New York Journal of Mathematics

New York J. Math. **12** (2006) 63–95.

T-homotopy and refinement of observation. IV. Invariance of the underlying homotopy type

Philippe Gaucher

Abstract. This series explores a new notion of T-homotopy equivalence of flows. The new definition involves embeddings of finite bounded posets preserving the bottom and the top elements and the associated cofibrations of flows. In this fourth part, it is proved that the generalized T-homotopy equivalences preserve the underlying homotopy type of a flow. The proof is based on Reedy model category techniques.

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1. Outline of the paper

The main feature of the two algebraic topological models of *higher-dimensional automata* (or HDA) introduced in [GG03] and in [Gau03] is to provide a framework

Received January 3, 2006.

Mathematics Subject Classification. 55U35, 55P99, 68Q85.

Key words and phrases. concurrency, homotopy, directed homotopy, model category, refinement of observation, poset, cofibration, Reedy category, homotopy colimit.

ISSN 1076-9803/06

for modelling continuous deformations of HDA corresponding to subdivision or refinement of observation. *Globular complexes* and *flows* are introduced in [GG03] and [Gau03] respectively for modelling a notion of dihomotopy equivalence between higher-dimensional automata [Pra91] [Gla05]. This equivalence relation preserves geometric properties like the *initial* or *final states*, and therefore computer-scientific properties like the presence or not of *deadlocks* or of *unreachable states* [Gou03]. More generally, dihomotopy is designed to preserve all computer-scientific properties invariant under refinement of observation (see Figure 2). The two settings are compared in [Gau05a] and are proved to be equivalent.

In the framework of flows, there are two kinds of dihomotopy equivalences [Gau00]: the *weak S-homotopy equivalences* (the spatial deformations of [Gau00]) which can be interpreted as the weak equivalences of a model structure [Gau03] and the *T-homotopy equivalences* (the temporal deformations of [Gau00]). The latter are considerably more difficult to model and to understand. The geometric explanations underlying the intuition of S-homotopy and T-homotopy are given in the first part of this series [Gau05b], but the reference [GG03] may be preferred.

The purpose of this paper is to prove that the notion of T-homotopy equivalence studied in this series preserves the *underlying homotopy type* of a flow. The underlying homotopy type of a flow is the topological space which is obtained after removing the temporal ordering. This underlying topological space is unique only up to weak homotopy equivalence. For example, the underlying homotopy type of the two flows of Figure 2 is a point. The main theorem of this paper is:

Theorem. *Let $f : X \longrightarrow Y$ be a generalized T-homotopy equivalence. Then the morphism of* $\mathbf{Ho}(\mathbf{Top})$ $|f| : |X| \longrightarrow |Y|$*, where $|-|$ is the underlying homotopy type functor, is an isomorphism.*

Section 4 recalls the notions of full directed ball and of generalized T-homotopy equivalence. Section 5 recalls the notion of globular complex. It is necessary for the definition of the underlying homotopy type of a flow. Section 6 gives the rigorous definition of the underlying homotopy type of a flow. Section 7 constructs a useful Reedy structure which will be crucial in the main proofs of the paper. Section 7 also establishes related lemmas. Section 8 proves that the underlying homotopy type of the full directed ball is contractible (Theorem 8.6). The latter result is important since a T-homotopy equivalence consists in replacing in a flow a full directed ball by a more refined full directed ball (see Figure 3), and in iterating this process transfinitely. Then Section 9 proves the theorem above.

Warning. This paper is the fourth part of a series of papers devoted to the study of T-homotopy. Several other papers explain the geometrical content of T-homotopy. The best reference is probably [GG03] (it does not belong to the series). However, the knowledge of the other parts is not required. In particular, this means that there are repetitions between the papers of this series. They are all of them collected in the appendices A, B and C which are already in the third part of this series. The proofs of these appendices are independent from the technical core of this part. The left properness of the weak S-homotopy model structure of **Flow** is not duplicated in this paper. It is available in [Gau05c]. This fact is used twice in the proof of Theorem 9.1.

2. Prerequisites and notations

The initial object (resp. the terminal object) of a category $\mathcal{C}$, if it exists, is denoted by $\varnothing$ (resp. $\mathbf{1}$).

Let $\mathcal{C}$ be a cocomplete category. If K is a set of morphisms of $\mathcal{C}$, then the class of morphisms of $\mathcal{C}$ that satisfy the RLP (*right lifting property*) with respect to any morphism of K is denoted by $\mathbf{inj}(K)$ and the class of morphisms of $\mathcal{C}$ that are transfinite compositions of pushouts of elements of K is denoted by $\mathbf{cell}(K)$. Denote by $\mathbf{cof}(K)$ the class of morphisms of $\mathcal{C}$ that satisfy the LLP (*left lifting property*) with respect to the morphisms of $\mathbf{inj}(K)$. This is a purely categorical fact that $\mathbf{cell}(K) \subset \mathbf{cof}(K)$. Moreover, every morphism of $\mathbf{cof}(K)$ is a retract of a morphism of $\mathbf{cell}(K)$ as soon as the domains of K are small relative to $\mathbf{cell}(K)$ ([Hov99] Corollary 2.1.15). An element of $\mathbf{cell}(K)$ is called a *relative K-cell complex*. If X is an object of $\mathcal{C}$, and if the canonical morphism $\varnothing \longrightarrow X$ is a relative K-cell complex, then the object X is called a *K-cell complex*.

Let $\mathcal{C}$ be a cocomplete category with a distinguished set of morphisms I. Then let $\mathbf{cell}(\mathcal{C}, I)$ be the full subcategory of $\mathcal{C}$ consisting of the objects X of $\mathcal{C}$ such that the canonical morphism $\varnothing \longrightarrow X$ is an object of $\mathbf{cell}(I)$. In other terms, $\mathbf{cell}(\mathcal{C}, I) = (\varnothing \downarrow \mathcal{C}) \cap \mathbf{cell}(I)$.

It is obviously impossible to read this paper without a strong familiarity with *model categories*. Possible references for model categories are [Hov99], [Hir03] and [DS95]. The original reference is [Qui67] but Quillen's axiomatization is not used in this paper. The axiomatization from Hovey's book is preferred. If $\mathcal{M}$ is a *cofibrantly generated* model category with set of generating cofibrations I, let $\mathbf{cell}(\mathcal{M}) := \mathbf{cell}(\mathcal{M}, I)$: this is the full subcategory of *cell complexes* of the model category $\mathcal{M}$. A cofibrantly generated model structure $\mathcal{M}$ comes with a *cofibrant replacement functor* $Q : \mathcal{M} \longrightarrow \mathbf{cell}(\mathcal{M})$. For any morphism f of $\mathcal{M}$, the morphism $Q(f)$ is a cofibration, and even an inclusion of subcomplexes ([Hir03] Definition 10.6.7) because the cofibrant replacement functor Q is obtained by the small object argument.

A *partially ordered set* $(P, \leqslant)$ (or *poset*) is a set equipped with a reflexive antisymmetric and transitive binary relation $\leqslant$. A poset is *locally finite* if for any $(x, y) \in P \times P$, the set $[x, y] = \{z \in P, x \leqslant z \leqslant y\}$ is finite. A poset $(P, \leqslant)$ is *bounded* if there exist $\widehat{0} \in P$ and $\widehat{1} \in P$ such that $P = [\widehat{0}, \widehat{1}]$ and such that $\widehat{0} \neq \widehat{1}$. Let $\widehat{0} = \min P$ (the bottom element) and $\widehat{1} = \max P$ (the top element). In a poset P, the interval $]\alpha, -]$ (the sub-poset of elements of P strictly bigger than α) can also be denoted by $P_{>\alpha}$.

A poset P, and in particular an ordinal, can be viewed as a small category denoted in the same way: the objects are the elements of P and there exists a morphism from x to y if and only if $x \leqslant y$. If λ is an ordinal, a *λ-sequence* in a cocomplete category $\mathcal{C}$ is a colimit-preserving functor X from λ to $\mathcal{C}$. We denote by X_λ the colimit $\varinjlim X$ and the morphism $X_0 \longrightarrow X_\lambda$ is called the *transfinite composition* of the $X_\mu \longrightarrow X_{\mu+1}$.

Let $\mathcal{C}$ be a category. Let α be an object of $\mathcal{C}$. The *latching category* $\partial(\mathcal{C} \downarrow \alpha)$ at α is the full subcategory of $\mathcal{C} \downarrow \alpha$ containing all the objects except the identity map of α. The *matching category* $\partial(\alpha \downarrow \mathcal{C})$ at α is the full subcategory of $\alpha \downarrow \mathcal{C}$ containing all the objects except the identity map of α.

Let $\mathcal{B}$ be a small category. A *Reedy structure* on $\mathcal{B}$ consists of two subcategories $\mathcal{B}_-$ and $\mathcal{B}_+$, a functor $d : \mathcal{B} \longrightarrow \lambda$ called the degree function for some ordinal λ, such that every nonidentity map in $\mathcal{B}_+$ raises the degree, every nonidentity map in $\mathcal{B}_-$ lowers the degree, and every map $f \in \mathcal{B}$ can be factored uniquely as $f = g \circ h$ with $h \in \mathcal{B}_-$ and $g \in \mathcal{B}_+$. A small category together with a Reedy structure is called a *Reedy category*.

Let $\mathcal{C}$ be a complete and cocomplete category. Let $\mathcal{B}$ be a Reedy category. Let i be an object of $\mathcal{B}$. The *latching space functor* is the composite $L_i : \mathcal{C}^{\mathcal{B}} \longrightarrow \mathcal{C}^{\partial(\mathcal{B}_+\downarrow i)} \longrightarrow \mathcal{C}$ where the latter functor is the colimit functor. The *matching space functor* is the composite $M_i : \mathcal{C}^{\mathcal{B}} \longrightarrow \mathcal{C}^{\partial(i\downarrow\mathcal{B}_-)} \longrightarrow \mathcal{C}$ where the latter functor is the limit functor.

If $\mathcal{C}$ is a small category and of $\mathcal{M}$ is a model category, the notation $\mathcal{M}^{\mathcal{C}}$ is the category of functors from $\mathcal{C}$ to $\mathcal{M}$, i.e., the category of diagrams of objects of $\mathcal{M}$ over the small category $\mathcal{C}$.

The category **Top** of *compactly generated topological spaces* (i.e., of weak Hausdorff k-spaces) is complete, cocomplete and cartesian closed (more details for this kind of topological spaces in [Bro88, May99], the appendix of [Lew78] and also the preliminaries of [Gau03]). For the sequel, all topological spaces will be supposed to be compactly generated. A *compact space* is always Hausdorff.

A model category is *left proper* if the pushout of a weak equivalence along a cofibration is a weak equivalence. The model categories **Top** and **Flow** (see below) are both left proper.

In this paper, the notation $\hookrightarrow$ means *cofibration*, the notation $\twoheadrightarrow$ means *fibration*, the notation $\simeq$ means *weak equivalence*, and the notation $\cong$ means *isomorphism*.

A categorical adjunction $\mathbb{L} : \mathcal{M} \leftrightarrows \mathcal{N} : \mathbb{R}$ between two model categories is a *Quillen adjunction* if one of the following equivalent conditions is satisfied:

(1) $\mathbb{L}$ preserves cofibrations and trivial cofibrations.
(2) $\mathbb{R}$ preserves fibrations and trivial fibrations.

In that case, $\mathbb{L}$ (resp. $\mathbb{R}$) preserves weak equivalences between cofibrant (resp. fibrant) objects.

If P is a poset, let us denote by $\Delta(P)$ the *order complex* associated with P. Recall that the order complex is a simplicial complex having P as underlying set and having the subsets $\{x_0, x_1, \ldots, x_n\}$ with $x_0 < x_1 < \cdots < x_n$ as n-simplices [Qui78]. Such a simplex will be denoted by $(x_0, x_1, \ldots, x_n)$. The order complex $\Delta(P)$ can be viewed as a poset ordered by the inclusion, and therefore as a small category. The corresponding category will be denoted in the same way. The opposite category $\Delta(P)^{\mathrm{op}}$ is freely generated by the morphisms $\partial_i : (x_0, \ldots, x_n) \longrightarrow (x_0, \ldots, \widehat{x_i}, \ldots, x_n)$ for $0 \leqslant i \leqslant n$ and by the simplicial relations $\partial_i\partial_j = \partial_{j-1}\partial_i$ for any $i < j$, where the notation $\widehat{x_i}$ means that x_i is removed.

If $\mathcal{C}$ is a small category, then the *classifying space* of $\mathcal{C}$ is denoted by $B\mathcal{C}$ [Seg68] [Qui73].

3. Reminder about the category of flows

The category **Top** is equipped with the unique model structure having the *weak homotopy equivalences* as weak equivalences and having the *Serre fibrations*[1] as fibrations.

The time flow of a higher-dimensional automaton is encoded in an object called a *flow* [Gau03]. A flow X consists of a set X^0 called the 0-*skeleton* and whose elements correspond to the *states* (or *constant execution paths*) of the higher-dimensional automaton. For each pair of states $(\alpha, \beta) \in X^0 \times X^0$, there is a topological space $\mathbb{P}_{\alpha,\beta}X$ whose elements correspond to the (*nonconstant*) *execution paths* of the higher-dimensional automaton *beginning at* α and *ending at* β. For $x \in \mathbb{P}_{\alpha,\beta}X$, let $\alpha = s(x)$ and $\beta = t(x)$. For each triple $(\alpha, \beta, \gamma) \in X^0 \times X^0 \times X^0$, there exists a continuous map $* : \mathbb{P}_{\alpha,\beta}X \times \mathbb{P}_{\beta,\gamma}X \longrightarrow \mathbb{P}_{\alpha,\gamma}X$ called the *composition law* which is supposed to be associative in an obvious sense. The topological space $\mathbb{P}X = \bigsqcup_{(\alpha,\beta)\in X^0\times X^0} \mathbb{P}_{\alpha,\beta}X$ is called the *path space* of X. The category of flows is denoted by **Flow**. A point α of X^0 such that there are no nonconstant execution paths ending at α (resp. starting from α) is called an *initial state* (resp. a *final state*). A morphism of flows f from X to Y consists of a set map $f^0 : X^0 \longrightarrow Y^0$ and a continuous map $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ preserving the structure. A flow is therefore "almost" a small category enriched in **Top**.

An important example is the flow $\mathrm{Glob}(Z)$ defined by the equations

$$\begin{aligned}
&\mathrm{Glob}(Z)^0 = \{\widehat{0}, \widehat{1}\}\\
&\mathbb{P}\mathrm{Glob}(Z) = Z\\
&s = \widehat{0}\\
&t = \widehat{1}
\end{aligned}$$

and a trivial composition law (cf. Figure 1).

The category **Flow** is equipped with the unique model structure such that [Gau03]:

- The weak equivalences are the *weak S-homotopy equivalences*, i.e., the morphisms of flows $f : X \longrightarrow Y$ such that $f^0 : X^0 \longrightarrow Y^0$ is a bijection and such that $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ is a weak homotopy equivalence.
- The fibrations are the morphisms of flows $f : X \longrightarrow Y$ such that $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ is a Serre fibration.

This model structure is cofibrantly generated. The set of generating cofibrations is the set $I^{\mathrm{gl}}_+ = I^{\mathrm{gl}} \cup \{R : \{0,1\} \longrightarrow \{0\}, C : \varnothing \longrightarrow \{0\}\}$ with

$$I^{\mathrm{gl}} = \left\{\mathrm{Glob}(\mathbf{S}^{n-1}) \subset \mathrm{Glob}(\mathbf{D}^n), n \geqslant 0\right\}$$

where $\mathbf{D}^n$ is the n-dimensional disk and $\mathbf{S}^{n-1}$ the $(n-1)$-dimensional sphere. The set of generating trivial cofibrations is

$$J^{\mathrm{gl}} = \left\{\mathrm{Glob}(\mathbf{D}^n \times \{0\}) \subset \mathrm{Glob}(\mathbf{D}^n \times [0,1]), n \geqslant 0\right\}.$$

If X is an object of **cell**(**Flow**), then a presentation of the morphism $\varnothing \longrightarrow X$ as a transfinite composition of pushouts of morphisms of I^{gl}_+ is called a *globular decomposition* of X.

