NYJM Monographs

Volume 1 2003

A Pext primer: Pure extensions and $\lim^1$ for infinite abelian groups

Claude L. Schochet

ABSTRACT. The abelian group $\mathrm{Pext}^1_{\mathbb{Z}}(G, H)$ of pure extensions has recently attracted the interest of workers in non-commutative topology, especially those using KK-theory, since under minimal hypotheses the closure of zero in the Kasparov group $KK_*(A, B)$ (for separable C^*-algebras A and B) is isomorphic to the group

$$\mathrm{Pext}^1_{\mathbb{Z}}(K_*(A), K_*(B)).$$

As $K_*(A)$ and $K_*(B)$ can take values in all countable abelian groups, assuming that G and H are countable is natural.

In this mostly expository work we survey the known (and not so well-known) properties of Pext and its relationship to $\lim^1$ and develop some new results on their computation.

Mathematics Subject Classification. Primary: 20K35, 19K35, 46L80. Secondary: 18E25, 18G15, 20K40, 20K45, 47L80, 55U99.

Key words and phrases. pure extensions, $\mathrm{Pext}^1_{\mathbb{Z}}(G, H)$, $\lim^1$, Jensen's Theorem, infinite abelian groups, quasidiagonality, phantom maps.

Contents

1. Introduction

The abelian group $\mathrm{Pext}^1_{\mathbb{Z}}(G,H)$ has recently appeared in non-commutative topology, specifically in the Kasparov KK-theory, since under minimal hypotheses the closure of zero in the Kasparov group

$$KK_*(A,B)$$

(for separable C^*-algebras A and B) is isomorphic to the group

$$\mathrm{Pext}^1_{\mathbb{Z}}(K_*(A),K_*(B))$$

and this subgroup of $KK_*(A,B)$ is the subgroup of quasidiagonal elements (see §13 for details.) The groups $K_*(A)$ and $K_*(B)$ range over all countable abelian groups, so assuming that G and H are countable is natural.

Coming from outside the world of infinite abelian groups, the functional analyst tends to ask elementary questions:

1. What are the typical examples of $\mathrm{Pext}^1_{\mathbb{Z}}(G,H)$?
2. When does the group vanish?
3. Which nonzero values does it take?
4. How does one compute the group?
5. Does this group appear elsewhere in mathematics as an obstruction group?

The present work is designed to answer these questions.

The most complete source for the theory of infinite abelian groups is the two-volume work by L. Fuchs [24], [25] which unfortunately is out of print. We have thus included more elementary material in this paper to compensate for the relative unavailability of these books. We also strongly recommend the little red book of Kaplansky [40] which has been so influential in this area.

C. U. Jensen's Theorem 6.1 is central to this paper. It asserts that if G is an abelian group written as the union of an increasing sequence of finitely generated subgroups G_i then there is a natural isomorphism

$$\varprojlim{}^1 \mathrm{Hom}_{\mathbb{Z}}(G_i,H) \;\cong\; \mathrm{Pext}^1_{\mathbb{Z}}(G,H).$$

This connects up Pext and the theory of infinite abelian groups with $\varprojlim^1$, a classic (and difficult) functor from homological algebra and algebraic topology.

Typically, $\varprojlim^1$ in topology detects phantom behavior. It notices maps $X \to Y$ (where X and Y are locally finite CW-complexes or, better, spectra) which are not null-homotopic but whose restriction to finite CW-complexes (or spectra) are null-homotopic. See §12 for details.

As mentioned before, Pext is also related to quasidiagonality for C^*-algebras (see §13 for details). There are in fact certain similarities between the two phenomena which we hope to examine in the future.

When writing a paper that is mostly expository, one must make a pedagogical decision: what level of generality is appropriate to the audience. We have written this paper with the specialists in operator algebras in mind as potential consumers. So we have stayed resolutely within the context of the category of abelian groups, even though we are well aware that much of this paper generalizes to categories of R-modules and further.

This decision may be somewhat short-sighted. For instance, one may well ask for equivariant versions of this work. If G is a locally compact group then the groups $KK_*^G(A, B)$ are modules over the ring $KK_*^G(\mathbb{C}, \mathbb{C})$ which (for G compact) is the complex representation ring $R(G)$. One must draw the line somewhere, though, so equivariant Pext will have to wait.

We have attempted to attribute each result in this work to the appropriate source. However, we are not expert in the area of pure homological algebra. It may be that all of the results in this work are known- possibly as trivial corollaries of major results in much more rarified contexts. We apologize in advance for incorrect attributions and we ask each reader's help in correcting these (and all other) errors. Errata will be posted at http://nyjm.albany.edu:8000/m/2003/1-info.htm.

We have been fortunate in having R. G. Bruner, John Irwin, and C. McGibbon as colleagues and we acknowledge with pleasure their continuing assistance. Likewise we thank Baruch Solel and the faculty of the Technion for their hospitality during our sabbatical visit. Finally, we gratefully acknowledge the assistance of H. P. Goeters.

All groups which appear in this paper are abelian unless specified otherwise.

2. First facts on Hom and Ext

We shall have occasion to use a number of terms from the theory of infinite abelian groups that were not familiar to the author and perhaps are not familiar to the reader. We begin with enough definitions to get started. We will place the rest as footnotes in the paper so as to facilitate easy retrieval.

Let $\mathbb{Z}_p$ denote the integers localized at the prime p (that is, all primes except p have been inverted) and let $\mathbb{Z}(p^\infty)$ denote the p-group $P/\mathbb{Z}$, where P is the group of rational numbers with denominator some power of the prime p. Let $\widehat{\mathbb{Z}}_p = \varprojlim \mathbb{Z}/p^n$ denote the p-adic integers. We note that

$$\widehat{\mathbb{Z}}_p \;\cong\; \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}(p^\infty), \mathbb{Z}(p^\infty)).$$

A group is *reduced* if it has no divisible subgroups other than 0. Let

$$nG = \{ng : g \in G\} \subseteq G.$$

A subgroup S of a group G is *pure* if

$$nS = S \cap nG$$

for all natural numbers n.

A group is *algebraically compact* ([24], §38) if it is a direct summand in every group that contains it as a pure subgroup. Equivalently ([24], §38.1) it is algebraically compact if and only if it is algebraically a direct summand in a group which admits a compact topology. (If the group is abelian then the compact group may be taken to be abelian). Examples include compact groups, divisible groups, and bounded groups. A group is algebraically compact if and only if it is of the form

$$D \oplus \Pi_p D^p$$

where D is divisible and for each prime p D^p is the completion in the p-adic topology of the direct sum of cyclic p-groups and groups of p-adic integers. If $\{G_i\}$ is any sequence of abelian groups, then the group $\Pi_i G_i / \oplus G_i$ is algebraically compact, by [37], and its structure is determined in [28].

We now give some basic and not so basic observations on the abelian groups $\mathrm{Hom}_{\mathbb{Z}}(G, H)$ and $\mathrm{Ext}^1_{\mathbb{Z}}(G, H)$.

Proposition 2.1. *Suppose that G and H are abelian groups.*

1. $\mathrm{Hom}_{\mathbb{Z}}(G, H)$ *is a functor to abelian groups, contravariant in G and covariant in H.*
2. Hom *is additive in each variable: there are natural isomorphisms*

$$\mathrm{Hom}_{\mathbb{Z}}(\oplus_i G_i, H) \;\cong\; \Pi_i \mathrm{Hom}_{\mathbb{Z}}(G_i, H)$$

and

$$\mathrm{Hom}_{\mathbb{Z}}(G, \Pi_i H_i) \;\cong\; \Pi_i \mathrm{Hom}_{\mathbb{Z}}(G, H_i).$$

3. *If G is divisible or H is torsionfree, then* $\mathrm{Hom}_{\mathbb{Z}}(G, H)$ *is torsionfree.*
4. *If G or H is both divisible and torsionfree, then* $\mathrm{Hom}_{\mathbb{Z}}(G, H)$ *is divisible and torsionfree.*
5. *If G and H are finitely generated, then* $\mathrm{Hom}_{\mathbb{Z}}(G, H)$ *is finitely generated.*
6. *If G is a torsion group, then* $\mathrm{Hom}_{\mathbb{Z}}(G, H)$ *is reduced and algebraically compact.*
7. *If H is algebraically compact then so is* $\mathrm{Hom}_{\mathbb{Z}}(G, H)$.
8. *If H is compact then $G \mapsto$* $\mathrm{Hom}_{\mathbb{Z}}(G, H)$ *is a contravariant functor from groups and homomorphisms to compact groups and continuous homomorphisms.*

Proof. Most of these results are elementary and are found in ([24], §§43 - 46). Part 6) is a theorem of Fuchs and Harrison (cf. ([24], §46.1). For Parts 7) and 8) we note that if H is compact then so is H^G with the product topology, and $\mathrm{Hom}_{\mathbb{Z}}(G, H)$ is a closed subset of H^G, hence compact. If H is algebraically compact, with $H \oplus H'$ compact, then $\mathrm{Hom}_{\mathbb{Z}}(G, H)$ is algebraically a direct summand of the compact group $\mathrm{Hom}_{\mathbb{Z}}(G, H \oplus H')$, hence algebraically compact. The rest is immediate. □

If G is a direct sum of cyclic groups C_i (for instance, if it is finitely generated), then of course the computation of $\mathrm{Hom}_{\mathbb{Z}}(G, H)$ is elementary:

$$\mathrm{Hom}_{\mathbb{Z}}(G, H) \cong \mathrm{Hom}_{\mathbb{Z}}(\oplus_i C_i, H) \cong \Pi_i \mathrm{Hom}_{\mathbb{Z}}(C_i, H)$$

and one then observes that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, H) \cong H$ and that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, H) \cong H[n]$.[1]

Definition 2.2. The *support* of a torsionfree group G, denoted $\mathrm{Supp}(G)$, is defined by

$$\mathrm{Supp}(G) = \{\text{primes } p \colon pG \neq G\}.$$

Thus $p \in \mathrm{Supp}(G)$ if and only if $p : G \to G$ is *not* an automorphism. We define $\mathbb{Z}_G$ to be the localization of $\mathbb{Z}$ obtained by inverting all primes $p \notin$

[1]The group $A[n]$ is defined by

$$A[n] = \{a \in A \colon na = 0\}.$$

Supp(G). (This is denoted $R(G)$ in the papers of Warfield and elsewhere.) Thus $\mathbb{Z}_G$ is the greatest subring of $\mathbb{Q}$ such that G is a $\mathbb{Z}_G$-module.

Warfield analyzes Hom as follows.

Proposition 2.3 (Warfield [62]). *Suppose that G is a torsionfree abelian group of finite rank*[2] *and H is a divisible, countable torsion group. Then:*

1. *The group $\mathrm{Hom}_{\mathbb{Z}}(G,H)$ is divisible, hence the sum of n_o copies of $\mathbb{Q}$ and for each p, n_p copies of $\mathbb{Z}(p^\infty)$.*
2. *$\mathrm{Hom}_{\mathbb{Z}}(G,H)$ is a torsion group if and only if $G \otimes \mathbb{Z}_H$ is a free $\mathbb{Z}_H$-module. Otherwise, $n_o = \mathfrak{c}$.*
3. *The p-torsion subgroup of the group $\mathrm{Hom}_{\mathbb{Z}}(G,H)$ is isomorphic to the direct sum of $r_p(G)$ copies*[3] *of the group H_p.*[4]

We turn now to elementary properties of Ext.

Proposition 2.4. *Let G and H be abelian groups.*

1. *If*
$$0 \to H' \to H \to H'' \to 0$$
is a short exact sequence of abelian groups then there is an associated natural six-term exact sequence
$$0 \to \mathrm{Hom}_{\mathbb{Z}}(G,H') \to \mathrm{Hom}_{\mathbb{Z}}(G,H) \to \mathrm{Hom}_{\mathbb{Z}}(G,H'') \to$$
$$\to \mathrm{Ext}^1_{\mathbb{Z}}(G,H') \to \mathrm{Ext}^1_{\mathbb{Z}}(G,H) \to \mathrm{Ext}^1_{\mathbb{Z}}(G,H'') \to 0$$
and similarly in the other variable.
2. *There are natural isomorphisms*
$$\mathrm{Ext}^1_{\mathbb{Z}}(\oplus G_i, H) \cong \Pi_i \mathrm{Ext}^1_{\mathbb{Z}}(G_i,H)$$
and
$$\mathrm{Ext}^1_{\mathbb{Z}}(G, \Pi_i H_i) \cong \Pi_i \mathrm{Ext}^1_{\mathbb{Z}}(G,H_i).$$

---

[2] If G is torsionfree then the *rank* of G is defined by
$$\mathrm{rank}(G) = \dim_{\mathbb{Q}}(G \otimes \mathbb{Q}).$$
Note that $G \cong G \otimes 1 \subset G \otimes \mathbb{Q}$ and hence G is isomorphic to a subgroup of a $\mathbb{Q}$-vector space of dimension rank(G). Thus every torsionfree group may be realized as a subgroup of a $\mathbb{Q}$-vector space. There is no general classification of torsionfree groups.

[3] For any torsion group G, let
$$r_p(G) = \dim_{\mathbb{Z}/p}(G/pG).$$
For example, if G is divisible and torsion, then $pG = G$, and hence $r_p(G) = 0$.

[4] Here, for any group G and prime p, its localization G_p is defined to be
$$G_p = G \otimes \mathbb{Z}_p.$$

3. *If G and H are finitely generated then* $\operatorname{Ext}^1_{\mathbb{Z}}(G,H)$ *is finite.*
4. *If G is torsionfree then* $\operatorname{Ext}^1_{\mathbb{Z}}(G,H)$ *is divisible.*
5. *If H is algebraically compact then* $\operatorname{Ext}^1_{\mathbb{Z}}(G,H)$ *is reduced and algebraically compact.*
6. *If H is compact then the group* $\operatorname{Ext}^1_{\mathbb{Z}}(G,H)$ *is reduced and compact, and* $\operatorname{Ext}^1_{\mathbb{Z}}(-,H)$ *is a functor from groups and homomorphisms to reduced compact groups and continuous homomorphisms.*
7. *If G and H are torsionfree and*

$$\operatorname{Supp}(G)\cap\operatorname{Supp}(H)=\phi$$

then $\operatorname{Ext}^1_{\mathbb{Z}}(G,H)$ *is torsionfree.*[5] *If* $\operatorname{Hom}_{\mathbb{Z}}(G,H)$ *is divisible then the converse is true.*

8. *If G is a torsion group then* $\operatorname{Ext}^1_{\mathbb{Z}}(G,H)$ *is reduced.*
9. *The group* $\operatorname{Ext}^1_{\mathbb{Z}}(G,H)=0$ *for all G if and only if H is divisible.*
10. *The group* $\operatorname{Ext}^1_{\mathbb{Z}}(G,H)=0$ *for all H if and only if G is free abelian.*

Proof. Parts 1) and 2) are due to Cartan-Eilenberg [8]. Part 3) comes down to the two cases

$$\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p,\mathbb{Z}/q)=\mathbb{Z}/r$$

where r is the gcd of p and q, and

$$\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p,\mathbb{Z})=\mathbb{Z}/p.$$

Part 4) is established in ([24], p. 223) and Part 5) in ([24], p. 225).

To prove the compactness conclusions of Part 6), argue as follows. Let

$$0\to F\xrightarrow{\iota}F'\to G\to 0$$

be a free resolution of the group G. Then

$$\operatorname{Hom}_{\mathbb{Z}}(F',H)\to\operatorname{Hom}_{\mathbb{Z}}(F,H)\to\operatorname{Ext}^1_{\mathbb{Z}}(G,H)\to 0$$

is exact by the Hom-Ext exact sequence. The groups $\operatorname{Hom}_{\mathbb{Z}}(F',H)$ and $\operatorname{Hom}_{\mathbb{Z}}(F,H)$ are compact since H is compact, and the map

$$\iota^*\colon\operatorname{Hom}_{\mathbb{Z}}(F',H)\to\operatorname{Hom}_{\mathbb{Z}}(F,H)$$

[5] The group $\operatorname{Ext}^1_{\mathbb{Z}}(G,H)$ may well have torsion in general even if G and H are torsionfree. Countable examples are not hard to find. For instance,

$$\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_p,\mathbb{Z})\;\cong\;\mathbb{Q}^{\aleph_o}\oplus\mathbb{Z}(p^\infty)$$

which has p^r-torsion for all r. Complicated necessary and sufficient conditions have been found to ensure that $\operatorname{Ext}^1_{\mathbb{Z}}(G,H)$ is torsionfree (cf. [26]). The conditions stated in 7) barely scratch the surface of the literature. We return to this point in Section 9.

is a continuous homomorphism, by Proposition 2.1. Thus the quotient group $\mathrm{Ext}^1_{\mathbb{Z}}(G,H)$ is compact.[6] Further, we see that a homomorphism $G \to G'$ induces a map of resolutions and hence a continuous homomorphism

$$\mathrm{Ext}^1_{\mathbb{Z}}(G',H) \to \mathrm{Ext}^1_{\mathbb{Z}}(G,H)$$

which completes the proof of Part 6). Part 8) is found in ([24], §55.3).

To prove Part 7) we follow [26]. Fix a prime p. Applying $\mathrm{Hom}_{\mathbb{Z}}(G,-)$ to the short exact sequence

$$0 \to H \xrightarrow{p} H \to H/pH \to 0$$

and identifying

$$\mathrm{Hom}_{\mathbb{Z}}(G,H/pH) \cong \mathrm{Hom}_{\mathbb{Z}}(G/pG,H/pH)$$

leads to the surjection

$$\mathrm{Hom}_{\mathbb{Z}}(G/pG,H/pH) \to \mathrm{Ext}^1_{\mathbb{Z}}(G,H)[p] \to 0$$

The support condition implies that $\mathrm{Hom}_{\mathbb{Z}}(G/pG,H/pH) = 0$ and then exactness implies that $\mathrm{Ext}^1_{\mathbb{Z}}(G,H)[p] = 0$. Thus $\mathrm{Ext}^1_{\mathbb{Z}}(G,H)[p] = 0$ for each prime p which implies that $\mathrm{Ext}^1_{\mathbb{Z}}(G,H)$ is torsionfree.

For the converse, note that if $\mathrm{Hom}_{\mathbb{Z}}(G,H)$ is divisible then multiplication by p is an isomorphism, which implies in turn that the natural map

$$\mathrm{Hom}_{\mathbb{Z}}(G/pG,H/pH) \to \mathrm{Ext}^1_{\mathbb{Z}}(G,H)[p]$$

is an isomorphism. So if $\mathrm{Ext}^1_{\mathbb{Z}}(G,H)$ is torsionfree then

$$\mathrm{Hom}_{\mathbb{Z}}(G/pG,H/pH) = 0$$

for each prime p, which is equivalent to the support condition

$$\mathrm{Supp}(G) \cap \mathrm{Supp}(H) = \phi.$$

The final two statements 9) and 10) may be proved directly, but they also are part of the axiomatic description of Ext thought of as the derived functor of Hom, since the divisible groups are the injective groups and the free abelian groups are the projective groups. See MacLane [41] for details. □

[6] Note that $\mathrm{Im}(\iota^*)$ is not necessarily a *closed* subgroup, and hence $\mathrm{Ext}^1_{\mathbb{Z}}(G,H)$ is not necessarily Hausdorff.

3. First facts on $\lim^1$

Next we record some well-known observations about $\varprojlim^1$ for abelian groups.[7] We review an important example, we define and explain the Mittag-Leffler condition, and we examine the behavior of $\varprojlim^1$ with respect to tensor product.

