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An unstable motivic null-Hopf relation

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ABSTRACT. We prove the unstable analogue of the relation $\eta h = h\eta = 0$ in stable motivic homotopy theory, where η is the first motivic Hopf map and h the hyperbolic plane. Using these relations we construct some non-trivial examples of Toda brackets in unstable motivic homotopy theory.

CONTENTS

1. Introduction

This paper is a part of the author's doctoral thesis [Don24] about Toda brackets in unstable motivic homotopy theory. Throughout the paper, we work over the base Spec $\mathbb{Z}$. Let $\mathcal{S}\mathrm{m}_{\mathbb{Z}}$ denote the category of smooth schemes of finite type over $\mathbb{Z}$. This category is equipped with the Nisnevich topology in the sense of [MV99]. The category of pointed motivic spaces is the category of pointed simplicial presheaves on $\mathcal{S}\mathrm{m}_{\mathbb{Z}}$. It is denoted by $\mathrm{sPre}(\mathbb{Z})_*$. For our purpose, we will use the $\mathbb{A}^1$-local injective model structure on $\mathrm{sPre}(\mathbb{Z})_*$ which is developed in [MV99]. The corresponding motivic homotopy category is denoted by $\mathcal{H}_\bullet(\mathbb{Z})$.

Let $S^{\alpha+(\beta)}$ denote the motivic sphere $S^\alpha \wedge \mathbb{G}_m^\beta$, where S^1 is the simplicial circle $\Delta^1/\partial\Delta^1$ and $\mathbb{G}_m$ the Tate circle based at 1. Maps from spheres to spheres are indexed by the bidegree of the target. Suspension from the right with $\mathbb{G}_m$ increases the weight (β) by 1. Suspension from the left by the simplicial circle S^1 increases the degree α by 1. We would like to smash S^1 from the left and $\mathbb{G}_m$ from the right as motivic spheres are of the form $S^\alpha \wedge \mathbb{G}_m^\beta$. If we smash in the other way around, we have to do some permutations to get spheres in the standard form and this causes some technical subtleties. Let $\eta_{1+(1)}$ denote the

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first algebraic Hopf map from $S^{1+(2)}$ to $S^{1+(1)}$. Let $\epsilon : \mathbb{G}_m \to \mathbb{G}_m$ be given by $x \mapsto x^{-1}$. Then, we define the hyperbolic plane $h_{1+(1)}$ to be $1_{1+(1)} - \epsilon_{1+(1)}$. Let $q_{(1)} : \mathbb{G}_m \to \mathbb{G}_m$ denote the map defined by $x \mapsto x^2$.

The aim of this paper is to prove that the composites $h_{1+(2)} \circ \eta_{1+(2)}$ and $\eta_{1+(2)} \circ h_{1+(3)}$ are $\mathbb{A}^1$-nullhomotopic. We note that these are the smallest bidegrees for which the two composites are nullhomotopic. The corresponding stable relation $\eta h = h\eta = 0$ was first proved by Morel [Mor04]. We also would like to mention that some other stable null-Hopf relations were proved by Dugger and Isaksen in [DuI13]. Our proof relies on the methods given by Cazanave in [Caz12]. In particular, we have to construct explicit sequences of naive $\mathbb{A}^1$-homotopies between certain maps. The key point for the whole proof is to show that the relation $q_{1+(1)} = 1_{1+(1)} - \epsilon_{1+(1)}$ holds (Proposition 5.3).

In the following, we give briefly the idea of the proof. The two morphisms $q_{1+(1)}$ and $1_{1+(1)} - \epsilon_{1+(1)}$ are endomorphisms of $S^1 \wedge \mathbb{G}_m$. We consider now the projective line $\mathbb{P}^1_{\mathbb{Z}}$ equipped with the base point $\infty := [1 : 0]$. Using a suitable isomorphism between $S^1 \wedge \mathbb{G}_m$ and $(\mathbb{P}^1_{\mathbb{Z}}, \infty)$ in the pointed $\mathbb{A}^1$-homotopy category, we can transform these two morphisms into two endomorphisms of $(\mathbb{P}^1_{\mathbb{Z}}, \infty)$. Let $[\mathbb{P}^1_{\mathbb{Z}}, \mathbb{P}^1_{\mathbb{Z}}]^{\mathrm{N}}$ be the set of pointed naive $\mathbb{A}^1$-homotopy classes of pointed scheme endomorphisms of $(\mathbb{P}^1_{\mathbb{Z}}, \infty)$ (Definition 2.1). In Proposition 2.5, we give a characterization of pointed scheme endomorphisms of $(\mathbb{P}^1_{\mathbb{Z}}, \infty)$. In Proposition 2.6, we give a characterization of pointed naive $\mathbb{A}^1$-homotopies of pointed scheme endomorphisms of $(\mathbb{P}^1_{\mathbb{Z}}, \infty)$. Let $\mathcal{H}_\bullet(\mathbb{Z})(\mathbb{P}^1_{\mathbb{Z}}, \mathbb{P}^1_{\mathbb{Z}})$ be the set of endomorphisms of $(\mathbb{P}^1_{\mathbb{Z}}, \infty)$ in the pointed $\mathbb{A}^1$-homotopy category $\mathcal{H}_\bullet(\mathbb{Z})$. We can equip $[\mathbb{P}^1_{\mathbb{Z}}, \mathbb{P}^1_{\mathbb{Z}}]^{\mathrm{N}}$ with a monoid structure and denote its monoid operation by $\oplus^{\mathrm{N}}$ (Definition 2.7). Furthermore, via the chosen isomorphism between $S^1 \wedge \mathbb{G}_m$ and $(\mathbb{P}^1_{\mathbb{Z}}, \infty)$, we can equip $(\mathbb{P}^1_{\mathbb{Z}}, \infty)$ with a cogroup structure such that $\mathcal{H}_\bullet(\mathbb{Z})(\mathbb{P}^1_{\mathbb{Z}}, \mathbb{P}^1_{\mathbb{Z}})$ is a group. We denote the induced group operation by $\oplus^{\mathbb{A}^1}$. In [Caz12, Appendix B], Cazanave shows that the canonical map

