussian systems. Call such factors of $\mathbf{Y}$ *spatial factors*. Note that the family of spatial factors is closed under intersections. By the spectral theorem we have a rich supply of such S-invariant subspaces of $\mathcal{H}$, hence a rich supply of splittings of $\mathbf{Y}$. (This can be made precise in several ways.)

Group factors of Gaussian systems

If K is a compact subgroup of $C(U)$ we may view it as a compact subgroup of $C(S)$ hence

$$\mathcal{F}(K) = \{A \in \mathcal{C} : kA = A \quad \forall k \in K\}$$

is a factor of $\mathbf{Y}$, which we will refer to as a *group factor*. Obviously $C(U)$ has plenty of compact subgroups. Since spatial factors of $\mathbf{Y}$ are again Gaussian they themselves have group factors which we will refer to as *classical factors* of $\mathbf{Y}$. We will see that in certain cases these are the only factors.

Kronecker Gaussian systems

A *Kronecker subset* F of $\mathbb{T}$ is a closed subset with the property that any continuous $f : \mathbb{T} \to \mathbb{T}$ is a uniform limit of powers of z. A *symmetric Kronecker measure* on $\mathbb{T}$ is a symmetric measure on $\mathbb{T}$ whose restriction to the upper half-circle is supported on a Kronecker set. Continuous symmetric Kronecker measures exist!

Theorem (Foias, Stratila). *Suppose $\{f_i\} \subset L^2(\nu)$ is an ergodic stationary process whose spectral measure σ is continuous and Kronecker. Then $\{f_i\}$ is Gaussian.*

Definition. A Kronecker Gaussian system $\mathbf{Y}$ is a Gaussian system with underlying unitary $(U, \mathcal{H})$ where the maximal spectral type $\sigma(U)$ is a symmetric (continuous) Kronecker measure.

Theorem (Foias, Stratila). *Any system spectrally equivalent to a Kronecker Gaussian system is isomorphic to it.*

Theorem (Thouvenot). *If $\mathbf{Y}$ is a Kronecker Gaussian then $C(\mathbf{Y})$ coincides with the Gaussian centralizer of $\mathbf{Y}$.*

Proof: This amounts to showing that any $R \in C(\mathbf{Y})$ preserves the underlying Gaussian subspace $\mathcal{H}$. $R\mathcal{H}$ is another S-invariant Gaussian subspace and $S|_{R\mathcal{H}}$ is unitarily equivalent to $U = S|_{\mathcal{H}}$. Let $\widehat{\mathcal{H}}$ be the closed subspace of $L^2_{\mathbb{R}}(\nu)$ spanned by $\mathcal{H}$ and $R\mathcal{H}$,

$$\widehat{\mathcal{H}} = \mathcal{H} \vee R\mathcal{H} \ .$$

If $f \in \widehat{\mathcal{H}}$ then $\sigma(\{S^i f\}$ is absolutely continuous with respect to

$$\sigma(S|_{\mathcal{H}}) + \sigma(S|_{R\mathcal{H}}) = \sigma(U) \ ,$$

so by Foias-Stratila $\{S^i f\}$ is Gaussian. In particular each such f is Gaussian so $\widehat{\mathcal{H}}$ is a Gaussian space containing the already generating space $\mathcal{H}$. This means $\widehat{\mathcal{H}} = \mathcal{H}$ since $\Sigma(\widehat{\mathcal{H}} \ominus \mathcal{H})$ is independent of $\Sigma(\mathcal{H})$.

Proposition (Thouvenot). *Suppose $\mathbf{Y}$ is a Kronecker Gaussian with underlying unitary $(U, \mathcal{H})$. Let*

$$\mathcal{H}_1 = \mathcal{H} \otimes 1 = \{f \otimes 1 : f \in \mathcal{H}\} \ ,$$

$$\mathcal{H}_2 = 1 \otimes \mathcal{H} \subseteq L^2(\lambda)$$

and let $\mathcal{H}_\lambda$ denote the closed subspace spanned by $\mathcal{H}_1$ and $\mathcal{H}_2$. Then $\mathcal{H}_\lambda$ is an $S \times S$-invariant generating Gaussian space in $L^2(\lambda)$ and $\sigma(S \times S\big|_{\mathcal{H}_\lambda}) = \sigma(U)$. In particular $(\lambda, S \times S)$ is again a Kronecker Gaussian.