An *inverse sequence* $\{G_i\}$ of abelian groups is a collection of abelian groups indexed by a countable[8] partially ordered set (which we may take to be the positive integers without loss of generality) together with a coherent family of maps $f_{ji}\colon G_j \to G_i$ for $j \geq i$. We let $f_i = f_{i,i-1}$. The functor $\varprojlim^1 G_i$ may be defined categorically as the first derived functor of $\varprojlim$, but for countable index sets the following description, due to Eilenberg [18], is available. Let

$$\Psi : \Pi_i G_i \to \Pi_i G_i$$

be defined by

$$\Psi(g_i) = (g_i - f_{i+1}(g_{i+1})) \tag{3.1}$$

so that $\mathrm{Ker}(\Psi) \cong \varprojlim G_i$. Then $\varprojlim^1 G_i$ is given by

$$\varprojlim{}^1 G_i = \mathrm{Coker}(\Psi). \tag{3.2}$$

Here are some of the resulting elementary properties. For proofs the reader may consult, e.g., [38] (which also deals with the much harder case of general index sets.) Some of these results were first established in the context of derived functors and abelian categories by Cartan-Eilenberg [8] and the rest by Yeh [64], Eilenberg-Moore [18], and Roos [52].

Proposition 3.3. 1. *The functors $\varprojlim$ and $\varprojlim^1$ are covariant functors from the category of inverse sequences of abelian groups to the category of abelian groups.*

2. *The functors $\varprojlim$ and $\varprojlim^1$ are left unchanged by passage to cofinal subsequences.*

---

[7] This is all the generality we need, but we note in passing that one could work in the context of modules over a commutative ring or even more generally in an abelian category.

[8] If one allows index sets of higher cardinality then the entire theory becomes very much more complex. The functor $\varprojlim^1$ is only the first of a sequence of derived functors $\varprojlim^n$ which are nonzero in general. The six term $\varprojlim$-$\varprojlim^1$ sequence of Part 5) of Proposition 3.3 becomes a long exact sequence. There is no explicit description of the functors $\varprojlim^n$ for $n > 1$ analogous to Eilenberg's description for $\varprojlim^1$. We shall have no need for those functors, as in our applications the index sets are in fact countable (corresponding to the fact that we concentrate upon *separable* C^*-algebras.)

3. *The functors* $\varprojlim$ *and* $\varprojlim^1$ *respect finite direct sums.*
4. *If each* $f_i : G_i \to G_{i-1}$ *is an isomorphism then* $\varprojlim G_i \cong G_1$ *and* $\varprojlim^1 G_i = 0$.
5. *If*

$$0 \to \{G_i'\} \to \{G_i\} \to \{G_i''\} \to 0$$

is a short exact sequence of inverse sequences then there is an associated $\varprojlim$ *-*$\varprojlim^1$ *exact sequence*

$$0 \to \varprojlim G_i' \to \varprojlim G_i \to \varprojlim G_i'' \to \varprojlim^1 G_i' \to \varprojlim^1 G_i \to \varprojlim^1 G_i'' \to 0$$

which is natural with respect to morphisms of short exact sequences of inverse sequences.[9]

The following proposition is due to Warfield [62], p. 434.

Proposition 3.4. *For any inverse sequence* $\{G_i\}$, *the group*

$$\varprojlim^1 G_i$$

is a cotorsion[10] *group.*

Proof. Let

$$\Psi : \Pi_i G_i \to \Pi_i G_i$$

be the Eilenberg map. It is easy to see that

$$\oplus_i G_i \subseteq \mathrm{Im}(\Psi)$$

and hence there is an exact sequence

$$\frac{\Pi_i G_i}{\oplus G_i} \longrightarrow \varprojlim^1 G_i \to 0.$$

Now the group $\frac{\Pi_i G_i}{\oplus G_i}$ is algebraically compact for any choice of $\{G_i\}$, by [37], (and see the material at the beginning of Section 2) hence cotorsion, and any quotient of a cotorsion group is again cotorsion. □

We mention in passing one general result on the size of $\varprojlim^1$, a theme to which we will return in connection with Pext.

---

[9]Formally we say that $\varprojlim^1$ is the first derived functor of $\varprojlim$.

[10]A group G is *cotorsion* if and only if $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, G) = 0$. For instance, algebraically compact groups are cotorsion, and any group of the form $G = \mathrm{Ext}^1_{\mathbb{Z}}(H, K)$ is cotorsion. A group is cotorsion if and only if it is the quotient of an algebraically compact group ([24], §54.1). On the other hand, $\mathbb{Z}$ is not cotorsion. See ([24] §§54-58.) Fuchs remarks that the concept of a cotorsion group is due to Harrison [31] and independently by Nunke [49] and Fuchs [23].

Proposition 3.5 (B. Gray [29]). *Suppose given an inverse sequence $\{G_i\}$ with each G_i finite or countable. Then the group*

$$\varprojlim{}^1 G_i$$

either is zero or uncountable.

Example 3.6. The simplest nontrivial example of $\varprojlim{}^1$ arises from the inverse sequence

$$\mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{2} \dots$$

denoted $\{\mathbb{Z}, 2\}$ where each map is just multiplication by 2. (This example arose in the first use of $\varprojlim{}^1$ by Steenrod; see our discussion after 3.12.)

We compute as follows. The collection of short exact sequences

$$0 \to \mathbb{Z} \xrightarrow{2^n} \mathbb{Z} \to \mathbb{Z}/2^n \to 0$$

combine to yield a natural short exact sequence of inverse sequences

$$0 \to \{\mathbb{Z}, 2\} \to \{\mathbb{Z}, 1\} \to \{\mathbb{Z}/2^n, \pi\} \to 0.$$

Take the associated $\varprojlim$-$\varprojlim{}^1$ sequence. Using the fact that $\varprojlim \{\mathbb{Z}, 1\} \cong \mathbb{Z}$, $\varprojlim \{\mathbb{Z}, 2\} = 0$, and $\varprojlim{}^1 \{\mathbb{Z}, 1\} = 0$, we obtain the sequence

$$0 \to \mathbb{Z} \to \varprojlim \mathbb{Z}/2^n \to \varprojlim{}^1 \{\mathbb{Z}, 2\} \to 0$$

and hence

$$\varprojlim{}^1 \{\mathbb{Z}, 2\} \cong \widehat{\mathbb{Z}}_2/\mathbb{Z}.$$

The example may be expanded as follows. Let $\mathbb{Z}[\frac{1}{2}]$ denote the subring of the rational numbers generated by $\mathbb{Z}$ and by $\frac{1}{2}$. Write $\mathbb{Z}[\frac{1}{2}] = \varinjlim G_i$, where $G_i \cong \mathbb{Z}$ and the maps are multiplication by 2. Then there is an obvious isomorphism of inverse sequences

$$\{\mathbb{Z}, 2\} \cong \{\mathrm{Hom}_{\mathbb{Z}}(G_i, \mathbb{Z})\}.$$

We may replace $\mathbb{Z}[\frac{1}{2}]$ by any subring R of the rational numbers which is not divisible [11] and obtain a similar isomorphism.

See the remarks at the end of Section 6 for further development of this example.

[11] To obtain nontrivial examples we must be in a situation where the map $R \to R$ is *not* an isomorphism, by 3.3. For instance, we cannot take $R = \mathbb{Q}$.

The following three examples are taken from McGibbon [44], p. 1238.

Example 3.7. Suppose that $\{G_i\}$ is an inverse sequence of abelian groups, p is a prime, and $G_{ip} \cong G_i \otimes \mathbb{Z}_p$ denotes the localization. Then there is a natural map

$$\delta_* : \varprojlim{}^1 G_i \longrightarrow \prod_p \varprojlim{}^1 G_{ip}.$$

McGibbon points out that this map is seldom an isomorphism. For instance, if

$$G_i = \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Z})$$

with $A = \varinjlim A_i$ and each A_i is finitely generated then this map corresponds via Jensen's isomorphism (6.1) to the natural map

$$\delta_* : \mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) \longrightarrow \prod_p \mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}_p).$$

This map is always surjective. If $\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) \neq 0$ then δ_* always has nontrivial kernel. For instance, in Example 3.6, $\mathrm{Ker}(\delta_*)$ is the countable group $\mathbb{Z}_2/\mathbb{Z}$.

Example 3.8. Suppose that $G_n = m_n\mathbb{Z}$ where m_n is the product of the first n primes. Then

$$\varprojlim{}^1 G_n \cong \mathbb{R} \oplus \mathbb{Q}/\mathbb{Z}$$

and $\delta_* = 0$.

Example 3.9. Suppose that $G_n = n!\mathbb{Z}$. Then $\varprojlim{}^1 G_n \cong \mathbb{R}$ and $\mathrm{Ker}(\delta_*)$ is isomorphic to the sum of uncountably many copies of $\mathbb{Q}$.

We use the following notation. If $\{G_i\}$ is an inverse sequence of abelian groups with structural maps

$$f_{ji} \colon G_j \to G_i$$

for each $j > i$, then define

$$G_{j,i} = \mathrm{Im}(f_{ji} \colon G_j \to G_i)$$

so that for each fixed i there is a sequence of subgroups

$$G_i \supseteq G_{i+1,i} \supseteq G_{i+2,i} \supseteq \cdots .$$

Definition 3.10. The inverse sequence [12] $\{G_i\}$ satisfies the *Mittag-Leffler condition* [30, 2, 3] if for each i there exists some integer $\phi(i) \geq i$ such that

$$G_{j,i} = G_{\phi(i),i}$$

[12] For inverse systems indexed by a directed set the Mittag-Leffler condition is not helpful: cf. [60], page 369.

for all $j \geq \phi(i)$.

For instance, if we are given an inverse system of finitely generated modules over a ring satisfying the descending chain condition, then the Mittag-Leffler condition is satisfied if and only if the original inverse sequence is isomorphic as a pro-object to an inverse sequence whose structural maps are surjective. It is then fairly easy to prove the first part of the following theorem.

Theorem 3.11. 1. *If an inverse sequence* $\{G_i\}$ *of abelian groups satisfies the Mittag-Leffler condition then* $\varprojlim^1 G_i = 0$.

2. *If an inverse sequence* $\{G_i\}$ *of countable abelian groups satisfies*

$$\varprojlim^1 G_i = 0$$

then it satisfies the Mittag-Leffler condition.

Proof. The first statement is a classic result established in [30], Ch. 0, §13. The second statement is due to B. Gray ([29] p. 242). □

Example 3.12 (McGibbon [44])**.** It is easy to see that the inverse sequence

$$\mathbb{Z} \xleftarrow{3} \mathbb{Z} \xleftarrow{5} \mathbb{Z} \xleftarrow{7} \mathbb{Z} \xleftarrow{11} \dots$$

is *not* Mittag-Leffler.

The earliest study of $\varprojlim^1$ was by Steenrod [59] who was interested in the difference between Vietoris (more or less the same as Čech) homology (which did not satisfy the exactness axiom, and hence was not really a homology theory) and what is now called Steenrod homology (which did indeed satisfy the axioms.) Steenrod homology maps onto Čech homology, and the kernel is a suitable $\varprojlim^1$ group which Steenrod computed in the case of a solenoid, demonstrating that the two theories were really different. He also introduced a topology for Steenrod homology and showed that the closure of zero was the $\varprojlim^1$ subgroup. At that time it was not clear that $\varprojlim^1$ was a derived functor (indeed, the word *functor* was not even in use mathematically), which makes Steenrod's contribution all the more impressive.

Eilenberg and Steenrod pursued this theme systematically in their seminal work [19]. They showed that the axioms *were* satisfied by $\check{H}_*(X; G)$ when G is compact or when G is a finite-dimensional vector space over a field. We understand now that the obstruction to exactness (the axiom that failed) is exactly the group $\varprojlim^1 H_*(X_\alpha; G)$.

The work of Eilenberg and Steenrod was followed by results of various people, particularly Yeh [64], Eilenberg and Moore [18], and Roos [52]. J.

Milnor [47], studied the behavior of limits on representable cohomology theories. If h^* is such a theory and if X is an infinite CW-complex with $h^*(X)$ of finite type then for each n there is a Milnor sequence of the form

$$0 \to \varprojlim{}^1 h^{n-1}(X_j) \to h^n(X) \to \varprojlim h^n(X_j) \to 0. \tag{3.13}$$

This is in a sense dual to Steenrod's initial use of $\varprojlim^1$. Steenrod dealt with a compact space written as an inverse limit of finite complexes. Milnor deals with an infinite CW-complex written as the direct limit of finite CW-complexes. Milnor's setting is typical in modern algebraic topology, whereas Steenrod's setting generalizes to C^*-algebras cf. [55].

Milnor [48] pursued Steenrod's original example in his paper "On the Steenrod homology theory" which was first distributed in 1961 and was published more than 30 years later.

Remark 3.14. C. McGibbon has observed that it is possible in general to have an inverse sequence which is *not* Mittag-Leffler and yet for which $\varprojlim^1$ vanishes. Here is his example. Let K be the direct product of countably many copies of $\mathbb{Z}/2$ where each factor has the discrete topology and K has the product topology. Let K_n be the kernel of the projection of K onto the first n factors. These projections are continuous, and the K_n are both open and closed (hence compact) subgroups of K. There are evident inclusion maps $K_n \to K_{n-1}$ and the associated sequence is *not* Mittag-Leffler. Nevertheless, $\varprojlim^1 K_n = 0$ by the following result.

Proposition 3.15. *An inverse sequence* $\{G_i\}$ *satisfies* $\varprojlim^1 G_i = 0$ *if either of the following conditions holds:*

1. $\{G_i\}$ *consists of compact Hausdorff (not necessarily abelian) groups and continuous homomorphisms.*
2. $\{G_i\}$ *consists of finite-dimensional vector spaces over a field and linear maps.*

Proof. This is established by Eilenberg-Steenrod [19] in the abelian setting and by McGibbon [44] in general. □

McGibbon notes that the Hausdorff assumption is essential. For instance, take any example with $\varprojlim^1 G_i \neq 0$ and place the indiscrete topology on each group. Then each G_i is compact (but not Hausdorff), maps are continuous, and obviously $\varprojlim^1 G_i \neq 0$!

Next we consider the relationship between the groups

$$\varprojlim{}^1 G_i \qquad \text{and} \qquad \varprojlim{}^1 (G_i \otimes M).$$

Proposition 3.16. *Let $\{G_i\}$ be an inverse sequence of abelian groups, and let M be an abelian group. Then:*

1. *If M is finitely generated, then*
$$(\varprojlim{}^1 G_i) \otimes M \cong \varprojlim{}^1 (G_i \otimes M).$$
2. *If $\{G_i\}$ is an inverse sequence of countable abelian groups and if*
$$\varprojlim{}^1 G_i = 0$$
then
$$\varprojlim{}^1 (G_i \otimes M) = 0.$$
3. *If $\{G_i\}$ is an inverse sequence of countable abelian groups, M is a countable, faithfully flat*[13] *abelian group, and if*
$$\varprojlim{}^1 (G_i \otimes M) = 0$$
then
$$\varprojlim{}^1 G_i = 0.$$

Proof. To establish Part 1), we note that since $\varprojlim{}^1$ respects finite sums this comes down to checking the case $M = \mathbb{Z}$, which is trivial, and the case $M = \mathbb{Z}/n$, which is very simple. For Part 2), suppose that $\varprojlim{}^1 G_i = 0$. Then the sequence $\{G_i\}$ satisfies the Mittag-Leffler condition, by Theorem 3.11, so that there is some function $\phi(i)$ such that
$$G_{j,i} = G_{\phi(i),i}$$
for all $j \geq \phi(i)$. Fix some index i. Then
$$\begin{aligned} \operatorname{Im}(G_{j,i} \otimes M \to G_i \otimes M) &\cong \frac{G_{j,i} \otimes M}{Tor^{\mathbb{Z}}(G_i/G_{j,i}, M)} \\ &\cong \frac{G_{\phi(i),i} \otimes M}{Tor^{\mathbb{Z}}(G_i/G_{\phi(i),i}, M)} \\ &\cong \operatorname{Im}(G_{\phi(i),i} \otimes M \to G_i \otimes M) \end{aligned}$$
and hence the inverse sequence $\{G_i \otimes M\}$ satisfies the Mittag-Leffler condition, so that $\varprojlim{}^1 (G_i \otimes M) = 0$. This proves Part 2).

For Part 3) we need the following fact:

Fact. If M is faithfully flat and $\alpha\colon G \to G'$ is a homomorphism such that
$$\alpha \otimes 1 : G \otimes M \to G' \otimes M$$

[13] An abelian group M is *faithfully flat* if for any group G, if $G \otimes M = 0$ then $G = 0$. For instance $\mathbb{Q}$ is not faithfully flat, while $\mathbb{Q} \oplus \mathbb{Z}$ *is* faithfully flat. This coincides with the usual definition when M is torsionfree.

is an isomorphism, then α itself is an isomorphism.

This fact is immediate from the definition of faithful flatness and the isomorphisms

$$0 \cong \mathrm{Ker}(\alpha \otimes 1) \cong \mathrm{Ker}(\alpha) \otimes M$$

and

$$0 \cong \mathrm{Coker}(\alpha \otimes 1) \cong \mathrm{Coker}(\alpha) \otimes M.$$

Let $H_i = G_i \otimes M$ and define

$$H_{j,i} = \mathrm{Im}(H_j \to H_i) \cong G_{j,i} \otimes M.$$

Suppose that $\varprojlim^1 H_i = 0$. Then Theorem 3.11(2) implies that $\{H_i\}$ satisfies the Mittag-Leffler condition. Let $\alpha : G_{j,i} \to G_{\phi(i),i}$ be the canonical map. We have a commuting diagram

$$\begin{array}{ccc} G_{j,i} \otimes M & \xrightarrow{\alpha\otimes 1} & G_{\phi(i),i} \otimes M \\ \downarrow \cong & & \downarrow \cong \\ H_{j,i} & \longrightarrow & H_{\phi(i),i} \end{array}$$

and the map $H_{j,i} \to H_{\phi(i),i}$ is an isomorphism by the Mittag-Leffler assumption. Thus $\alpha \otimes 1$ is an isomorphism and hence α is itself an isomorphism. Thus $\{G_i\}$ satisfies the Mittag-Leffler condition. □

I. Emmanouil [21] gives the definitive result in this direction. His result is in the context of modules over a fixed ring; we state it for abelian groups.

Proposition 3.17. *For an inverse sequence $\{G_i\}$ the following are equivalent:*

1. *$\{G_i\}$ satisfies the Mittag-Leffler condition.*
2. *For all abelian groups M,*
$$\varprojlim^1 (G_i \otimes M) = 0.$$
3. *There is some free abelian group F of infinite rank such that*
$$\varprojlim^1 (G_i \otimes F) = 0.$$

Finally, anticipating our study of Pext and Jensen's theorem, we record the following.

Proposition 3.18. *Suppose that $G = \varinjlim G_i$ is a direct limit of finitely generated abelian groups G_i and suppose that H is also a finitely generated abelian group. Then:*

1.
$$\varprojlim{}^1 \mathrm{Hom}_{\mathbb{Z}}(G_i, H) \;\cong\; [\varprojlim{}^1 \mathrm{Hom}_{\mathbb{Z}}(G_i, \mathbb{Z})] \otimes H.$$

2. *If*
$$\varprojlim{}^1 \mathrm{Hom}_{\mathbb{Z}}(G_i, \mathbb{Z}) = 0$$
then
$$\varprojlim{}^1 \mathrm{Hom}_{\mathbb{Z}}(G_i, H) = 0.$$

3. *If* $fH \neq 0$ [14] *and*
$$\varprojlim{}^1 \mathrm{Hom}_{\mathbb{Z}}(G_i, H) = 0$$
then $\varprojlim{}^1 \mathrm{Hom}_{\mathbb{Z}}(G_i, \mathbb{Z}) = 0.$

Proof. Since each G_i is free there are natural isomorphisms
$$\mathrm{Hom}_{\mathbb{Z}}(G_i, H) \;\cong\; \mathrm{Hom}_{\mathbb{Z}}(G_i, \mathbb{Z}) \otimes H$$
for each i and hence Proposition 3.16 implies Part 1). Part 2) is a consequence of Part 1). For Part 3) we note that a finitely generated abelian group H is faithfully flat if and only if $fH \neq 0$. □

[14] We let tG denote the torsion subgroup of G and $fG = G/tG$ denote the maximal torsionfree quotient.

4. Exact sequences, divisibility results

In this section we search for general results which imply that $\varprojlim^1 = 0$ or that $\varprojlim^1$ is divisible. We pay special attention to algebraically compact groups, and we show that if $\{G_i\}$ is an inverse sequence of abelian groups such that $\{tG_i\}$ is a continuous algebraically compact inverse sequence and each fG_i is torsion-free of finite rank, then $\varprojlim^1 G_i$ is divisible. This implies that in many cases of interest the Milnor $\varprojlim^1$ sequence 3.13 splits unnaturally.