[1] That is a continuous map having the RLP with respect to the inclusion $\mathbf{D}^n \times \{0\} \subset \mathbf{D}^n \times [0,1]$ for any $n \geqslant 0$ where $\mathbf{D}^n$ is the n-dimensional disk.

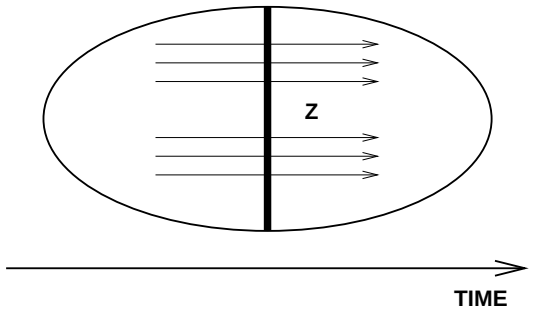


FIGURE 1. Symbolic representation of Glob(Z) for some topological space Z

4. Generalized T-homotopy equivalences

Definition 4.1. A flow X is loopless if for any $\alpha \in X^0$, the space $\mathbb{P}_{\alpha,\alpha}X$ is empty.

Recall that a flow is a small category without identity morphisms enriched over a category of topological spaces. So the preceding definition is meaningful.

Lemma 4.2. *A flow X is loopless if and only if the transitive closure of the set $\{(\alpha,\beta) \in X^0 \times X^0$ such that $\mathbb{P}_{\alpha,\beta}X \neq \varnothing\}$ induces a partial ordering on X^0.*

Proof. If (α,β) and (β,α) with $\alpha \neq \beta$ belong to the transitive closure, then there exists a finite sequence $(x_1,\dots,x_\ell)$ of elements of X^0 with $x_1 = \alpha$, $x_\ell = \alpha$, $\ell > 1$ and for any m, $\mathbb{P}_{x_m,x_{m+1}}X$ is nonempty. So the space $\mathbb{P}_{\alpha,\alpha}X$ is nonempty because of the existence of the composition law of X: contradiction. □

Definition 4.3. A full directed ball is a flow $\overrightarrow{D}$ such that:

- The 0-skeleton $\overrightarrow{D}^0$ is finite.
- $\overrightarrow{D}$ has exactly one initial state $\widehat{0}$ and one final state $\widehat{1}$ with $\widehat{0} \neq \widehat{1}$.
- Each state α of $\overrightarrow{D}^0$ is between $\widehat{0}$ and $\widehat{1}$, that is there exists an execution path from $\widehat{0}$ to α, and another execution path from α to $\widehat{1}$.
- $\overrightarrow{D}$ is loopless.
- For any $(\alpha,\beta) \in \overrightarrow{D}^0 \times \overrightarrow{D}^0$, the topological space $\mathbb{P}_{\alpha,\beta}\overrightarrow{D}$ is empty if $\alpha \geqslant \beta$ and weakly contractible if $\alpha < \beta$.

Let $\overrightarrow{D}$ be a full directed ball. Then by Lemma 4.2, the set $\overrightarrow{D}^0$ can be viewed as a finite bounded poset. Conversely, if P is a finite bounded poset, let us consider the *flow $F(P)$ associated with P*: it is of course defined as the unique flow $F(P)$ such that $F(P)^0 = P$ and $\mathbb{P}_{\alpha,\beta}F(P) = \{u_{\alpha,\beta}\}$ if $\alpha < \beta$ and $\mathbb{P}_{\alpha,\beta}F(P) = \varnothing$ otherwise. Then $F(P)$ is a full directed ball and for any full directed ball $\overrightarrow{D}$, the two flows $\overrightarrow{D}$ and $F(\overrightarrow{D}^0)$ are weakly S-homotopy equivalent.

Let $\overrightarrow{E}$ be another full directed ball. Let $f : \overrightarrow{D} \longrightarrow \overrightarrow{E}$ be a morphism of flows preserving the initial and final states. Then f induces a morphism of posets from $\overrightarrow{D}^0$ to $\overrightarrow{E}^0$ such that $f(\min \overrightarrow{D}^0) = \min \overrightarrow{E}^0$ and $f(\max \overrightarrow{D}^0) = \max \overrightarrow{E}^0$. Hence the following definition:

Definition 4.4. Let $\mathcal{T}$ be the class of morphisms of posets $f : P_1 \longrightarrow P_2$ such that:

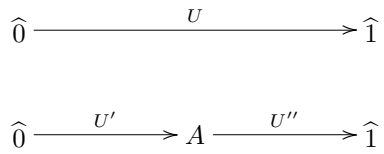


FIGURE 2. The simplest example of refinement of observation

(1) The posets P_1 and P_2 are finite and bounded.
(2) The morphism of posets $f : P_1 \longrightarrow P_2$ is one-to-one; in particular, if x and y are two elements of P_1 with $x < y$, then $f(x) < f(y)$.
(3) One has $f(\min P_1) = \min P_2$ and $f(\max P_1) = \max P_2$.

Then a generalized T-homotopy equivalence is a morphism of

$$\mathbf{cof}(\{Q(F(f)), f \in \mathcal{T}\})$$

where Q is the cofibrant replacement functor of **Flow**.

One can choose a *set* of representatives for each isomorphism class of finite bounded posets. One obtains a *set* of morphisms $\overline{\mathcal{T}} \subset \mathcal{T}$ such that there is the equality of classes $\mathbf{cof}(\{Q(F(f)), f \in \overline{\mathcal{T}}\}) = \mathbf{cof}(\{Q(F(f)), f \in \mathcal{T}\})$. By [Gau03] Proposition 11.5, the set of morphisms $\{Q(F(f)), f \in \overline{\mathcal{T}}\}$ permits the small object argument. Thus, the class of morphisms $\mathbf{cof}(\{Q(F(f)), f \in \mathcal{T}\})$ contains exactly the retracts of the morphisms of

$$\mathbf{cell}(\{Q(F(f)), f \in \mathcal{T}\})$$

by [Hov99] Corollary 2.1.15.

The inclusion of posets $\{\widehat{0} < \widehat{1}\} \subset \{\widehat{0} < 2 < \widehat{1}\}$ corresponds to the case of Figure 2.

A T-homotopy consists in locally replacing in a flow a full directed ball by a more refined one (cf. Figure 3), and in iterating the process transfinitely.

5. Globular complex

The technical reference is [Gau05a]. A *globular complex* is a topological space together with a structure describing the sequential process of attaching *globular cells*. A general globular complex may require an arbitrary long transfinite construction. We restrict our attention in this paper to globular complexes whose globular cells are morphisms of the form $\mathrm{Glob}^{\mathrm{top}}(\mathbf{S}^{n-1}) \longrightarrow \mathrm{Glob}^{\mathrm{top}}(\mathbf{D}^n)$.

Definition 5.1. A multipointed topological space (X, X^0) is a pair of topological spaces such that X^0 is a discrete subspace of X. A morphism of multipointed topological spaces $f : (X, X^0) \longrightarrow (Y, Y^0)$ is a continuous map $f : X \longrightarrow Y$ such that $f(X^0) \subset Y^0$. The corresponding category is denoted by $\mathbf{Top}^m$. The set X^0 is called the 0-skeleton of (X, X^0). The space X is called the underlying topological space of (X, X^0).

The category of multipointed spaces is cocomplete.

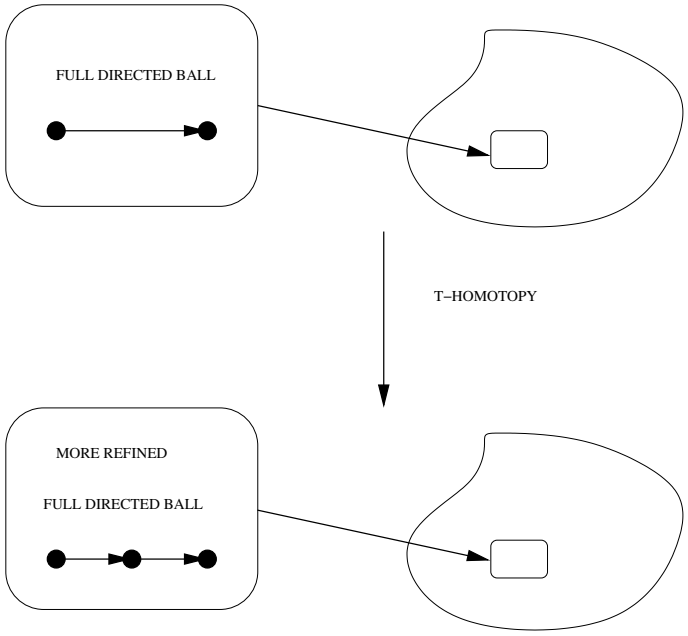


FIGURE 3. Replacement of a full directed ball by a more refined one

Definition 5.2. Let Z be a topological space. The globe of Z, which is denoted by $\mathrm{Glob}^{\mathrm{top}}(Z)$, is the multipointed space

$$\Big(|\mathrm{Glob}^{\mathrm{top}}(Z)|, \{\widehat{0}, \widehat{1}\}\Big)$$

where the topological space $|\mathrm{Glob}^{\mathrm{top}}(Z)|$ is the quotient of $\{\widehat{0}, \widehat{1}\} \sqcup (Z \times [0,1])$ by the relations $(z,0) = (z',0) = \widehat{0}$ and $(z,1) = (z',1) = \widehat{1}$ for any $z, z' \in Z$. In particular, $\mathrm{Glob}^{\mathrm{top}}(\varnothing)$ is the multipointed space $(\{\widehat{0}, \widehat{1}\}, \{\widehat{0}, \widehat{1}\})$.

If Z is not empty, then the space $|\mathrm{Glob}^{\mathrm{top}}(Z)|$ is the unpointed suspension of Z. If Z is the empty space, then the space $|\mathrm{Glob}^{\mathrm{top}}(Z)|$ is the discrete two-point space.

Notation 5.3. Let Z be a singleton. The globe of Z is denoted by $\overrightarrow{I}^{\mathrm{top}}$.

Definition 5.4. Let $I^{\mathrm{gl,top}} := \{\mathrm{Glob}^{\mathrm{top}}(\mathbf{S}^{n-1}) \longrightarrow \mathrm{Glob}^{\mathrm{top}}(\mathbf{D}^n), n \geqslant 0\}$. A relative globular precomplex is a relative $I^{\mathrm{gl,top}}$-cell complex in the category of multipointed topological spaces.

Definition 5.5. A globular precomplex is a λ-sequence of multipointed topological spaces $X : \lambda \longrightarrow \mathbf{Top}^m$ such that X is a relative globular precomplex and such that $X_0 = (X^0, X^0)$ with X^0 a discrete space. This λ-sequence is characterized by a presentation ordinal λ, and for any $\beta < \lambda$, an integer $n_\beta \geqslant 0$ and an attaching map $\phi_\beta : \mathrm{Glob}^{\mathrm{top}}(\mathbf{S}^{n_\beta - 1}) \longrightarrow X_\beta$. The family $(n_\beta, \phi_\beta)_{\beta<\lambda}$ is called the globular decomposition of X.

Let X be a globular precomplex. The 0-skeleton of $\varinjlim X$ is equal to X^0.

Definition 5.6. A morphim of globular precomplexes $f : X \longrightarrow Y$ is a morphism of multipointed spaces still denoted by f from $\varinjlim X$ to $\varinjlim Y$.

Notation 5.7. If X is a globular precomplex, then the underlying topological space of the multipointed space $\varinjlim X$ is denoted by $|X|$ and the 0-skeleton of the multipointed space $\varinjlim X$ is denoted by X^0.

Definition 5.8. Let X be a globular precomplex. The space $|X|$ is called the underlying topological space of X. The set X^0 is called the 0-skeleton of X.

Definition 5.9. Let X be a globular precomplex. A morphism of globular precomplexes $\gamma : \overrightarrow{I}^{\mathrm{top}} \longrightarrow X$ is a nonconstant execution path of X if there exists $t_0 = 0 < t_1 < \cdots < t_n = 1$ such that:

(1) $\gamma(t_i) \in X^0$ for any $0 \leqslant i \leqslant n$.
(2) $\gamma(]t_i, t_{i+1}[) \subset \mathrm{Glob}^{\mathrm{top}}(\mathbf{D}^{n_{\beta_i}} \backslash \mathbf{S}^{n_{\beta_i}-1})$ for some $(n_{\beta_i}, \phi_{\beta_i})$ of the globular decomposition of X.
(3) for $0 \leqslant i < n$, there exists $z^i_\gamma \in \mathbf{D}^{n_{\beta_i}} \backslash \mathbf{S}^{n_{\beta_i}-1}$ and a strictly increasing continuous map $\psi^i_\gamma : [t_i, t_{i+1}] \longrightarrow [0, 1]$ such that $\psi^i_\gamma(t_i) = 0$ and $\psi^i_\gamma(t_{i+1}) = 1$ and for any $t \in [t_i, t_{i+1}]$, $\gamma(t) = (z^i_\gamma, \psi^i_\gamma(t))$.

In particular, the restriction $\gamma \restriction_{]t_i,t_{i+1}[}$ of γ to $]t_i, t_{i+1}[$ is one-to-one. The set of nonconstant execution paths of X is denoted by $\mathbb{P}^{\mathrm{top}}(X)$.

Definition 5.10. A morphism of globular precomplexes $f : X \longrightarrow Y$ is nondecreasing if the canonical set map $\mathbf{Top}([0, 1], |X|) \longrightarrow \mathbf{Top}([0, 1], |Y|)$ induced by composition by f yields a set map $\mathbb{P}^{\mathrm{top}}(X) \longrightarrow \mathbb{P}^{\mathrm{top}}(Y)$. In other terms, one has the commutative diagram of sets

$$\begin{array}{ccc} \mathbb{P}^{\mathrm{top}}(X) & \longrightarrow & \mathbb{P}^{\mathrm{top}}(Y) \\ \subset\downarrow & & \subset\downarrow \\ \mathbf{Top}([0, 1], |X|) & \longrightarrow & \mathbf{Top}([0, 1], |Y|). \end{array}$$

Definition 5.11. A globular complex (resp. a relative globular complex) X is a globular precomplex (resp. a relative globular precomplex) such that the attaching maps ϕ_β are nondecreasing. A morphism of globular complexes is a morphism of globular precomplexes which is nondecreasing. The category of globular complexes together with the morphisms of globular complexes as defined above is denoted by **glTop**.

Definition 5.12. Let X be a globular complex. A point α of X^0 such that there are no nonconstant execution paths ending at α (resp. starting from α) is called initial state (resp. final state). More generally, a point of X^0 will be sometime called a state as well.

Theorem 5.13 ([Gau05a] Theorem III.3.1)**.** *There exists a unique functor*

$$\mathrm{cat} : \mathbf{glTop} \longrightarrow \mathbf{Flow}$$

such that:

(1) *If $X = X^0$ is a discrete globular complex, then* $\mathrm{cat}(X)$ *is the achronal flow X^0 (“achronal” meaning with an empty path space).*
(2) *If $Z = \mathbf{S}^{n-1}$ or $Z = \mathbf{D}^n$ for some integer $n \geqslant 0$, then* $\mathrm{cat}(\mathrm{Glob}^{\mathrm{top}}(Z)) = \mathrm{Glob}(Z)$.

(3) *For any globular complex X with globular decomposition $(n_\beta, \phi_\beta)_{\beta<\lambda}$, for any limit ordinal $\beta \leqslant \lambda$, the canonical morphism of flows*

$$\varinjlim{}_{\alpha<\beta} \mathrm{cat}(X_\alpha) \longrightarrow \mathrm{cat}(X_\beta)$$

is an isomorphism of flows.