$$[\mathbb{P}^1_k, \mathbb{P}^1_k]^{\mathrm{N}} \to \mathcal{H}_\bullet(k)(\mathbb{P}^1_k, \mathbb{P}^1_k)$$

is a homomorphism of monoids for any field k. In this paper, we extend this result partially to the base Spec $\mathbb{Z}$. We show that for certain pointed $\mathbb{A}^1$-homotopy classes of pointed scheme endomorphisms of $(\mathbb{P}^1_{\mathbb{Z}}, \infty)$ their $\oplus^{\mathrm{N}}$-sums coincide with the $\oplus^{\mathbb{A}^1}$-sums, This result can be found in the proof of Proposition 5.3.

In order to get Proposition 5.3, we also have to fix a gap in Cazanave's paper on the cogroup structure on $\mathbb{P}^1$. In his paper, Cazanave gives only a codiagonal morphism for $\mathbb{P}^1$ using some geometry for the projective line (see [Caz12, Lemma B.4]). We are able to show that his codiagonal morphism actually comes from the chosen isomorphism with $S^1 \wedge \mathbb{G}_m$ and therefore really defines a cogroup structure (Proposition 3.1).

We consider now the pointed endomorphisms f and g of $(\mathbb{P}^1_{\mathbb{Z}}, \infty)$ which correspond to $q_{1+(1)}$ and $-\epsilon_{1+(1)}$, respectively. In particular, $1_{1+(1)} - \epsilon_{1+(1)}$ corresponds to $\mathrm{id} \oplus^{\mathbb{A}^1} g$. And, we can apply the extended result to $\mathrm{id} \oplus^{\mathbb{A}^1} g$ and get

$\mathrm{id} \oplus^{\mathbb{A}^1} g = \mathrm{id} \oplus^{\mathrm{N}} g$. Thus, we can determine the $\mathbb{A}^1$-sum in this case explicitly. Then we can give an explicit sequence of pointed naive $\mathbb{A}^1$-homotopies between f and $\mathrm{id} \oplus^{\mathrm{N}} g$. Therefore, we get $q_{1+(1)} = 1_{1+(1)} - \epsilon_{1+(1)}$. Using this relation, we can then show that $h_{1+(2)} \circ \eta_{1+(2)}$ and $\eta_{1+(2)} \circ h_{1+(3)}$ are nulhomotopic (Proposition 5.8).

Especially, we can use these nullhomotopic composites to get nontrivial Toda brackets in unstable motivic homotopy theory. Unstable motivic Toda brackets are constructed in their doctoral thesis [Don24, Section 2]. We recall here quickly the construction. Suppose we are given a sequence of three composable morphisms of pointed motivic spaces

$$\mathcal{W} \xrightarrow{\gamma} \mathcal{X} \xrightarrow{\beta} \mathcal{Y} \xrightarrow{\alpha} \mathcal{Z}$$

such that the composites $\alpha \circ \beta$ and $\beta \circ \gamma$ are $\mathbb{A}^1$-nullhomotopic. Then we choose a nullhomotopy $A : C(\mathcal{X}) = \Delta^1 \wedge \mathcal{X} \to \mathcal{Z}$ for $\alpha \circ \beta$ and a nullhomotopy $B : C(\mathcal{W}) = \Delta^1 \wedge \mathcal{W} \to \mathcal{Y}$ for $\beta \circ \gamma$ where Δ^1 is based at 1. In particular, we also get the morphisms $\alpha \circ B$ and $A \circ C(\gamma)$, where $C(\gamma)$ is the morphism between the cones induced by γ. Hence, we obtain a morphism