Definition 4.1. An inverse sequence $\{G_i\}$ of algebraically compact abelian groups is *continuous* if there exists an inverse sequence $\{K_i\}$ of compact abelian groups and continuous maps such that the inverse sequence $\{G_i\}$ is a direct summand [15] of $\{K_i\}$.

Proposition 4.2. *If $\{G_i\}$ is a continuous inverse sequence of algebraically compact abelian groups then $\varprojlim^1 G_i = 0$.*

Proof. Since $\{G_i\}$ is a direct summand of an inverse sequence $\{K_i\}$ of compact groups and continuous homomorphisms, there is an isomorphism

$$\varprojlim^1 K_i \cong \varprojlim^1 G_i \oplus \varprojlim^1 K_i/G_i$$

and $\varprojlim^1 K_i = 0$ by Proposition 3.15. This implies that $\varprojlim^1 G_i = 0$. □

The following example shows that the assumption that the structural maps be continuous is essential.

Example 4.3 (C. McGibbon)**.** Let B_o be a vector space over $\mathbb{Z}/2$ of countably infinite dimension, with basis $\{e_1, e_2, \dots, e_n, \dots\}$. Let C_n be the span of $\{e_1, \dots, e_n\}$ and let $B_o \to C_n$ be the canonical projection with kernel A_n. Then there is a natural commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_n & \longrightarrow & B_n & \longrightarrow & C_n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_{n-1} & \longrightarrow & B_{n-1} & \longrightarrow & C_{n-1} & \longrightarrow & 0 \end{array}$$

(where in the middle sequence $B_n = B_o$ and the structural map is the identity) and hence a short exact sequence of inverse sequences. This gives

[15] as inverse sequences. That is, not only is $\{G_i\}$ an inverse subsequence of $\{K_i\}$ and each G_i a direct summand of K_i, but the retraction maps $K_i \to G_i$ must respect the structural maps of the inverse sequences. See Example 4.3 for an illustration of what can go wrong otherwise.

an exact sequence

$$0 \to \varprojlim B_n \to \varprojlim C_n \to \varprojlim{}^1 A_n \to 0$$

since $\varprojlim A_n = 0$ and $\varprojlim{}^1 B_n = 0$. This implies that

$$\varprojlim{}^1 A_n \cong \frac{\varprojlim C_n}{\varprojlim B_n} \cong \frac{\Pi_1^\infty(\mathbb{Z}/2)}{\oplus_1^\infty(\mathbb{Z}/2)} \cong (\mathbb{Z}/2)^{\aleph_o}$$

which is the product of countably many copies of $\mathbb{Z}/2$ and is isomorphic as a vector space to the sum of uncountably many copies of $\mathbb{Z}/2$. Thus $\varprojlim{}^1 A_n$ is an uncountable reduced group with every element of order 2. This shows that although each group A_n is algebraically compact (it embeds as a pure subgroup of the compact group $\Pi_1^\infty(\mathbb{Z}/2)$), the structural maps are not continuous (in the sense of the definition above). The inverse sequence $\{A_n\}$ is *not* a direct summand of a compact inverse sequence and $\varprojlim{}^1$ is highly nontrivial.[16]

Note that in the example above that the group $\mathbb{Z}/2$ may be replaced by any commutative ring R, in which case

$$\varprojlim{}^1 A_n \cong \frac{\Pi_1^\infty R}{\oplus_1^\infty R}.$$

For instance, if we insert $\mathbb{Q}$ then we obtain an example of an inverse sequence of divisible groups with nontrivial $\varprojlim{}^1$. In fact, $\varprojlim{}^1 = \mathbb{Q}^{\aleph_o}$.

Proposition 4.4. *Let $\{G_i\}$ be an inverse sequence of abelian groups satisfying the following two conditions:*

1. *The inverse sequence $\{tG_i\}$ is a continuous inverse sequence of algebraically compact*[17] *groups (e.g., $tG_i = 0$ for all i).*
2. *For each prime p,*

$$\varprojlim{}^1 \frac{G_i/pG_i}{tG_i/ptG_i} = 0.$$

Then the group $\varprojlim{}^1 G_i$ is divisible.

---

[16] It is instructive to experiment with this directly. Take $A_n \subseteq B_n \subseteq K_n$ where $B_n = B_o \subseteq K = K_n$ is the canonical inclusion. Then it is not hard to see that there is no "chain map" $s : K_* \to A_*$ which is a retraction for the canonical inclusion.

[17] An algebraically compact group which is also torsion is the direct sum of cyclic groups and of groups of the type $\mathbb{Z}(p^\infty)$, by results of Fuchs [24], §40.3, Prüfer [51] , and Baer [4].

Condition 2) of Proposition 4.4 is satisfied whenever fG_i is torsionfree of finite rank since any torsionfree group G of finite rank has the property (cf. [1] 0.3) that for each prime p,

$$\dim_{\mathbb{Z}/p}(G/pG) \leq \operatorname{rank}(G)$$

which implies that for each i, G_i/nG_i is a finite group.

Both Conditions 1) and 2) are satisfied in each of the following special cases:

1. Each G_i is finitely generated.[18]
2. Each G_i is torsionfree, and for each prime p, the group G_i/pG_i is finite.
3. Each G_i is torsionfree and divisible.
4. Each G_i is torsionfree of finite rank.

Proof. The first hypothesis implies that $\varprojlim^1 tG_i = 0$ by Proposition 4.2 and hence

$$\varprojlim^1 G_i \cong \varprojlim^1 G_i/tG_i$$

by the long exact $\varprojlim$-$\varprojlim^1$ sequence. Further,

$$\frac{fG_i}{pfG_i} \cong \frac{G_i/tG_i}{p(G_i/tG_i)} \cong \frac{G_i/pG_i}{tG_i/ptG_i}$$

for each i by the Snake Lemma, and hence

$$\varprojlim^1 \frac{G_i/tG_i}{p(G_i/tG_i)} \cong \varprojlim^1 \frac{G_i/pG_i}{tG_i/ptG_i}.$$

So without loss of generality we may assume that each G_i is torsionfree.

Fix some prime p and let $\zeta : G_i \to G_i$ denote multiplication by p. This induces a short exact sequence of inverse sequences of the form

$$0 \to \{G_i\} \xrightarrow{\zeta} \{G_i\} \to \{G_i/pG_i\} \to 0$$

and hence a six term $\varprojlim$-$\varprojlim^1$ sequence, the last three terms of which are

$$\varprojlim^1 G_i \xrightarrow{\zeta_*} \varprojlim^1 G_i \longrightarrow \varprojlim^1 G_i/pG_i \longrightarrow 0.$$

It is easy to show that ζ_* is still multiplication by p. We have assumed that $\varprojlim^1(G_i/pG_i) = 0$, and hence the map

$$\zeta_* \colon \varprojlim^1 G_i \longrightarrow \varprojlim^1 G_i$$

is surjective. Thus

$$p(\varprojlim^1 G_i) = \varprojlim^1 G_i.$$

[18]This case of the Proposition is well-known and is proved, e.g., in [38].

This is true for each prime p and hence $\varprojlim^1 G_i$ is divisible. □

Note that if each G_i is a divisible group then $\varprojlim^1 G_i$ is divisible, since it is a quotient of the divisible group $\Pi_i G_i$. We may extend this result slightly as follows. Let R_i denote the maximal reduced quotient of G_i.

Corollary 4.5 (C. McGibbon). *Suppose that $\{G_i\}$ is an inverse sequence of abelian groups. Then the following conditions are equivalent:*

1. $\varprojlim^1 G_i$ *is divisible.*
2. *For each prime p,* $\varprojlim^1 (G_i/pG_i) = 0$.
3. *For each prime p,* $\varprojlim^1 (R_i/pR_i) = 0$.

Proof. The sequence

$$0 \to pG_i \to G_i \to G_i/pG_i \to 0 \tag{4.6}$$

is exact and there is an associated short exact sequence of inverse sequences obtained by varying i. The $\varprojlim$-$\varprojlim^1$ sequence concludes with

$$\varprojlim^1 pG_i \longrightarrow \varprojlim^1 G_i \longrightarrow \varprojlim^1 G_i/pG_i \to 0. \tag{4.7}$$

Suppose that $\varprojlim^1 G_i$ is divisible. Then $\varprojlim^1 (G_i/pG_i)$ is divisible as well, by the exact sequence 4.7. In particular, division by p is possible. On the other hand, $\varprojlim^1 (G_i/pG_i)$ is a quotient of the $\mathbb{Z}/p$-module $\Pi_i(G_i/pG_i)$ and these together imply that $\varprojlim^1 (G_i/pG_i) = 0$. Thus Part 1) implies Part 2).

To prove the converse, suppose that for each prime p we have

$$\varprojlim^1 (G_i/pG_i) = 0.$$

Sequence 4.7 becomes

$$\varprojlim^1 pG_i \longrightarrow \varprojlim^1 G_i \longrightarrow 0. \tag{4.8}$$

The short exact sequences

$$0 \to Tor_1^{\mathbb{Z}}(G_i, \mathbb{Z}/p) \to G_i \xrightarrow{\pi} pG_i \to 0$$

give rise to a short exact sequence of inverse sequences and hence a $\varprojlim$-$\varprojlim^1$ sequence which concludes as

$$\varprojlim^1 G_i \xrightarrow{\pi_*} \varprojlim^1 pG_i \to 0$$

and hence the composite map

$$\varprojlim^1 G_i \xrightarrow{\pi_*} \varprojlim^1 pG_i \longrightarrow \varprojlim^1 G_i$$

is surjective. This composite is induced by the natural maps

$$G_i \xrightarrow{\pi} G_i[p] \to G_i$$

and this is the canonical factorization of the multiplication by p map. Hence

$$\zeta_* : \varprojlim{}^1 G_i \to \varprojlim{}^1 G_i$$

is surjective. This simply says that

$$p(\varprojlim{}^1 G_i) = \varprojlim{}^1 G_i$$

for each p, which implies that $\varprojlim{}^1 G_i$ is divisible. Thus Part 2) implies Part 1).

Finally, the equivalence of Part 2) and Part 3) is immediate from the natural isomorphism

$$G_i/pG_i \cong R_i/pR_i.$$

□

Let us examine Condition 2) more closely. Note that the group G_i/pG_i is a $\mathbb{Z}/p$-module, that is, a vector space over the field $\mathbb{Z}/p$, and hence it is determined up to isomorphism by its $\mathbb{Z}/p$-dimension.

Theorem 4.9. *Suppose that $\{G_i\}$ is an inverse sequence of abelian groups such that the following conditions hold:*

1. *For each i, tG_i is algebraically compact and the sequence $\{tG_i\}$ is continuous.*
2. *Each G_i/tG_i is torsionfree of finite rank.*

Then

$$\varprojlim{}^1 G_i \cong \varprojlim{}^1 (G_i/tG_i)$$

and this group is divisible.

Proof. The first assumption implies that

$$\varprojlim{}^1 tG_i = 0.$$

The $\varprojlim$-$\varprojlim^1$sequence implies that

$$\varprojlim{}^1 G_i \cong \varprojlim{}^1 G_i/tG_i$$

and the group $\varprojlim^1 G_i/tG_i$ is divisible by Proposition 4.4. □

Remark 4.10. It is possible that these assumptions may fail even if the G_i are countable. A countable abelian group is algebraically compact if and only if it is a direct sum of a divisible group and a bounded group [24] §40; this follows from the general classification of algebraic compact groups. For instance, the group $\oplus(\mathbb{Z}/p)$ (where the sum runs over all primes) is countable, torsion, and yet not algebraically compact. Similarly, it is easy to construct examples of countable torsionfree abelian groups of infinite rank — a countable-dimensional $\mathbb{Q}$-vector space provides the simplest example.

Corollary 4.11. *Suppose that $\{G_i\}$ is an inverse sequence of abelian groups and that one of the following conditions holds:*

1. *Each G_i is divisible.*
2. *Each G_i is torsionfree of finite rank.*
3. *Each G_i is finitely generated.*

Then $\varprojlim^1 G_i$ is divisible. In particular, any short exact sequence of the form

$$0 \to \varprojlim{}^1 G_i \to K \to L \to 0$$

splits.

Short exact sequences of this sort are very common in classical topology as well as in non-commutative topology. So the corollary gives us very general circumstances under which they split.

Proof. Part 1) follows as indicated before, since $\varprojlim^1 G_i$ is a quotient of $\Pi_i G_i$, and quotients and products of divisible groups are again divisible. Parts 2) and 3) follow from Theorem 4.9. □

Remark 4.12. We note in passing that if the groups $\{G_i\}$ are allowed to be *nonabelian* then the terrain is quite different. (This is the situation of greatest interest when studying homotopy classes of phantom maps in algebraic topology.) The interested reader should consult [46, 44].

5. First facts on Pext

In this section we recall the basic definitions and elementary properties of the Ulm subgroups and Pext, and we show that Ext = Pext in certain special cases.

Definition 5.1. Suppose that G is an abelian group. The *first Ulm subgroup* of G is defined by

$$U^1(G) = \cap_n nG$$

and the *higher Ulm subgroups* are defined inductively by

$$U^{n+1}(G) = U^1(U^n(G))$$

and by taking limits for limit ordinals. These form a decreasing family of subgroups of G. The *Ulm length* of G is the first ordinal τ such that

$$U^{\tau+1}(G) = U^\tau(G).$$

Recall that a subgroup G' of a group G is said to be *pure* if $G' \cap nG = nG'$ for all positive integers n.

Definition 5.2. An extension

$$0 \to G' \to G \to G'' \to 0$$

is said to be *pure* if G' is a pure subgroup of G. If an extension is pure then any other extension in its equivalence class in Ext is also pure. Equivalence classes of pure extensions form a subgroup of $\mathrm{Ext}^1_{\mathbb{Z}}(G,H)$ denoted $\mathrm{Pext}^1_{\mathbb{Z}}(G,H)$.

If G' is a direct summand of G then G' is a pure subgroup, but not conversely. For example, the torsion subgroup tG is always a pure subgroup of G (this is a routine check) but it is not necessarily a direct summand of G. (See §8).

Proposition 5.3. *Suppose that G is an abelian group. Then:*

1. *If G is divisible then $U^\sigma(G) = G$ for all σ.*
2. *If τ is the Ulm length of G, then $U^\tau(G)$ is the maximal divisible subgroup of G.*
3. *If G is torsionfree then*

$$U^\tau(G) = U^1(G)$$

so that $U^1(G)$ is the maximal divisible subgroup of G.

4. *If G is algebraically compact then $U^\tau(G) = U^1(G)$ so that $U^1(G)$ is the maximal divisible subgroup of G.*

5. *For any abelian group H,*

$$\mathrm{Pext}^1_{\mathbb{Z}}(G,H) \cong U^1(\mathrm{Ext}^1_{\mathbb{Z}}(G,H)).$$

Proof. An abelian group A is divisible if and only if $A = nA$ for each positive integer n. This condition is clearly satisfied by $U^\tau(G)$. Any divisible group $H \subseteq G$ lies in $U^1(G)$, since

$$H = nH \subseteq nG$$

for each n, and using induction we see that $H \subseteq U^\tau(G)$. Thus $U^\tau(G)$ contains all divisible subgroups of G. This proves Parts 1) and 2). Part 3) is stated in ([24], Page 101, Exer. 2.) Part 4) is stated in ([24], Page 162, Exer. 7). Part 5) is a theorem of Nunke and Fuchs (cf. [24], §53.3) and it will follow from Jensen's Theorem 6.1. □

Proposition 5.4. 1. *For each pair of abelian groups G and H, the group $\mathrm{Pext}^1_{\mathbb{Z}}(G,H)$ is a subgroup of the group $\mathrm{Ext}^1_{\mathbb{Z}}(G,H)$. The functor is contravariant in G and covariant in H.*

2. *A pure short exact sequence induces long exact* Hom-Pext *sequences in either variable.*
3. $\mathrm{Pext}^1_{\mathbb{Z}}(G,H) = 0$ *for all G if and only if H is algebraically compact.*
4. $\mathrm{Pext}^1_{\mathbb{Z}}(G,H) = 0$ *for all H if and only if G is a direct sum of cyclic groups.*
5. *There is a natural isomorphism*

$$\mathrm{Pext}^1_{\mathbb{Z}}(\oplus_i G_i, H) \cong \Pi_i \mathrm{Pext}^1_{\mathbb{Z}}(G_i, H).$$

6. *There is a natural isomorphism*

$$\mathrm{Pext}^1_{\mathbb{Z}}(G, \Pi_i H_i) \cong \Pi_i \mathrm{Pext}^1_{\mathbb{Z}}(G, H_i).$$

Proof. These are foundational facts found in [24], §53. Part 1) is due to Fuchs [22] and Baer [5]. Part 2) is a result of Harrison [31]. Part 3) follows immediately by the definition of an algebraically compact group. Part 4) is due to Maranda [24]. Parts 5) and 6) are exercises in Fuchs [24]. If one has already shown that Pext is really defined in terms of relative homological algebra then all of these properties follow at once for elementary reasons. □

Here is an easy corollary of our previous work.

Corollary 5.5. *If G is a torsionfree abelian group then*

$$\mathrm{Pext}^1_{\mathbb{Z}}(G,H) \cong \mathrm{Ext}^1_{\mathbb{Z}}(G,H).$$

In particular, $\mathrm{Pext}^1_{\mathbb{Z}}(G,H)$ *is divisible.*

Proof. If G is torsionfree then $\mathrm{Ext}^1_{\mathbb{Z}}(G,H)$ is divisible, by Proposition 2.4, and then Proposition 5.3 implies the result. □

Proposition 5.6. *Let G and H be abelian groups. Then:*

1. *The group* $\operatorname{Pext}^1_{\mathbb{Z}}(G,H)$ *satisfies*

$$\operatorname{Pext}^1_{\mathbb{Z}}(G,H) \;\cong\; \bigcap_{G'} \operatorname{Ker}[\operatorname{Ext}^1_{\mathbb{Z}}(G,H) \longrightarrow \operatorname{Ext}^1_{\mathbb{Z}}(G',H)]$$

   where G' runs over all finitely generated subgroups of G.

2. *The group* $\operatorname{Pext}^1_{\mathbb{Z}}(G,H)$ *satisfies*

$$\operatorname{Pext}^1_{\mathbb{Z}}(G,H) \;\cong\; \bigcap_{G'} \operatorname{Ker}[\operatorname{Ext}^1_{\mathbb{Z}}(G,H) \longrightarrow \operatorname{Ext}^1_{\mathbb{Z}}(G',H)]$$

   where G' runs over all finite subgroups of G.

3. *If G is written as an increasing union of finitely generated subgroups G_i, then there is a natural short exact sequence*

$$0 \to \operatorname{Pext}^1_{\mathbb{Z}}(G,H) \longrightarrow \operatorname{Ext}^1_{\mathbb{Z}}(G,H) \longrightarrow \varprojlim \operatorname{Ext}^1_{\mathbb{Z}}(G_i,H) \to 0. \tag{5.7}$$

Proof. Fuchs [24], §20.3 proves that an extension

$$0 \to H \to J \to G \to 0$$

is pure exact if and only if for each cyclic group C and for each homomorphism $\zeta\colon C \to G$ the induced extension

$$0 \to H \to J' \to C \to 0$$

splits. Translated into terms of Ext, this says that

$$\operatorname{Pext}^1_{\mathbb{Z}}(G,H) \;\cong\; \bigcap_{G'} \operatorname{Ker}[\operatorname{Ext}^1_{\mathbb{Z}}(G,H) \longrightarrow \operatorname{Ext}^1_{\mathbb{Z}}(G',H)]$$

where G' runs over all cyclic subgroups of G. Equivalently, we may let G' run over all finitely generated subgroups of G, or over all finite subgroups of G. This establishes Parts 1) and 2). Part 3) is immediate from Part 1) and elementary definitions. □

Let $\theta\colon tG \to G$ denote the natural inclusion of the torsion subgroup, and let

$$\theta^*_E : \operatorname{Ext}^1_{\mathbb{Z}}(G,H) \to \operatorname{Ext}^1_{\mathbb{Z}}(tG,H)$$

denote the induced map. Similarly, let

$$\theta^*_P : \operatorname{Pext}^1_{\mathbb{Z}}(G,H) \to \operatorname{Pext}^1_{\mathbb{Z}}(tG,H)$$

denote the induced map.

