New York Journal of Mathematics

New York J. Math. **12** (2006) 269–273.

Real structure in unital separable simple C^*-algebras with tracial rank zero and with a unique tracial state

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Abstract. Let A be a simple unital C^*-algebra with tracial rank zero and with a unique tracial state and let Φ be an involutory $*$-antiautomorphism of A. It is shown that the associated real algebra $A_\Phi = \{a \in A : \Phi(a) = a^*\}$ also has tracial rank zero.

Let A be a unital simple separable C^*-algebra with tracial rank zero and suppose that A has a unique tracial state. If Φ is an involutory $*$-antiautomorphism of A, then it is clear that the associated real algebra $A_\Phi = \{a \in A : \Phi(a) = a^*\}$ is unital and simple with a unique tracial state, but it is not clear that it has tracial rank zero, even when A is approximately finite-dimensional.

The purpose of the present note is to show that techniques recently developed by Phillips [14] and Osaka and Phillips [12], [13] can be used to show that A_Φ does have tracial rank zero. This raises the possibility of classifying all real structures in the algebras under consideration by developing a real analogue of Lin's classification [10] of C^*-algebras of tracial rank zero. Previously all classifications of real structures in non-type I simple C^*-algebras, such as [2], [3], [15] for AF algebras and [5], [16] for irrational rotation algebras, have assumed very specific forms for the real algebras.

The key step in showing that A_Φ has tracial rank zero is to show that Φ has the tracial Rokhlin property, defined below, as introduced in Definition 1.1 of [14].

Definition 1. Let A be a stably finite simple unital C^*-algebra and let Φ be an involutory $*$-antiautomorphism of A. Then Φ has the *tracial Rokhlin property* if for every finite set $F \subset A$, every $\epsilon > 0$, every $N \in \mathbb{N}$ and every nonzero positive element $x \in A$ there are mutually orthogonal projections $e_0, e_1 \in A$ such that:

(1) $\|\Phi(e_0) - e_1\| < \epsilon$.
(2) $\|e_j a - a e_j\| < \epsilon$ for $0 \leq j \leq 1$ and all $a \in F$.
(3) The projection $1 - e_0 - e_1$ is Murray–von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x.
(4) For every $0 \leq j \leq 1$ there are N mutually orthogonal projections

$$f_1, \ldots, f_N \leq e_j,$$

Received September 8, 2005.

Mathematics Subject Classification. 46L35,46L40,46L05.

Key words and phrases. Real C^*-algebras, tracial rank, tracial Rokhlin property, involutory antiautomorphism.

ISSN 1076-9803/06

each of which is Murray–von Neumann equivalent to the projection $1-e_0-e_1$.

Remark 2. As in Lemma 1.4 of [12], with ϵ chosen sufficiently small to ensure (4), if A has real rank zero and the order on projections is determined by traces then conditions (3) and (4) in Definition 1 can be replaced by:

(3′) $\tau(1-e_0-e_1)<\epsilon$ for all tracial states τ.

Following Theorem 2.14 of [13], a further reformulation of the definition can be given in terms of the L^2-seminorm associated with a tracial state τ, defined by $\|a\|_{2,\tau}=\tau(a^*a)^{1/2}$.

Lemma 3. *Let A be a simple separable unital C^*-algebra with tracial rank zero, let Φ be an involutory $*$-antiautomorphism of A and let $T(A)$ be the tracial state space of A. Then Φ has the tracial Rokhlin property if and only if for every $\epsilon>0$ and every finite subset S of A there exists a projection $e\in A$ such that:*

(1) $\|\Phi(e)+e-1\|_{2,\tau}<\epsilon$ *for all* $\tau\in T(A)$.
(2) $\|[e,a]\|_{2,\tau}<\epsilon$ *for all* $a\in S$ *and all* $\tau\in T(A)$.

Proof. The proof directly follows that of Theorem 2.14 in [13], although the present situation is considerably simpler. As noted there, A satisfies the conditions of Remark 2. Thus, if Φ satisfies the tracial Rokhlin property, there exist e_0,e_1 with $\|\Phi(e_0)-e_1\|<\frac{1}{2}\epsilon$, $\tau(1-e_0-e_1)<\frac{1}{4}\epsilon^2$ and $\|[e_0,a]\|<\epsilon$ for each $a\in S$. For $\tau\in T(A)$ and $a\in S$ it follows that $\|[e_0,a]\|_{2,\tau}<\epsilon$ and

$$\begin{aligned}\|\Phi(e_0)+e_0-1\|_{2,\tau}&\le\|\Phi(e_0)-e_1\|_{2,\tau}+\|1-e_0-e_1\|_{2,\tau}\\&<\frac{1}{2}\epsilon+\tau(1-e_0-e_1)^{1/2}<\epsilon.\end{aligned}$$

The converse holds by the argument in Theorem 2.14 in [13], applied to the real-linear automorphism $\alpha=\Phi\circ *$ (and $n=1$): the Lemma 2.13 used in the proof holds equally well for a real linear action. □

The following result now follows as in Theorem 2.17 of [13] , using a property of involutory $*$-antiautomorphisms of continuous von Neumann algebras from [1].