(4) *For any globular complex X with globular decomposition $(n_\beta, \phi_\beta)_{\beta<\lambda}$, for any $\beta < \lambda$, one has the pushout of flows*

$$\begin{array}{ccc} \mathrm{Glob}(\mathbf{S}^{n_\beta-1}) & \xrightarrow{\mathrm{cat}(\phi_\beta)} & \mathrm{cat}(X_\beta) \\ \downarrow & & \downarrow \\ \mathrm{Glob}(\mathbf{D}^{n_\beta}) & \longrightarrow & \mathrm{cat}(X_{\beta+1}). \end{array}$$

6. The underlying homotopy type of a flow

Theorem 6.1. *The functor cat induces a functor, still denoted by cat from* **glTop** *to* **cell**(**Flow**)*. For any flow X of* **cell**(**Flow**)*, there exists a globular complex Y such that* $\mathrm{cat}(Y) = X$*. It is constructed by using the globular decomposition of X. If two globular complexes Y_1 and Y_2 satisfy* $\mathrm{cat}(Y_1) = \mathrm{cat}(Y_2) = X$*, then the two topological spaces $|Y_1|$ and $|Y_2|$ are homotopy equivalent.*

Proof. The construction of Y is made in the proof of [Gau05a] Theorem V.4.1. If two globular complexes Y_1 and Y_2 satisfy $\mathrm{cat}(Y_1) = \mathrm{cat}(Y_2) = X$, then they are S-homotopy equivalent by [Gau05a] Theorem IV.4.9. And the S-homotopy equivalence between the globular complexes Y_1 and Y_2 yields an homotopy equivalence between the underlying topological spaces $|Y_1|$ and $|Y_2|$ by [Gau05a] Proposition VII.2.2. □

The recipe to obtain the underlying homotopy type of a flow X is as follows [Gau05a]:

(1) Take a flow X.
(2) Take its cofibrant replacement $Q(X) \in$ **cell**(**Flow**).
(3) By Theorem 6.1, there exists a globular complex X^{top} such that

$$\mathrm{cat}(X^{\mathrm{top}}) = Q(X).$$

(4) The cofibrant topological space $|X^{\mathrm{top}}|$ is unique up to homotopy and is the underlying homotopy type $|X|$ of X.

This yields a well-defined functor $|-|$: **Flow** $\longrightarrow$ **Ho**(**Top**) from the category of flows to the homotopy category of topological spaces ([Gau05a] Part VII.2).

Roughly speaking, the underlying homotopy type of a flow X consists in factoring the morphism of flows $X^0 \longrightarrow Q(X)$ as a transfinite composition of pushouts of elements of I^{gl}; and then replacing this transfinite composition by a transfinite composition of pushouts of the continuous maps

$$\left\{ |\mathrm{Glob}^{\mathrm{top}}(\mathbf{S}^{n-1})| \subset |\mathrm{Glob}^{\mathrm{top}}(\mathbf{D}^n)|, n \geqslant 0 \right\};$$

and then calculating this transfinite composition in **Top**: the result is a cofibrant topological space which is unique up to homotopy.

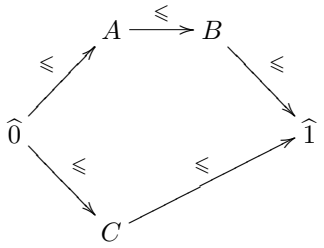
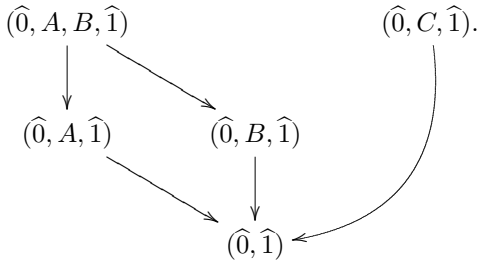


FIGURE 4. Example of finite bounded poset

7. A useful Reedy category and related lemmas

Let P be a finite bounded poset with bottom element $\widehat{0}$ and with top element $\widehat{1}$. Let us denote by $\Delta^{\mathrm{ext}}(P)$ the full subcategory of $\Delta(P)$ consisting of the simplices $(\alpha_0, \ldots, \alpha_p)$ such that $\widehat{0} = \alpha_0$ and $\widehat{1} = \alpha_p$. If $P = \{\widehat{0} < A < B < \widehat{1}, \widehat{0} < C < \widehat{1}\}$ is the poset of Figure 4, then the small category $\Delta^{\mathrm{ext}}(P)^{\mathrm{op}}$ looks as follows:



The simplex $(\widehat{0}, \widehat{1})$ is always a terminal object of $\Delta^{\mathrm{ext}}(P)^{\mathrm{op}}$.

Notation 7.1. Let X be a loopless flow such that $(X^0, \leqslant)$ is locally finite. Let (α, β) be a 1-simplex of $\Delta(X^0)$. We denote by $\ell(\alpha, \beta)$ the maximum of the set of integers

$$\big\{p \geqslant 1, \exists(\alpha_0, \ldots, \alpha_p)\ p\text{-simplex of } \Delta(X^0) \text{ s.t. } (\alpha_0, \alpha_p) = (\alpha, \beta)\big\}$$

One always has $1 \leqslant \ell(\alpha, \beta) \leqslant \mathrm{card}(]\alpha, \beta])$.

Lemma 7.2. *Let X be a loopless flow such that $(X^0, \leqslant)$ is locally finite. Let (α, β, γ) be a 2-simplex of $\Delta(X^0)$. Then one has*

$$\ell(\alpha, \beta) + \ell(\beta, \gamma) \leqslant \ell(\alpha, \gamma).$$

Proof. Let $\alpha = \alpha_0 < \cdots < \alpha_{\ell(\alpha,\beta)} = \beta$. Let $\beta = \beta_0 < \cdots < \beta_{\ell(\beta,\gamma)} = \gamma$. Then

$$(\alpha_0, \ldots, \alpha_{\ell(\alpha,\beta)}, \beta_1, \ldots, \beta_{\ell(\beta,\gamma)})$$

is a simplex of $\Delta(X^0)$ with $\alpha = \alpha_0$ and $\beta_{\ell(\beta,\gamma)} = \gamma$. So $\ell(\alpha, \beta) + \ell(\beta, \gamma) \leqslant \ell(\alpha, \gamma)$. □

Proposition 7.3. *Let P be a finite bounded poset. Let*

$$d(\alpha_0, \ldots, \alpha_p) = \ell(\alpha_0, \alpha_1)^2 + \cdots + \ell(\alpha_{p-1}, \alpha_p)^2$$

where ℓ is the function of Notation 7.1. Then d yields a functor $\Delta^{\mathrm{ext}}(P)^{\mathrm{op}} \longrightarrow \mathbb{N}$ making $\Delta^{\mathrm{ext}}(P)^{\mathrm{op}}$ a direct category, that is a Reedy category with

$$\Delta^{\mathrm{ext}}(P)^{\mathrm{op}}_{+} := \Delta^{\mathrm{ext}}(P)^{\mathrm{op}} \quad \textit{and} \quad \Delta^{\mathrm{ext}}(P)^{\mathrm{op}}_{-} = \varnothing.$$

Proof. Let $\partial_i : (\alpha_0, \dots, \alpha_p) \longrightarrow (\alpha_0, \dots, \widehat{\alpha_i}, \dots, \alpha_p)$ be a morphism of $\Delta^{\mathrm{ext}}(P)^{\mathrm{op}}$ with $p \geqslant 2$ and $0 < i < p$. Then

$$\begin{aligned} d(\alpha_0, \dots, \alpha_p) &= \ell(\alpha_0, \alpha_1)^2 + \cdots + \ell(\alpha_{p-1}, \alpha_p)^2 \\ d(\alpha_0, \dots, \widehat{\alpha_i}, \dots, \alpha_p) &= \ell(\alpha_0, \alpha_1)^2 + \cdots + \ell(\alpha_{i-1}, \alpha_{i+1})^2 + \cdots + \ell(\alpha_{p-1}, \alpha_p)^2. \end{aligned}$$

So one obtains

$$d(\alpha_0, \dots, \alpha_p) - d(\alpha_0, \dots, \widehat{\alpha_i}, \dots, \alpha_p) = \ell(\alpha_{i-1}, \alpha_i)^2 + \ell(\alpha_i, \alpha_{i+1})^2 - \ell(\alpha_{i-1}, \alpha_{i+1})^2.$$

By Lemma 7.2, one has

$$(\ell(\alpha_{i-1}, \alpha_i) + \ell(\alpha_i, \alpha_{i+1}))^2 \leqslant \ell(\alpha_{i-1}, \alpha_{i+1})^2.$$

Therefore, one obtains

$$\ell(\alpha_{i-1}, \alpha_i)^2 + \ell(\alpha_i, \alpha_{i+1})^2 < \ell(\alpha_{i-1}, \alpha_{i+1})^2$$

since $2\ell(\alpha_{i-1}, \alpha_i)\ell(\alpha_i, \alpha_{i+1}) \geqslant 2$. Thus, every morphism of $\Delta^{\mathrm{ext}}(P)^{\mathrm{op}}$ raises the degree. □

Corollary 7.4. *Let P be a finite bounded poset. Then the colimit functor*

$$\varinjlim : \mathbf{Top}^{\Delta^{\mathrm{ext}}(P)^{\mathrm{op}} \setminus \{(\widehat{0}, \widehat{1})\}} \longrightarrow \mathbf{Top}$$

is a left Quillen functor if the category of diagrams $\mathbf{Top}^{\Delta^{\mathrm{ext}}(P)^{\mathrm{op}} \setminus \{(\widehat{0}, \widehat{1})\}}$ is equipped with the Reedy model structure.

Indeed, the fact that the colimit functor is a left Quillen functor will be actually applied for $\Delta^{\mathrm{ext}}(P)^{\mathrm{op}} \setminus \{(\widehat{0}, \widehat{1})\}$. Recall that the pair $(\widehat{0}, \widehat{1})$ is a terminal object of $\Delta^{\mathrm{ext}}(P)^{\mathrm{op}}$. Therefore, it is not particularly interesting to calculate the colimit of a diagram of spaces over the whole category $\Delta^{\mathrm{ext}}(P)^{\mathrm{op}}$. Note also that there is an isomorphism of small categories

$$\boxed{\Delta^{\mathrm{ext}}(P)^{\mathrm{op}} \setminus \{(\widehat{0}, \widehat{1})\} \cong \partial(\Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}}_{+} \downarrow (\widehat{0}, \widehat{1}))}.$$

Proof. The Reedy structure on $\Delta^{\mathrm{ext}}(P)^{\mathrm{op}} \setminus \{(\widehat{0}, \widehat{1})\}$ provides a model structure on the category $\mathbf{Top}^{\Delta^{\mathrm{ext}}(P)^{\mathrm{op}} \setminus \{(\widehat{0}, \widehat{1})\}}$ of diagrams of topological spaces over the small category $\Delta^{\mathrm{ext}}(P)^{\mathrm{op}} \setminus \{(\widehat{0}, \widehat{1})\}$ such that a morphism of diagrams $f : D \longrightarrow E$ is:

(1) a weak equivalence if and only if for every object $\underline{\alpha}$ of $\Delta^{\mathrm{ext}}(P)^{\mathrm{op}} \setminus \{(\widehat{0}, \widehat{1})\}$, the morphism $D_{\underline{\alpha}} \longrightarrow E_{\underline{\alpha}}$ is a weak equivalence of **Top** (we will use the term objectwise weak equivalence to describe this situation);
(2) a cofibration if and only if for every object $\underline{\alpha}$ of $\Delta^{\mathrm{ext}}(P)^{\mathrm{op}} \setminus \{(\widehat{0}, \widehat{1})\}$, the morphism $D_{\underline{\alpha}} \sqcup_{L_{\underline{\alpha}}D} L_{\underline{\alpha}}E \longrightarrow E_{\underline{\alpha}}$ is a cofibration of **Top**;
(3) a fibration if and only if for every object $\underline{\alpha}$ of $\Delta^{\mathrm{ext}}(P)^{\mathrm{op}} \setminus \{(\widehat{0}, \widehat{1})\}$, the morphism $D_{\underline{\alpha}} \longrightarrow E_{\underline{\alpha}} \times_{M_{\underline{\alpha}}E} M_{\underline{\alpha}}D$ is a fibration of **Top**.

For every object $\underline{\alpha}$ of $\Delta^{\mathrm{ext}}(P)^{\mathrm{op}}\backslash\{(\widehat{0},\widehat{1})\}$, the matching category $\partial(\underline{\alpha}\downarrow\Delta^{\mathrm{ext}}(P)^{\mathrm{op}}_{-})$ is empty. So for every object A of the diagram category $\mathbf{Top}^{\Delta^{\mathrm{ext}}(P)^{\mathrm{op}}}$ and every object $\underline{\alpha}$ of the small category $\Delta^{\mathrm{ext}}(P)^{\mathrm{op}}\backslash\{(\widehat{0},\widehat{1})\}$, there is an isomorphism $M_{\underline{\alpha}}A \cong \mathbf{1}$. So a Reedy fibration is an objectwise fibration. Therefore, the diagonal functor Diag of the adjunction $\varinjlim : \mathbf{Top}^{\Delta^{\mathrm{ext}}(P)^{\mathrm{op}}\backslash\{(\widehat{0},\widehat{1})\}} \leftrightarrows \mathbf{Top} : \mathrm{Diag}$ is a right Quillen functor. □

Proposition 7.5. *Let $\overrightarrow{D}$ be a full directed ball. There exists one and only one functor*

$$\mathcal{F}\overrightarrow{D} : \Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}} \longrightarrow \mathbf{Top}$$

satisfying the following conditions:

1. *$\mathcal{F}\overrightarrow{D}_{(\alpha_0,\ldots,\alpha_p)} = \mathbb{P}_{\alpha_0,\alpha_1}\overrightarrow{D} \times \ldots \times \mathbb{P}_{\alpha_{p-1},\alpha_p}\overrightarrow{D}$ (recall that necessarily, one has the equalities $\alpha_0 = \widehat{0}$ and $\alpha_p = \widehat{1}$ by definition of the small category $\Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}}$).*
2. *The unique morphism $\partial_i : \mathcal{F}\overrightarrow{D}_{(\alpha_0,\ldots,\alpha_p)} \longrightarrow \mathcal{F}\overrightarrow{D}_{(\alpha_0,\ldots,\widehat{\alpha_i},\ldots,\alpha_p)}$ for $0 < i < p$ is induced by the composition law $\mathbb{P}_{\alpha_{i-1},\alpha_i}\overrightarrow{D} \times \mathbb{P}_{\alpha_i,\alpha_{i+1}}\overrightarrow{D} \longrightarrow \mathbb{P}_{\alpha_{i-1},\alpha_{i+1}}\overrightarrow{D}$.*

Proof. The uniqueness on objects is exactly the first assertion. The uniqueness on morphisms comes from the fact that any morphism of $\Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}}$ is a composite of ∂_i. We have to prove the existence. The diagram of topological spaces

$$\begin{array}{ccc}
\mathcal{F}\overrightarrow{D}_{(\alpha_0,\ldots,\alpha_p)} & \xrightarrow{\partial_i} & \mathcal{F}\overrightarrow{D}_{(\alpha_0,\ldots,\widehat{\alpha_i},\ldots,\alpha_p)} \\
\downarrow{\scriptstyle \partial_j} & & \downarrow{\scriptstyle \partial_{j-1}} \\
\mathcal{F}\overrightarrow{D}_{(\alpha_0,\ldots,\widehat{\alpha_j},\ldots,\alpha_p)} & \xrightarrow{\partial_i} & \mathcal{F}\overrightarrow{D}_{(\alpha_0,\ldots,\widehat{\alpha_i},\ldots,\widehat{\alpha_j},\ldots,\alpha_p)}
\end{array}$$

is commutative for any $0 < i < j < p$ and any $p \geqslant 2$. Indeed, if $i < j-1$, then one has

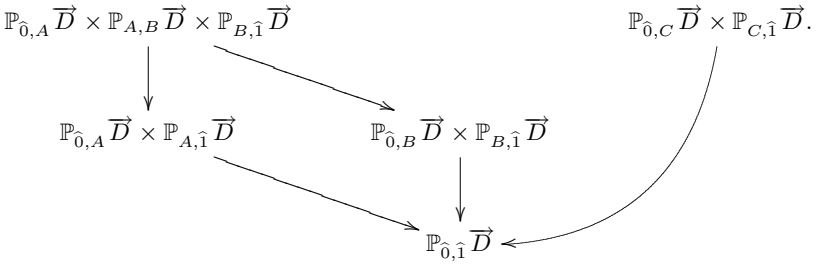
$$\boxed{\partial_i\partial_j(\gamma_1,\ldots,\gamma_p) = \partial_{j-1}\partial_i(\gamma_1,\ldots,\gamma_p) = (\gamma_1,\ldots,\gamma_i\gamma_{i+1},\ldots,\gamma_j\gamma_{j+1},\ldots,\gamma_p)}$$

and if $i = j-1$, then one has

$$\boxed{\partial_i\partial_j(\gamma_1,\ldots,\gamma_p) = \partial_{j-1}\partial_i(\gamma_1,\ldots,\gamma_p) = (\gamma_1,\ldots,\gamma_{j-1}\gamma_j\gamma_{j+1},\ldots,\gamma_p)}$$

because of the associativity of the composition law of X. In other terms, the ∂_i maps satisfy the simplicial identities. Hence the result. □

Take again the poset P of Figure 4, and the corresponding full directed ball $\overrightarrow{D} = F(P)$. The diagram $\mathcal{F}\overrightarrow{D}$ looks as follows:



Definition 7.6. Let X be a flow. Let A be a subset of X^0. Then the *restriction* $X\restriction_A$ of X over A is the unique flow such that $(X\restriction_A)^0 = A$, such that $\mathbb{P}_{\alpha,\beta}(X\restriction_A) = \mathbb{P}_{\alpha,\beta}X$ for $(\alpha,\beta) \in A \times A$ and such that the inclusions $A \subset X^0$ and $\mathbb{P}(X\restriction_A) \subset \mathbb{P}X$ induce a morphism of flows $X\restriction_A \longrightarrow X$.