$$\Sigma \mathcal{W} \xrightarrow{\sim} C(\mathcal{W}) \sqcup_{\mathcal{W}} C(\mathcal{W}) \longrightarrow \mathcal{Z}$$

in the pointed motivic homotopy category. The identification of $\Sigma \mathcal{W}$ with the pushout $C(\mathcal{W}) \sqcup_{\mathcal{W}} C(\mathcal{W})$ is canonical. The Toda bracket

$$\{\alpha, \beta, \gamma\} \subset \mathcal{H}_\bullet(\mathbb{Z})(\Sigma \mathcal{W}, \mathcal{Z})$$

is defined to be the set of all morphisms obtained in this way by choosing all possible nullhomotopies for $\alpha \circ \beta$ and $\beta \circ \gamma$.

As an application of the results of this paper, we get the Toda brackets

$$\{h_{1+(2)}, \eta_{1+(2)}, h_{1+(3)}\}$$

and

$$\{\eta_{1+(2)}, h_{1+(3)}, \eta_{1+(3)}\}.$$

Moreover, these two Toda brackets are not trivial, in the sense that they do not contain the homotopy classes of constant morphisms (Proposition 5.9). This can be proved by using the complex realization.

Additionally, we also construct another Toda bracket

$$\{\Delta_{1+(3)}, h_{1+(2)}, \eta_{1+(2)}\}$$

over the base Spec $\mathbb{Z}$, where $\Delta_{1+(3)}$ is a suspension of the diagonal map $\Delta_{(2)} : \mathbb{G}_m \to \mathbb{G}_m \wedge \mathbb{G}_m$ defined by $x \mapsto x \wedge x$. The interesting point is that the complex and real realization of this Toda bracket are trivial. The complex realization of this Toda bracket is $\{0, 2\mathrm{id}_{S^3}, \Sigma \eta_{\mathrm{top}}\}$, where $\Sigma \eta_{\mathrm{top}}$ is the suspension of the first topological Hopf map and the third map is nullhomotopic as it is a map from S^3 to S^4. Since the third map is nullhomotopic, the Toda bracket is trivial. The real realization of $\{\Delta_{1+(3)}, h_{1+(2)}, \eta_{1+(2)}\}$ is a subset of $\pi_2(S^1)$; therefore the realization is also trivial. But we can show that the Toda bracket $\{\Delta_{1+(3)}, h_{1+(2)}, \eta_{1+(2)}\}$ itself is actually not trivial.

At the end of the paper, we provide an appendix which is a short summary of the geometric realization functor which the author introduced in the doctoral thesis [Don24, Section 2.1]. We will use this functor for some technical lemmas in the paper.

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2. Rational functions and naive $\mathbb{A}^1$-homotopies

For the paper, we need some results from [Caz12], so we first recall some basic facts about pointed naive $\mathbb{A}^1$-homotopies. We work in this section over the base Spec S where S is either a field or the ring of integers $\mathbb{Z}$. Let $\mathcal{S}\mathrm{m}_S$ denote the category of smooth schemes of finite type over S. The category of pointed motivic spaces over S is denoted by $\mathrm{sPre}(S)_*$. It is equipped with the $\mathbb{A}^1$-local injective model structure. The corresponding motivic homotopy category is denoted by $\mathcal{H}_\bullet(S)$

Definition 2.1. Let $\mathcal{X}$ and $\mathcal{Y}$ be two pointed motivic spaces in $\mathrm{sPre}(S)_*$. Let f and g be two pointed morphisms from $\mathcal{X}$ to $\mathcal{Y}$. A pointed naive $\mathbb{A}^1$-homotopy is a morphism $F : \mathcal{X} \wedge \mathbb{A}^1_+ \to \mathcal{Y}$ such that $F|_{\mathcal{X}\times\{0\}}$ is f and $F|_{\mathcal{X}\times\{1\}}$ is g. We define the set $[\mathcal{X}, \mathcal{Y}]^{\mathrm{N}}$ of pointed naive homotopy classes of morphisms from $\mathcal{X}$ to $\mathcal{Y}$ as the quotient of the set of pointed morphisms by the equivalence relation generated by pointed naive $\mathbb{A}^1$-homotopies.

If there is a pointed naive $\mathbb{A}^1$-homotopy from f to g, then f is equal to g in $\mathcal{H}_\bullet(S)$. Therefore there is a canonical map

$$[\mathcal{X}, \mathcal{Y}]^{\mathrm{N}} \to \mathcal{H}_\bullet(S)(\mathcal{X}, \mathcal{Y}).$$

In general, this map is far from being a bijection. Examples where this map is not a bijection can be found in [BHS15, Section 4].