Theorem 5.8. *Let G and H be abelian groups. Then*

$$\mathrm{Ker}(\theta_E^*) \;\cong\; \mathrm{Ker}(\theta_P^*).$$

Further,

$$\mathrm{Ker}(\theta_E^*) \;\cong\; \mathrm{Pext}^1_{\mathbb{Z}}(G,H) \Longleftrightarrow \mathrm{Pext}^1_{\mathbb{Z}}(tG,H) = 0. \tag{5.9}$$

Thus if H is algebraically compact then θ_E^* is a monomorphism. Suppose further that one of the following hypotheses hold:

1. *tG is the direct sum of cyclic groups.*
2. *tG is countable and has no nonzero elements of infinite height*[19] *(for instance, if tG is bounded).*

Then

$$\mathrm{Ker}(\theta_E^*) \;\cong\; \mathrm{Pext}^1_{\mathbb{Z}}(G,H). \tag{5.10}$$

Proof. The short exact sequence

$$0 \to tG \to G \to G/tG \to 0$$

is pure and hence there is a commutative diagram with exact columns of the form

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbb{Z}}(tG,H) & \xrightarrow{\;1\;} & \mathrm{Hom}_{\mathbb{Z}}(tG,H) \\
\big\downarrow & & \big\downarrow \\
\mathrm{Pext}^1_{\mathbb{Z}}(G/tG,H) & \xrightarrow{\;\cong\;} & \mathrm{Ext}^1_{\mathbb{Z}}(G/tG,H) \\
\big\downarrow \lambda_P & & \big\downarrow \lambda_E \\
\mathrm{Pext}^1_{\mathbb{Z}}(G,H) & \longrightarrow & \mathrm{Ext}^1_{\mathbb{Z}}(G,H) \\
\big\downarrow \theta_P^* & & \big\downarrow \theta_E^* \\
\mathrm{Pext}^1_{\mathbb{Z}}(tG,H) & \longrightarrow & \mathrm{Ext}^1_{\mathbb{Z}}(tG,H) \\
\big\downarrow & & \big\downarrow \\
0 & & 0.
\end{array} \tag{5.11}$$

The inclusion

$$\mathrm{Pext}^1_{\mathbb{Z}}(G/tG,H) \to \mathrm{Ext}^1_{\mathbb{Z}}(G/tG,H)$$

[19] An element $g \in G$ has *infinite height* if for all primes p and natural numbers r the equation $p^r x = g$ has a solution $x \in G$.

is an isomorphism by Corollary 5.5, since G/tG is torsionfree. A standard diagram chase shows that

$$\mathrm{Ker}(\theta_P^*) \;\cong\; \mathrm{Im}(\lambda_P) \;\cong\; \mathrm{Im}(\lambda_E) \;\cong\; \mathrm{Ker}(\theta_E^*)$$

which proves the first assertion.

If H is algebraically compact then $\mathrm{Pext}^1_{\mathbb{Z}}(-,H) = 0$ and hence θ_E^* is mono.

Moving to the last part of the Theorem, suppose that $tG \cong \oplus_i C_i$ where each C_i is a finite cyclic group. Then

$$\mathrm{Ext}^1_{\mathbb{Z}}(tG,H) \;\cong\; \mathrm{Ext}^1_{\mathbb{Z}}(\oplus_i C_i,H) \;\cong\; \Pi_i\,\mathrm{Ext}^1_{\mathbb{Z}}(C_i,H)$$

so that

$$\theta_E^* \;\cong\; \Pi_i\,\theta_i^* :\; \mathrm{Ext}^1_{\mathbb{Z}}(G,H) \longrightarrow \Pi_i\mathrm{Ext}^1_{\mathbb{Z}}(C_i,H).$$

Thus

$$\mathrm{Pext}^1_{\mathbb{Z}}(G,H) \;\cong\; \bigcap \mathrm{Ker}[\mathrm{Ext}^1_{\mathbb{Z}}(G,H) \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(G',H)]$$

with intersection over all finite subgroups $G' \subseteq G$

$$\subseteq\; \bigcap \mathrm{Ker}[\mathrm{Ext}^1_{\mathbb{Z}}(G,H) \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(C_i,H)]$$

with intersection over all cyclic summands C_i,

$$\subseteq\; \bigcap \mathrm{Ker}(\theta_E^*) \;\cong\; \mathrm{Ker}(\theta_E^*)$$

and hence

$$\mathrm{Pext}^1_{\mathbb{Z}}(G,H) \;\cong\; \mathrm{Ker}(\theta_E^*)$$

so Equation 5.10 holds.

Any countable abelian group with no nonzero elements of infinite height is a sum of cyclic groups, by a theorem of Prüfer (cf. [24], §17.2). Hence Part 2) is implied by Part 1). □

Remark 5.12. Pext may be brought into the context of relative homological algebra. Let $\mathcal{S}$ be the class of pure short exact sequences. This is a *proper class* in the sense of Buchsbaum-MacLane [41] where $\mathcal{S}$-projectives are direct sums of cyclic groups and $\mathcal{S}$-injectives are algebraically compact groups. It is not hard to prove (cf. [61]) that

$$\mathrm{Ext}^1_{\mathcal{S}}(G,H) \;\cong\; \mathrm{Pext}^1_{\mathbb{Z}}(G,H).$$

This is the beginning of a long and deep story which goes under the rubric of "pure homological algebra". There are extensive results on pure extensions over commutative rings. As indicated in the Introduction, we do not intend to discuss these results, as they take us too far afield. Pure global dimension and the Ziegler spectrum of indecomposable pure-injective modules are active areas of research. See the survey article by Ivo Herzog [32] for more information.

6. Jensen's theorem and Roos's theorem

This section is devoted to Jensen's theorem and its application to a proof of a well-known result of Roos.

C. U. Jensen [38] has discovered a remarkable connection between homological algebra and infinite abelian groups. Here is his result, specialized to our situation.

Theorem 6.1 ([38]). *Let G be a countable abelian group written as an increasing union of finitely generated subgroups G_i and let H be any abelian group. Then there is a natural isomorphism*

$$\varprojlim{}^1 \operatorname{Hom}_{\mathbb{Z}}(G_i, H) \;\cong\; \operatorname{Pext}^1_{\mathbb{Z}}(G, H).$$

We believe that this theorem deserves more attention, since anything which sheds light upon $\varprojlim^1$ is welcome, and especially because of the cross-cultural nature of the result.

Sklaryenko observes [60] that the theorem has as a consequence the Stein-Serre theorem: If G is a countable abelian group with $\operatorname{Ext}^1_{\mathbb{Z}}(G, \mathbb{Z}) = 0$, then G is free abelian.

Jensen proves Theorem 6.1 in a much more general context. He shows that if R is a ring, G_α and H are R-modules with $G = \varinjlim\ G_\alpha$ where α runs over an arbitrary directed set, then there is a spectral sequence with

$$E_2^{p,q} \;\cong\; \varprojlim{}^p \operatorname{Pext}^q_R(G_\alpha, H)$$

and which converges to

$$\operatorname{Pext}^n_R(G, H).$$

If the R-modules G_α are of finite type then

$$\operatorname{Pext}^q_R(G_\alpha, H) = 0$$

for all $q > 0$ and all α and hence the spectral sequence collapses, with the edge homomorphism producing an isomorphism

$$\operatorname{Pext}^n_R(G, H) \;\cong\; \varprojlim{}^n \operatorname{Hom}_R(G_\alpha, H).$$

Taking $R = \mathbb{Z}$ and $n = 1$ yields the theorem.

We present here a very simple alternate proof of Jensen's Theorem, using only the Eilenberg definition of $\varprojlim^1$ and the most elementary facts about Pext. (Our longer argument was rendered obsolete by an excellent suggestion of an anonymous referee.)

Proof. Suppose that G is a countable abelian group written as the union of an increasing sequence of finitely generated subgroups

$$G_1 \xrightarrow{\xi_1} G_2 \xrightarrow{\xi_2} G_3 \longrightarrow \cdots \longrightarrow G.$$

Define $\phi_i : G_i \to \oplus G_n$ by

$$\phi_i(g_i) = (0, \dots, 0, g_i, -\xi_i(g_i), 0, \dots).$$

Then $\{\phi_i\}$ coalesce to form $\phi : \oplus_n G_n \to \oplus_n G_n$ and there is a natural short exact sequence

$$0 \to \oplus_n G_n \xrightarrow{\phi} \oplus_n G_n \longrightarrow G \to 0$$

expressing G as the direct limit of the groups $\{G_n\}$. This is a pure short exact sequence. Apply the functor $\mathrm{Hom}_{\mathbb{Z}}(-, H)$ to the short exact sequence. The resulting Hom-Pext long exact sequence degenerates to give an exact sequence whose final terms are

$$\mathrm{Hom}_{\mathbb{Z}}(\oplus_n G_n, H) \xrightarrow{\phi^*} \mathrm{Hom}_{\mathbb{Z}}(\oplus_n G_n, H) \longrightarrow \mathrm{Pext}^1_{\mathbb{Z}}(G, H) \to 0.$$

The diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{Z}}(\oplus_n G_n, H) & \xrightarrow{\phi^*} & \mathrm{Hom}_{\mathbb{Z}}(\oplus_n G_n, H) \\ \downarrow \cong & & \downarrow \cong \\ \Pi_n \mathrm{Hom}_{\mathbb{Z}}(G_n, H) & \xrightarrow{\psi} & \Pi_n \mathrm{Hom}_{\mathbb{Z}}(G_n, H) \end{array}$$

commutes by a direct check, and so

$$\mathrm{Pext}^1_{\mathbb{Z}}(G, H) \cong \mathrm{Coker}(\phi^*) \cong \mathrm{Coker}(\psi) \cong \varprojlim{}^1 \mathrm{Hom}_{\mathbb{Z}}(G_n, H)$$

which proves Theorem 6.1. □

Theorem 6.1 has as a consequence an important theorem[20] of Roos [52]. We supply two proofs. The first (of the first part of the theorem) uses Jensen's Theorem. The second proof is a sketch of Roos's high-tech proof.

Theorem 6.2 (Roos)**.** *Let $\{G_i\}$ be a direct sequence of abelian groups with $G = \varinjlim G_i$ and let H be an abelian group. Then there is a natural short exact sequence*

(6.3) $$0 \to \varprojlim{}^1 \mathrm{Hom}_{\mathbb{Z}}(G_i, H) \to \mathrm{Ext}^1_{\mathbb{Z}}(G, H) \xrightarrow{\Gamma} \varprojlim \mathrm{Ext}^1_{\mathbb{Z}}(G_i, H) \to 0$$

and in addition

(6.4) $$\varprojlim{}^1 \mathrm{Ext}^1_{\mathbb{Z}}(G_i, H) \cong 0.$$

[20]Roos actually establishes this result in the context of inverse systems in an abelian category; the result below is a simple and very special case of his result.

Proof (low tech)**.** Equation 5.7 yields a natural short exact sequence

$$0 \to \operatorname{Pext}^1_{\mathbb{Z}}(G,H) \to \operatorname{Ext}^1_{\mathbb{Z}}(G,H) \xrightarrow{\Gamma} \varprojlim \operatorname{Ext}^1_{\mathbb{Z}}(G_i,H) \to 0$$

and Jensen's Theorem 6.1 yields a natural isomorphism

$$\varprojlim{}^1 \operatorname{Hom}_{\mathbb{Z}}(G_i,H) \;\cong\; \operatorname{Pext}^1_{\mathbb{Z}}(G,H).$$

Combining these two results yields the first part (6.3) of Roos's theorem. □

Proof (high tech)**.** Roos shows that for any ring R, left R-module H, and directed system (countable or not) of left R-modules G_i, that there is a spectral sequence which converges to $\operatorname{Ext}^n_R(G,H)$ with

$$E_2^{p,q} \;\cong\; \varprojlim{}^p \operatorname{Ext}^q_R(G_*,H).$$

Take $R = \mathbb{Z}$. This is a principal ideal domain, and hence $\operatorname{Ext}^q_{\mathbb{Z}} = 0$ for $q > 1$. Further, insist that the index set be countable. Then $\varprojlim^p = 0$ for $p > 1$. Thus there are at most four nontrivial terms at the E_2-level. There are no further differentials since $E_2^{p,*} = 0$ for $p > 1$. Hence $E_2 = E_\infty$. Here is E_∞:

In total degree $p + q = 0$ there is the term

$$E_\infty^{0,0} \;\cong\; \varprojlim \operatorname{Hom}_{\mathbb{Z}}(G_i,H) \;\cong\; \operatorname{Hom}_{\mathbb{Z}}(G,H)$$

as required.

In total degree one there are two terms, namely

$$E_\infty^{1,0} \;\cong\; \varprojlim{}^1 \operatorname{Hom}_{\mathbb{Z}}(G_i,H)$$

and

$$E_\infty^{0,1} \;\cong\; \varprojlim \operatorname{Ext}^1_{\mathbb{Z}}(G_i,H)$$

and these give a composition series for $\operatorname{Ext}^1_{\mathbb{Z}}(G,H)$, namely sequence 6.3, where Γ is the edge homomorphism in the spectral sequence.

In total degree two there is only one[21] possible nonzero term, namely

$$E_\infty^{1,1} \;\cong\; \varprojlim{}^1 \operatorname{Ext}^1_{\mathbb{Z}}(G_i,H).$$

---

[21] Here is a good example where sticking to countable index sets really pays off. If the index set were allowed to be uncountable then the term

$$E_2^{2,0} \;\cong\; \varprojlim{}^2 \operatorname{Hom}_{\mathbb{Z}}(G_i,H)$$

would also be present. There would be a possible differential

$$d^2 : E_2^{2,0} \to E_2^{0,1}.$$

The kernel of this differential would survive to $E_\infty^{2,0}$ to contribute a possibly nontrivial component to $\varprojlim^1 \operatorname{Ext}^1_{\mathbb{Z}}(G_i,H)$ to cause trouble at this point, and the cokernel would replace $\varprojlim \operatorname{Ext}^1_{\mathbb{Z}}(G_i,H)$ in the sequence 6.3.

However, $E_\infty^{1,1}$ must be the associated graded group associated to some filtration of the group $\mathrm{Ext}^2_{\mathbb{Z}}(G,H)$, since the spectral sequence converges. This group vanishes, since

$$\mathrm{Ext}^2_{\mathbb{Z}}(-,-) \cong 0$$

and this implies that

$$\varprojlim{}^1 \mathrm{Ext}^1_{\mathbb{Z}}(G_i,H) = 0$$

as required. This verifies 6.4 and completes the proof. □

We conclude this section by reexamining Example 3.6, in which we demonstrated that

$$\varprojlim{}^1 \{\mathbb{Z}, 2\} \;\cong\; \widehat{\mathbb{Z}}_2/\mathbb{Z}.$$

Let us revisit this calculation using Jensen's Theorem 6.1. Let $\{G_i\}$ be an increasing sequence of copies of $\mathbb{Z}$ in $\mathbb{Z}[\frac{1}{2}]$ whose union is equal to $\mathbb{Z}[\frac{1}{2}]$. Apply $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Z})$ and we obtain an inverse sequence $\{\mathrm{Hom}_{\mathbb{Z}}(G_i,\mathbb{Z})\}$ which is obviously isomorphic to a cofinal subsequence of $\{\mathbb{Z}, 2\}$ and hence has the same $\varprojlim{}^1$. Thus

$$\varprojlim{}^1 \{\mathbb{Z}, 2\} \;\cong\; \varprojlim{}^1 \{\mathrm{Hom}_{\mathbb{Z}}(G_i,\mathbb{Z})\} \;\cong\; \mathrm{Pext}^1_{\mathbb{Z}}(\mathbb{Z}[\tfrac{1}{2}],\mathbb{Z})$$

by Jensen's Theorem 6.1

$$\cong\; \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}[\tfrac{1}{2}],\mathbb{Z})$$

since $\mathbb{Z}[\frac{1}{2}]$ is torsionfree, by Corollary 5.5. This is a divisible group, and by Jensen's calculations it is the direct sum of $2^{\aleph_o}$ copies of $\mathbb{Q}$ and for each odd prime p one copy of $\mathbb{Z}(p^\infty)$ ([38], and see 9.7). This is a very round-about way of determining the underlying structure of $\widehat{\mathbb{Z}}_2/\mathbb{Z}$ as an abelian group!

7. Ranges and inverse limits

In this section we consider certain classes of abelian groups determined by $\varprojlim^1$ and by Pext. Then we turn our attention briefly to an examination of $\varprojlim G_i$.

Which abelian groups can be realized as $\varprojlim^1$ groups? Here is a good answer to that question.

Proposition 7.1 ([63], Thm. 1)**.** *The following are equivalent for an abelian group L:*

1. *L is a cotorsion group.*
2. *There exists abelian groups $\{G_i\}_{i\in\mathbb{N}}$ such that $\varprojlim^1 G_i = L$.*
3. *There exists a sequence $\{G_i\}_{i\in\mathbb{N}}$ such that L is a homomorphic image of the group $\Pi G_n / \oplus G_n$.*
4. *There exists an abelian group H and a countable abelian group G with $\mathrm{Pext}^1_{\mathbb{Z}}(G, H) = L$.*

We note that cotorsion groups have some interesting properties that help in various applications. For example, a countable cotorsion group is a direct sum of a divisible group and a bounded group. For more, see [24], §§54, 55.

If one relaxes the assumption on the index set and allows inverse systems which are indexed by the set of all ordinals less than the smallest uncountable ordinal, then Warfield and Huber also show ([63], Theorem 3) that *every* abelian group is of the form $\varprojlim^1 G_i$.

Proposition 7.1 shows that the collection of abelian groups of the form $\varprojlim^1 G_i$ is exactly the cotorsion groups. It is an open problem (cited, for instance by McGibbon [44]) to determine which cotorsion groups are of the form $\varprojlim^1 G_i$ when the G_i must be *countable*. Denote this collection by $\mathcal{G}$. Of course any $G \in \mathcal{G}$ must be cotorsion, and it also must be uncountable, by Proposition 3.5.

Similarly, let $\mathcal{P}$ denote those groups which are of the form $\mathrm{Pext}^1_{\mathbb{Z}}(G, H)$ with both G and H countable. Then $\mathcal{P} \subseteq \mathcal{G}$ by Jensen's Theorem 6.1. It is an open problem to determine this class as well. Determination of $\mathcal{P}$ would also determine under weak hypotheses which groups can appear as the closure of zero in the Kasparov group $KK_*(A, B)$.

Is studying Pext more restrictive than studying arbitrary $\varprojlim^1$? Any inverse sequence $\{L_i\}$ can be written as $\{\mathrm{Hom}_{\mathbb{Z}}(G_i, H)\}$ simply by taking G_i to be the character group of L_i and $H = \mathbb{R}/\mathbb{Z}$, and hence

$$\varprojlim^1 L_i \;\cong\; \varprojlim^1 \mathrm{Hom}_{\mathbb{Z}}(G_i, H) \;\cong\; \mathrm{Pext}^1_{\mathbb{Z}}(G, H)$$

so, for instance, the class of abelian groups of the form $\varprojlim^1 L_i$ is the same as the class of abelian group of the form $\mathrm{Pext}^1_{\mathbb{Z}}(G,H)$, namely the cotorsion groups, by Proposition 7.1.

Suppose that we require L_i, G and H to be *countable* abelian groups. Then there are two possible questions.

One possibility would be to ask that given any inverse system $\{L_i\}$ of *countable* abelian groups, does there exist some countable abelian groups G and H such that

$$\varprojlim{}^1 L_i \;\cong\; \mathrm{Pext}^1_{\mathbb{Z}}(G,H).$$

We don't know the answer to this question.