The following proposition is immediate:

Proposition 7.7. *Let $\overrightarrow{D}$ be a full directed ball. Let (α,β) be a simplex of $\Delta(\overrightarrow{D}^0)$. Then $\overrightarrow{D}\restriction_{[\alpha,\beta]}$ is a full directed ball with initial state α and with final state β.*

Proposition 7.8. *Let $\overrightarrow{D}$ and $\overrightarrow{D'}$ be two full directed balls. Then the flow $\overrightarrow{D} * \overrightarrow{D'}$ obtained by identifying the final state $\widehat{1}$ of $\overrightarrow{D}$ with the initial state $\widehat{0}$ of $\overrightarrow{D'}$ is a full directed ball.*

Proof. The condition which is less easy to verify than the other ones is: for any $(\alpha,\beta) \in (\overrightarrow{D} * \overrightarrow{D'})^0 \times (\overrightarrow{D} * \overrightarrow{D'})^0$, the topological space $\mathbb{P}_{\alpha,\beta}(\overrightarrow{D} * \overrightarrow{D'})$ is weakly contractible if $\alpha < \beta$. Let m be the point of $\overrightarrow{D} * \overrightarrow{D'}$ corresponding to the final state of $\overrightarrow{D}$ and the initial state of $\overrightarrow{D'}$. If $\alpha < \beta \leqslant m$, then one has the isomorphism of spaces $\mathbb{P}_{\alpha,\beta}(\overrightarrow{D} * \overrightarrow{D'}) \cong \mathbb{P}_{\alpha,\beta}\overrightarrow{D}$. If $m \leqslant \alpha < \beta$, then one has the isomorphism of spaces $\mathbb{P}_{\alpha,\beta}(\overrightarrow{D} * \overrightarrow{D'}) \cong \mathbb{P}_{\alpha,\beta}\overrightarrow{D'}$. At last, if $\alpha < m < \beta$, then one has the isomorphism of spaces $\mathbb{P}_{\alpha,\beta}(\overrightarrow{D} * \overrightarrow{D'}) \cong \mathbb{P}_{\alpha,m}\overrightarrow{D} \times \mathbb{P}_{m,\beta}\overrightarrow{D'}$. So in each case, the space $\mathbb{P}_{\alpha,\beta}(\overrightarrow{D} * \overrightarrow{D'})$ is weakly contractible. □

Proposition 7.9. *Let $\overrightarrow{D}$ be a full directed ball. There exists one and only one functor*

$$\mathcal{G}\overrightarrow{D} : \Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}} \longrightarrow \mathbf{Flow}$$

satisfying the following conditions:

(1) *For any object $(\alpha_0,\dots,\alpha_p)$ of $\Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}}$,*

$$\mathcal{G}\overrightarrow{D}(\alpha_0,\dots,\alpha_p) = \overrightarrow{D}\restriction_{[\alpha_0,\alpha_1]} * \cdots * \overrightarrow{D}\restriction_{[\alpha_{p-1},\alpha_p]} .$$

(2) *The unique morphism $\mathcal{G}\overrightarrow{D}(\alpha_0,\dots,\alpha_p) \longrightarrow \mathcal{G}\overrightarrow{D}(\alpha_0,\dots,\widehat{\alpha_i},\dots,\alpha_p)$ for $0 < i < p$ is induced by the composition law*

$$\overrightarrow{D}\restriction_{[\alpha_{i-1},\alpha_i]} * \overrightarrow{D}\restriction_{[\alpha_i,\alpha_{i+1}]} \longrightarrow \overrightarrow{D}\restriction_{[\alpha_{i-1},\alpha_{i+1}]} .$$

Notice that $\overrightarrow{D}\restriction_{[\widehat{0},\widehat{1}]} = \overrightarrow{D}$.

Proof. This comes from the associativity of the composition law of a flow. □

Proposition 7.10. *Let $\overrightarrow{D}$ be a full directed ball. Let $(\alpha_0,\dots,\alpha_p) \in \Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}}$ be a simplex. Then there exists a unique morphism of flows*

$$u_{(\alpha_0,\dots,\alpha_p)} : \mathrm{Glob}(\mathbb{P}_{\alpha_0,\alpha_1}\overrightarrow{D} \times \dots \times \mathbb{P}_{\alpha_{p-1},\alpha_p}\overrightarrow{D}) \longrightarrow \mathrm{Glob}(\mathbb{P}_{\alpha_0,\alpha_1}\overrightarrow{D}) * \cdots * \mathrm{Glob}(\mathbb{P}_{\alpha_{p-1},\alpha_p}\overrightarrow{D})$$

such that $u_{(\alpha_0,\dots,\alpha_p)}(x_1,\dots,x_p) = x_1 * \cdots * x_p$. With $(\alpha_0,\dots,\alpha_p)$ running over the set of simplices of $\Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}}$, one obtains a morphism of diagrams of flows

$$\mathrm{Glob}(\mathcal{F}\overrightarrow{D}) \longrightarrow \mathcal{G}\overrightarrow{D}.$$

Proof. Obvious. □

Proposition 7.11. *Let $\overrightarrow{D}$ be a full directed ball. Then one has the pushout diagram of flows:*

$$\begin{array}{ccc} \mathrm{Glob}(L_{(\widehat{0},\widehat{1})}\mathcal{F}\overrightarrow{D}) & \longrightarrow & L_{(\widehat{0},\widehat{1})}\mathcal{G}\overrightarrow{D} \\ \downarrow & & \downarrow \\ \mathrm{Glob}(\mathbb{P}_{\widehat{0},\widehat{1}}\overrightarrow{D}) & \longrightarrow & \overrightarrow{D}. \end{array}$$

This statement remains true when the 1-simplex $(\widehat{0},\widehat{1})$ is replaced by another 1-simplex (α,β) of $\Delta(\overrightarrow{D}^0)$. This statement above becomes false in general when the 1-simplex $(\widehat{0},\widehat{1})$ is replaced by a p-simplex of $\Delta(\overrightarrow{D}^0)$ with $p \geqslant 2$.

Let us illustrate this proposition in the case of $\overrightarrow{D}^0 = \{\widehat{0} < A < \widehat{1}\}$. One then has:

(1) $L_{(\widehat{0},\widehat{1})}\mathcal{F}\overrightarrow{D} = \mathbb{P}_{\widehat{0},A}\overrightarrow{D} \times \mathbb{P}_{A,\widehat{1}}\overrightarrow{D}$.

(2) $L_{(\widehat{0},\widehat{1})}\mathcal{G}\overrightarrow{D} = \overrightarrow{D} \restriction_{[\widehat{0},A]} * \overrightarrow{D} \restriction_{[A,\widehat{1}]} = \mathrm{Glob}(\mathbb{P}_{\widehat{0},A}\overrightarrow{D}) * \mathrm{Glob}(\mathbb{P}_{A,\widehat{1}}\overrightarrow{D})$; the last equality is due to the fact that $]\widehat{0},A[=]A,\widehat{1}[=\varnothing$.

(3) The pushout above is equivalent to the following pushout:

$$\begin{array}{ccc} \mathrm{Glob}(\mathbb{P}_{\widehat{0},A}\overrightarrow{D} \times \mathbb{P}_{A,\widehat{1}}\overrightarrow{D}) & \longrightarrow & \mathrm{Glob}(\mathbb{P}_{\widehat{0},A}\overrightarrow{D}) * \mathrm{Glob}(\mathbb{P}_{A,\widehat{1}}\overrightarrow{D}) \\ \downarrow & & \downarrow \\ \mathrm{Glob}(\mathbb{P}_{\widehat{0},\widehat{1}}\overrightarrow{D}) & \longrightarrow & \overrightarrow{D}. \end{array}$$

Proof. One already has the commutative diagram

$$\begin{array}{ccc} \mathrm{Glob}(L_{(\widehat{0},\widehat{1})}\mathcal{F}\overrightarrow{D}) & \longrightarrow & L_{(\widehat{0},\widehat{1})}\mathcal{G}\overrightarrow{D} \\ \downarrow & & \downarrow \\ \mathrm{Glob}(\mathbb{P}_{\widehat{0},\widehat{1}}\overrightarrow{D}) & \longrightarrow & \overrightarrow{D}. \end{array}$$

Therefore, one only has to check that $\overrightarrow{D}$ satisfies the same universal property as the pushout.

Consider a commutative diagram of flows of the form:

$$\begin{array}{ccc} \mathrm{Glob}(L_{(\widehat{0},\widehat{1})}\mathcal{F}\overrightarrow{D}) & \longrightarrow & L_{(\widehat{0},\widehat{1})}\mathcal{G}\overrightarrow{D} \\ \downarrow & & \downarrow \\ \mathrm{Glob}(\mathbb{P}_{\widehat{0},\widehat{1}}\overrightarrow{D}) & \longrightarrow & X. \end{array}$$

The morphism of flows $\mathrm{Glob}(\mathbb{P}_{\widehat{0},\widehat{1}}\overrightarrow{D}) \longrightarrow X$ induces a continuous map $\mathbb{P}_{\widehat{0},\widehat{1}}\overrightarrow{D} \longrightarrow \mathbb{P}X$. The morphism of flows $L_{(\widehat{0},\widehat{1})}\mathcal{G}\overrightarrow{D} \longrightarrow X$ induces a continuous map $\mathbb{P}_{\alpha,\beta}\overrightarrow{D} \longrightarrow \mathbb{P}X$ for any 1-simplex (α,β) of $\Delta(\overrightarrow{D}^0)$ with $(\alpha,\beta) \neq (\widehat{0},\widehat{1})$. The existence of the morphism of flows $L_{(\widehat{0},\widehat{1})}\mathcal{G}\overrightarrow{D} \longrightarrow X$ ensures the compatibility of the continuous maps $\mathbb{P}_{\alpha,\beta}\overrightarrow{D} \longrightarrow \mathbb{P}X$ for $(\alpha,\beta) \in \Delta(\overrightarrow{D}^0)$ with the composition of execution paths involving a triple (α,β,γ) such that $(\alpha,\gamma) \neq (\widehat{0},\widehat{1})$. And the commutativity of the

diagram with X ensures the compatibility of the continuous maps $\mathbb{P}_{\alpha,\beta}\overrightarrow{D} \longrightarrow \mathbb{P}X$ for $(\alpha,\beta) \in \Delta(\overrightarrow{D}^0)$ with the composition of execution paths involving a triple (α,β,γ) such that $(\alpha,\gamma) = (\widehat{0},\widehat{1})$. Hence the existence and uniqueness of the morphism $\overrightarrow{D} \longrightarrow X$. □

Theorem 7.12 ([ML98] Theorem 1, p. 213)**.** *Let $L : J' \longrightarrow J$ be a final functor between small categories, i.e., such that for any $k \in J$, the comma category $(k \downarrow L)$ is nonempty and connected. Let $F : J \longrightarrow \mathcal{C}$ be a functor from J to a cocomplete category $\mathcal{C}$. Then L induces a canonical morphism $\varinjlim F \circ L \longrightarrow \varinjlim F$ which is an isomorphism.*

Notation 7.13. Let X be a loopless flow. Let $\underline{\alpha} = (\alpha_0,\dots,\alpha_p)$ be a simplex of the order complex $\Delta(X^0)$ of the poset X^0. Let $\alpha < \alpha_0$. Then the notation $\alpha.\underline{\alpha}$ represents the simplex $(\alpha,\alpha_0,\dots,\alpha_p)$ of $\Delta(X^0)$.

Theorem 7.14. *Let $\overrightarrow{D}$ be a full directed ball. Let $\underline{\alpha} = (\alpha_0,\dots,\alpha_p)$ be a simplex of $\Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}}$. Let $i_{(\alpha_0,\dots,\alpha_p)} : L_{(\alpha_0,\dots,\alpha_p)}\mathcal{F}\overrightarrow{D} \longrightarrow \mathcal{F}\overrightarrow{D}_{(\alpha_0,\dots,\alpha_p)}$. Then one has*

$$i_{(\alpha_0,\dots,\alpha_p)} = i_{(\alpha_0,\alpha_1)} \square \dots \square i_{(\alpha_{p-1},\alpha_p)}$$

where $\square$ is the pushout product (cf. Notation B.2*).*

Proof. Let $\underline{\alpha} = (\alpha_0,\dots,\alpha_p)$ be a fixed object of $\Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}}$. The latching category

$$\partial(\Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}}_{+} \downarrow \underline{\alpha})$$

is the full subcategory of $\Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}}$ consisting of the simplices $\underline{\beta} = (\beta_0,\dots,\beta_q)$ such that there is a strict inclusion

$$\{\alpha_0,\dots,\alpha_p\} \subsetneqq \{\beta_0,\dots,\beta_q\},$$

that is $\{\alpha_0,\dots,\alpha_p\} \subset \{\beta_0,\dots,\beta_q\}$ and $\{\alpha_0,\dots,\alpha_p\} \neq \{\beta_0,\dots,\beta_q\}$. Recall that by definition of the category $\Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}}$, one necessarily has $\alpha_0 = \beta_0 = \widehat{0}$ and $\alpha_p = \beta_q = \widehat{1}$. Such a simplex $\underline{\beta} = (\beta_0,\dots,\beta_q)$ can be written as an expression of the form

$$\alpha_0.\underline{\delta_1}.\underline{\delta_2}\dots\underline{\delta_p}$$

with $\alpha_i.\underline{\delta_{i+1}} \supseteq (\alpha_i,\alpha_{i+1})$ for all $0 \leqslant i \leqslant p-1$ and such that at least for one i, one has $\alpha_i.\underline{\delta_{i+1}} \supsetneqq (\alpha_i,\alpha_{i+1})$. And since the small category $\Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}}$ only contains commutative diagrams, one obtains the homeomorphism

$$L_{(\alpha_0,\dots,\alpha_p)}\mathcal{F}\overrightarrow{D} \cong \varinjlim_{\{\alpha_0,\dots,\alpha_p\} \subsetneqq \{\beta_0,\dots,\beta_q\}} \mathcal{F}\overrightarrow{D}_{(\beta_0,\dots,\beta_q)}. \tag{1}$$

Let $\mathcal{E}$ be the set of subsets S of $\{0,\dots,p-1\}$ such that $S \neq \{0,\dots,p-1\}$. Let $I(S)$ be the full subcategory of $\Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}}$ consisting of the objects $\underline{\beta} = (\beta_0,\dots,\beta_q)$ such that:

(1) $\{\alpha_0,\dots,\alpha_p\} \subsetneqq \{\beta_0,\dots,\beta_q\}$.
(2) For any $i \notin S$, one has $\alpha_i.\underline{\delta_{i+1}} \supsetneqq (\alpha_i,\alpha_{i+1})$.