Let S for now be a field k. We equip the projective line $\mathbb{P}^1_k = \mathrm{Proj}\, k[T_0, T_1]$ over k with the base point $\infty = [1 : 0]$. We are interested in the set $[\mathbb{P}^1_k, \mathbb{P}^1_k]^{\mathrm{N}}$. A morphism from $\mathbb{P}^1_k$ to $\mathbb{P}^1_k$ in $\mathrm{sPre}(k)_*$ is uniquely determined by a pointed scheme endomorphism of $\mathbb{P}^1_k$, therefore we can restrict ourselves to scheme morphisms. In particular, there is a classical correspondence between pointed scheme endomorphisms of $(\mathbb{P}^1_k, \infty)$ and pointed rational functions with coefficients in k. Furthermore, we also have a description of pointed naive homotopies of pointed scheme endomorphisms of $(\mathbb{P}^1_k, \infty)$ in terms of pointed rational functions with coefficients in the polynomial ring $k[X]$. For both correspondences, we need the notion of the resultant of two polynomials.

Definition 2.2. Let R be a commutative ring. Let f and g be two polynomials in $R[X]$ of the form

$$f = \alpha_n X^n + \alpha_{n-1} X^{n-1} + \cdots + \alpha_0$$

and

$$g = \beta_n X^n + \beta_{n-1} X^{n-1} + \cdots + \beta_0$$

of degree n. We do not require here that $\alpha_n, \beta_n \neq 0$, so we call n the formal degree of f and g. Then we define the resultant $res_{n,n}(f, g)$ to be the determinant of the $(n+n) \times (n+n)$-matrix.

$$\begin{pmatrix} \alpha_n & 0 & \cdots & 0 & \beta_n & 0 & \cdots & 0 \\ \alpha_{n-1} & \alpha_n & \cdots & 0 & \beta_{n-1} & \beta_n & \cdots & 0 \\ \alpha_{n-2} & \alpha_{n-1} & \ddots & 0 & \beta_{n-2} & \beta_{n-1} & \ddots & 0 \\ \vdots & \vdots & \ddots & \alpha_n & \vdots & \vdots & \ddots & \beta_n \\ \alpha_0 & \alpha_1 & \cdots & \vdots & \beta_0 & \beta_1 & \cdots & \vdots \\ 0 & \alpha_0 & \ddots & \vdots & 0 & \beta_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \alpha_1 & \vdots & \vdots & \ddots & \beta_1 \\ 0 & 0 & \cdots & \alpha_0 & 0 & 0 & \cdots & \beta_0 \end{pmatrix}.$$

Proposition 2.3 (cf. [Caz12, Proposition 2.3]). *Any pointed endomorphism of $\mathbb{P}^1_k$ is given uniquely by a pair of polynomials $(f, g) \in k[X]$ with $X := \frac{T_0}{T_1}$, where*

- *f is monic of degree n,*
- *g is of degree strictly less than n,*
- *$res_{n,n}(f, g)$ is invertible in k.*

We abuse the notation and denote such a pair in the following by $\frac{f}{g}$.

Proposition 2.4. *Any pointed naive $\mathbb{A}^1$-homotopy of $\mathbb{P}^1_k$ is given uniquely by a pair of polynomials $\frac{f}{g}$ with $f, g \in k[T][X]$ and $X := \frac{T_0}{T_1}$, where*

- *f is monic of degree n,*
- *g is of degree strictly less than n,*
- *$res_{n,n}(f, g)$ is invertible in $k[T]$.*

Next, we consider $\mathbb{P}^1_{\mathbb{Z}} = \operatorname{Proj} \mathbb{Z}[T_0, T_1]$ where it is equipped with a morphism $\infty : \operatorname{Spec} \mathbb{Z} \to \mathbb{P}^1_{\mathbb{Z}}$ in $\mathcal{S}\mathrm{m}_{\mathbb{Z}}$. It is given by $\mathbb{Z}[\frac{T_1}{T_0}] \to \mathbb{Z}; \frac{T_1}{T_0} \mapsto 0$. A pointed endomorphism of $\mathbb{P}^1_{\mathbb{Z}}$ is a scheme morphism $f : \mathbb{P}^1_{\mathbb{Z}} \to \mathbb{P}^1_{\mathbb{Z}}$ such that the diagram

$$\begin{array}{ccc} & \operatorname{Spec} \mathbb{Z} & \\ {}^{\infty}\swarrow & & \searrow^{\infty} \\ \mathbb{P}^1_{\mathbb{Z}} & \xrightarrow{\ f\ } & \mathbb{P}^1_{\mathbb{Z}} \end{array}$$

commutes. This condition is equivalent to $f(p, T_1) = (T_1)$ where p is either 0 or runs over the prime numbers in $\mathbb{Z}$. The same arguments used for proving Proposition 2.3 also work for pointed endomorphisms of $(\mathbb{P}^1_{\mathbb{Z}}, \infty)$. Again, we work in coordinates $X := \frac{T_0}{T_1}$. We get the following characterization.