Proof. For $h \in \mathcal{H}_\lambda$ the spectral measure of $\{(S \times S)^i h\}$ is absolutely continuous with respect to the sum of the spectral types of $S \times S\big|_{\mathcal{H}_1}$ and $S \times S\big|_{\mathcal{H}_2}$, which are both the Kronecker measure $\sigma(U)$. By Foias-Stratila h is a Gaussian random variable. $\square$

Theorem (Thouvenot). *Every factor $\mathcal{A}$ of a Kronecker Gaussian $\mathbf{Y}$ is classical.*

Proof. Since the family of spatial factors of $\mathbf{Y}$ is closed under intersections there is a smallest spatial factor $\hat{\mathcal{A}}$ containing $\mathcal{A}$. We shall prove that $\mathcal{A}$ is a group factor of $\hat{\mathcal{A}}$, and there is no loss of generality in assuming that $\hat{\mathcal{A}} = \mathcal{C}$. By Veech's theorem we need only check that if $\lambda \in J_e^2(\mathbf{Y})$ and λ is diagonal on $\mathcal{A} \times \mathcal{A}$ then λ is graphic. Let

$$\mathcal{H}_1 = \mathcal{H} \otimes 1 \ , \ \mathcal{H}_2 = 1 \otimes \mathcal{H} \subseteq \mathcal{H}_\lambda$$

and

$$\mathcal{C}_1 = \mathcal{C} \times Y \ , \ \mathcal{C}_2 = Y \times \mathcal{C}$$

so

$$\mathcal{C}_1 = \Sigma(\mathcal{H}_1) \ , \ \mathcal{C}_2 = \Sigma(\mathcal{H}_2) \ .$$

Then $\mathcal{C}_1 \cap \mathcal{C}_2 = \Sigma(\mathcal{H}_1 \cap \mathcal{H}_2)$ since $\mathcal{H}_\lambda$ is Gaussian. Now $\mathcal{H}_1 \cap \mathcal{H}_2$ is a closed $S \times S$-invariant subspaces of $\mathcal{H}_1$ so $\mathcal{H}_1 \cap \mathcal{H}_2 = E \otimes 1$ where E is a closed S-invariant subspace of $\mathcal{H}$. On the other hand

$$\mathcal{A} \times Y \subset \mathcal{C}_1 \cap \mathcal{C}_2 \ ,$$

by the assumption that λ is diagonal on $\mathcal{A} \otimes \mathcal{A}$. Thus we have

$$\mathcal{A} \times Y \subset \Sigma(E \otimes 1) = \Sigma(E) \times Y$$

so $\mathcal{A} \subset \Sigma(E)$. Since $\hat{\mathcal{A}} = \mathcal{C}$ we conclude that $E = \mathcal{H}$, that is $\mathcal{H}_1 = \mathcal{H}_2$ which means that $\mathcal{C}_1 = \mathcal{C}_2$, i.e. λ is graphic.

Although Thouvenot's result is for a very special class of Gaussians one can use it to get some weak but nonetheless useful information about factors of an arbitrary Gaussian.