The full subcategory $\bigcup_{S\in\mathcal{E}} I(S)$ is exactly the subcategory of $\Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}}$ consisting of the objects $\underline{\beta}$ such that $\{\alpha_0,\dots,\alpha_p\} \subsetneqq \{\beta_0,\dots,\beta_q\}$, that is to say the

subcategory calculating $L_{(\alpha_0,\ldots,\alpha_p)}\mathcal{F}\overrightarrow{D}$. In other terms, one has the isomorphism of spaces

$$\varinjlim_{\bigcup_{S\in\mathcal{E}} I(S)} \mathcal{F}\overrightarrow{D} \cong L_{\underline{\alpha}}\mathcal{F}\overrightarrow{D}. \tag{2}$$

The full subcategory $I(S)$ of $\Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}}$ has a final subcategory $\overline{I(S)}$ consisting of the $\underline{\beta} = (\beta_0, \ldots, \beta_q)$ such that:

(1) $\{\alpha_0, \ldots, \alpha_p\} \subsetneqq \{\beta_0, \ldots, \beta_q\}$.
(2) For any $i \notin S$, one has $\alpha_i.\underline{\delta_{i+1}} \supsetneqq (\alpha_i, \alpha_{i+1})$.
(3) For any $i \in S$, one has $\alpha_i.\underline{\delta_{i+1}} = (\alpha_i, \alpha_{i+1})$.

The subcategory $\overline{I(S)}$ is final in $I(S)$ because for any object $\underline{\beta}$ of $I(S)$, there exists a unique $\underline{\gamma}$ of $\overline{I(S)}$ and a unique arrow $\underline{\beta} \longrightarrow \underline{\gamma}$. Therefore, by Theorem 7.12, one has the isomorphism

$$\varinjlim_{I(S)} \mathcal{F}\overrightarrow{D} \cong \varinjlim_{\overline{I(S)}} \mathcal{F}\overrightarrow{D} \tag{3}$$

since the comma category $(\underline{\beta} \downarrow \overline{I(S)})$ is the one-object category. For any object $\underline{\beta}$ of $\overline{I(S)}$, one has

$$\begin{aligned}
&\mathcal{F}\overrightarrow{D}_{\underline{\beta}} \\
&= \prod_{i=0}^{i=q-1} \mathbb{P}_{\beta_i,\beta_{i+1}}\overrightarrow{D} && \text{by definition of } \mathcal{F}\overrightarrow{D} \\
&= \prod_{i=0}^{i=p-1} \mathcal{F}\overrightarrow{D}_{\alpha_i.\underline{\delta_{i+1}}} && \text{by definition of } \mathcal{F}\overrightarrow{D} \\
&= \left(\prod_{i\in S} \mathcal{F}\overrightarrow{D}_{(\alpha_i,\alpha_{i+1})}\right) \times \left(\prod_{i\notin S} \mathcal{F}\overrightarrow{D}_{\alpha_i.\underline{\delta_{i+1}}}\right) && \text{by definition of } S.
\end{aligned}$$

Thus, since the category **Top** is cartesian closed, one obtains

$$\begin{aligned}
&\varinjlim_{\overline{I(S)}} \mathcal{F}\overrightarrow{D} \\
&\cong \varinjlim_{\overline{I(S)}} \left(\left(\prod_{i\in S} \mathcal{F}\overrightarrow{D}_{(\alpha_i,\alpha_{i+1})}\right) \times \left(\prod_{i\notin S} \mathcal{F}\overrightarrow{D}_{\alpha_i.\underline{\delta_{i+1}}}\right)\right) \\
&\cong \left(\prod_{i\in S} \mathcal{F}\overrightarrow{D}_{(\alpha_i,\alpha_{i+1})}\right) \times \varinjlim_{\substack{i\notin S \\ \alpha_i.\underline{\delta_{i+1}} \supsetneqq (\alpha_i,\alpha_{i+1})}} \left(\prod_{i\notin S} \mathcal{F}\overrightarrow{D}_{\alpha_i.\underline{\delta_{i+1}}}\right) \\
&\cong \left(\prod_{i\in S} \mathcal{F}\overrightarrow{D}_{(\alpha_i,\alpha_{i+1})}\right) \times \left(\prod_{i\notin S} \varinjlim_{\alpha_i.\underline{\delta_{i+1}} \supsetneqq (\alpha_i,\alpha_{i+1})} \mathcal{F}\overrightarrow{D}_{\alpha_i.\underline{\delta_{i+1}}}\right) && \text{by Lemma B.1.}
\end{aligned}$$

Therefore, one obtains the isomorphism of topological spaces

$$\varinjlim_{\overline{I(S)}} \mathcal{F}\overrightarrow{D} \cong \left(\prod_{i\in S} \mathcal{F}\overrightarrow{D}_{(\alpha_i,\alpha_{i+1})}\right) \times \left(\prod_{i\notin S} L_{(\alpha_i,\alpha_{i+1})}\mathcal{F}\overrightarrow{D}\right). \tag{4}$$

thanks to Isomorphism (1).

If S and T are two elements of $\mathcal{E}$ such that $S \subset T$, then there exists a canonical morphism of diagrams $I(S) \longrightarrow I(T)$ inducing a canonical morphism of topological spaces

$$\varinjlim_{\underline{\beta} \in I(S)} \mathcal{F}\overrightarrow{D}_{\underline{\beta}} \longrightarrow \varinjlim_{\underline{\beta} \in I(T)} \mathcal{F}\overrightarrow{D}_{\underline{\beta}}.$$

Therefore, by Equation (3) and Equation (4), the double colimit

$$\varinjlim_{S \in \mathcal{E}} \left(\varinjlim_{I(S)} \mathcal{F}\overrightarrow{D} \right)$$

calculates the source of the morphism $i_{(\alpha_0,\alpha_1)} \square \dots \square i_{(\alpha_{p-1},\alpha_p)}$ by Theorem B.3. It then suffices to prove the isomorphism

$$\varinjlim_{S \in \mathcal{E}} \left(\varinjlim_{I(S)} \mathcal{F}\overrightarrow{D} \right) \cong \varinjlim_{\{\alpha_0,\dots,\alpha_p\} \subsetneq \{\beta_0,\dots,\beta_q\}} \mathcal{F}\overrightarrow{D}_{\underline{\beta}}$$

to complete the proof. For that purpose, it suffices to construct two canonical morphisms

$$\varinjlim_{S \in \mathcal{E}} \left(\varinjlim_{I(S)} \mathcal{F}\overrightarrow{D} \right) \longrightarrow \varinjlim_{\{\alpha_0,\dots,\alpha_p\} \subsetneq \{\beta_0,\dots,\beta_q\}} \mathcal{F}\overrightarrow{D}_{\underline{\beta}}$$

and

$$\varinjlim_{\{\alpha_0,\dots,\alpha_p\} \subsetneq \{\beta_0,\dots,\beta_q\}} \mathcal{F}\overrightarrow{D}_{\underline{\beta}} \longrightarrow \varinjlim_{S \in \mathcal{E}} \left(\varinjlim_{I(S)} \mathcal{F}\overrightarrow{D} \right).$$

The first morphism comes from the isomorphism of Equation (2). As for the second morphism, let us consider a diagram of flows of the form:

$$\begin{array}{ccc} \mathcal{F}\overrightarrow{D}_{\underline{\beta}} & \longrightarrow & \varinjlim_{S \in \mathcal{E}} \left(\varinjlim_{I(S)} \mathcal{F}\overrightarrow{D} \right). \\ \downarrow & \nearrow & \\ \mathcal{F}\overrightarrow{D}_{\underline{\gamma}} & & \end{array}$$

One has to prove that it is commutative. Since one has $\bigcup_{S \in \mathcal{E}} I(S) = \Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}}$, there exists $S \in \mathcal{E}$ such that $\underline{\gamma}$ is an object of $I(S)$. So $\underline{\beta}$ is an object of $I(S)$ as well and there exists a commutative diagram

$$\begin{array}{ccc} \mathcal{F}\overrightarrow{D}_{\underline{\beta}} & \longrightarrow & \varinjlim_{I(S)} \mathcal{F}\overrightarrow{D} \\ \downarrow & \nearrow & \\ \mathcal{F}\overrightarrow{D}_{\underline{\gamma}} & & \end{array}$$

since the subcategory $\Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}}$ is commutative. Hence the result. □

8. Calculating the underlying homotopy type

Theorem 8.1. *Let $\overrightarrow{D}$ be a full directed ball. Then the diagram of spaces $\mathcal{F}Q(\overrightarrow{D})$ (where Q is the cofibrant replacement functor of* **Flow**) *is Reedy cofibrant.*

Proof. By Proposition A.3 and since the model category **Top** is monoidal, one deduces that for any object $\underline{\alpha}$ of $\mathcal{F}Q(\overrightarrow{D})$, the topological space $\mathcal{F}Q(\overrightarrow{D})_{\underline{\alpha}}$ is cofibrant. By Theorem 7.14 and by induction on the cardinal of the set $\overrightarrow{D}^0$, it then suffices

to prove that the continuous map $L_{(\widehat{0},\widehat{1})}\mathcal{F}Q(\overrightarrow{D}) \longrightarrow \mathcal{F}Q(\overrightarrow{D})_{(\widehat{0},\widehat{1})}$ is a cofibration of topological spaces.

Let X be an object of **cell**(**Flow**) such that $X^0 = \overrightarrow{D}^0$ and such that the continuous map $L_{(\widehat{0},\widehat{1})}\mathcal{F}X \longrightarrow \mathcal{F}X_{(\widehat{0},\widehat{1})}$ is a cofibration of topological spaces. Consider a pushout diagram of flows with $n \geqslant 0$ as follows:

$$\begin{array}{ccc} \mathrm{Glob}(\mathbf{S}^{n-1}) & \xrightarrow{\phi} & X \\ \downarrow & & \downarrow \\ \mathrm{Glob}(\mathbf{D}^n) & \longrightarrow & Y. \end{array}$$

One wants to prove that the continuous map $L_{(\widehat{0},\widehat{1})}\mathcal{F}Y \longrightarrow \mathcal{F}Y_{(\widehat{0},\widehat{1})}$ is a cofibration of topological spaces as well. One has the equality $X^0 = Y^0$ since the morphism $\mathrm{Glob}(\mathbf{S}^{n-1}) \longrightarrow \mathrm{Glob}(\mathbf{D}^n)$ restricts to the identity of $\{\widehat{0},\widehat{1}\}$ on the 0-skeletons and since the 0-skeleton functor $X \mapsto X^0$ preserves colimits.[2] So one has the commutative diagram

$$\begin{array}{ccc} L_{(\widehat{0},\widehat{1})}\mathcal{F}X & \longrightarrow & L_{(\widehat{0},\widehat{1})}\mathcal{F}Y \\ \downarrow & & \downarrow \\ \mathcal{F}X_{(\widehat{0},\widehat{1})} & \longrightarrow & \mathcal{F}Y_{(\widehat{0},\widehat{1})}. \end{array}$$

There are two mutually exclusive cases:

(1) $(\phi(\widehat{0}),\phi(\widehat{1})) = (\widehat{0},\widehat{1})$. One then has the situation

$$\begin{array}{ccc} L_{(\widehat{0},\widehat{1})}\mathcal{F}X & \xrightarrow{=} & L_{(\widehat{0},\widehat{1})}\mathcal{F}Y \\ \downarrow & & \downarrow \\ \mathcal{F}X_{(\widehat{0},\widehat{1})} & \hookrightarrow & \mathcal{F}Y_{(\widehat{0},\widehat{1})} \end{array}$$

where the bottom horizontal arrow is a cofibration since it is a pushout of the morphism of flows $\mathrm{Glob}(\mathbf{S}^{n-1}) \longrightarrow \mathrm{Glob}(\mathbf{D}^n)$. So the continuous map $L_{(\widehat{0},\widehat{1})}\mathcal{F}Y \longrightarrow \mathcal{F}Y_{(\widehat{0},\widehat{1})}$ is a cofibration.

(2) $(\phi(\widehat{0}),\phi(\widehat{1})) \neq (\widehat{0},\widehat{1})$. Then, one has the pushout diagram of flows

$$\begin{array}{ccc} L_{(\widehat{0},\widehat{1})}\mathcal{F}X & \longrightarrow & L_{(\widehat{0},\widehat{1})}\mathcal{F}Y \\ \downarrow & & \downarrow \\ \mathcal{F}X_{(\widehat{0},\widehat{1})} & \longrightarrow & \mathcal{F}Y_{(\widehat{0},\widehat{1})}. \end{array}$$

So the continuous map $L_{(\widehat{0},\widehat{1})}\mathcal{F}Y \longrightarrow \mathcal{F}Y_{(\widehat{0},\widehat{1})}$ is again a cofibration. In this situation, it may happen that $L_{(\widehat{0},\widehat{1})}\mathcal{F}X = L_{(\widehat{0},\widehat{1})}\mathcal{F}Y$.

The proof is complete with Proposition C.1, and because the canonical morphism of flows $\overrightarrow{D}^0 \longrightarrow \overrightarrow{D}$ is a relative I^{gl}-cell complex, and at last because the property above is clearly satisfied for $X = \overrightarrow{D}^0$. □

[2] One has the canonical bijection $\mathbf{Set}(X^0, Z) \cong \mathbf{Flow}(X, T(Z))$ where $T(Z)$ is the flow defined by $T(Z)^0 = Z$ and for any $(\alpha,\beta) \in Z \times Z$, $\mathbb{P}_{\alpha,\beta}T(Z) = \{0\}$.

Theorem 8.2. *Let $\overrightarrow{D}$ be a full directed ball. Then the diagram of spaces*

$$|\mathrm{Glob}^{\mathrm{top}}(\mathcal{F}Q(\overrightarrow{D}))|$$

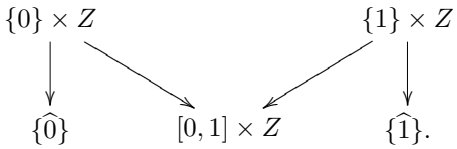
(*where Q is the cofibrant replacement functor of* **Flow**) *is Reedy cofibrant.*

Proof. The endofunctor of **Top** defined by the mapping $Z \mapsto |\mathrm{Glob}^{\mathrm{top}}(Z)|$ preserves colimits. Therefore, one has the isomorphism

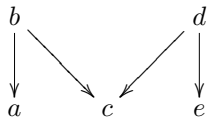
$$L_{(\widehat{0},\widehat{1})}|\mathrm{Glob}^{\mathrm{top}}(\mathcal{F}Q(\overrightarrow{D}))| \cong \left|\mathrm{Glob}^{\mathrm{top}}\left(L_{(\widehat{0},\widehat{1})}\mathcal{F}Q(\overrightarrow{D})\right)\right|.$$

It remains to prove that this endofunctor preserves cofibrations.[3] The proof will then be complete thanks to Theorem 8.1.

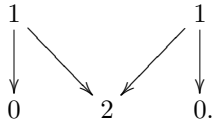
The space $|\mathrm{Glob}^{\mathrm{top}}(Z)|$ is equal to the colimit of the diagram of spaces $\mathcal{D}(Z)$



Let us consider the small category $\mathcal{C}$



equipped with the Reedy structure



If D is an object of the diagram category $\mathbf{Top}^{\mathcal{C}}$, then the latching spaces and the matching spaces of D are equal to:

(1) $L_aD = L_bD = L_dD = L_eD = \varnothing$.
(2) $L_cD = D_b \sqcup D_d$.
(3) $M_aD = M_eD = M_cD = \mathbf{1}$.
(4) $M_bD = D_a$.
(5) $M_dD = D_e$.