Proposition 2.5. *Any pointed endomorphism of* $(\mathbb{P}^1_{\mathbb{Z}}, \infty)$ *is given uniquely by a pair of polynomials* $\frac{f}{g}$ *with* $f, g \in \mathbb{Z}[X]$, *where*

- *f is monic of degree n,*
- *g is of degree strictly less than n,*
- *$res_{n,n}(f, g)$ is invertible in $\mathbb{Z}$.*

Finally, we also can consider pointed naive $\mathbb{A}^1$-homotopies of $\mathbb{P}^1_{\mathbb{Z}}$. A pointed naive $\mathbb{A}^1$-homotopy can be viewed as a scheme morphism $f : \mathbb{P}^1_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}} = \operatorname{Proj} \mathbb{Z}[T][T_0, T_1] \to \mathbb{P}^1_{\mathbb{Z}}$ such that the composition

$$\operatorname{Spec} \mathbb{Z} \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}} \to \mathbb{P}^1_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}} \to \mathbb{P}^1_{\mathbb{Z}}$$

factors through the structure morphism $\operatorname{Spec} \mathbb{Z} \to \mathbb{P}^1_{\mathbb{Z}}$. This is equivalent to $f((\rho, T_1)) = (T_1)$ where (T_1) is the homogeneous prime ideal in $\mathbb{Z}[T][T_0, T_1]$ generated by T_1 and ρ runs over the prime ideals of $\mathbb{Z}[T]$. As before we can apply the arguments for pointed $\mathbb{A}^1$-homotopies of $\mathbb{P}^1_k$ to pointed $\mathbb{A}^1$-homotopies of $\mathbb{P}^1_{\mathbb{Z}}$. We take here $X := \frac{T_0}{T_1}$, then we obtain the characterization below.

Proposition 2.6. *Any pointed* $\mathbb{A}^1$*-homotopy of* $(\mathbb{P}^1_{\mathbb{Z}}, \infty)$ *is given uniquely by a pair of polynomials* $\frac{f}{g}$ *with* $f, g \in \mathbb{Z}[T][X]$, *where*

- *f is monic of degree n,*
- *g is of degree strictly less than n,*
- *$res_{n,n}(f, g)$ is invertible in $\mathbb{Z}[T]$.*

In [Caz12], Cazanave gives the set $[\mathbb{P}^1_k, \mathbb{P}^1_k]^{\mathrm{N}}$ a monoid structure. Actually, his method also works over $\mathbb{Z}$, so we introduce the monoid structure for $[\mathbb{P}^1_{\mathbb{Z}}, \mathbb{P}^1_{\mathbb{Z}}]^{\mathrm{N}}$. Let $\frac{f}{g}$ be a pair of polynomials which determines a pointed endomorphism of $\mathbb{P}^1_{\mathbb{Z}}$ such that $deg(f) = n$. Then there exist polynomials $p, q \in \mathbb{Z}[X]$ with $deg(p) < n - 1$ and $deg(q) < n$ such that $1 = pf + qg$, since $res_{n,n}(f, g)$ is invertible in $\mathbb{Z}$. Furthermore, p and q are unique.

Definition 2.7. Let $\frac{f_1}{g_1}, \frac{f_2}{g_2}$ be two pairs of polynomials which determine pointed endomorphisms of $\mathbb{P}^1_{\mathbb{Z}}$ with $deg(f_1) = n_1$ and $deg(f_2) = n_2$. Then there are unique polynomials $p_1, q_1, p_2, q_2 \in \mathbb{Z}[X]$ with $deg(p_1) < n_1 - 1, deg(q_1) < n_1, deg(p_2) < n_2 - 1, deg(q_2) < n_2$ such that $1 = p_1 f_1 + q_1 g_1$ and $1 = p_2 f_2 + q_2 g_2$. We define polynomials f_3, g_3, p_3 and q_3 by setting

$$\begin{pmatrix} f_3 & -q_3 \\ g_3 & p_3 \end{pmatrix} := \begin{pmatrix} f_1 & -q_1 \\ g_1 & p_1 \end{pmatrix} \cdot \begin{pmatrix} f_2 & -q_2 \\ g_2 & p_2 \end{pmatrix}.$$

The matrices $\begin{pmatrix} f_1 & -q_1 \\ g_1 & p_1 \end{pmatrix}$ and $\begin{pmatrix} f_2 & -q_2 \\ g_2 & p_2 \end{pmatrix}$ are in $\mathrm{SL}_2(\mathbb{Z}[X])$, hence it is also true for $\begin{pmatrix} f_3 & -q_3 \\ g_3 & p_3 \end{pmatrix}$. By definition $f_3 = f_1 f_2 - q_1 g_2$ is monic of degree $n_1 + n_2$ and $g_3 = g_1 f_2 + p_1 g_2$ is of degree strictly less than $n_1 + n_2$. Therefore $\frac{f_3}{g_3}$ defines a pointed endomorphism of $\mathbb{P}^1_{\mathbb{Z}}$ by Proposition 2.5. We define the sum $\frac{f_1}{g_1} \oplus^{\mathrm{N}} \frac{f_2}{g_2}$ to be the pair of polynomials $\frac{f_3}{g_3}$. The neutral element for this addition is the pair of polynomials $\frac{1}{0}$ which represents the constant morphism.