Theorem 4. *Let A be a simple separable unital C^*-algebra with tracial rank zero and with a unique tracial state τ. Then any involutory $*$-antiautomorphism Φ of A has the tracial Rokhlin property.*

Proof. The conditions of Lemma 3 will be demonstrated, so let $\epsilon>0$ and let S be a finite subset of A with $\|a\|\le 1$ for each $a\in S$. If τ is the unique tracial state, let $N=\pi_\tau(A)''$ and, for $\omega\in\beta\mathbb{N}\backslash\mathbb{N}$, let N_ω be the central sequence algebra and let Φ_ω be the involutory antiautomorphism of N_ω arising from Φ. N_ω is a continuous von Neumann algebra (being a type II_1 factor, as observed in the proof of Theorem 2.17 of [13]) so, by Lemme 1.8 of [1], there exists a 2×2 set of matrix units $\{e_{i,j}\}_{1\le i,j\le 2}$ with $\Phi_\omega(e_{i,j})=e_{j,i}$. Then $f=\frac{1}{2}(1+ie_{1,2}-ie_{2,1})$ is a projection with $\Phi_\omega(f)=1-f$. As in the proof of Theorem 2.17 of [13], represent f by a sequence $(f_\ell)_{\ell\in\mathbb{N}}$ in $\ell^\infty(N)$ such that each f_ℓ is a projection and let U be a neighbourhood of ω in $\beta\mathbb{N}$ such that $\ell\in U$ implies $\|[a,f_\ell]\|_{2,\tau}<\frac{1}{3}\epsilon$ for $a\in S$. Let $\ell_0\in\mathbb{N}$ satisfy $\ell_0\in U$ and $\|\overline{\Phi}(f_{\ell_0})-(1-f_{\ell_0})\|_{2,\tau}<\frac{1}{3}\epsilon$, where $\overline{\Phi}$ is the extension of Φ to N. By Lemma 2.15

of [13] there exists a projection $e \in A$ with $\|e - f_{\ell_0}\|_{2,\tau} < \frac{1}{3}\epsilon$ and therefore

$$\begin{aligned}\|e + \Phi(e) - 1\|_{2,\tau} &\leq \|e - f_{\ell_0}\|_{2,\tau} + \|f_{\ell_0} - \overline{\Phi}(1 - f_{\ell_0})\|_{2,\tau} \\ &\quad + \|\overline{\Phi}(1 - f_{\ell_0}) - \Phi(1 - e)\|_{2,\tau} \\ &< \frac{1}{3}\epsilon + \frac{1}{3}\epsilon + \frac{1}{3}\epsilon = \epsilon.\end{aligned}$$

Also

$$\|[e, a]\|_{2,\tau} \leq 2\|e - f_{\ell_0}\|_{2,\tau}\|a\| + \|[f_{\ell_0}, a]\|_{2,\tau} < \frac{2}{3}\epsilon + \frac{1}{3}\epsilon = \epsilon. \qquad \square$$

Theorem 4 will be applied as in Theorem 2.7 of [14], but this invokes a result of Jeong and Osaka from [6] which in turn invokes a result of Kishimoto [7] on outer automorphisms. An analogue of Kishimoto's result for antiautomorphisms has been obtained in [4], but this result is not quite strong enough to establish the analogue of Jeong and Osaka's result. However Tomohiro Hayashi has informed the author of the following strengthening of his result from [4] which serves the required purpose.

Theorem 5 (Hayashi)**.** *Let A be a non-type I separable simple unital C^*-algebra and let α be an antiautomorphism on A. Then, for any hereditary C^*-subalgebra $B \subset A$ and for any $a \in A$, we have*

$$\inf\left\{||xa\alpha(x)|| \ : \ x \in B^+, \ ||x|| = 1\right\} = 0.$$

Proof. For the proof, it is enough to show that for any unitaries $u_1, \cdots, u_n \in A$, any hereditary C^*-subalgebra $B \subset A$ and any positive number $\epsilon > 0$, we can find an element $x \in B^+$ such that $||x|| = 1$ and $||xu_i\alpha(x)|| < \epsilon$ $(i = 1, \cdots, n)$. (The element a can be expressed as a linear combination of unitaries.) By Theorem 2.1 of [4] we can find an element $x_1 \in B^+$ such that $||x_1|| = 1$ and $||x_1u_1\alpha(x_1)|| < \epsilon$ since $x \mapsto u_1\alpha(x)u_1^*$ is an antiautomorphism. Moreover, replacing x_1 by a suitable function $f(x_1)$ if necessary, we may assume that there exists a positive element $c_1 \in B$ such that $||c_1|| = 1$ and $c_1x_1 = x_1c_1 = c_1$. Then applying Theorem 2.1 of [4] again, we can find an element $x_2 \in \overline{(c_1Ac_1)}^{||\cdot||}(\subset B)$ such that $||x_2|| = 1$ and $||x_2u_2\alpha(x_2)|| < \epsilon$. Here we remark that

$$||x_2u_1\alpha(x_2)|| = ||x_2x_1u_1\alpha(x_2x_1)|| \leq ||x_1u_1\alpha(x_1)|| < \epsilon$$

because of the choice of c_1. Therefore, by induction we get the desired element $x = x_n$. $\square$