A morphism of diagrams $D \longrightarrow E$ is a Reedy fibration if:

(1) $D_a \longrightarrow E_a \times_{M_aE} M_aD = E_a$ is a fibration.
(2) $D_e \longrightarrow E_e \times_{M_eE} M_eD = E_e$ is a fibration.
(3) $D_c \longrightarrow E_c \times_{M_cE} M_cD = E_c$ is a fibration.
(4) $D_b \longrightarrow E_b \times_{M_bE} M_bD = E_b \times_{E_a} D_a$ is a fibration.
(5) $D_d \longrightarrow E_d \times_{M_dE} M_dD = E_d \times_{E_e} D_e$ is a fibration.

[3]This functor is of course very close to the pointed suspension functor. But it is not known how to view it as a left adjoint, and therefore as a left Quillen functor.

Consider the categorical adjunction $\varinjlim : \mathbf{Top}^{\mathcal{C}} \leftrightarrows \mathbf{Top} : \mathrm{Diag}$. By the calculations above, if $X \longrightarrow Y$ is a (resp. trivial) fibration of spaces, then $\mathrm{Diag}(X) \longrightarrow \mathrm{Diag}(Y)$ is a (trivial) Reedy fibration. The colimit functor from $\mathbf{Top}^{\mathcal{C}}$ to **Top** is therefore a left Quillen functor.

A morphism of diagrams $D \longrightarrow E$ is a Reedy cofibration if:

(1) $D_a = D_a \sqcup_{L_a D} L_a E \longrightarrow E_a$ is a cofibration.
(2) $D_b = D_b \sqcup_{L_b D} L_b E \longrightarrow E_b$ is a cofibration.
(3) $D_d = D_d \sqcup_{L_d D} L_d E \longrightarrow E_d$ is a cofibration.
(4) $D_e = D_e \sqcup_{L_e D} L_e E \longrightarrow E_e$ is a cofibration.
(5) $D_c \sqcup_{(D_b \sqcup D_d)} (E_b \sqcup E_d) = D_c \sqcup_{L_c D} L_c E \longrightarrow E_c$ is a cofibration.

Now take a cofibration $Z_1 \longrightarrow Z_2$. Since the colimit functor $\varinjlim : \mathbf{Top}^{\mathcal{C}} \longrightarrow \mathbf{Top}$ preserves cofibrations, it then suffices to check that the morphism of diagrams $\mathcal{D}(Z_1) \longrightarrow \mathcal{D}(Z_2)$ is a Reedy cofibration. It then suffices to check the fifth condition above, that is to say it suffices to prove that the continuous map

$$([0,1] \times Z_1) \sqcup_{\{0\} \times Z_1 \sqcup \{1\} \times Z_1} (\{0\} \times Z_2 \sqcup \{1\} \times Z_2) \longrightarrow ([0,1] \times Z_2)$$

is a cofibration of topological spaces. It turns out that the latter map is the pushout product (cf. Notation B.2) of the two cofibrations $\{0,1\} \longrightarrow [0,1]$ and $Z_1 \longrightarrow Z_2$. The proof is then complete because **Top** is a monoidal model category. □

Theorem 8.3. *Let $\overrightarrow{D}$ be a full directed ball. Then the diagram of flows $\mathcal{G}Q(\overrightarrow{D})$ (where Q is the cofibrant replacement functor of* **Flow**) *is Reedy cofibrant.*

Proof. The argument is different from the one of Theorem 8.1. The flow $Q(\overrightarrow{D})$ is an object of **cell**(**Flow**). Therefore, the canonical morphism of flows $\overrightarrow{D}^0 \longrightarrow Q(\overrightarrow{D})$ is a priori a transfinite composition of pushouts of elements of $I^{\mathrm{gl}}_+ = I^{\mathrm{gl}} \cup \{R, C\}$. Since there is a bijection of sets $\overrightarrow{D}^0 \cong Q(\overrightarrow{D})^0$, a pushout of $R : \{0,1\} \longrightarrow \{0\}$ or of $C : \varnothing \longrightarrow \{0\}$ in the globular decomposition of the relative I^{gl}_+-cell complex $\overrightarrow{D}^0 \longrightarrow Q(\overrightarrow{D})$ is necessarily without effect on $\overrightarrow{D}^0$. Thus, the canonical morphism of flows $\overrightarrow{D}^0 \longrightarrow Q(\overrightarrow{D})$ is a transfinite composition of pushouts of elements of I^{gl}. So there exists an ordinal λ and a λ-sequence $M : \lambda \longrightarrow \mathbf{Flow}$ such that $M_0 = \overrightarrow{D}^0$, $M_\lambda = Q(\overrightarrow{D})$ and for any $\mu < \lambda$, the morphism of flows $M_\mu \longrightarrow M_{\mu+1}$ is a pushout of the inclusion of flows $e_\mu : \mathrm{Glob}(\mathbf{S}^{n_\mu - 1}) \subset \mathrm{Glob}(\mathbf{D}^{n_\mu})$ for some $n_\mu \geqslant 0$, that is one has the pushout diagram of flows:

$$\begin{array}{ccc} \mathrm{Glob}(\mathbf{S}^{n_\mu - 1}) & \xrightarrow{\phi_\mu} & M_\mu \\ \downarrow & & \downarrow \\ \mathrm{Glob}(\mathbf{D}^{n_\mu}) & \longrightarrow & M_{\mu+1}. \end{array}$$

Let $(\alpha_0, \ldots, \alpha_p)$ be a simplex of $\Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}}$. The relative I^{gl}-cell complex

$$\overrightarrow{D}^0 \longrightarrow \mathcal{G}Q(\overrightarrow{D})_{(\alpha_0, \ldots, \alpha_p)} = \overrightarrow{D} \restriction_{[\alpha_0, \alpha_1]} * \cdots * \overrightarrow{D} \restriction_{[\alpha_{p-1}, \alpha_p]}$$

is a relative I^{gl}-cell subcomplex which is the union of the globular cells e_μ such that $[\phi_\mu(\widehat{0}), \phi_\mu(\widehat{1})] \subset [\alpha_i, \alpha_{i+1}]$ for some $0 \leqslant i < p$.[4] So the subcomplex $\overrightarrow{D}^0 \longrightarrow$

[4] A I^{gl}-cell subcomplex is characterized by its cells since any morphism of I^{gl} is an effective monomorphism of flows by [Gau03] Theorem 10.6 and by [Hir03] Proposition 12.2.1.

$\mathcal{G}Q(\overrightarrow{D})_{(\alpha_0,\dots,\alpha_p)}$ contains the globular cells e_μ such that $[\phi_\mu(\widehat{0}),\phi_\mu(\widehat{1})]\subset[\alpha_0,\alpha_1]\sqcup\cdots\sqcup[\alpha_{p-1},\alpha_p]$ ($\sqcup$ meaning the disjoint union !).

We then deduce that all morphisms of the diagram $\mathcal{G}Q(\overrightarrow{D})$ are inclusions of relative I^{gl}-cell subcomplexes. Thus, the canonical morphism of flows

$$L_{(\alpha_0,\dots,\alpha_p)}\mathcal{G}Q(\overrightarrow{D})\longrightarrow\mathcal{G}Q(\overrightarrow{D})_{(\alpha_0,\dots,\alpha_p)}$$

is an inclusion of relative I^{gl}-cell subcomplexes as well. More precisely, it is equal to the transfinite composition of the inclusions of flows $\mathrm{Glob}(\mathbf{S}^{n_\mu-1})\subset\mathrm{Glob}(\mathbf{D}^{n_\mu})$ such that $[\phi_\mu(\widehat{0}),\phi_\mu(\widehat{1})]\subset[\alpha_0,\alpha_1]\sqcup\cdots\sqcup[\alpha_{p-1},\alpha_p]$ and such that there does not exist any state α such that $[\phi_\mu(\widehat{0}),\phi_\mu(\widehat{1})]\subset[\alpha_0,\alpha_1]\sqcup\cdots\sqcup[\alpha_i,\alpha]\sqcup[\alpha,\alpha_{i+1}]\sqcup\cdots\sqcup[\alpha_{p-1},\alpha_p]$ and $\alpha_i<\alpha<\alpha_{i+1}$. □

The proof of Theorem 8.3 also has the following consequences:

Corollary 8.4. *Let $\overrightarrow{D}$ be a full directed ball. Then there exists a diagram of globular complexes*

$$\mathcal{G}^{\mathrm{top}}Q(\overrightarrow{D}):\Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}}\longrightarrow\mathbf{glTop}$$

such that the composition by the functor cat : $\mathbf{glTop}\longrightarrow\mathbf{Flow}$

$$\Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}}\longrightarrow\mathbf{glTop}\longrightarrow\mathbf{Flow}$$

is exactly the diagram $\mathcal{G}Q(\overrightarrow{D})$.

Proof. First of all, consider the flow $Q(\overrightarrow{D})$ and using Theorem 6.1, construct a globular complex $Q(\overrightarrow{D})^{\mathrm{top}}$ such that $\mathrm{cat}(Q(\overrightarrow{D})^{\mathrm{top}})=Q(\overrightarrow{D})$. Let $(\alpha_0,\dots,\alpha_p)$ be a simplex of $\Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}}$. Then the globular complex

$$\mathcal{G}^{\mathrm{top}}Q(\overrightarrow{D})_{(\alpha_0,\dots,\alpha_p)}$$

is defined as the globular subcomplex containing the globular cells of $Q(\overrightarrow{D})^{\mathrm{top}}$ such that the attaching map ϕ satisfies $[\phi(\widehat{0}),\phi(\widehat{1})]\subset[\alpha_0,\alpha_1]\sqcup\cdots\sqcup[\alpha_{p-1},\alpha_p]$. □

Let $(\alpha_0,\dots,\alpha_p)$ be a simplex of $\Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}}$. The category of multipointed topological spaces being cocomplete, one can consider the multipointed topological space

$$L_{(\alpha_0,\dots,\alpha_p)}\mathcal{G}^{\mathrm{top}}Q(\overrightarrow{D}).$$

It consists of the globular subcomplexes of $Q(\overrightarrow{D})^{\mathrm{top}}$ containing the globular cells such that the attaching map ϕ satisfies $[\phi(\widehat{0}),\phi(\widehat{1})]\subset[\alpha_0,\alpha_1]\sqcup\cdots\sqcup[\alpha_{p-1},\alpha_p]$ and such that there exists a state α such that $[\phi(\widehat{0}),\phi(\widehat{1})]\subset[\alpha_0,\alpha_1]\sqcup\cdots\sqcup[\alpha_i,\alpha]\sqcup[\alpha,\alpha_{i+1}]\sqcup\cdots\sqcup[\alpha_{p-1},\alpha_p]$ and $\alpha_i<\alpha<\alpha_{i+1}$. So the multipointed topological space $L_{(\alpha_0,\dots,\alpha_p)}\mathcal{G}^{\mathrm{top}}Q(\overrightarrow{D})$ is a globular complex. And one obtains the equality

$$\mathrm{cat}(L_{(\alpha_0,\dots,\alpha_p)}\mathcal{G}^{\mathrm{top}}Q(\overrightarrow{D}))=L_{(\alpha_0,\dots,\alpha_p)}\mathcal{G}\overrightarrow{D}.$$

Corollary 8.5. *With the choices of Corollary* 8.4. *Let $\overrightarrow{D}$ be a full directed ball. Then the diagram of spaces $|\mathcal{G}^{\mathrm{top}}Q(\overrightarrow{D})|$ (where Q is the cofibrant replacement functor of* **Flow**) *is Reedy cofibrant.*

Proof. Let $(\alpha_0, \ldots, \alpha_p)$ be a simplex of $\Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}}$. The continuous map

$$|L_{(\alpha_0,\ldots,\alpha_p)}\mathcal{G}^{\mathrm{top}}Q(\overrightarrow{D})| \longrightarrow |\mathcal{G}^{\mathrm{top}}Q(\overrightarrow{D})_{(\alpha_0,\ldots,\alpha_p)}|$$

is a transfinite composition of pushouts of continuous maps of the form

$$|\mathrm{Glob}^{\mathrm{top}}(\mathbf{S}^{n-1})| \longrightarrow |\mathrm{Glob}^{\mathrm{top}}(\mathbf{D}^n)|$$

with $n \geqslant 0$. The proof is complete thanks to the proof of Theorem 8.2. □

Theorem 8.6. *Let $\overrightarrow{D}$ be a full directed ball. Then its underlying homotopy type $|\overrightarrow{D}|$ is that of a point.*

Proof. We are going to make an induction on the cardinal of the poset $\overrightarrow{D}^0$. If $\overrightarrow{D}^0 = \{\widehat{0} < \widehat{1}\}$, then $Q(\overrightarrow{D}) = \mathrm{Glob}(Z)$ for some topological space Z. By hypothesis, the space $Z = \mathbb{P}_{\widehat{0},\widehat{1}}\overrightarrow{D}$ is contractible (and cofibrant). Therefore, the flows $\mathrm{Glob}(Z)$ and $\mathrm{Glob}(\{0\})$ are S-homotopy equivalent. Thus, the globular complexes $\mathrm{Glob}^{\mathrm{top}}(Z)$ and $\mathrm{Glob}^{\mathrm{top}}(\{0\})$ are S-homotopy equivalent as well. Hence the topological spaces $|\mathrm{Glob}^{\mathrm{top}}(Z)|$ and $|\mathrm{Glob}^{\mathrm{top}}(\{0\})|$ are homotopy equivalent by Theorem 6.1. Now suppose that $\overrightarrow{D}^0 \backslash \{\widehat{0} < \widehat{1}\}$ is nonempty and suppose the theorem proved for any full directed ball $\overrightarrow{E}$ such that $\mathrm{card}(\overrightarrow{E}^0) < \mathrm{card}(\overrightarrow{D}^0)$.

By Proposition 7.11 applied to the full directed ball $Q(\overrightarrow{D})$, one has the pushout diagram of flows:

$$\begin{array}{ccc} \mathrm{Glob}(L_{(\widehat{0},\widehat{1})}\mathcal{F}Q(\overrightarrow{D})) & \longrightarrow & L_{(\widehat{0},\widehat{1})}\mathcal{G}Q(\overrightarrow{D}) \\ \downarrow & & \downarrow \\ \mathrm{Glob}(\mathbb{P}_{\widehat{0},\widehat{1}}Q(\overrightarrow{D})) & \longrightarrow & Q(\overrightarrow{D}). \end{array}$$

One obtains the commutative diagram of globular complexes:

$$\begin{array}{ccc} \mathrm{Glob}^{\mathrm{top}}(L_{(\widehat{0},\widehat{1})}\mathcal{F}Q(\overrightarrow{D})) & \longrightarrow & L_{(\widehat{0},\widehat{1})}\mathcal{G}^{\mathrm{top}}Q(\overrightarrow{D}) \\ \downarrow & & \downarrow \\ \mathrm{Glob}^{\mathrm{top}}(\mathbb{P}_{\widehat{0},\widehat{1}}Q(\overrightarrow{D})) & \longrightarrow & \mathcal{G}^{\mathrm{top}}Q(\overrightarrow{D}) \end{array}$$

which must be a pushout of multipointed topological spaces by Corollary 8.5. One can now pass to the underlying topological spaces of all of these globular complexes and one obtains the pushout diagram of topological spaces:

$$\begin{array}{ccc} L_{(\widehat{0},\widehat{1})}|\mathrm{Glob}^{\mathrm{top}}(\mathcal{F}Q(\overrightarrow{D}))| = |\mathrm{Glob}^{\mathrm{top}}(L_{(\widehat{0},\widehat{1})}\mathcal{F}Q(\overrightarrow{D}))| & \longrightarrow & L_{(\widehat{0},\widehat{1})}|\mathcal{G}^{\mathrm{top}}Q(\overrightarrow{D})| \\ \downarrow & & \downarrow \\ |\mathrm{Glob}^{\mathrm{top}}(\mathbb{P}_{\widehat{0},\widehat{1}}Q(\overrightarrow{D}))| & \longrightarrow & |\mathcal{G}^{\mathrm{top}}Q(\overrightarrow{D})|(\widehat{0},\widehat{1}). \end{array}$$

The top horizontal arrow is induced by the morphism of diagrams of spaces

$$|\mathrm{Glob}^{\mathrm{top}}(\mathcal{F}Q(\overrightarrow{D}))| \longrightarrow |\mathcal{G}^{\mathrm{top}}Q(\overrightarrow{D})|.$$

If we can prove that the top horizontal arrow is a weak homotopy equivalence of topological spaces, and since the continuous map $L_{(\widehat{0},\widehat{1})}|\mathrm{Glob}^{\mathrm{top}}(\mathcal{F}Q(\overrightarrow{D}))| \longrightarrow$

$|\mathrm{Glob}^{\mathrm{top}}(\mathbb{P}_{\widehat{0},\widehat{1}}Q(\overrightarrow{D}))|$ is a cofibration of spaces by Theorem 8.2, then one will be able to deduce the weak homotopy equivalence of spaces $|\mathrm{Glob}^{\mathrm{top}}(\mathbb{P}_{\widehat{0},\widehat{1}}Q(\overrightarrow{D}))| \simeq |\mathcal{G}^{\mathrm{top}}Q(\overrightarrow{D})|(\widehat{0},\widehat{1})$ since the model category **Top** is left proper. Since the topological space $|\mathrm{Glob}^{\mathrm{top}}(\mathbb{P}_{\widehat{0},\widehat{1}}Q(\overrightarrow{D}))|$ is contractible, one will be then able to deduce that the space $|\mathcal{G}^{\mathrm{top}}Q(\overrightarrow{D})|(\widehat{0},\widehat{1}) \simeq |\overrightarrow{D}|$ is weakly contractible. And the proof will be finished.