3. Cogroup structure on $\mathbb{P}^1_{\mathbb{Z}}$

In this section, we would like to study the cogroup structure on $\mathbb{P}^1_{\mathbb{Z}}$ in some detail. From now on, notions from algebraic geometry are taken from [Liu06]. In particular, $\mathbb{P}^1_{\mathbb{Z}}$ has a standard open covering by the principal open subsets $\mathrm{D}_+(T_0)$ and $\mathrm{D}_+(T_1)$ which are both isomorphic to $\mathbb{A}^1_{\mathbb{Z}}$ (see [Liu06, Section 2.3.3]). The intersection $\mathrm{D}_+(T_0T_1)$ of these two open subsets is isomorphic to $\mathbb{G}_m$.

We can equip the presheaf $\mathbb{P}^1_{\mathbb{Z}}$ with three base points which are given by the following scheme morphisms:

$$\infty : \operatorname{Spec} \mathbb{Z} \to \mathbb{P}^1_{\mathbb{Z}}$$

induced by $\mathbb{Z}[\frac{T_1}{T_0}] \to \mathbb{Z}; \frac{T_1}{T_0} \mapsto 0$;

$$0 : \operatorname{Spec} \mathbb{Z} \to \mathbb{P}^1_{\mathbb{Z}}$$

induced by $\mathbb{Z}[\frac{T_0}{T_1}] \to \mathbb{Z}; \frac{T_0}{T_1} \mapsto 0$ and

$$1 : \operatorname{Spec} \mathbb{Z} \to \mathbb{P}^1_{\mathbb{Z}}$$

induced by $\mathbb{Z}[\frac{T_1}{T_0}, \frac{T_0}{T_1}] \to \mathbb{Z}; \frac{T_1}{T_0} \mapsto 1,\ \frac{T_0}{T_1} \mapsto 1.$

Next, we recall the standard elementary distinguished square for $\mathbb{P}^1_{\mathbb{Z}}$

$$\begin{array}{ccc} \mathbb{G}_m & \xrightarrow{t_0} & \mathbb{A}^1_{\mathbb{Z}} \\ {\scriptstyle t_\infty}\downarrow & & \downarrow{\scriptstyle j_0} \\ \mathbb{A}^1_{\mathbb{Z}} & \xrightarrow[j_\infty]{} & \mathbb{P}^1_{\mathbb{Z}} . \end{array}$$

The morphism j_0 is induced by the canonical ring isomorphism $\mathbb{Z}[T] \cong \mathbb{Z}[\frac{T_0}{T_1}]$ and j_∞ is induced by $\mathbb{Z}[T] \cong \mathbb{Z}[\frac{T_1}{T_0}]$. The open immersion t_0 is defined by $\mathbb{Z}[T] \to \mathbb{Z}[T, T^{-1}]$; $T \mapsto T$ and t_∞ is defined by $\mathbb{Z}[T] \to \mathbb{Z}[T, T^{-1}]$; $T \mapsto T^{-1}$. In particular, the canonical morphism from the pushout of the diagram above to $\mathbb{P}^1_{\mathbb{Z}}$ is a motivic weak equivalence. If we equip $\mathbb{G}_m$, $\mathbb{A}^1_{\mathbb{Z}}$ and $\mathbb{P}^1_{\mathbb{Z}}$ with the base point 1, this weak equivalence becomes a weak equivalence of pointed motivic spaces. Note that the pushout of the diagram is also the pushout of the following diagram:

$$\begin{array}{ccc} (\mathbb{G}_m, 1) \vee (\mathbb{G}_m, 1) & \xrightarrow{(\mathrm{id},\mathrm{id})} & (\mathbb{G}_m, 1) \\ {\scriptstyle t_\infty \vee t_0}\big\downarrow & & \\ (\mathbb{A}^1_{\mathbb{Z}}, 1) \vee (\mathbb{A}^1_{\mathbb{Z}}, 1)\,. & & \end{array}$$