Suppose that we restrict further: we can insist that the entire inverse system $\{L_i\}$ coincide with the inverse system $\{\mathrm{Hom}_{\mathbb{Z}}(G_i,H)\}$. Here we have an answer: in general this cannot be done. In fact, Example 4.3 does not arise in this way. Recall that it was of the form $\varprojlim^1 A_n$, with each A_n a countable sum of copies of $\mathbb{Z}/2$ and each map $A_n \to A_{n-1}$ was mono and had cokernel one copy of $\mathbb{Z}/2$. Here is a proof that this cannot be realized.

Proposition 7.2. *Example* 4.3 *cannot be realized as an inverse sequence* $\mathrm{Hom}_{\mathbb{Z}}(G_i,H)$ *with each* G_i *finitely generated, each* $G_{i-1} \to G_i$ *an inclusion, and* H *a countable group.*

Proof. Let $\{A_i\}$ denote the sequence defined in 4.3. Suppose that the increasing sequence $\{G_i\}$ and countable group H satisfied

$$A_i \;\cong\; \mathrm{Hom}_{\mathbb{Z}}(G_i,H)$$

and the connecting maps respected the isomorphisms so that

$$\varprojlim{}^1 A_i \;\cong\; \varprojlim{}^1 \mathrm{Hom}_{\mathbb{Z}}(G_i,H).$$

We have

$$\oplus_1^\infty(\mathbb{Z}/2) \cong A_i \cong \mathrm{Hom}_{\mathbb{Z}}(G_i,H)$$

for each i. Write $G_i = tG_i \oplus fG_i$, the sum of its torsion subgroup and torsionfree quotient. Then

$$\oplus_1^\infty(\mathbb{Z}/2) \cong \mathrm{Hom}_{\mathbb{Z}}(tG_i,H) \oplus \mathrm{Hom}_{\mathbb{Z}}(fG_i,H).$$

If $fG_i = \mathbb{Z}^{t(i)}$ with $t(i) > 0$ for at least one value of i then

$$\mathrm{Hom}_{\mathbb{Z}}(fG_i,H) \cong H^{t(i)} \subseteq \oplus_1^\infty(\mathbb{Z}/2)$$

which implies that H is also a direct sum of a countable number of copies of $\mathbb{Z}/2$. On the other hand, if tG_i is nontrivial then without loss of generality we may assume that it is a sum of $u(i) < \infty$ copies of $\mathbb{Z}/2$, and hence

$$\mathrm{Hom}_{\mathbb{Z}}(tG_i,H) \cong \oplus_1^{u(i)} H[2] \subseteq \oplus_1^\infty(\mathbb{Z}/2).$$

In either situation it is clear that we may assume without loss of generality that H is a torsion group with all torsion of order 2. Thus H is a vector space over $\mathbb{Z}/2$. It must be of infinite dimension for us to have a chance of success, and so without loss of generality we may assume $H = \oplus_1^\infty(\mathbb{Z}/2)$.

So we are reduced down to consideration of the inverse sequence

$$A_i \cong \mathrm{Hom}_{\mathbb{Z}}(G_i, \oplus_1^\infty(\mathbb{Z}/2))$$

as the only remaining possibility.

Consider the commuting diagram

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ 0 & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(G_i/G_{i-1}, \oplus_1^\infty(\mathbb{Z}/2)) \\ \downarrow & & \downarrow \\ A_i & \xrightarrow{\cong} & \mathrm{Hom}_{\mathbb{Z}}(G_i, \oplus_1^\infty(\mathbb{Z}/2)) \\ \downarrow & & \downarrow \\ A_{i-1} & \xrightarrow{\cong} & \mathrm{Hom}_{\mathbb{Z}}(G_{i-1}, \oplus_1^\infty(\mathbb{Z}/2)). \end{array}$$

The right column is exact as it is part of the 6-term Hom-Ext sequence. The left column is exact by the definition of the A_i and the two lower horizontal arrows are isomorphisms making the diagram commute by assumption. Thus

$$\mathrm{Hom}_{\mathbb{Z}}(G_i/G_{i-1}, \oplus_1^\infty(\mathbb{Z}/2)) = 0$$

for each i, and since the G_i are finitely generated abelian groups we may assume that each G_i/G_{i-1} is a finite group of odd order. By induction we see that G_i/G_0 must also be a finite group of odd order. So $G_i = G_0 \oplus K_i$ with each K_i of odd order. Then

$$\mathrm{Hom}_{\mathbb{Z}}(K_i, H) \cong 0$$

since H has only 2-torsion. So there is a natural isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(G_i, H) \cong \mathrm{Hom}_{\mathbb{Z}}(G_0, H)$$

which respects the given maps $G_{i-1} \to G_i$. Thus

$$\varprojlim{}^1 A_i \cong \varprojlim{}^1 \mathrm{Hom}_{\mathbb{Z}}(G_i, H) \cong \varprojlim{}^1 \mathrm{Hom}_{\mathbb{Z}}(G_0, H) = 0$$

because the inverse system $\mathrm{Hom}_{\mathbb{Z}}(G_0, H)$ is constant. □

In the remainder of this section we focus upon the structure of the groups

$$\varprojlim \mathrm{Ext}^1_{\mathbb{Z}}(G_i, H).$$

These groups arise in applications together with $\varprojlim^1$ as in Jensen's Theorem.

We note first that the group depends only upon tG and not upon G itself, since $\mathrm{Ext}^1_{\mathbb{Z}}(G,H) \cong \mathrm{Ext}^1_{\mathbb{Z}}(tG,H)$ whenever G is finitely generated:

Proposition 7.3. *Suppose that $G = \varinjlim\ G_i$ with each G_i finitely generated. Then*

$$\varprojlim \mathrm{Ext}^1_{\mathbb{Z}}(G_i,H) \ \cong\ \varprojlim \mathrm{Ext}^1_{\mathbb{Z}}(tG_i,H).$$

Proposition 7.4. *Suppose that*

$$G = \varinjlim\ G_i$$

with each G_i finitely generated and that H is finitely generated. Then

$$\varprojlim \mathrm{Ext}^1_{\mathbb{Z}}(G_i,H)$$

is profinite and reduced.

Proof. The group

$$\varprojlim \mathrm{Ext}^1_{\mathbb{Z}}(G_i,H)$$

is a subgroup of the group

$$\Pi_i \mathrm{Ext}^1_{\mathbb{Z}}(G_i,H).$$

Thus to prove the result it is enough to show that if G and H are cyclic groups then $\mathrm{Ext}^1_{\mathbb{Z}}(G,H)$ is finite and reduced, and this is clear. □

Proposition 7.5. *Suppose that $\{G_i\}$ is a direct sequence of abelian groups and H is an (algebraically) compact abelian group. Then the group*

$$\varprojlim \mathrm{Ext}^1_{\mathbb{Z}}(G_i,H)$$

is reduced and (algebraically) compact.

Proof. Suppose first that H is compact. Then each $\mathrm{Ext}^1_{\mathbb{Z}}(G_i,H)$ is compact and the maps

$$\mathrm{Ext}^1_{\mathbb{Z}}(G_{i+1},H) \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(G_i,H)$$

are continuous, by Proposition 2.4(4). A continuous inverse limit of compact groups is again compact. If H is only algebraically compact then we may write $H \oplus H' \cong K$ for some compact group K. Then

$$\mathrm{Ext}^1_{\mathbb{Z}}(G_i,H) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(G_i,H') \ \cong\ \mathrm{Ext}^1_{\mathbb{Z}}(G_i,K).$$

As before, the groups $\mathrm{Ext}^1_{\mathbb{Z}}(G_i,K)$ are compact and the structural maps are continuous, so that $\varprojlim \mathrm{Ext}^1_{\mathbb{Z}}(G_i,K)$ is compact. Further, the projection $K \to H$ induces a continuous retraction of the inclusion

$$\{\mathrm{Ext}^1_{\mathbb{Z}}(G_i,H)\} \hookrightarrow \{\mathrm{Ext}^1_{\mathbb{Z}}(G_i,K)\}$$

and hence $\{\mathrm{Ext}^1_{\mathbb{Z}}(G_i, H)\}$ is a continuous inverse sequence of algebraically compact groups. Arguing as in Proposition 4.2 we see that $\varprojlim \mathrm{Ext}^1_{\mathbb{Z}}(G_i, H)$ is reduced and algebraically compact. (See also [24], §39.4). $\square$

8. Decoupling

In this section we consider the computation of $\mathrm{Pext}^1_{\mathbb{Z}}(G,H)$ when G and H are general abelian groups. The goal is to reduce down to calculations that only involve torsion subgroups and torsionfree quotient groups, and, up to group extension, this is the result. Theorem 8.11 is the complete statement.

If G and H are split [22] groups, (and this is, of course, a common situation), then these results are unnecessary; one proceeds at once to the subsequent sections.

We believe that the non-attributed results in Sections 8-11 are new, but please keep in mind our general caution regarding attribution.

Definition 8.1. Given abelian groups G and H, let

$$D(G,H) \subset \mathrm{Pext}^1_{\mathbb{Z}}(G,H)$$

denote the maximal divisible subgroup of $\mathrm{Pext}^1_{\mathbb{Z}}(G,H)$ and let

$$R(G,H) = \mathrm{Pext}^1_{\mathbb{Z}}(G,H)/D(G,H)$$

denote the maximal reduced quotient group of $\mathrm{Pext}^1_{\mathbb{Z}}(G,H)$.

The canonical short exact sequence

$$0 \to D(G,H) \to \mathrm{Pext}^1_{\mathbb{Z}}(G,H) \to R(G,H) \to 0$$

splits (unnaturally), of course, and so $\mathrm{Pext}^1_{\mathbb{Z}}(G,H)$ is determined up to (unnatural) isomorphism by the groups $D(G,H)$ and $R(G,H)$.

The short exact sequence

$$0 \to tG \to G \to fG \to 0$$

is pure and hence it induces both long exact Hom-Ext and Hom-Pext sequences. Let ∂_E and ∂_P denote the connecting homomorphisms in the two sequences respectively.

[22] A group is said to be *split* if its torsion subgroup is a direct summand. For instance, finitely generated groups, divisible groups, and (obviously) torsionfree groups have this property. Kaplansky has shown that if tG is bounded then G is split. (Actually he proved this for modules over Dedekind domains.) It is a major open problem to characterize and classify all such groups (cf. [25], §100.) Here is an example of a non-split group from Fuchs [25]. Let $\mathcal{P}$ be an infinite collection of distinct primes and define

$$G = \prod_{p\in\mathcal{P}} \mathbb{Z}/p$$

Then

$$tG = \bigoplus_{p\in\mathcal{P}} \mathbb{Z}/p$$

and G is not split.

Proposition 8.2. *For all groups G and H,*

$$D(G,H) \;\cong\; \operatorname{Coker}(\partial_P) \cong \operatorname{Coker}(\partial_E) \tag{8.3}$$

$$R(G,H) \;\cong\; \operatorname{Pext}^1_{\mathbb{Z}}(tG,H) \;\cong\; \operatorname{Pext}^1_{\mathbb{Z}}(tG,tH) \;\cong\; R(tG,tH). \tag{8.4}$$

Proof. Consider the following commutative diagram, where the horizontal arrows indicate the inclusion of Pext into Ext:

$$\begin{array}{ccc}
\operatorname{Hom}_{\mathbb{Z}}(tG,H) & \xrightarrow{\cong} & \operatorname{Hom}_{\mathbb{Z}}(tG,H) \\
\Big\downarrow \partial_P & & \Big\downarrow \partial_E \\
\operatorname{Pext}^1_{\mathbb{Z}}(fG,H) & \xrightarrow{\cong} & \operatorname{Ext}^1_{\mathbb{Z}}(fG,H) \\
\Big\downarrow & & \Big\downarrow \\
\operatorname{Pext}^1_{\mathbb{Z}}(G,H) & \longrightarrow & \operatorname{Ext}^1_{\mathbb{Z}}(G,H) \\
\Big\downarrow \theta_P^* & & \Big\downarrow \theta_E^* \\
\operatorname{Pext}^1_{\mathbb{Z}}(tG,H) & \longrightarrow & \operatorname{Ext}^1_{\mathbb{Z}}(tG,H) \\
\Big\downarrow & & \Big\downarrow \\
0 & & 0.
\end{array} \tag{8.5}$$

Each column is exact, and so it is immediate that

$$\operatorname{Coker}(\partial_P) \;\cong\; \operatorname{Coker}(\partial_E).$$

Furthermore, we may unsplice the left column to obtain the short exact sequence

$$0 \to \operatorname{Coker}(\partial_P) \to \operatorname{Pext}^1_{\mathbb{Z}}(G,H) \xrightarrow{\theta_P^*} \operatorname{Pext}^1_{\mathbb{Z}}(tG,H) \to 0. \tag{8.6}$$

The group $\operatorname{Ext}^1_{\mathbb{Z}}(tG,H)$ is reduced by Proposition 2.4, since tG is a torsion group. The group $\operatorname{Pext}^1_{\mathbb{Z}}(tG,H)$ is a subgroup of a reduced group, hence reduced. The group

$$\operatorname{Pext}^1_{\mathbb{Z}}(fG,H) \;\cong\; \operatorname{Ext}^1_{\mathbb{Z}}(fG,H)$$

is divisible, since fG is torsionfree. The group $\operatorname{Coker}(\partial_P)$ is thus divisible, since it is a quotient of the divisible group $\operatorname{Pext}^1_{\mathbb{Z}}(fG,H)$. Furthermore, it must be the maximal divisible subgroup of the group $\operatorname{Pext}^1_{\mathbb{Z}}(G,H)$, since the quotient group

$$\frac{\operatorname{Pext}^1_{\mathbb{Z}}(G,H)}{\operatorname{Coker}(\partial_P)} \;\cong\; \operatorname{Pext}^1_{\mathbb{Z}}(tG,H)$$

is reduced. This implies that

$$D(G,H) \cong \mathrm{Coker}(\partial_P)$$

and that

$$R(G,H) \cong \mathrm{Pext}^1_{\mathbb{Z}}(tG,H)$$

as required. The final isomorphism is immediate from the long exact sequence for Pext in the second variable and the fact that

$$\mathrm{Pext}^1_{\mathbb{Z}}(tG,fH) = 0$$

for all groups G and H. □

We note as a consequence that $\mathrm{Pext}^1_{\mathbb{Z}}(G,H)$ is divisible if and only if

$$\mathrm{Pext}^1_{\mathbb{Z}}(tG,H) = 0,$$

which is a theorem of Huber and Meier [34].

Proposition 8.7. *For all groups G and H, suppose that one of the following conditions holds:*

1. *G or H is torsionfree.*
2. *tG is the sum of cyclic groups.*
3. *tG is countable with no elements of infinite height.*

Then

$$\mathrm{Pext}^1_{\mathbb{Z}}(G,H) \;\cong\; D(G,H).$$

In general,

$$D(G,H) \;\cong\; \mathrm{Ker}(\theta_E^*) \;\cong\; \mathrm{Ker}(\theta_P^*).$$

Proof. Part 1) is immediate from (8.4). For Part 2) we note that

$$R(G,H) = R(tG,tH) \subseteq \mathrm{Pext}^1_{\mathbb{Z}}(tG,tH)$$

which is trivial by Proposition 5.4. Part 3) follows from 2) and the fact (due to Prüfer [51] and Baer [4] and demonstrated in [24], §17.2) that a countable group with no elements of infinite height must be a sum of cyclic groups.

In the general situation we refer to Diagram 8.5. Then

$$\begin{aligned} D(G,H) &\cong \mathrm{Coker}(\partial_E) && \text{by Proposition 8.2} \\ &\cong \mathrm{Ker}(\theta_P^*) && \text{by Diagram 8.5} \\ &\cong \mathrm{Ker}(\theta_E^*) && \text{by Theorem 5.8} \end{aligned}$$

as required. □

Proposition 8.8. *For all abelian groups G and H, there is a natural long exact sequence*

$$0 \to \frac{\mathrm{Hom}_{\mathbb{Z}}(G, fH)}{\mathrm{Im}\,\mathrm{Hom}_{\mathbb{Z}}(G, H)} \to D(G, tH) \xrightarrow{\sigma_D} D(G, H) \to D(G, fH) \to 0.$$

If H is split then σ_D is a monomorphism which splits unnaturally.

Proof. The commutative diagram

$$\begin{array}{ccccccccc}
 & & & & \mathrm{Hom}_{\mathbb{Z}}(G, H) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \mathrm{Hom}_{\mathbb{Z}}(G, fH) & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & D(G, tH) & \longrightarrow & \mathrm{Pext}^1_{\mathbb{Z}}(G, tH) & \longrightarrow & R(G, tH) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sigma_D} & & \downarrow{\scriptstyle \sigma_P} & & \downarrow{\scriptstyle \sigma_R} & & \\
0 & \longrightarrow & D(G, H) & \longrightarrow & \mathrm{Pext}^1_{\mathbb{Z}}(G, H) & \longrightarrow & R(G, H) & \longrightarrow & 0 \\
 & & & & \downarrow & & & & \\
 & & & & \mathrm{Pext}^1_{\mathbb{Z}}(G, fH) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

has exact rows and columns, and the map σ_R is an isomorphism. The six term Snake sequence degenerates to the isomorphisms

$$\mathrm{Ker}(\sigma_D) \xrightarrow{\cong} \mathrm{Ker}(\sigma_P)$$

and

$$\mathrm{Coker}(\sigma_D) \xrightarrow{\cong} \mathrm{Coker}(\sigma_P).$$

Now

$$\mathrm{Ker}(\sigma_P) \;\cong\; \frac{\mathrm{Hom}_{\mathbb{Z}}(G, fH)}{\mathrm{Im}\ \mathrm{Hom}_{\mathbb{Z}}(G, H)}$$

and

$$\mathrm{Coker}(\sigma_P) \;\cong\; \mathrm{Pext}^1_{\mathbb{Z}}(G, fH) \;\cong\; D(G, fH)$$

by Proposition 8.2 and hence after making identifications the natural long exact sequence

$$0 \to \mathrm{Ker}(\sigma_D) \to D(G, tH) \xrightarrow{\sigma_D} D(G, H) \to \mathrm{Coker}(\sigma_D) \to 0$$

becomes the sequence 8.8 as desired. □

Proposition 8.9. *For all abelian groups G and H there is a short exact sequence*

$$0 \longrightarrow \frac{\mathrm{Hom}_{\mathbb{Z}}(tG,tH)}{\mathrm{Im}\ \mathrm{Hom}_{\mathbb{Z}}(G,tH)} \longrightarrow D(fG,tH) \xrightarrow{\tau_D} D(G,tH) \longrightarrow 0,$$

and in addition there is a natural isomorphism

$$D(G,fH) \;\cong\; D(fG,fH).$$

Proof. Consider the commuting diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D(fG,tH) & \longrightarrow & \mathrm{Pext}^1_{\mathbb{Z}}(fG,tH) & \longrightarrow & R(fG,tH) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tau_D} & & \downarrow{\scriptstyle \tau_P} & & \downarrow{\scriptstyle \tau_R} & & \\
0 & \longrightarrow & D(G,tH) & \longrightarrow & \mathrm{Pext}^1_{\mathbb{Z}}(G,tH) & \longrightarrow & R(G,tH) & \longrightarrow & 0.
\end{array}$$

The group $R(fG,tH) = 0$ since fG is torsionfree and hence the associated six term Snake sequence degenerates to the isomorphism

$$\mathrm{Ker}(\tau_D) \;\cong\; \mathrm{Ker}(\tau_P)$$

and the short exact sequence

$$(8.10) \qquad 0 \to \mathrm{Coker}(\tau_D) \to \mathrm{Coker}(\tau_P) \to R(G,tH) \to 0.$$

As τ_P lies in the long exact Hom-Pext sequence, both of the groups $\mathrm{Ker}(\tau_P)$ and $\mathrm{Coker}(\tau_P)$ are easily identified:

$$\mathrm{Ker}(\tau_P) \;\cong\; \frac{\mathrm{Hom}_{\mathbb{Z}}(tG,tH)}{\mathrm{Im}\ \mathrm{Hom}_{\mathbb{Z}}(G,tH)}$$

and

$$\mathrm{Coker}(\tau_P) \;\cong\; \mathrm{Pext}^1_{\mathbb{Z}}(tG,tH).$$

Thus the sequence 8.10 is isomorphic to the sequence

$$0 \to \mathrm{Coker}(\tau_D) \to \mathrm{Pext}^1_{\mathbb{Z}}(tG,tH) \xrightarrow{\tau'} R(G,tH) \to 0.$$

The map τ' is an isomorphism by Proposition 8.2 and this implies that

$$\mathrm{Coker}(\tau_D) \;\cong\; \mathrm{Coker}(\tau_P) \;\cong\; 0$$

so the natural sequence

$$0 \to \mathrm{Ker}(\tau_D) \to D(fG,tH) \xrightarrow{\tau_D} D(G,tH) \to 0$$

becomes the exact sequence

$$0 \to \frac{\mathrm{Hom}_{\mathbb{Z}}(tG,tH)}{\mathrm{Im}\ \mathrm{Hom}_{\mathbb{Z}}(G,tH)} \to D(fG,tH) \xrightarrow{\tau_D} D(G,tH) \to 0,$$

as required. □

The results above combine as follows:

Theorem 8.11. *Let G and H be abelian groups. Define obstruction groups $\Psi_1(G,H)$ and $\Psi_2(G,H)$ by*

$$\Psi_1(G,H) \;=\; \frac{\mathrm{Hom}_{\mathbb{Z}}(tG,tH)}{\mathrm{Im}\,\mathrm{Hom}_{\mathbb{Z}}(G,tH)}$$

and

$$\Psi_2(G,H) \;=\; \frac{\mathrm{Hom}_{\mathbb{Z}}(G,fH)}{\mathrm{Im}\,\mathrm{Hom}_{\mathbb{Z}}(G,H)}\;.$$

Then there is a diagram with exact rows and columns of the form

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 & & \Psi_1(G,H) & & & & \\
 & & \downarrow & & & & \\
 & & D(fG,tH) & & 0 & & D(fG,fH) \\
 & & \downarrow & & \downarrow & & \downarrow \cong \\
0 \to \Psi_2(G,H) & \longrightarrow & D(G,tH) & \xrightarrow{\sigma_D} & D(G,H) & \longrightarrow & D(G,fH) \to 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & \mathrm{Pext}^1_{\mathbb{Z}}(G,H) & & \\
 & & & & \downarrow & & \\
 & & & & R(tG,tH) & & \\
 & & & & \downarrow & & \\
 & & & & 0 & &
\end{array}$$

and thus $\mathrm{Pext}^1_{\mathbb{Z}}(G,H)$ is determined up to group extension by the three groups

$$D(fG,fH), \qquad D(fG,tH), \qquad \textit{and} \qquad R(tG,tH)$$

and by the two obstruction groups $\Psi_1(G,H)$ and $\Psi_2(G,H)$.