The diagrams of topological spaces $|\mathrm{Glob}^{\mathrm{top}}(\mathcal{F}Q(\overrightarrow{D}))|$ and $|\mathcal{G}^{\mathrm{top}}Q(\overrightarrow{D})|$ are both Reedy cofibrant by Theorem 8.2 and Corollary 8.5. So their restriction to the full subcategory $\partial(\Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}}_+\downarrow(\widehat{0},\widehat{1})) \cong \Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}}\backslash\{(\widehat{0},\widehat{1})\}$ of $\Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}}$ is Reedy cofibrant as well. Thus, one obtains

$$\begin{aligned} L_{(\widehat{0},\widehat{1})}|\mathrm{Glob}^{\mathrm{top}}(\mathcal{F}Q(\overrightarrow{D}))| &\cong \varinjlim{}_{\partial(\Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}}_+\downarrow(\widehat{0},\widehat{1}))}|\mathrm{Glob}^{\mathrm{top}}(\mathcal{F}Q(\overrightarrow{D}))| \\ &\qquad \text{by definition of the latching space} \\ &\simeq \operatorname*{holim}_{\partial(\Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}}_+\downarrow(\widehat{0},\widehat{1}))}|\mathrm{Glob}^{\mathrm{top}}(\mathcal{F}Q(\overrightarrow{D}))| \\ &\qquad \text{by Corollary 7.4 and by Theorem 8.2} \end{aligned}$$

and

$$\begin{aligned} L_{(\widehat{0},\widehat{1})}|\mathcal{G}^{\mathrm{top}}Q(\overrightarrow{D})| &\cong \varinjlim{}_{\partial(\Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}}_+\downarrow(\widehat{0},\widehat{1}))}|\mathcal{G}^{\mathrm{top}}Q(\overrightarrow{D})| \\ &\qquad \text{by definition of the latching space} \\ &\simeq \operatorname*{holim}_{\partial(\Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}}_+\downarrow(\widehat{0},\widehat{1}))}|\mathcal{G}^{\mathrm{top}}Q(\overrightarrow{D})| \\ &\qquad \text{by Corollary 7.4 and by Corollary 8.5.} \end{aligned}$$

It then suffices to prove that for any simplex $(\alpha_0,\dots,\alpha_p)$ of the latching category $\partial(\Delta^{\mathrm{ext}}(\overrightarrow{D}^0)^{\mathrm{op}}_+\downarrow(\widehat{0},\widehat{1}))$, the morphism of diagrams

$$|\mathrm{Glob}^{\mathrm{top}}(\mathcal{F}Q(\overrightarrow{D}))| \longrightarrow |\mathcal{G}^{\mathrm{top}}Q(\overrightarrow{D})|$$

induces a weak homotopy equivalence

$$|\mathrm{Glob}^{\mathrm{top}}(\mathcal{F}Q(\overrightarrow{D}))|(\alpha_0,\dots,\alpha_p) \simeq |\mathcal{G}^{\mathrm{top}}Q(\overrightarrow{D})|(\alpha_0,\dots,\alpha_p).$$

The topological space $|\mathcal{G}^{\mathrm{top}}Q(\overrightarrow{D})|_{(\alpha_0,\dots,\alpha_p)}$ is the "concatenation"

$$|\mathcal{G}^{\mathrm{top}}Q(\overrightarrow{D})|_{(\alpha_0,\alpha_1)} * \cdots * |\mathcal{G}^{\mathrm{top}}Q(\overrightarrow{D})|_{(\alpha_{p-1},\alpha_p)}$$

of p topological spaces, that is where the final state of $\mathcal{G}^{\mathrm{top}}Q(\overrightarrow{D})_{(\alpha_i,\alpha_{i+1})}$ is identified with the initial state of $\mathcal{G}^{\mathrm{top}}Q(\overrightarrow{D})_{(\alpha_{i+1},\alpha_{i+2})}$ for any $i+2 \leqslant p$. The latter space is contractible by induction hypothesis and since a finite join of well-pointed cofibrant contractible spaces is contractible. The topological space

$$|\mathrm{Glob}^{\mathrm{top}}(\mathcal{F}Q(\overrightarrow{D}))|_{(\alpha_0,\dots,\alpha_p)}$$

is contractible since the product of spaces

$$\mathbb{P}_{\alpha_0,\alpha_1}Q(\overrightarrow{D}) \times \ldots \times \mathbb{P}_{\alpha_{p-1},\alpha_p}Q(\overrightarrow{D})$$

is contractible since $\overrightarrow{D}$ is a full directed ball and since a finite product of cofibrant contractible spaces is contractible. □

The proof of Theorem 8.6 implies the following theorem:

Corollary 8.7. *Let $\overrightarrow{D}$ be a loopless flow such that:*

(1) *The poset $\overrightarrow{D}^0$ is finite and bounded with initial state $\widehat{0}$ and with final state $\widehat{1}$.*
(2) *For any $(\alpha, \beta) \in \overrightarrow{D}^0$ such that $\alpha < \beta$ and $(\alpha, \beta) \neq (\widehat{0}, \widehat{1})$, the topological space $\mathbb{P}_{\alpha,\beta}\overrightarrow{D}$ is weakly contractible.*

Then the underlying homotopy type of $\overrightarrow{D}$ is homotopy equivalent to the underlying homotopy type of $\mathrm{Glob}(\mathbb{P}_{\widehat{0},\widehat{1}}\overrightarrow{D})$*: in other words, one has $|\overrightarrow{D}| \simeq |\mathrm{Glob}(\mathbb{P}_{\widehat{0},\widehat{1}}\overrightarrow{D})|$.*

9. Preservation of the underlying homotopy type

Theorem 9.1. *Let $f : X \longrightarrow Y$ be a generalized T-homotopy equivalence. Then the morphism $|f| : |X| \longrightarrow |Y|$ is an isomorphism of* **Ho**(**Top**)*.*

Proof. First of all, let us suppose that f is a pushout diagram of flows of the form

$$\begin{array}{ccc} Q(F(P_1)) & \longrightarrow & X \\ {\scriptstyle Q(F(u))}\downarrow & & \downarrow{\scriptstyle f} \\ Q(F(P_2)) & \longrightarrow & Y \end{array}$$

where P_1 and P_2 are two finite bounded posets and where $u : P_1 \longrightarrow P_2$ belongs to $\mathcal{T}$. Let us factor the morphism of flows $Q(F(P_1)) \longrightarrow X$ as a composite of a relative I^{gl}_+-cell complex $Q(F(P_1)) \longrightarrow W$ followed by a trivial fibration $W \longrightarrow X$. Then one obtains the commutative diagram of flows

$$\begin{array}{ccccc} Q(F(P_1)) & \hookrightarrow & W & \xrightarrow{\simeq} & X \\ {\scriptstyle Q(F(u))}\downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ Q(F(P_2)) & \hookrightarrow & T & \xrightarrow{\simeq} & Y. \end{array}$$

The morphism $T \longrightarrow Y$ of the diagram above is a weak S-homotopy equivalence since the model category **Flow** is left proper by [Gau05c] Theorem 6.4. So the flows W and X (resp. T and Y) have the same underlying homotopy types by [Gau05a] Proposition VII.2.2 and we are reduced to the following situation:

$$\begin{array}{ccc} Q(F(P_1)) & \hookrightarrow & X \\ {\scriptstyle Q(F(u))}\downarrow & & \downarrow{\scriptstyle f} \\ Q(F(P_2)) & \hookrightarrow & Y. \end{array}$$

The four morphisms of the diagram above are inclusions of I^{gl}_+-cell complexes. So using the globular decompositions of the flows $Q(F(P_1))$, $Q(F(P_2))$, X and Y,

there exist four globular complexes $Q^{\mathrm{top}}(F(P_1))$, $Q^{\mathrm{top}}(F(P_2))$, X^{top} and Y^{top} and a commutative diagram of globular complexes

$$\begin{array}{ccc} Q^{\mathrm{top}}(F(P_1)) & \hookrightarrow & X^{\mathrm{top}} \\ \downarrow & & \downarrow \\ Q^{\mathrm{top}}(F(P_2)) & \hookrightarrow & Y^{\mathrm{top}} \end{array}$$

which is a pushout diagram of multipointed spaces and whose image by the functor $cat : \mathbf{glTop} \longrightarrow \mathbf{Flow}$ gives back the diagram of flows above. Now by passing to the underlying topological spaces, one obtains the pushout diagram of topological spaces

$$\begin{array}{ccc} |Q^{\mathrm{top}}(F(P_1))| & \hookrightarrow & |X^{\mathrm{top}}| \\ \downarrow & & \downarrow \\ |Q^{\mathrm{top}}(F(P_2))| & \hookrightarrow & |Y^{\mathrm{top}}|. \end{array}$$

The continuous map $|Q^{\mathrm{top}}(F(P_1))| \longrightarrow |Q^{\mathrm{top}}(F(P_2))|$ is a trivial cofibration of topological spaces since the morphism of posets $u : P_1 \longrightarrow P_2$ is one-to-one. Thus, the continuous map $|X^{\mathrm{top}}| \longrightarrow |Y^{\mathrm{top}}|$ is a trivial cofibration as well.

Now let us suppose that $f : X \longrightarrow Y$ is a transfinite composition of morphisms as above. Then there exists an ordinal λ and a λ-sequence $Z : \lambda \longrightarrow \mathbf{Flow}$ with $Z_0 = X$, $Z_\lambda = Y$ and the morphism $Z_0 \longrightarrow Z_\lambda$ is equal to f. Since for any $u \in \mathcal{T}$, the morphism of flows $Q(F(u))$ is a cofibration, the morphism $Z_\mu \longrightarrow Z_{\mu+1}$ is a cofibration for any $\mu < \lambda$. Since the model category **Flow** is left proper by [Gau05c] Theorem 6.4, there exists by [Hir03] Proposition 17.9.4 a λ-sequence $\widetilde{Z} : \lambda \longrightarrow \mathbf{Flow}$ and a morphism of λ-sequences $\widetilde{Z} \longrightarrow Z$ such that for any $\mu \leqslant \lambda$, the flow $\widetilde{Z}_\mu$ is an object of $\mathbf{cell}(\mathbf{Flow})$, such that each morphism $\widetilde{Z}_\mu \longrightarrow \widetilde{Z}_{\mu+1}$ is a relative I^{gl}_+-cell complex, and such that the morphism $\widetilde{Z}_\mu \longrightarrow Z_\mu$ is a weak S-homotopy equivalence. Using the globular decomposition of $\widetilde{Z}_0$, construct a globular complex $\widetilde{Z}^{\mathrm{top}}_0$ such that $\mathrm{cat}(\widetilde{Z}^{\mathrm{top}}_0) = \widetilde{Z}_0$. And by transfinite induction on μ, since each morphism $\widetilde{Z}_\mu \longrightarrow \widetilde{Z}_{\mu+1}$ is a relative I^{gl}_+-cell complex, construct a globular complex $\widetilde{Z}^{\mathrm{top}}_\mu$ such that $\mathrm{cat}(\widetilde{Z}^{\mathrm{top}}_\mu) = \widetilde{Z}_\mu$. Then one obtains a λ-sequence of topological spaces $\mu \mapsto |\widetilde{Z}^{\mathrm{top}}_\mu|$ whose colimit is the underlying topological space of $\widetilde{Z}^{\mathrm{top}}_\lambda$.

For any $\mu < \lambda$, the continuous map $|\widetilde{Z}^{\mathrm{top}}_\mu| \longrightarrow |\widetilde{Z}^{\mathrm{top}}_{\mu+1}|$ is a trivial cofibration of topological spaces. So the transfinite composition $|\widetilde{Z}^{\mathrm{top}}_0| \longrightarrow |\widetilde{Z}^{\mathrm{top}}_\lambda|$ is a trivial cofibration as well.

The case remains where f is a retract of a generalized T-equivalence of the preceding kinds. The result follows from the fact that everything is functorial and that the retract of a weak homotopy equivalence is a weak homotopy equivalence. □

10. Conclusion

This new definition of T-homotopy equivalence seems to be well-behaved because it preserves the underlying homotopy type of flows. For an application of this new

approach of T-homotopy, see the proof of an analogue of Whitehead's theorem for the full dihomotopy relation in [Gau06].

Appendix A. Elementary remarks about flows

This is a reminder of results of [Gau05d].

Proposition A.1 ([Gau03] Proposition 15.1)**.** *If one has the pushout of flows*

$$\begin{array}{ccc} \mathrm{Glob}(\partial Z) & \xrightarrow{\phi} & A \\ \downarrow & & \downarrow \\ \mathrm{Glob}(Z) & \longrightarrow & M \end{array}$$

then the continuous map $\mathbb{P}A \longrightarrow \mathbb{P}M$ is a transfinite composition of pushouts of continuous maps of the form $\mathrm{Id} \times \ldots \times \mathrm{Id} \times f \times \mathrm{Id} \times \ldots \times \mathrm{Id}$ where $f : \mathbb{P}_{\phi(\widehat{0}),\phi(\widehat{1})}A \longrightarrow T$ is the canonical inclusion obtained with the pushout diagram of topological spaces

$$\begin{array}{ccc} \partial Z & \longrightarrow & \mathbb{P}_{\phi(\widehat{0}),\phi(\widehat{1})}A \\ \downarrow & & \downarrow \\ Z & \longrightarrow & T. \end{array}$$

Proposition A.2. *Let Y be a flow such that $\mathbb{P}Y$ is a cofibrant topological space. Let $f : Y \longrightarrow Z$ be a pushout of a morphism of I^{gl}_+. Then the topological space $\mathbb{P}Z$ is cofibrant.*

Proof. By hypothesis, f is the pushout of a morphism of flows $g \in I^{\mathrm{gl}}_+$. So one has the pushout of flows

$$\begin{array}{ccc} A & \xrightarrow{\phi} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[\psi]{} & Z. \end{array}$$

If f is a pushout of $C : \varnothing \subset \{0\}$, then $\mathbb{P}Z = \mathbb{P}Y$. Therefore, the space $\mathbb{P}Z$ is cofibrant. If f is a pushout of $R : \{0,1\} \to \{0\}$ and if $\phi(0) = \phi(1)$, then $\mathbb{P}Z = \mathbb{P}Y$ again. Therefore, the space $\mathbb{P}Z$ is cofibrant again. If f is a pushout of $R : \{0,1\} \to \{0\}$ and if ϕ is one-to-one, then one has the homeomorphism

$$\begin{aligned} \mathbb{P}Z \cong \mathbb{P}Y \sqcup \bigsqcup_{r \geqslant 0} \left(\mathbb{P}_{.,\phi(0)}Y \times \mathbb{P}_{\phi(1),\phi(0)}Y \times \mathbb{P}_{\phi(1),\phi(0)}Y \times \ldots (r \text{ times}) \times \mathbb{P}_{\phi(1),.}Y \right) \\ \sqcup \bigsqcup_{r \geqslant 0} \left(\mathbb{P}_{.,\phi(1)}Y \times \mathbb{P}_{\phi(0),\phi(1)}Y \times \mathbb{P}_{\phi(0),\phi(1)}Y \times \ldots (r \text{ times}) \times \mathbb{P}_{\phi(0),.}Y \right). \end{aligned}$$

Therefore, the space $\mathbb{P}Z$ is again cofibrant since the model category **Top** is monoidal. It remains the case where g is the inclusion $\mathrm{Glob}(\mathbf{S}^{n-1}) \subset \mathrm{Glob}(\mathbf{D}^n)$ for some $n \geqslant 0$.