Let I be the pointed space $(\Delta^1, 1) \vee (\Delta^1, 0)$. Then I is a simplicial model of the interval admitting a mid-point. We denote the glueing point of I by $\frac{1}{2}$. There is a canonical weak equivalence from I to Δ^1 by projecting $(\Delta^1, 1)$ to the point 0. We consider now the comparison maps between pushouts from [Lev10, Lemma 4.1] (In the left diagram, $\mathbb{G}_m$ and $\mathbb{A}^1_{\mathbb{Z}}$ are equipped with the base point 1):

$$\begin{array}{ccccc} \mathbb{G}_m \vee \mathbb{G}_m \xrightarrow{(\mathrm{id},\mathrm{id})} \mathbb{G}_m & & 0_+ \wedge \mathbb{G}_m \vee 1_+ \wedge \mathbb{G}_m & \xrightarrow{(l_0,l_1)} & I_+ \wedge \mathbb{G}_m \\ {\scriptstyle t_\infty \vee t_0}\big\downarrow \qquad\qquad & & \big\downarrow{\scriptstyle t_\infty \vee t_0} & & \\ \mathbb{A}^1_{\mathbb{Z}} \vee \mathbb{A}^1_{\mathbb{Z}} \qquad\qquad & \longleftarrow & 0_+ \wedge \mathbb{A}^1_{\mathbb{Z}} \vee 1_+ \wedge \mathbb{A}^1_{\mathbb{Z}} & & \\ & & \big\downarrow & & \\ & & 0_+ \wedge \mathbb{G}_m \vee 1_+ \wedge \mathbb{G}_m & \xrightarrow{(l_0,l_1)} & I_+ \wedge \mathbb{G}_m \\ & & \big\downarrow & & \\ & & * & & . \end{array}$$

The first map is induced by the canonical projections and the second by collapsing $\mathbb{A}^1_{\mathbb{Z}}$ to a point. Both comparison maps are motivic weak equivalences. We denote the pushout of the middle diagram by $\mathcal{X}$. There is a canonical weak equivalence from the pushout of the diagram:

$$\begin{array}{ccc} 0_+ \wedge \mathbb{G}_m \vee 1_+ \wedge \mathbb{G}_m & \longrightarrow & I_+ \wedge \mathbb{G}_m \\ \big\downarrow & & \\ * & & \end{array}$$

to $S^1 \wedge \mathbb{G}_m$, which is induced by the weak equivalence from I to Δ^1. We can equip $\mathcal{X}$ with the base point ∞ which comes from the point $0 : \operatorname{Spec} \mathbb{Z} \to 0_+ \wedge \mathbb{A}^1_{\mathbb{Z}}$. It is defined by $\mathbb{Z}[T] \to \mathbb{Z}; T \mapsto 0$. Now, we also equip $\mathbb{A}^1_{\mathbb{Z}} \vee \mathbb{A}^1_{\mathbb{Z}}$ with the base point

$$\operatorname{Spec} \mathbb{Z} \xrightarrow{0} \mathbb{A}^1_{\mathbb{Z}} \xhookrightarrow{i_1} \mathbb{A}^1_{\mathbb{Z}} \vee \mathbb{A}^1_{\mathbb{Z}}$$

where i_1 is the inclusion into the first copy of $\mathbb{A}^1_{\mathbb{Z}}$. Then the canonical morphism $\mathbb{A}^1_{\mathbb{Z}} \vee \mathbb{A}^1_{\mathbb{Z}} \to \mathbb{P}^1_{\mathbb{Z}}$ becomes a morphism of the pointed motivic spaces $(\mathbb{A}^1_{\mathbb{Z}} \vee \mathbb{A}^1_{\mathbb{Z}}, 0) \to (\mathbb{P}^1_{\mathbb{Z}}, \infty)$. Hence, the first comparison map above between pushouts also induces a weak equivalence of pointed motivic spaces $(\mathcal{X}, \infty) \simeq (\mathbb{P}^1_{\mathbb{Z}}, \infty)$. The second comparison map also induces a weak equivalence $(\mathcal{X}, \infty) \simeq S^1 \wedge \mathbb{G}_m$, since $0_+ \wedge \mathbb{A}^1_{\mathbb{Z}}$ is sent to the point $*$. Thus, we get here an isomorphism $\alpha : (\mathbb{P}^1_{\mathbb{Z}}, \infty) \to S^1 \wedge \mathbb{G}_m$ in $\mathcal{H}_\bullet(\mathbb{Z})$ and $(\mathbb{P}^1_{\mathbb{Z}}, \infty)$ inherits a cogroup structure from $S^1 \wedge \mathbb{G}_m$ via α.

In [Caz12, Lemma B.4], Cazanave gives a co-diagonal morphism for $(\mathbb{P}^1_k, \infty)$. Actually, his method also works over the base Spec $\mathbb{Z}$. In the following, we write down this morphism in detail. Again we consider the commutative diagram:

$$\begin{array}{ccc} \mathbb{G}_m \vee \mathbb{G}_m & \xrightarrow{\mathrm{id} \vee \mathrm{id}} & \mathbb{G}_m \\ {\scriptstyle t_\infty \vee t_0}\downarrow & & \downarrow \\ \mathbb{A}^1_{\mathbb{Z}} \vee \mathbb{A}^1_{\mathbb{Z}} & \xrightarrow[j_\infty \vee j_0]{} & \mathbb{P}^1_{\mathbb{Z}}\,. \end{array}$$