The relevant consequence of this result is the following simple variant of Theorem 4.2 of [6].

Lemma 6. *Let A be a simple unital C^*-algebra in which every nonzero hereditary C^*-subalgebra has a nonzero projection, let Φ be an involutory $*$-antiautomorphism of A and let $\alpha = \Phi \circ *$. Then any nonzero hereditary C^*-subalgebra of the crossed product $A \times_\alpha \mathbb{Z}_2$ contains a nonzero projection which is equivalent to a projection in A.*

Proof. As in the proof of Theorem 4.2 of [6], let a be a positive element of norm 1 in $A \times_\alpha \mathbb{Z}_2$ and write $a = a_0 + a_1\delta_1$ where δ_1 is the unitary implementing α. Note that Theorem 5 enables Lemma 3.2 of [7] to be applied, given $\epsilon > 0$, to produce a positive element $x \in A$ of norm 1 with $\|xa_0x\| > (1 - \epsilon)||a_0||$ and $||xa_1\alpha(x)|| < \epsilon/4$

and hence with $||xa_1\delta_1 x|| < \epsilon/4$. The proof of Theorem 4.2 of [6] then applies directly (with $N = \{1\}$ and $a = b^*b$) to produce the required projection. □

The required analogue of Theorem 2.7 of [14] can now be obtained. The criterion used here for a simple real C^*-algebra to have tracial rank zero is the real analogue of Proposition 2.1 of [14], as follows:

Definition 7. A simple separable unital real C^*-algebra is said to have *tracial topological rank zero* if the following holds. For every finite subset S of A, every $\epsilon > 0$, every nonzero positive $x \in A$ and every $N \in \mathbb{N}$, there exists a projection $p \in A$ and a finite-dimensional unital subalgebra E of pAp such that:

(1) $\|pa - ap\| < \epsilon$ for all $a \in S$.
(2) For every $a \in S$ there exists $b \in E$ such that $\|pap - b\| < \epsilon$.
(3) $1 - p$ is Murray–von Neumann equivalent to a projection in $\overline{xAx}$.
(4) There are N mutually orthogonal projections in pAp, each Murray–von Neumann equivalent to $1 - p$.

Remark 8. As in Proposition 2.1 of [14], it can be shown that if in addition $a_0 \in A$ is a given nonzero element, then E and p can be chosen so that $||pa_0p|| < ||a_0|| - \epsilon$.

Theorem 9. *Let A be a simple unital C^*-algebra with tracial rank zero and let Φ be an involutory $*$-antiautomorphism of A with the tracial Rokhlin property. Then the associated real algebra $A_\Phi = \{a \in A : \Phi(a) = a^*\}$ has tracial rank zero.*

Proof. Recall firstly that, with $\alpha = \Phi \circ *$, $A \times_\alpha \mathbb{Z}_2$ is isomorphic to the algebra $M_2(A_\Phi)$ of 2×2 matrices over A_Φ, with the element $a + ib$ of $A = A_\Phi + iA_\Phi$ corresponding to the element $a(e_{11}+e_{22})+b(e_{12}-e_{21})$ of $M_2(A_\Phi)$ and the canonical unitary to $e_{12} + e_{21}$. If it can be shown that $A \times_\alpha \mathbb{Z}_2$ has tracial rank zero then, as in Lemma 3.6.5 of [11], it follows that $e_{11}M_2(A_\Phi)e_{11}$ has tracial rank zero, giving the required result.

A considerably simpler version of the argument in Theorem 2.7 of [14], applied to $\alpha = \Phi \circ *$, shows that $A \times_\alpha \mathbb{Z}_2$ does indeed have tracial rank zero. The only result quoted in that proof which does not immediately carry through to the current context is Theorem 4.2 of [6], which is replaced by Lemma 6 above. □

Corollary 10. *Let A be a simple unital C^*-algebra with tracial rank zero and with a unique tracial state and let Φ be an involutory $*$-antiautomorphism of A. Then the associated real algebra $A_\Phi = \{a \in A : \Phi(a) = a^*\}$ has tracial rank zero.*

Proof. This is immediate from Theorems 4 and 9. □

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This paper is available via http://nyjm.albany.edu/j/2006/12-17.html.