Remark 8.12. We note that the following are equivalent:

1. The group G is split.
2. The group $\Psi_1(G,G) = 0$.
3. For all groups H, $\Psi_1(G,H) = 0$.

Similarly that the following are equivalent:

1. The group H is split.
2. The group $\Psi_2(H, H) = 0$.
3. For all groups G, $\Psi_2(G, H) = 0$.

Thus the two obstruction groups vanish if and only if both G and H are split.

Proof of Theorem 8.11. The horizontal long exact sequence is the conclusion of Proposition 8.8 and the left vertical sequence is an application of Proposition 8.9. The middle vertical sequence is exact by the definition of D and R. Finally, the right vertical isomorphism is immediate from the fact that $R(G, K) = 0$ whenever K is torsionfree and similarly $D(fG, fH) = 0$. □

Finally, we note that Goeters and Keef have found complicated necessary and sufficient conditions for $\mathrm{Pext}^1_{\mathbb{Z}}(G, H) = 0$. We note a nice corollary of their result.

Theorem 8.13 ([27]). *Suppose that G and H are countable abelian groups. Then the following are equivalent:*

1. $\mathrm{Pext}^1_{\mathbb{Z}}(G, H) = 0$.
2. $\mathrm{Pext}^1_{\mathbb{Z}}(G, fH) = 0$ *and* $\mathrm{Pext}^1_{\mathbb{Z}}(G, tH) = 0$.

The implication 2) $\Rightarrow$ 1) is immediate from the long exact sequence for Pext. If 1) holds then $\mathrm{Pext}^1_{\mathbb{Z}}(G, fH) = 0$ also by the long exact sequence, so it suffices to show that $\mathrm{Pext}^1_{\mathbb{Z}}(G, tH) = 0$. Our Theorem 8.11 implies that

$$D(tG, H) = D(fG, tH) \qquad \text{and} \qquad R(G, tH) = R(tG, tH) = 0$$

so it suffices to show that $D(fG, tH) = 0$. Theorem 10.1 implies that

$$D(fG, tH) \cong \mathbb{Q}^n$$

with n either zero or countably infinite. To show that $n = 0$ is harder, and for this we refer the reader to [27].

9. $\mathrm{Pext}^1_{\mathbb{Z}}(G, H)$ for H torsionfree

This section is devoted to the calculation of the group $\mathrm{Pext}^1_{\mathbb{Z}}(G, H)$ when H is assumed to be torsionfree. In that case,

$$\mathrm{Pext}^1_{\mathbb{Z}}(G, H) \cong D(G, H) \cong D(fG, H)$$

by Theorem 8.11. Since fG is torsionfree, we have

$$\mathrm{Pext}^1_{\mathbb{Z}}(fG, H) \cong \mathrm{Ext}^1_{\mathbb{Z}}(fG, H)$$

by Corollary 5.5. Finally, any divisible group must be a direct sum of copies of the groups $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$. Thus we have established the following result:

Theorem 9.1. *Suppose that G and H are abelian groups with H torsionfree. Then*

$$\mathrm{Pext}^1_{\mathbb{Z}}(G, H) \cong \mathrm{Ext}^1_{\mathbb{Z}}(fG, H) \cong \left(\bigoplus_{n_o} \mathbb{Q}\right) \bigoplus_p \left(\bigoplus_{n_p} \mathbb{Z}(p^\infty)\right)$$

where p runs over all primes and the cardinals n_o and n_p count the multiplicities of the summands $\mathbb{Q}$ and the various $\mathbb{Z}(p^\infty)$.

Thus the computation of $\mathrm{Pext}^1_{\mathbb{Z}}(G, H)$ for H torsionfree reduces down to the determination of $\mathrm{Ext}^1_{\mathbb{Z}}(G, H)$ where both G and H are torsionfree. Unfortunately this is an unsolved problem in general.[23]

Suppose that G and H are torsionfree, G is countable, and

$$\mathrm{Ext}^1_{\mathbb{Z}}(G, H) = \mathrm{Pext}^1_{\mathbb{Z}}(G, H) \neq 0.$$

Then this group must be uncountable. This follows by combining Jensen's Theorem 6.1 with Brayton Gray's result 3.5. There is a parallel, very overlapping result for $\mathrm{Ext}^1_{\mathbb{Z}}(G, H)$ under somewhat different hypotheses due to Warfield [62], Theorem 4.

Proposition 9.2 (Warfield)**.** *Suppose that G and H are torsionfree groups of finite rank and $\mathrm{Ext}^1_{\mathbb{Z}}(G, H) \neq 0$. Then this group has cardinality $\geq \mathfrak{c}$.*

We want more specific results, and we start with some examples (all found in [24] (or assigned there as homework)).

Proposition 9.3.
1. $\mathrm{Pext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \cong \mathbb{Q}^{\aleph_o}$.
2. $\mathrm{Pext}^1_{\mathbb{Z}}(\mathbb{Z}_p, \mathbb{Z}) \cong \mathbb{Q}^{\aleph_o} \oplus \mathbb{Z}(p^\infty)$.
3. $\mathrm{Pext}^1_{\mathbb{Z}}(\widehat{\mathbb{Z}}_p, \mathbb{Z}) \cong \mathbb{Q}^{2^{\aleph_o}} \oplus \mathbb{Z}(p^\infty)$.
4. $\mathrm{Pext}^1_{\mathbb{Z}}(\widehat{\mathbb{Z}}_p, \widehat{\mathbb{Z}}_p) \cong \mathbb{Q}^{2^{\aleph_o}} \oplus \mathbb{Z}(p^\infty)$.

[23] One should keep in mind that torsionfree groups themselves are not classified!

5. $\text{Pext}^1_{\mathbb{Z}}(\mathbb{Z}_p, \mathbb{Z}_p) \;=\; 0.$
6. $\text{Pext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}_p) \;\cong\; \mathbb{Q}^{\aleph_o}.$

Proof. We shall prove 1), leaving the others as exercises (cf. [24], p. 226). As $\mathbb{Q}$ is torsionfree,

$$\text{Pext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \;\cong\; \text{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$$

and this group is divisible. The injective resolution of $\mathbb{Z}$

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0 \tag{9.4}$$

yields a long exact sequence

$$0 \to \text{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}) \to \text{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \to \text{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \to 0$$

and since $\text{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}) \cong \mathbb{Q}$,

$$\text{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \;\cong\; \frac{\text{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z})}{\mathbb{Q}}.$$

The group $\text{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z})$ is divisible by Proposition 2.1, hence the group $\text{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$ is torsionfree and divisible, and so

$$\text{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \;\cong\; \bigoplus_m \mathbb{Q}$$

and it remains to determine m.

Using the sequence 9.4 in the other variable yields a short exact sequence

$$0 \to \text{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \to \text{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \to \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \to 0$$

which simplifies to

$$0 \to \Pi_p \widehat{\mathbb{Z}}_p \to \text{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z} \to 0.$$

and since $\Pi_p \widehat{\mathbb{Z}}_p$ has cardinality $2^{\aleph_o}$ it follows that $m = 2^{\aleph_o}$. □

There are two general results which improve the early result of B. Gray 3.11.

Theorem 9.5 (Warfield [62])**.** *Suppose that G and H are countable abelian groups, H is torsionfree, and*

$$\text{Pext}^1_{\mathbb{Z}}(G, H) \neq 0.$$

Then $\text{Pext}^1_{\mathbb{Z}}(G, H)$ *contains $n_o = 2^{\aleph_o}$ summands of the group* $\mathbb{Q}$*.*

Theorem 9.6 (Warfield, [62], Theorem 2)**.** *Suppose that G and H are torsionfree abelian groups of finite rank. Then*

1. *The group* $\text{Ext}^1_{\mathbb{Z}}(G, H) = 0$ *if and only if $G \otimes \mathbb{Z}_H$ is a free $\mathbb{Z}_H$-module. Otherwise, the group has $n_o = c$.*

2. *The p-torsion subgroup of* $\mathrm{Ext}^1_{\mathbb{Z}}(G,H)$ *has p-rank*

$$r_p(G)r_p(H) - r_p(\mathrm{Hom}_{\mathbb{Z}}(G,H)).$$

The case of $\mathrm{Pext}^1_{\mathbb{Z}}(G,\mathbb{Z})$ for G countable has been analyzed completely by C. U. Jensen, as follows.

Theorem 9.7 ([38]). *Suppose that* $\{\Gamma_i\}$ *is an inverse sequence of finitely generated abelian groups. Let*

$$G = \varinjlim \mathrm{Hom}_{\mathbb{Z}}(\Gamma_i/t\Gamma_i, \mathbb{Z}).$$

Then G *is countable, torsionfree, and*

$$\varprojlim{}^1 \Gamma_i \;\cong\; \mathrm{Pext}^1_{\mathbb{Z}}(G,\mathbb{Z}) \;\cong\; \mathrm{Ext}^1_{\mathbb{Z}}(G,\mathbb{Z}).$$

Further:

1. *For any countable torsionfree group* G*, if* $\mathrm{Pext}^1_{\mathbb{Z}}(G,\mathbb{Z}) \neq 0$*, then the number* n_o *of direct summands of type* $\mathbb{Q}$ *that it contains is* $2^{\aleph_o}$*.*
2. *For each prime* p*, the number* n_p *of summands of type* $\mathbb{Z}(p^\infty)$ *that* $\mathrm{Pext}^1_{\mathbb{Z}}(G,H)$ *contains is either finite or* $2^{\aleph_o}$*.*

Conversely, given cardinals n_o*,* n_p *which satisfy Conditions* 1) *and* 2)*, there is a countable torsionfree abelian group* G *such that* $\mathrm{Pext}^1_{\mathbb{Z}}(G,H)$ *has this decomposition.*

The following result generalizes Jensen's result.

Theorem 9.8 (Huber and Warfield [35]). *Suppose that* G *and* H *are torsionfree abelian groups with* G *of countable rank and* H *of rank* 1*. If*

$$\mathrm{Ext}^1_{\mathbb{Z}}(G,H) \neq 0$$

then for each prime p*, the number* $r_p(\mathrm{Ext}^1_{\mathbb{Z}}(G,H))$ *is either finite or* $2^{\aleph_o}$*.*

The case $\mathrm{Ext}^1_{\mathbb{Z}}(G,\mathbb{Z})$ for G uncountable has been analyzed by Hiller, Huber, and Shelah [33]. Their work uses similar tools to those used by Shelah in his solution to Whitehead's problem using Gödel's Axiom of Constructibility. For instance, they show:

Proposition 9.9. *Suppose that* G *is an abelian group with* $\mathrm{Ext}^1_{\mathbb{Z}}(G,\mathbb{Z})$ *nontrivial and divisible. Then its torsionfree rank is* 2^μ *for some infinite cardinal* μ*.*

Shelah also constructs an example of an uncountable group G such that $\mathrm{Ext}^1_{\mathbb{Z}}(G,\mathbb{Z})$ does not satisfy the conditions of Theorem 9.7.

Finally, we note their compactness results. The assumption $V = L$ is a logic assumption.

Theorem 9.10 ([33]). 1. *Suppose that* $V = L$. *If* $\mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Z}) = 0$ (*or, equivalently, if* G *has no free summand*), *then the group* $\mathrm{Ext}^1_{\mathbb{Z}}(G, \mathbb{Z})$ *admits a compact topology.*

2. *If* A *is a compact abelian group then there exists an abelian group* G *with* $\mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Z}) = 0$ *and* $\mathrm{Ext}^1_{\mathbb{Z}}(G, \mathbb{Z}) = A$.

Harrison showed [31] that any compact abelian group is a product of groups of the form $\mathbb{Z}/n$, $\widehat{\mathbb{Z}}_p$, $\mathbb{R}$, $\mathbb{Z}(p^\infty)$, with the condition that for each prime p the number of copies of $\mathbb{Z}(p^\infty)$ must not exceed the number of copies of $\mathbb{R}$.

10. $\mathrm{Pext}^1_{\mathbb{Z}}(G,H)$ for G torsionfree and H torsion

If the group G is torsionfree then we know (Theorem 8.11) that

$$\mathrm{Pext}^1_{\mathbb{Z}}(G,H) \;\cong\; \mathrm{Ext}^1_{\mathbb{Z}}(G,H) \;\cong\; D(G,H)$$

and hence once again is determined by the multiplicity numbers. If G is of finite rank[24] then the following result gives a good general picture of the situation.

Theorem 10.1. *Suppose that G is a torsionfree abelian group and that H is a torsion group. In addition, suppose either:*

a) *G is of finite rank, or*
b) *H is countable.*

Then

$$Pext^1_{\mathbb{Z}}(G,H) \;\cong\; \mathbb{Q}^n$$

with either $n = 0$ or $n \geq \aleph_o$.

Proof. For Part a), See [24], p. 232, Ex. 14bc. For Part b), see [20], p. 448, who credit Baer. □

Eklof and Huber also give the following sharp information:

Theorem 10.2. *Suppose that G is a countable torsionfree abelian group and H is an abelian group. Then the following are equivalent:*

1. $\mathrm{Ext}^1_{\mathbb{Z}}(G,H) = 0$.
2. *There exists an ascending sequence of pure subgroups of G*

$$G_0 \subseteq G_i \subseteq \cdots \subseteq G_n \cdots, \qquad \underset{i}{\cup} G_i = G,$$

with $G_0 = 0$, $\mathrm{rank}(G_{n+1}/G_n) = 1$, and

$$\mathrm{Ext}^1_{\mathbb{Z}}(G_{n+1}/G_n, H) = 0.$$

Eklof and Huber note the following corollary.

Corollary 10.3. *Suppose that G is a countable torsionfree group and H is arbitrary. If*

$$\mathrm{Pext}^1_{\mathbb{Z}}(G', H) = 0$$

for all finite rank subgroups $G' \subset G$, then

$$\mathrm{Pext}^1_{\mathbb{Z}}(G,H) = 0.$$

[24] Recall that if G is torsionfree then its (torsionfree) rank is the $\mathbb{Q}$-dimension of the vector space $G \otimes \mathbb{Q}$. The map $\mathbb{Q} \to G \otimes \mathbb{Q}$ is an inclusion, and the rank may also be described as the dimension of the $\mathbb{Q}$-vector space of minimal dimension which contains a copy of G as a subgroup.

It is striking that the structure of H seems to play no role at this level. Of course we have no way in Theorem 10.1 to determine which of the extreme alternatives occurs — perhaps this depends on some deep properties of H.

There is an obvious conjecture that emerges from these results which would clarify the general situation.

Conjecture 10.4. *Suppose that G is a torsionfree abelian group and H is a torsion group. Then*

$$\mathrm{Pext}^1_{\mathbb{Z}}(G,H) \;\cong\; \mathbb{Q}^n$$

with either $n = 0$ or $n \geq \aleph_o$.

We are not expert in these matters, and so perhaps there is an elementary counterexample. What we would need to discover is some information concerning the map

$$\Theta : \mathrm{Pext}^1_{\mathbb{Z}}(G,H) \longrightarrow \varprojlim \mathrm{Pext}^1_{\mathbb{Z}}(G',H)$$

where G' runs over all finite rank subgroups of G. This might be of some interest for all abelian groups G and H. Note that Corollary 10.3 tells us that if G is torsionfree then Θ is mono.

11. $\mathrm{Pext}^1_{\mathbb{Z}}(G, H)$ for both G and H torsion

The remaining case is when G and H are both torsion groups. Our principal result is Theorem 11.7 which determines $\mathrm{Pext}^1_{\mathbb{Z}}(G, H)$ when G and H are p-groups and G has finite Ulm length.

If G is a torsion group then $\mathrm{Ext}^1_{\mathbb{Z}}(G, H)$ is reduced by Proposition 2.4 which implies that $\mathrm{Pext}^1_{\mathbb{Z}}(G, H)$ is reduced. Thus we have the following elementary fact.

Proposition 11.1. *If G is a torsion group then*

$$\mathrm{Pext}^1_{\mathbb{Z}}(G, H) = R(G, H).$$

Assume that G and H are torsion groups, and let $G(p)$ and $H(p)$ denote their p-primary subgroups. Then

$$\begin{aligned}\mathrm{Pext}^1_{\mathbb{Z}}(G, H) &\cong \mathrm{Pext}^1_{\mathbb{Z}}(\oplus_p G(p), H)\\ &\cong \Pi_p \mathrm{Pext}^1_{\mathbb{Z}}(G(p), H)\\ &\cong \Pi_p \mathrm{Pext}^1_{\mathbb{Z}}(G(p), H(p))\end{aligned}$$

so that the determination of the group $\mathrm{Pext}^1_{\mathbb{Z}}(G, H)$ comes down to a determination of the various groups $\mathrm{Pext}^1_{\mathbb{Z}}(G(p), H(p))$. Thus without loss of generality we may focus attention upon the case where both G and H are torsion groups with all torsion having orders various powers of some fixed prime p. For brevity we refer to such a group as a *p-group.*

Let

$$\widehat{H} \;\cong\; \varprojlim H/p^n H$$

denote the p-adic completion of the abelian group H. Note that the p-adic and $\mathbb{Z}$-adic topologies coincide on p-groups, so that $\widehat{H}$ is isomorphic to the $\mathbb{Z}$-adic completion of H (and the notation is not ambiguous). Let

$$\mu : H \to \widehat{H}$$

denote the natural map. It is elementary that

$$\mathrm{Ker}(\mu) \;\cong\; \cap_n p^n H = H^1$$

where H^1 denotes the first Ulm subgroup of H. Thus $\mu : H \to \widehat{H}$ is an inclusion if and only if H is Hausdorff. If so then the resulting short exact sequence

$$0 \to H \to \widehat{H} \to \widehat{H}/H \to 0$$

is pure ([24], §39.5) and $\widehat{H}/H$ is divisible. For any abelian group G, let

$$\Delta = \Delta_G \colon \mathrm{Hom}_{\mathbb{Z}}(G, \widehat{H}/H) \longrightarrow \mathrm{Pext}^1_{\mathbb{Z}}(G, H)$$

denote the resulting connecting homomorphism.