Consider the pushout of topological spaces

$$\begin{array}{ccc} \mathbf{S}^{n-1} & \xrightarrow{\mathbb{P}\phi} & \mathbb{P}_{\phi(\widehat{0}),\phi(\widehat{1})}Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{D}^n & \xrightarrow[\mathbb{P}\psi]{} & T. \end{array}$$

By Proposition A.1, the continuous map $\mathbb{P}Y \longrightarrow \mathbb{P}Z$ is a transfinite composition of pushouts of continuous maps of the form $\mathrm{Id} \times \mathrm{Id} \times \ldots \times f \times \ldots \times \mathrm{Id} \times \mathrm{Id}$ where f is a cofibration and the identities maps are the identity maps of cofibrant topological spaces. So it suffices to notice that if k is a cofibration and if X is a cofibrant topological space, then $\mathrm{Id}_X \times k$ is still a cofibration since the model category **Top** is monoidal. □

Proposition A.3. *Let X be a cofibrant flow. Then for any $(\alpha, \beta) \in X^0 \times X^0$, the topological space $\mathbb{P}_{\alpha,\beta}X$ is cofibrant.*

Proof. A cofibrant flow X is a retract of a I^{gl}_+-cell complex Y and $\mathbb{P}X$ becomes a retract of $\mathbb{P}Y$. So it suffices to show that $\mathbb{P}Y$ is cofibrant. Proposition A.2 completes the proof. □

Appendix B. Calculating pushout products

This is a reminder of results of [Gau05d].

Lemma B.1. *Let $D : I \longrightarrow$ **Top** and $E : J \longrightarrow$ **Top** be two diagrams in a complete cocomplete cartesian closed category. Let $D \times E : I \times J :\longrightarrow$ **Top** be the diagram of topological spaces defined by $(D \times E)(x, y) := D(x) \times E(y)$ if (x, y) is either an object or an arrow of the small category $I \times J$. Then one has $\varinjlim(D \times E) \cong (\varinjlim D) \times (\varinjlim E)$.*

Proof. One has $\varinjlim(D \times E) \cong \varinjlim_i(\varinjlim_j D(i) \times E(j))$ by [ML98]. And one has $\varinjlim_j(D(i) \times E(j)) \cong D(i) \times (\varinjlim E)$ since the category is cartesian closed. So $\varinjlim(D \times E) \cong \varinjlim_i(D(i) \times (\varinjlim E)) \cong (\varinjlim D) \times (\varinjlim E)$. □

Notation B.2. If $f : U \longrightarrow V$ and $g : W \longrightarrow X$ are two morphisms of a complete cocomplete category, then let us denote by $f \square g : (U \times X) \sqcup_{(U \times W)} (V \times W) \longrightarrow V \times X$ the pushout product of f and g. The notation $f_0 \square \ldots \square f_p$ is defined by induction on p by $f_0 \square \ldots \square f_p := (f_0 \square \ldots \square f_{p-1}) \square f_p$.

Theorem B.3. (*Calculating a pushout product of several morphisms*): *Let $f_i : A_i \longrightarrow B_i$ for $0 \leqslant i \leqslant p$ be $p + 1$ morphisms of a complete cocomplete cartesian closed category $\mathcal{C}$. Let $S \subset \{0, \ldots, p\}$. Let*

$$C_p(S) := \left(\prod_{i \in S} B_i \right) \times \left(\prod_{i \notin S} A_i \right).$$

If S and T are two subsets of $\{0, \ldots, p\}$ such that $S \subset T$, let $C_p(i^T_S) : C_p(S) \longrightarrow C_p(T)$ be the morphism

$$\left(\prod_{i \in S} \mathrm{Id}_{B_i} \right) \times \left(\prod_{i \in T \setminus S} f_i \right) \times \left(\prod_{i \notin T} \mathrm{Id}_{A_i} \right).$$

Then:

(1) *The mappings* $S \mapsto C_p(S)$ *and* $i_S^T \mapsto C_p(i_S^T)$ *give rise to a functor from* $\Delta(\{0,\dots,p\})$ (*the order complex of the poset* $\{0,\dots,p\}$) *to* $\mathcal{C}$.

(2) *There exists a canonical morphism*

$$\varinjlim_{\substack{S \subset \{0,\dots,p\} \\ S \neq \{0,\dots,p\}}} C_p(S) \longrightarrow C_p(\{0,\dots,p\}).$$

and it is equal to the morphism $f_0 \square \dots \square f_p$.

Proof. The first assertion is clear. Moreover, for any subset S and T of $\{0,\dots,p\}$ such that $S \subset T$, the diagram

$$\begin{array}{ccc} S & \longrightarrow & \{0,\dots,p\} \\ \downarrow & \nearrow & \\ T & & \end{array}$$

is commutative since there is at most one morphism between two objects of the order complex $\Delta(\{0,\dots,p\})$, hence the existence of the morphism

$$\varinjlim_{\substack{S \subset \{0,\dots,p\} \\ S \neq \{0,\dots,p\}}} C_p(S) \longrightarrow C(\{0,\dots,p\}).$$

The second assertion is clear for $p = 0$ and $p = 1$. We are going to prove it by induction on p. By definition, the morphism $f_0 \square \dots \square f_{p+1}$ is the canonical morphism from

$$\left(\left(\varinjlim_{\substack{S \subset \{0,\dots,p\} \\ S \neq \{0,\dots,p\}}} C_p(S)\right) \times B_{p+1}\right) \sqcup_{\left(\left(\varinjlim_{\substack{S \subset \{0,\dots,p\} \\ S \neq \{0,\dots,p\}}} C_p(S)\right) \times A_{p+1}\right)} (C_p(\{0,\dots,p\}) \times A_{p+1})$$

to $B_0 \times \dots \times B_{p+1}$. Since the underlying category is supposed to be cartesian closed, the functors $M \mapsto M \times B_{p+1}$ and $M \mapsto M \times A_{p+1}$ both preserve colimits. So the source of the morphism $f_0 \square \dots \square f_{p+1}$ is equal to

$$\left(\varinjlim_{\substack{S \subset \{0,\dots,p\} \\ S \neq \{0,\dots,p\}}} (C_p(S) \times B_{p+1})\right) \sqcup_{\left(\varinjlim_{\substack{S \subset \{0,\dots,p\} \\ S \neq \{0,\dots,p\}}} (C_p(S) \times A_{p+1})\right)} (C_p(\{0,\dots,p\}) \times A_{p+1})$$

or in other terms to

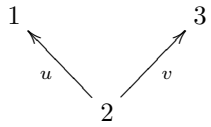
$$\left(\varinjlim_{\substack{S \subset \{0,\dots,p\} \\ S \neq \{0,\dots,p\}}} C_{p+1}(S \cup \{p+1\})\right) \sqcup_{\left(\varinjlim_{\substack{S \subset \{0,\dots,p\} \\ S \neq \{0,\dots,p\}}} C_{p+1}(S)\right)} C_{p+1}(\{0,\dots,p\})$$

or at last to

$$\left(\varinjlim_{\substack{S \subset \{0,\dots,p+1\} \\ S \neq \{0,\dots,p\} \\ p+1 \in S}} C_{p+1}(S)\right) \sqcup_{\left(\varinjlim_{\substack{S \subset \{0,\dots,p\} \\ S \neq \{0,\dots,p\}}} C_{p+1}(S)\right)} C_{p+1}(\{0,\dots,p\}).$$

The notation $\partial\Delta(\{0,\dots,p+1\})$ will represent the simplicial order complex $\Delta(\{0,\dots,p+1\})$ with the simplex $(0,\dots,p+1)$ removed.

Let us consider the small category $\mathcal{D}$



and the composition of functors:

$$\partial\Delta(\{0,\dots,p+1\}) \longrightarrow \mathcal{D} \longrightarrow *$$

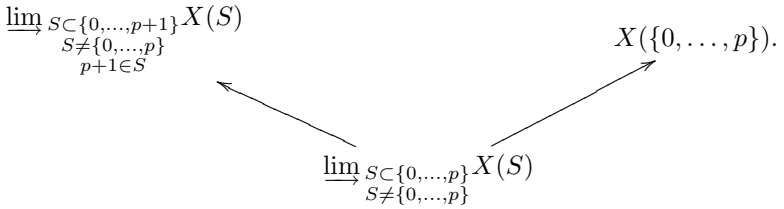
where $*$ is the category with one object and one morphism and where the functor $F : \partial\Delta(\{0,\dots,p+1\}) \longrightarrow \mathcal{D}$ is defined as follows:

(1) The full subcategory of $\partial\Delta(\{0,\dots,p+1\})$ of S such that $S \subset \{0,\dots,p+1\}$, $S \neq \{0,\dots,p\}$ and $p+1 \in S$ is mapped to 1 and the identity morphism Id_1 of 1.
(2) The full subcategory of $\partial\Delta(\{0,\dots,p+1\})$ of S such that $S \subset \{0,\dots,p\}$ and $S \neq \{0,\dots,p\}$ is mapped to 2 and the identity morphism Id_2 of 2.
(3) $F(\{0,\dots,p\}) = 3$.
(4) Any morphism from $F^{-1}(2)$ to $F^{-1}(1)$ is mapped to u.
(5) Any morphism from $F^{-1}(2)$ to $F^{-1}(3)$ is mapped to v.

The functor F gives rise to the adjunction between diagram categories:

$$\boxed{F_* : \mathcal{C}^{\partial\Delta(\{0,\dots,p+1\})\setminus\{\{0,\dots,p+1\}\}} \leftrightarrows \mathcal{C}^{\mathcal{D}} : F^*}$$

where $F^*(X) = X \circ F$. It is easily seen that its left adjoint F_* (i.e., the left Kan extension) sends a diagram X of $\mathcal{C}^{\partial\Delta(\{0,\dots,p+1\})}$ to the diagram:



The functor $\mathcal{D} \longrightarrow *$ gives rise to the adjunction

$$\boxed{\varinjlim_{\mathcal{D}} : \mathcal{C}^{\mathcal{D}} \leftrightarrows \ C : \mathrm{Diag}_{\mathcal{D}}}$$

where $\mathrm{Diag}_{\mathcal{D}}$ is the diagonal functor. By composition of the two adjunctions, one obtains the isomorphism

$$\varinjlim_{\substack{S\subset\{0,\dots,p+1\}\\ S\neq\{0,\dots,p+1\}}} X \cong \left(\varinjlim_{\substack{S\subset\{0,\dots,p+1\}\\ S\neq\{0,\dots,p\}\\ p+1\in S}} X(S)\right) \sqcup \left(\varinjlim_{\substack{S\subset\{0,\dots,p\}\\ S\neq\{0,\dots,p\}}} X(S)\right) X(\{0,\dots,p\}).$$

This completes the induction. □

Appendix C. Mixed transfinite composition of pushouts and cofibrations

This is a reminder of results of [Gau05d].

Proposition C.1. *Let $\mathcal{M}$ be a model category. Let λ be an ordinal. Let $(f_\mu : A_\mu \longrightarrow B_\mu)_{\mu<\lambda}$ be a λ-sequence of morphisms of $\mathcal{M}$. Let us suppose that for any $\mu < \lambda$, the diagram of objects of $\mathcal{M}$*

$$\begin{array}{ccc} A_\mu & \longrightarrow & A_{\mu+1} \\ {\scriptstyle f_\mu}\downarrow & & \downarrow \\ B_\mu & \hookrightarrow & B_{\mu+1} \end{array}$$

is either a pushout diagram, or $A_\mu \to A_{\mu+1}$ is an isomorphism and such that for any $\mu < \lambda$, $B_\mu \longrightarrow B_{\mu+1}$ is a cofibration. Then: if $f_0 : A_0 \longrightarrow B_0$ is a cofibration, then $f_\lambda : A_\lambda \longrightarrow B_\lambda$ is a cofibration as well, where of course $A_\lambda := \varinjlim A_\mu$ and $B_\lambda := \varinjlim B_\mu$.

Proof. It is clear that if $f_\mu : A_\mu \longrightarrow B_\mu$ is a cofibration, then $f_{\mu+1} : A_{\mu+1} \longrightarrow B_{\mu+1}$ is a cofibration as well. It then suffices to prove that if $\nu \leqslant \lambda$ is a limit ordinal such that $f_\mu : A_\mu \longrightarrow B_\mu$ is a cofibration for any $\mu < \nu$, then $f_\nu : A_\nu \longrightarrow B_\nu$ is a cofibration as well. Consider a commutative diagram

$$\begin{array}{ccc} A_\nu & \longrightarrow & C \\ {\scriptstyle f_\nu}\downarrow & \overset{k}{\nearrow} & \downarrow \\ B_\nu & \longrightarrow & D \end{array}$$

where $C \longrightarrow D$ is a trivial fibration of $\mathcal{M}$. Then one has to find $k : B_\nu \longrightarrow C$ making both triangles commutative. Recall that by hypothesis, $f_\nu = \varinjlim_{\mu<\nu} f_\mu$. Since f_0 is a cofibration, there exists a map k_0 making both triangles of the diagram

$$\begin{array}{ccc} A_0 & \longrightarrow & C \\ {\scriptstyle f_0}\downarrow & \overset{k_0}{\nearrow} & \downarrow \\ B_0 & \longrightarrow & D \end{array}$$

commutative. Let us suppose k_μ constructed. There are two cases. Either the diagram

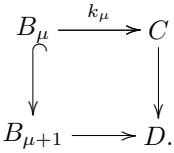
$$\begin{array}{ccc} A_\mu & \longrightarrow & A_{\mu+1} \\ {\scriptstyle f_\mu}\downarrow & & \downarrow \\ B_\mu & \hookrightarrow & B_{\mu+1} \end{array}$$

is a pushout, and one can construct a morphism $k_{\mu+1}$ making both triangles of the diagram

$$\begin{array}{ccc} A_{\mu+1} & \longrightarrow & C \\ {\scriptstyle f_{\mu+1}}\downarrow & \overset{k_{\mu+1}}{\nearrow} & \downarrow \\ B_{\mu+1} & \longrightarrow & D \end{array}$$

commutative and such that the composite $B_\mu \longrightarrow B_{\mu+1} \longrightarrow C$ is equal to k_μ by using the universal property satisfied by the pushout. Or the morphism $A_\mu \to A_{\mu+1}$

is an isomorphism. In that latter case, consider the commutative diagram



Since the morphism $B_\mu \longrightarrow B_{\mu+1}$ is a cofibration, there exists $k_{\mu+1} : B_{\mu+1} \longrightarrow C$ making the two triangles of the latter diagram commutative. So, once again, the composite $B_\mu \longrightarrow B_{\mu+1} \longrightarrow C$ is equal to k_μ.

The map $k := \varinjlim_{\mu<\nu} k_\mu$ is a solution. □

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Preuves Programmes et Systèmes, Université Paris 7–Denis Diderot, Case 7014, 2 Place Jussieu, 75251 PARIS Cedex 05, France
gaucher@pps.jussieu.fr http://www.pps.jussieu.fr/~gaucher/

This paper is available via http://nyjm.albany.edu/j/2006/12-5.html.