Lemma. *If U is a real unitary operator with continuous spectrum on a separable real Hilbert space then $C(U)$ contains a copy of any unitary operator V with continuous spectrum on a separable real Hilbert space.*

Sketch of proof for complex $\mathcal{H}$. Suppose for "simplicity" that U and V have simple spectrum so $U \simeq M_z$ on $L^2(\sigma)$ where σ is a finite measure on $\mathbb{T}$ and M_z is multiplication by z. Then $C(M_z)$ is the set of all multiplications M_f where $f : \mathbb{T} \to \mathbb{T}$ is measurable. If f happens to be one-to-one σ-a.e. then M_f has simple spectrum and spectral type $f(\sigma)$. $f(\sigma)$ can be essentially any continuous measure on the circle. □

Corollary. *If $\mathbf{Y}$ is a Gaussian system then the Gaussian centralizer of $\mathbf{Y}$ contains an isomorphic copy of any Gaussian system.*

Proposition (del Junco, Lemanczyk). *If $\mathbf{Y}$ is Gaussian and $\mathbb{Z}$ is a factor then Y has a countable self-joining which is isomorphic to a classical factor of Y.*

Proof: Let $\mathcal{A}$ be the relevant factor algebra, choose any Kronecker Gaussian $R \in C(\mathbf{Y})$ having simple spectrum and form the algebra

$$\mathcal{A} = \bigvee_{i \in \mathbb{Z}} R^i \mathcal{A} \ ,$$

which is both S and R-invariant. Since the action of S on each $R^i\mathcal{A}$ is isomorphic to its action on $\mathcal{A}$ the system $(Y, \hat{\mathcal{A}}, \nu, S)$ may be viewed as a countable self-joining of Z. On the other hand, since $\hat{\mathcal{A}}$ is R-invariant, Thouvenot's result tells us that $\hat{\mathcal{A}}$ is a classical factor of R. One can then use the fact that R has simple spectrum to argue that it is a classical factor of S as well.

■

The above result plays a key role in the proof of:

Theorem (del Junco, Lemanczyk). *A simple system* $\mathbf{X}$ *is disjoint from any Gaussian system* $\mathbf{Y}$.

By the del Junco-Rudolph result (talk #1) this is equivalent to showing that no factor $\mathbf{Y}^0$ of $\mathbf{Y}$ can be isomorphic to some $(X/K)^{n\odot} = \mathbf{X}^0$. If this were the case then by the above proposition X^0 has a countable self-joining $\mathbf{X}'$ which is isomorphic to a classical factor Y' of Y. Using simplicity it is easy to see that $\mathbf{X}'$ is isomorphic to a factor of the countable product $\mathbf{X}^{\mathbb{N}}$. Summarizing, what has to be shown is:

Theorem. *No factor* $\mathbf{X}$ *of* $X^{\mathbb{N}}$ *is isomorphic to a classical factor* $\mathbf{Y}'$ *of* $\mathbf{Y}$.

In case $\mathbf{X}' = \mathbf{X}^{\mathbb{N}}$ and $Y' = \mathbf{Y}$ this can be done as follows.

Definition. A system $\mathbf{Y}$ is (infinitely) *divisible* if it has a system of factor algebras

$$\{C_\delta : \delta \in \bigcup_{n=0}^{\infty} \{0,1\}^n\}$$

(we write $\{0,1\}^0 = \{\phi\}$, the "empty" sequence) such that $(C_{\delta_0}, C_{\delta_1})$ is a splitting of C_δ for each δ, and for all $x \in \{0,1\}^{\mathbb{N}}$ the intersection $\bigcap_n C_{x_1,\ldots,x_n}$ is trivial.

Remark. This notion has been studied by Katok and Thouvenot, who constructed an example of a divisible system with discrete spectrum.

Proposition. *Every Gaussian system is divisible.*

Proposition. $\mathbf{X}^{\mathbb{N}}$ *has only the "obvious" splittings, in particular it is not divisible.*

To deal with factors we introduce a weaker notion "virtual divisibility" which is, more or less, "divisibility up to distal extension".

Thouvenot's analysis of factors of a Kronecker Gaussian can be extended to Gaussians all of whose ergodic self joinings are Gaussian (Lemanczyk, Parreau, Thouvenot). This is a considerably larger class which includes all Gaussian systems with simple spectrum.

Questions:

1. Is there a zero entropy Gaussian with non-classical factors?
2. Is there a zero entropy Gaussian with non-Gaussian elements in its centralizer?