Proposition 11.2. *Let H be a p-group which is Hausdorff in the p-adic topology. Then*

$$\Delta_{\mathbb{Q}/\mathbb{Z}} \colon \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \widehat{H}/H) \longrightarrow \mathrm{Pext}^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, H)$$

is an isomorphism, and for each prime q the map

$$\Delta_{\mathbb{Z}(q^\infty)} \colon \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}(q^\infty), \widehat{H}/H) \longrightarrow \mathrm{Pext}^1_{\mathbb{Z}}(\mathbb{Z}(q^\infty), H)$$

is an isomorphism. Thus the maximal divisible subgroup of $\widehat{H}/H$ has no q-torsion except possibly for $p = q$.

Proof. The map $\Delta_{\mathbb{Q}/\mathbb{Z}}$ is an isomorphism by [24], §53.8. The natural isomorphism

$$\mathbb{Q}/\mathbb{Z} \;\cong\; \oplus_q \mathbb{Z}(q^\infty)$$

induces the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \widehat{H}/H) & \xrightarrow{\cong} & \Pi_q \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}(q^\infty, \widehat{H}/H) \\
{\scriptstyle \Delta_{\mathbb{Q}/\mathbb{Z}}}\big\downarrow & & \big\downarrow{\scriptstyle \Pi_q \Delta_{\mathbb{Z}(q^\infty)}} \\
\mathrm{Pext}^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, H) & \xrightarrow{\cong} & \Pi_q \mathrm{Pext}^1_{\mathbb{Z}}(\mathbb{Z}(q^\infty), H)
\end{array}$$

and the naturality of Δ implies that the map

$$\Pi_q \Delta_{\mathbb{Z}(q^\infty)}$$

is an isomorphism. It follows that $\Delta_{\mathbb{Z}(q^\infty)}$ is an isomorphism for each q. □

Next we consider the analogous question when H is not necessarily Hausdorff. We may assume without loss of generality that H is a reduced p-group. Then H has a *p-basic subgroup* B (cf. [24], §33). Such a subgroup is characterized up to isomorphism as a subgroup which satisfies the following three conditions:

1. B is a direct sum of cyclic groups.
2. B is a pure subgroup of H.
3. H/B is divisible.

Since H is a p-group, B has the form

$$B = \oplus_n B_n$$

where $n = 1, 2, 3, \ldots$, and B_n is the sum of $k(n)$ copies of the group $\mathbb{Z}/p^n$. (For convenience of notation we allow the case $k(n) = 0$.) Note that the group B is bounded if and only if $k(n) = 0$ for all but finitely many values of n.

The group H/B must be a direct sum of j copies of the group $\mathbb{Z}(p^\infty)$. However, even though B is uniquely determined up to isomorphism, the copy of B that we choose in H is not: in general there are many choices for B. The effect of this is that the group H/B depends upon our choice of B (see [24], Page 150). Thus the number j depends upon our choice of B.

Proposition 11.3. *Let B be a sum of cyclic p-groups as above. Then:*

1. *If B is bounded, then*

$$\mathrm{Pext}^1_{\mathbb{Z}}(\mathbb{Z}(p^\infty), B) \cong 0.$$

2. *If B is unbounded then*

$$\mathrm{Pext}^1_{\mathbb{Z}}(\mathbb{Z}(p^\infty), B) \;\cong\; (\oplus_{2^{\aleph_o}} \widehat{\mathbb{Z}}_p)\widehat{\ } \;\cong\; (\widehat{\mathbb{Z}}_p)^{\aleph_o}.$$

Proof. Any direct sum of cyclic groups is Hausdorff, so by Proposition 11.2 it suffices to determine the group

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}(p^\infty), \widehat{B}/B).$$

Fuchs ([24], p. 166, ex. 7) shows that $\widehat{B}/B$ is the maximal divisible subgroup of the group

$$\frac{\Pi_n B_n}{\oplus_n B_n}.$$

Fortunately this situation has been analyzed in great generality by Golema and Hulanicki [28] and we use their analysis below.

If B is bounded then $\widehat{B} = B$ and $\widehat{B}/B = 0$, establishing Part 1).

Now suppose that B is unbounded. Let

$$b_n = (1, 0, 0, \dots) \in B_n.$$

(If $B_n = 0$ then set $b_n = 0$.) The element b_n has order exactly p^n whenever $B_n \neq 0$. Thus for infinitely many n the element $p^{n-1}b_n$ has order exactly p and it is divisible by p^{n-1}. Define

$$b = \Pi p^{n-1} b_n \in \Pi_{n\in N} B_n.$$

Then b has order p and for any fixed t all but a finite number of the entries of b are divisible by p^t. Let $\bar{b}$ denote the image of b in the group

$$\frac{\Pi_n B_n}{\oplus_n B_n}.$$

We see that $\bar{b}$ has order exactly p and that for any t, $\bar{b}$ is divisible by p^t. It follows easily that in fact $\bar{b}$ is divisible by any positive integer. This forces the group $\frac{\Pi_n B_n}{\oplus_n B_n}$ to contain at least one copy of the group $\mathbb{Z}(p^\infty)$. In fact we have the following proposition.

Proposition 11.4 ([28]). *With the above notation, suppose that the group B is unbounded. Then:*

$$\widehat{B}/B \;\cong\; \left(\underset{2^{\aleph_o}}{\oplus}\mathbb{Q}\right) \oplus \left(\underset{2^{\aleph_o}}{\oplus}\mathbb{Z}(p^\infty)\right).$$

Thus

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}(p^\infty), \widehat{B}/B) \;\cong\; \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}(p^\infty), \oplus_{2^{\aleph_o}}\mathbb{Z}(p^\infty))$$

and hence (*by Fuchs* [24] 44.3)

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}(p^\infty), \widehat{B}/B) \;\cong\; \left(\bigoplus_{2^{\aleph_o}} \widehat{\mathbb{Z}}_p\right)^{\widehat{}}.$$

Finally,

$$\left(\bigoplus_{2^{\aleph_o}} \widehat{\mathbb{Z}}_p\right)^{\widehat{}} \cong (\widehat{\mathbb{Z}}_p)^{\aleph_o}$$

We note that the final conclusion follows since both groups are torsionfree, p-adic algebraically compact, and with the same basic p-adic module.

This completes the proof of Part 2). □

With this preparation we are able to establish the following proposition.

Proposition 11.5. *Suppose that H is a reduced p-group with p-basic subgroup B and $H/B \cong \oplus_j \mathbb{Z}(p^\infty)$. Then:*

1. *If B is bounded then*

$$\mathrm{Pext}^1_{\mathbb{Z}}(\mathbb{Z}(p^\infty), H) = 0.$$

2. *If B is unbounded, then*

$$\mathrm{Pext}^1_{\mathbb{Z}}(\mathbb{Z}(p^\infty), H) \;\cong\; \frac{\left(\bigoplus_{2^{\aleph_o}} \widehat{\mathbb{Z}}_p\right)^{\widehat{}}}{\Phi_B\left(\left(\bigoplus_j \widehat{\mathbb{Z}}_p\right)^{\widehat{}}\right)}$$

where Φ_B is a homomorphism defined below.

Proof. Applying the functor $\mathrm{Pext}^1_{\mathbb{Z}}(\mathbb{Z}(p^\infty), -)$ to the pure short exact sequence

$$0 \to B \to H \to H/B \to 0$$

yields a long exact Hom-Pext sequence. However, the group

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}(p^\infty), H) = 0$$

since H is reduced and

$$\mathrm{Pext}^1_{\mathbb{Z}}(\mathbb{Z}(p^\infty), H/B) = 0$$

since H/B is divisible. Thus the sequence degenerates to the exact sequence

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}(p^\infty), H/B) \to \mathrm{Pext}^1_{\mathbb{Z}}(\mathbb{Z}(p^\infty), B) \to \mathrm{Pext}^1_{\mathbb{Z}}(\mathbb{Z}(p^\infty), H) \to 0.$$

Using Proposition 11.3 and the natural isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}(p^\infty), H/B) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}(p^\infty), \oplus_j \mathbb{Z}(p^\infty)) \cong (\oplus_j \widehat{\mathbb{Z}}_p)\widehat{\ }$$

gives us the exact sequence

$$\left(\bigoplus_j \widehat{\mathbb{Z}}_p\right)^{\widehat{}} \xrightarrow{\Phi_B} \left(\bigoplus_{2^{\aleph_o}} \widehat{\mathbb{Z}}_p\right)^{\widehat{}} \longrightarrow \mathrm{Pext}^1_{\mathbb{Z}}(\mathbb{Z}(p^\infty), H) \to 0$$

as required. □

We emphasize that j and the map Φ_B depend upon the choice of the subgroup B in H. On the other hand, the group

$$\mathrm{Coker}(\Phi_B) \cong \mathrm{Pext}^1_{\mathbb{Z}}(\mathbb{Z}(p^\infty), H)$$

is obviously independent of the choice of B.

We recall from Definition 5.1 that each p-torsion group G has an Ulm sequence

$$\cdots G^3 \subseteq G^2 \subseteq G^1 \subseteq G$$

and G is said to have *finite Ulm length* if the sequence stabilizes after a finite number τ steps, in which case G^τ is the maximal divisible subgroup of G. For example, if G is algebraically compact then G has Ulm length 1, by Proposition 5.3.

There exist countable abelian groups of every finite Ulm length as well as countable abelian groups with infinite Ulm length (cf. [25], §76). Note that each G^s/G^{s+1} is a direct sum of cyclic groups. Thus

$$\mathrm{Pext}^1_{\mathbb{Z}}(G^s/G^{s+1}, H) = 0$$

for all s. It follows easily from the long exact sequence

$$\mathrm{Pext}^1_{\mathbb{Z}}(G^s/G^{s+1}, H) \to \mathrm{Pext}^1_{\mathbb{Z}}(G^s, H) \to \mathrm{Pext}^1_{\mathbb{Z}}(G^{s+1}, H) \to 0$$

that

$$\mathrm{Pext}^1_{\mathbb{Z}}(G^s, H) \cong \mathrm{Pext}^1_{\mathbb{Z}}(G^{s+1}, H)$$

via the natural map. Thus we have established the following theorem.

Theorem 11.6. *Let G be a p-group with finite Ulm length and with maximal divisible subgroup $\oplus_i \mathbb{Z}(p^\infty)$. Let H be a reduced p-group. Then*

$$\mathrm{Pext}^1_{\mathbb{Z}}(G,H) \;\cong\; \Pi_i \mathrm{Pext}^1_{\mathbb{Z}}(\mathbb{Z}(p^\infty),H).$$

In particular, if G is a reduced p-group with finite Ulm length and H is a reduced p-group, then

$$\mathrm{Pext}^1_{\mathbb{Z}}(G,H) \;\cong\; 0.$$

These results on the torsion-torsion case may be gathered up into the following omnibus theorem. The theorem is a generalization of a result of Harrison (cf. [24], §57.3) which deals with the case where H is Hausdorff.

Theorem 11.7. *Let G be a p-group with finite Ulm length. Let H be a reduced p-group with p-basic subgroup B and with*

$$H/B \;\cong\; \oplus_j \mathbb{Z}(p^\infty).$$

1. *If B is bounded or if G is reduced then*
$$\mathrm{Pext}^1_{\mathbb{Z}}(G,H) = 0.$$
2. *On the other hand, if B is unbounded and if G has maximal divisible subgroup $\oplus_i \mathbb{Z}(p^\infty) \neq 0$ then*
$$\mathrm{Pext}^1_{\mathbb{Z}}(G,H) \;\cong\; \prod_i \frac{\left(\bigoplus_{2^{\aleph_o}} \widehat{\mathbb{Z}}_p\right)^{\widehat{}}}{\Phi_B\left(\left(\bigoplus_j \widehat{\mathbb{Z}}_p\right)^{\widehat{}}\right)}.$$

We note that if we regard $\widehat{\mathbb{Z}}_p$ as a commutative ring then its indecomposable modules are $\widehat{\mathbb{Z}}_p$, $\widehat{\mathbb{Z}}_p/p^n$, the p-adic numbers $\widehat{\mathbb{Q}}_p$, and $\widehat{\mathbb{Q}}_p/\widehat{\mathbb{Z}}_p$ (cf. [40], p. 53). We suppose that the group

$$\prod_i \frac{\left(\bigoplus_{2^{\aleph_o}} \widehat{\mathbb{Z}}_p\right)^{\widehat{}}}{\Phi_B\left(\left(\bigoplus_j \widehat{\mathbb{Z}}_p\right)^{\widehat{}}\right)}$$

is decomposable as a ring (with its obvious ring structure) and (based on naive optimism) we expect that it will be the sum of (many!) copies of $\widehat{\mathbb{Z}}_p$.

Proof. As G has finite Ulm length, Proposition 11.6 reduces the computation down to the computation of the group $\mathrm{Pext}^1_{\mathbb{Z}}(\mathbb{Z}(p^\infty),H)$, and this group is dealt with in Proposition 11.5. □

The assumption that H is reduced is not really restrictive, since

$$\mathrm{Pext}^1_{\mathbb{Z}}(G,H) \cong \mathrm{Pext}^1_{\mathbb{Z}}(G,H/D)$$

where D is the maximal divisible subgroup of H.

The assumption that G has finite Ulm length is a real restriction. We know of no examples of explicit computations of Pext for groups G of infinite Ulm length.

We conclude this section with our strongest divisibility result on Pext.

Theorem 11.8. *Let G and H be countable abelian groups. Assume that $(tG)_p$ has finite Ulm length for each prime p and that tH is a reduced group. Let $B(p)$ denote the p-basic subgroup of $(tH)_p$. Then the following are equivalent:*

1. $\mathrm{Pext}^1_{\mathbb{Z}}(G,H)$ *is divisible.*
2. *For each prime p, either $B(p)$ is bounded or $(tG)_p$ is reduced.*

Proof. Fix some prime p. Then the second condition is equivalent to

$$\mathrm{Pext}^1_{\mathbb{Z}}((tG)_p,(tH)_p) = 0$$

by Theorem 11.7. From Section 8 and elementary considerations there are isomorphisms

$$R(G,H) \cong R(tG,tH) \cong \mathrm{Pext}^1_{\mathbb{Z}}(G,H) \cong \Pi_p \mathrm{Pext}^1_{\mathbb{Z}}((tG)_p,(tH)_p)$$

and the corollary follows at once. □

Finally, we note that perhaps the earliest study of these problems is represented by the work of Kulikoff and Papp (cf. [22], §34.6.)

12. $\mathrm{Pext}^1_{\mathbb{Z}}(G, H)$ and phantom maps

This section is a very brief introduction to phantom maps. It is based upon McGibbon's fine survey article [44] and we encourage the reader to consult that article for a much better and more extensive discussion. We aim here simply to give a taste of the subject. Below all maps and homotopies are understood to be base-point preserving.

Let X be a connected locally finite[25] CW-complex with n-skeleton X_n. A continuous map $f : X \to Y$ is said to be a *phantom map* if its restriction to each X_n is null-homotopic. If X is infinite-dimensional and Y has infinitely many nontrivial homotopy groups then there sometimes exist phantom maps which are nontrivial in homotopy. For example, there are uncountably many homotopy classes of phantom maps from $\mathbb{C}P^\infty$ to S^3.

Phantom maps are difficult to detect. For instance, if $f : X \to Y$ is a phantom map and $\alpha : S^k \to X$ is any map, then α is homotopic to a map that factors through some X_n by the cellular approximation theorem, and hence $f\alpha \simeq *$. This implies that

$$f_* = 0 : \pi_*(X) \longrightarrow \pi_*(Y).$$

Similarly, $f_* = 0$ when a generalized homology theory is used.

Let $[X, Y]$ denote homotopy classes of maps from X to Y and let $\mathcal{P}(X, Y)$ denote homotopy classes of phantom maps from X to Y. Then it is clear that $\mathcal{P}(X, Y)$ is by definition the "kernel" of the natural map

$$[X, Y] \longrightarrow \varprojlim\, [X_n, Y].$$

The quotations are appropriate since in general these are not groups, but it still makes sense to take inverse limits of sets and to talk about the kernel of the map as the subset of $[X, Y]$ sent to the trivial class. Generalizing the work of Milnor [47], Bousfield and Kan [7] identified the kernel of this map in terms of $\varprojlim^1$. For this purpose they extended the definition of $\varprojlim^1$ to nonabelian groups. Note that $[\Sigma X, Y]$ is a group, where Σ denotes pointed suspension.

Theorem 12.1. *(Bousfield-Kan) For any pointed complexes X and Y,*

$$\mathcal{P}(X, Y) \cong \varprojlim{}^1 [\Sigma X_n, Y].$$

[25] If X is not locally finite then this definition is incorrect. In the general situation one says that $f : X \to Y$ is a *phantom map* if for every finite CW-complex X' and for every map $X' \to X$ the composite $X' \to X \to Y$ is null-homotopic. If X is locally finite then this is equivalent to the definition above.

For example, take $Y = BU \times \mathbb{Z}$ so that

$$[X, Y] = K^0(X)$$

for any compact space. Then we have

$$\mathcal{P}(X, BU \times \mathbb{Z}) \cong \varprojlim{}^1 K^0(\Sigma X_n) \cong \varprojlim{}^1 K^1(X_n).$$

If we take $X = \mathbb{C}P^\infty$ (or, more generally, if X has cells only in even dimensions), then

$$\mathcal{P}(\mathbb{C}P^\infty, BU \times \mathbb{Z}) \cong \varprojlim{}^1 K^1(\mathbb{C}P^n) \cong \varprojlim{}^1 0 \cong 0$$

so there are no nontrivial phantom maps from $\mathbb{C}P^\infty$ to $BU \times \mathbb{Z}$.

Huber and Meier [34] relate the group Pext to phantom maps in a way which is very similar to our study of quasidiagonality in [55]; see below. Suppose that X is a spectrum, E_* is a homology theory of finite type, F^* is a cohomology theory, and H is some abelian group. Suppose that there is a natural exact UCT sequence for E_* and F^* of the form

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(E_{n-1}(X), H) \to F^n(X) \to \mathrm{Hom}_{\mathbb{Z}}(E_n(X), H) \to 0. \tag{12.2}$$

Then the collection of phantom classes in $F^n(X)$ is isomorphic to the subgroup $\mathrm{Pext}^1_{\mathbb{Z}}(E_{n-1}(X), H)$.

There are analogous definitions and theorems at the level of spectra. For instance, if A and B are abelian groups with associated Eilenberg-MacLane spectra $\underline{H}(A)$, $\underline{H}(B)$, then Christensen-Strickland [10] show that phantom maps

$$\Sigma^{-1}\underline{H}(A) \longrightarrow \underline{H}(B)$$

correspond bijectively to elements of $\mathrm{Pext}^1_{\mathbb{Z}}(A, B)$.

There is a *universal phantom map*

$$\Theta : X \longrightarrow \bigvee_{n=1}^{\infty} \Sigma X_n$$

defined as follows (cf. [44], Pages 1214-5). First construct the mapping telescope $\mathrm{Tel}(X)$ of X as

$$\mathrm{Tel}(X) = \bigcup_n X_n \times [n-1, n]/\sim$$

where each $X_n \times n$ is identified with its image in $X_{n+1} \times n$. We may identify X with this space up to homotopy type. There is a natural collapse map

$$\mathrm{Tel}(X) \longrightarrow \bigvee_{n=1}^{\infty} \Sigma X_n$$

obtained by collapsing each subset $X_n \times n$ to a point. It is easy to see that the resulting map Θ is a phantom map. In fact, under modest hypotheses every phantom map from X to some other space Y must factor through Θ up to homotopy.

We note in the following section that there is an analogous construction for C^*-algebras.

13. $\mathrm{Pext}^1_{\mathbb{Z}}(G, H)$ and quasidiagonality

As we mentioned in the introduction, our interest in Pext arose from our study of the topological structure of the Kasparov groups in C^*-algebras. We discuss this matter in depth in [58]. Here we give a semi-heuristic introduction, showing by example how some knowledge of Pext helps in functional analysis. (For undefined terms in this section please see [58].)

A bounded operator T on a (complex separable) Hilbert space is said to be *quasidiagonal* if for some choice of basis the operator has the form $T = D + K$ where D is a block-diagonal operator (with each block of finite dimension) and K is a compact operator.

For example, let the Hilbert space have basis $\{e_o, e_1, e_2, \dots\}$ and define the unilateral shift by $Te_n = e_{n+1}$. This operator is *not* quasidiagonal. Here is a proof. Suppose that T were quasidiagonal, so that $T = D + K$ for some choice of block diagonal matrix D and compact operator K. The operator T is Fredholm, and

$$\mathrm{index}(T) = \dim\ker(T) - \dim\ker(T^*) = 0 - 1 = -1.$$

On the other hand,

$$\mathrm{index}(D + K) = \mathrm{index}(D) = 0.$$

Thus T cannot be quasidiagonal. This argument is related to the γ invariant below.

A C^*-algebra A is said to be *quasidiagonal* if it has a faithful representation on a Hilbert space such that for some choice of basis each of its elements is quasidiagonal. For instance, any commutative C^*-algebra is quasidiagonal, and obviously the compact operators $\mathcal{K}$ themselves are quasidiagonal. However, no C^*-algebra containing the unilateral shift can be quasidiagonal.

Let $\mathcal{S}$ be the C^*-algebra generated by the identity, the compact operators, and the unilateral shift. It is not quasidiagonal, by the remarks above. Then

$$\mathcal{S}/\mathcal{K} \cong C(S^1)$$

and hence there is a natural short exact sequence

$$0 \to \mathcal{K} \to \mathcal{S} \to C(S^1) \to 0.$$

We see from this example that a extension of quasidiagonal C^*-algebras is not necessarily quasidiagonal.

Davidson, Herrero, and Salinas established the following theorem. Let $\mathcal{L}$ denote the bounded operators on a separable Hilbert space and let $\mathcal{K}$ denote the compacts in $\mathcal{L}$ as is customary.

Theorem 13.1 (Davidson, Herrero, Salinas [17]). *Suppose given the essential extension*

$$0 \to \mathcal{K} \longrightarrow E_\tau \longrightarrow A \to 0$$

with associated faithful representation

$$e_\tau : E_\tau \longrightarrow \mathcal{L}$$

with E_τ separable and nuclear. Suppose that e_τ is a quasidiagonal representation. Then $A \cong E/\mathcal{K}$ is quasidiagonal.

The example of the extension generated by the unilateral shift demonstrates that the converse of this theorem is false. Here is the complete story within the category $\widetilde{\mathcal{N}}$ of separable nuclear C^*-algebras which satisfy the UCT.

Theorem 13.2 ([58]). *Suppose given the essential extension*

$$0 \to \mathcal{K} \longrightarrow E_\tau \longrightarrow A \to 0$$

with associated faithful representation

$$e_\tau : E_\tau \longrightarrow \mathcal{L}$$

with $A \in \widetilde{\mathcal{N}}$ and quasidiagonal. Then e_τ is a quasidiagonal representation (that is, the set $\tau(E_\tau)$ is a quasidiagonal set) if and only if the following two conditions hold:

1. *The K-theory boundary homomorphism*

   $$\gamma : K_1(A) \to K_0(\mathcal{K}) = \mathbb{Z}$$

   is the zero homomorphism.
2. *The resulting K-theory short exact sequence*

   $$0 \to \mathbb{Z} \longrightarrow K_0(E_\tau) \longrightarrow K_0(A) \to 0$$

   is a pure exact sequence.

If in addition $K_0(A)$ is torsionfree then Condition 2) is automatically satisfied, so that E_τ is quasidiagonal if and only if Condition 1) holds.

The analysis of this problem goes as follows: one considers all [26] extensions of the form

$$0 \to \mathcal{K} \longrightarrow D \longrightarrow A \to 0.$$

These form a topological group called the Kasparov group. The group is denoted $KK_1(A, \mathcal{K})$ or just $K^1(A)$. Salinas shows [54] that the quasidiagonal extensions form a subgroup, denoted $QD(A, \mathcal{K})$ and this subgroup is just the

[26] Well, not all — there are technical conditions.

closure of zero. The C^*-algebra A satisfies the UCT, which says that there is a natural short exact UCT (Universal Coefficient Theorem) sequence [53]

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K_0(A), \mathbb{Z}) \to K^1(A) \xrightarrow{\gamma} \mathrm{Hom}_{\mathbb{Z}}(K_1(A), \mathbb{Z}) \to 0. \tag{13.3}$$

Under this identification we demonstrate in [58] that

$$QD(A, \mathcal{K}) \cong \mathrm{Pext}^1_{\mathbb{Z}}(K_0(A), \mathbb{Z})$$

from which the theorem follows easily.

In fact we consider the more general situation where $\mathcal{K}$ is replaced by $\mathcal{K} \otimes B$ for some other C^*-algebra B. Extensions of C^*-algebras of the form

$$0 \to \mathcal{K} \otimes B \longrightarrow E \longrightarrow A \to 0$$

form the Kasparov group $KK_1(A, B)$. This is a topological group and its closure of zero is again isomorphic to the "relatively quasidiagonal" extensions $QD(A, B)$. For $A \in \mathcal{N}$ and quasidiagonal relative to B, we demonstrate [58] that

$$QD(A, B) \cong \mathrm{Pext}^1_{\mathbb{Z}}(K_*(A), K_*(B))_0$$

as abelian groups.

If $A \in \mathcal{N}$ is *not* relatively quasidiagonal then Manuilov and Thomsen [42] have found it possible to identify the group $\mathrm{Pext}^1_{\mathbb{Z}}(K_*(A), K_*(B))$ as "weakly quasidiagonal" extensions.

We should like to emphasize the formal similarity between sequences 12.2 and 13.3. It is our hope that there is some deeper connection here between phantom behavior and quasidiagonality which remains to be uncovered.

Here is one more example of this connection. We have explained that in the context of phantom maps there is a universal phantom map

$$\Theta : X \longrightarrow \bigvee_{n=1}^{\infty} \Sigma X_n.$$

It turns out that there is a natural analogue of this map in the context of C^*-algebras. We learned the construction from L.G. Brown and used it in our general study of homology theories for C^*-algebras [55], pages 431-432. One is given a direct sequence of C^*-algebras A_i with $A = \varinjlim A_i$. One constructs a contractible mapping telescope $T = T(\{A_i\})$ with a natural map $e : T(\{A_i\}) \to A$. Inside of T one has the subalgebra $\oplus_i SA_i$, where SA denotes the suspension of A, and the induced map

$$\Theta : \bigoplus_{n=1}^{\infty} SA_i \longrightarrow A \tag{13.4}$$

is very analogous to the universal phantom map.

References

[1] D. M. Arnold, *Finite Rank Torsion Free Groups and Rings*, Lecture Notes in Mathematics, **931**, Springer Verlag, New York, 1972, MR 84d:20002, Zbl 0493.20034.

[2] M. F. Atiyah, *Characters and cohomology of finte groups*, Inst. Hautes Études Sci. Publ. Math. **9** (1961), 23–64, MR 26:6228, Zbl 0107.02303.

[3] M. F. Atiyah and G. B. Segal, *Equivariant K-Theory and Completion*, J. Differential Geom. **9** (1961), 1–18, MR 41:4575, Zbl 0215.24403.

[4] R. Baer, *Der Kern, eine charakteristische Untergruppe*, Compositio Math. **1** (1934), 254–283, Zbl 0107.02303.

[5] R. Baer, *Die Torsionsuntergruppe einer abelschen Gruppe*, Math Ann. **135** (1958), 219–234, MR 20:6460, Zbl 0081.25704.

[6] B. Blackadar, *K-Theory for Operator Algebras*, Math. Sci. Res. Inst. **5**, 2nd edition, Cambridge University Press, New York, 1998, MR 99g:46104, Zbl 0081.25704.

[7] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions, and localizations*, Lecture Notes in Mathematics **304**, Springer Verlag, New York, 1972, MR 51:1825, Zbl 0259.55004.

[8] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press, Princeton, 1956, MR 17:1040e, Zbl 0075.24305.

[9] J. D. Christensen, B. Keller, and A. Neeman, *Failure of Brown representability in derived categories*, Topology, **40** (2001), 1339–1361, MR 1867248, Zbl 0997.18007.

[10] J. D. Christensen and N. P. Strickland, *Phantom maps and homology theories*, Topology **37** (1998), 339–364, MR 99k:55008, Zbl 1001.55009.

[11] M. Dădărlat, *Quasidiagonal morphisms and homotopy*, J. Functional Analysis **151** (1997), 213–233, MR 99a:46102, Zbl 0992.46042.

[12] M. Dădărlat, *On Approximation of quasidiagonal C^*-algebras*, J. Functional Analysis 167 (1999), no. 1, 69–78, MR 2000f:46069, Zbl 0936.46043.

[13] M. Dădărlat, *Approximate unitary equivalence and the topology of* $\mathrm{Ext}(A, B)$, C^*-algebras (Münster, 1999), 42–60, Springer, Berlin, 2000, MR 2001j:46080, Zbl 0977.46046.

[14] M. Dădărlat and S. Eilers, *On the classification of nuclear C^*-algebras*, Proc. London Math. Soc. (3) **85** (2002), 168–210, MR 2003d:19006.

[15] M. Dădărlat and S. Eilers, *Asymptotic unitary equivalence in KK-theory*, K-Theory **23** (2001), 305–322, MR 2002h:19005, Zbl 0991.19002.

[16] M. Dădărlat and T. Loring, *A* universal multicoefficient theorem for the Kasparov groups, Duke Math J. **84** (1996), 355–377, MR 97f:46109, Zbl 0881.46048.

[17] K. R. Davidson, D. A. Herrero, and N. Salinas, *Quasidiagonal operators, approximations, and C^*-algebras*, Indiana U. Math. J. **38** (1989), 973–998, MR 91b:47033, Zbl 0675.41052.

[18] S. Eilenberg and J. C. Moore, *Limits and spectral sequences*, Topology **1** (1961), 1–22, MR 26:6229, Zbl 0104.39603.

[19] S. Eilenberg and N. Steenrod, *Foundations of Algebraic Topology*, Princeton University Press, Princeton, N. J., 1952, MR 14:398b, Zbl 0047.41402.

[20] P. Eklof and M. Huber, *On the p-ranks of* $\mathrm{Ext}^1_{\mathbb{Z}}(A, G)$ *assuming CH*, Lecture Notes in Mathematics **874**, Springer-Verlag, New York, 1981, 93–108, MR 84e:20060, Zbl 0468.20046.

[21] I. Emmanouil, *Mittag-Leffler condition and the vanishing of* $\varprojlim^1$, Topology **35** (1996), 267–271, MR 96m:18004, Zbl 0864.18005.

[22] Lászlo Fuchs, *Infinite Abelian Groups*, Pergamon Press, Oxford, 1960 reprint of 1958 edition, MR 21:5672.

[23] Lászlo Fuchs, *Notes on Abelian Groups.* I, Ann. Univ. Sci. Budapest **2** (1959), 5–23; II, Acta. Math. Acad. Sci. Hungar. **11** (1960), 117–125, MR 23:A933, Zbl 0090.02002.

[24] Lászlo Fuchs, *Infinite Abelian Groups*, Pure and Applied Mathematics **36**, Vol. **1**, Academic Press, New York, 1970, MR 41:333, Zbl 0209.05503.

[25] Lászlo Fuchs, *Infinite Abelian Groups*, Pure and Applied Mathematics **36**, Vol. **2**, Academic Press, New York, 1973, MR 50:2362, Zbl 0257.20035.

[26] H. P. Goeters, *When is* $\mathrm{Ext}^1_{\mathbb{Z}}(A,B)$ *torsion-free? and related problems*, Comm. in Alg. **16** (1988), 1605–1619, MR 90d:20097, Zbl 0653.20055.

[27] H. P. Goeters and P. Keef, *When must pure extensions of countable abelian groups necessarily split?*, Proc. Amer. Math. Soc. **126** (1998), 1321–1329, MR 98j:20082, Zbl 0981.20044.

[28] K. Golema and A. Hulanicki, *The structure of the factor group of the unrestricted sum by the restricted sum of Abelian groups* II, Fund. Math. **53** (1964), 177–185, MR 29:5901, Zbl 0122.03503.

[29] B. Gray, *Spaces of the same* n*-type for all* n, Topology **5** (1966), 241–243, MR 33:4929, Zbl 0149.20102.

[30] A. Grothendieck and J. Dieudonné, *Eléments de géométrie algébrique* III. *Étude cohomologique des faisceaux cohérents*, I, Inst. Hautes Études Sci. Publ. Math. **11** (1961), 5–167, MR 36:177c, Zbl 0118.36206.

[31] D. K. Harrison, *Infinite abelian groups and homological methods*, Annals. of Math. (2) **69**, 1959, 366–391, MR 21:3481, Zbl 0100.02901.

[32] Ivo Herzog, *The Ziegler spectrum of a locally coherent Grothendieck category*, Proc. London Math. Soc. (3) **74** (1997), 503–558, MR 98j:18017, Zbl 0909.16004.

[33] H. L. Hiller, M. Huber, and S. Shelah, *The structure of* $\mathrm{Ext}^1_{\mathbb{Z}}(A,\mathbb{Z})$ *and* $V = L$, Math. Z. **162** (1978), 39–50, MR 58 #11171, Zbl 0379.20045.

[34] M. Huber and W. Meier, *Cohomology theories and infinite CW-complexes*, Comment. Math. Helv. **53** (1978), 239–257, MR 58 #2796, Zbl 0432.55002.

[35] M. Huber and R. B. Warfield, *On the torsion subgroup of* $\mathrm{Ext}^1_{\mathbb{Z}}(A,G)$, Arch. Math. (Basel) **32** (1979), 5–9, MR 80h:20077, Zbl 0402.20043.

[36] A. Hulanicki, *Algebraic characterization of abelian divisible groups which admit compact topologies*, Fund. Math. **44** (1957), 192–197, MR 19,1063e, Zbl 0082.02604.

[37] A. Hulanicki, *The structure of the factor group of the unrestricted sum by the restricted sum of Abelian groups*, Bull. Acad. Polonaise des Sci., Ser. des. Sci. Math. Astr. et Physics **10** (1962), 77–80, MR 26 #5060.

[38] C. U. Jensen, *Les Foncteurs Dérivés de* $\varprojlim$ *et leur Applications en Théorie des Modules*, Lecture Notes in Mathematics, Vol. **254**, Springer-Verlag, New York, 1972, MR 53 #10874, Zbl 0238.18007.

[39] J. Kaminker and C. Schochet, *K-Theory and Steenrod homology: Applications to the Brown-Douglas-Fillmore theory of operator algebras*, Trans. Amer. Math. Soc. **227** (1977), 63–107, MR 55 #4138, Zbl 0368.46054.

[40] I. Kaplansky, *Infinite Abelian Groups*, Revised Edition, The University of Michigan Press, Ann Arbor, MI., 1969, MR 38 #2208, Zbl 0194.04402.

[41] S. MacLane, *Homology*, Springer-Verlag, New York, 1963, MR 28 #122, Zbl 0133.26502.

[42] V. M. Manuilov and K. Thomsen, *Quasidiagonal extensions and sequentially trivial asymptotic homomorphisms*, Adv. Math. **154** (2000), 258–279, MR 2002k:19006, Zbl 0974.46052.

[43] J. M. Maranda, *On pure subgroups of abelian groups*, Arch. Math. **11** (1960), 1–13, MR 22 #5672, Zbl 0109.01701.

[44] C. A. McGibbon, *Phantom maps*, Chapter 25, *Handbook of Algebraic Topology*, ed. I. M. James, Elsevier, NY, 1995, 1209–1257, MR 96i:55021, Zbl 0867.55013.

[45] C. A. McGibbon and J. M. Møller, *On spaces with the same n-type for all n*, Topology 31 (1992), 177–201, MR 92m:55008, Zbl 0765.55010.

[46] C. A. McGibbon and R. Steiner, *Some questions about the first derived functor of the inverse limit*, J. Pure Appl. Algebra **103** (1995), no. 3, 325–340, MR 98c:20098, Zbl 0837.55005.

[47] J. W. Milnor, *On axiomatic homology theory*, Pacific J. Math. **12** (1962), 337–341, MR 28 #2544, Zbl 0114.39604.

[48] J. W. Milnor, *On the Steenrod homology theory*, Novikov Conjectures, Index Theorems and Rigidity, Vol. 1., S. C. Ferry, A. Ranicki, J. Rosenberg, eds., London Math. Soc. Lecture Note Series **226** (1993), 79–96, MR 98d:55005, Zbl 0954.55004.

[49] R. J. Nunke, *Modules of extensions over Dedekind rings*, Illinois J. Math. **3** (1959), 222–241, MR 21 #1329, Zbl 0087.26603.

[50] H. Prüfer, *Unendliche abelsche Gruppen von Elementen endlicher Ordnung*, Dissertation, Berlin, 1921, Zbl 48.1150.03.

[51] H. Prüfer, *Untersuchungen über die Zerlegbarkeit der abzählbaren primären abelschen Gruppen*, Math. Z. **17** (1923), 35–61, Zbl 49.0084.03.

[52] J. E. Roos, *Sur les dérivés de* $\varprojlim$. *Applications* C. R. Acad. Sci. Paris **252** (1961), 3702–3704, MR 24 #A1938, Zbl 0102.02501.

[53] J. Rosenberg and C. Schochet, *The Künneth Theorem and the Universal Coefficient Theorem for Kasparov's generalized K-functor*, Duke Math. J. **55** (1987), 431–474, MR 88i:46091, Zbl 0644.46051.

[54] N. Salinas, *Relative quasidiagonality and KK-theory*, Houston J. Math. **18** (1992), 97–116, MR 94c:19005, Zbl 0772.46039.

[55] C. L. Schochet, *Topological methods for C^*-algebras* III: *Axiomatic homology*, Pacific J. Math. **114** (1984), 399–445, MR 86g:46102, Zbl 0491.46061.

[56] C. L. Schochet, *The fine structure of the Kasparov groups* I: *Continuity of the KK-pairing*, J. Functional Analysis **186** (2001), 25–61, MR 2003d:19009, Zbl 0990.19003.

[57] C. L. Schochet, *The fine structure of the Kasparov groups* II: *Topologizing the* UCT, J. Functional Analysis **194** (2002), 263–287, MR 1 934 604.

[58] C. L. Schochet, *The fine structure of the Kasparov groups* III: *Relative quasidiagonality*, preprint.

[59] N.E. Steenrod, *Regular cycles of compact metric spaces*, Annals of Math. **41** (1940), 833–851, MR 2,73c, Zbl 0025.23405.

[60] E. G. Sklyarenko, *Some applications of the functor* $\varprojlim^1$, Math. USSR Sbornik **51** (1985), 367–387, MR 87d:18017, Zbl 0573.55005.

[61] C. P. Walker, *Relative homological algebra and abelian groups*, Illinois J. Math. **10** (1966), 186–209, MR 32 #7624, Zbl 0136.25601.

[62] R. B. Warfield, *Extensions of torsion-free Abelian groups of finite rank*, Arch. Math. **23** (1972), 145–150, Zbl 0244.20064.

[63] R. B. Warfield, Jr., and M. Huber, *On the values of the functor* $\varprojlim^1$, Arch. Math. (Basel) **33** (1979), 430–436, MR 82b:20077, Zbl 0427.20048.

[64] Z. Z. Yeh, *Higher inverse limits and homology*, Thesis, Princeton University, 1959.

Department of Mathematics, Wayne State University, Detroit, MI 48202
claude@math.wayne.edu

This monograph is available via http://nyjm.albany.edu:8000/m/2003/1.htm.