We denote by $(\mathbb{P}^1_{\mathbb{Z}}, \mathbb{G}_m)$ the cofiber of the inclusion $\mathbb{G}_m \hookrightarrow \mathbb{P}^1_{\mathbb{Z}}$ above. Similarly, we denote by $(\mathbb{A}^1_{\mathbb{Z}}, \mathbb{G}_m)_{t_\infty}$ the cofiber of the inclusion t_∞ and by $(\mathbb{A}^1_{\mathbb{Z}}, \mathbb{G}_m)_{t_0}$ the cofiber of t_0. Then from the first comparison map

$$\begin{array}{ccccc} \mathbb{G}_m \vee \mathbb{G}_m \xrightarrow{\mathrm{id} \vee \mathrm{id}} \mathbb{G}_m & & & & 0_+ \wedge \mathbb{G}_m \vee 1_+ \wedge \mathbb{G}_m \longrightarrow I_+ \wedge \mathbb{G}_m \\ {\scriptstyle t_\infty \vee t_0}\downarrow \qquad\qquad & & & & {\scriptstyle t_\infty \vee t_0}\downarrow \qquad\qquad\qquad \\ \mathbb{A}^1_{\mathbb{Z}} \vee \mathbb{A}^1_{\mathbb{Z}} \qquad\qquad & & \longleftarrow & & 0_+ \wedge \mathbb{A}^1_{\mathbb{Z}} \vee 1_+ \wedge \mathbb{A}^1_{\mathbb{Z}} \qquad\qquad \end{array}$$

we obtain a motivic weak equivalence

$$\mathcal{X}/(I_+ \wedge \mathbb{G}_m) = (\mathbb{A}^1_{\mathbb{Z}}, \mathbb{G}_m)_{t_\infty} \vee (\mathbb{A}^1_{\mathbb{Z}}, \mathbb{G}_m)_{t_0} \to (\mathbb{P}^1_{\mathbb{Z}}, \mathbb{G}_m)$$

which is induced by $j_\infty \vee j_0$.

From the elementary distinguished square $\mathbb{P}^1_{\mathbb{Z}}$,

$$\begin{array}{ccc} \mathbb{G}_m & \xrightarrow{t_0} & \mathbb{A}^1_{\mathbb{Z}} \\ {\scriptstyle t_\infty}\downarrow & & \downarrow{\scriptstyle j_0} \\ \mathbb{A}^1_{\mathbb{Z}} & \xrightarrow[j_\infty]{} & \mathbb{P}^1_{\mathbb{Z}} \end{array}$$

we get two motivic weak equivalences

$$(\mathbb{A}^1_{\mathbb{Z}}, \mathbb{G}_m)_{t_\infty} \to (\mathbb{P}^1_{\mathbb{Z}}, \mathbb{A}^1_{\mathbb{Z}})_{j_0}$$

and

$$(\mathbb{A}^1_{\mathbb{Z}}, \mathbb{G}_m)_{t_0} \to (\mathbb{P}^1_{\mathbb{Z}}, \mathbb{A}^1_{\mathbb{Z}})_{j_\infty}.$$

Next, we recall that there is a path $j : \mathbb{A}^1_{\mathbb{Z}} \to \mathbb{P}^1_{\mathbb{Z}}$ from ∞ to 0. In [Caz12, Appendix B], Cazanave calls this path the canonical path from ∞ to 0. In " homogeneous coordinates", it is given by $[1-T : T]$. Moreover, we also can give the precise definition of j. We first define an automorphism ψ of $\mathbb{P}^1_{\mathbb{Z}}$ which is induced by the ring isomorphism

$$\mathbb{Z}[T_0,T_1] \to \mathbb{Z}[T_0,T_1]; T_0 \mapsto T_0 - T_1, T_1 \mapsto T_1.$$

Recall that $j_\infty : \mathbb{A}^1_{\mathbb{Z}} \hookrightarrow \mathbb{P}^1_{\mathbb{Z}}$ is the open embedding into $\mathrm{D}_+(T_0)$. Then we define the path j to be the composition $\psi \circ j_\infty$. Therefore, this path is an open embedding. We denote the cofiber of j just simply by $\mathbb{P}^1_{\mathbb{Z}}/\mathbb{A}^1_{\mathbb{Z}}$. The canonical projection $\theta : \mathbb{P}^1_{\mathbb{Z}} \to \mathbb{P}^1_{\mathbb{Z}}/\mathbb{A}^1_{\mathbb{Z}}$ is a weak equivalence and it induces two pointed weak equivalences $\theta_0 : (\mathbb{P}^1_{\mathbb{Z}},0) \to \mathbb{P}^1_{\mathbb{Z}}/\mathbb{A}^1_{\mathbb{Z}}$ and $\theta_\infty : (\mathbb{P}^1_{\mathbb{Z}},\infty) \to \mathbb{P}^1_{\mathbb{Z}}/\mathbb{A}^1_{\mathbb{Z}}$.

Finally, we can write down the co-diagonal morphism for $(\mathbb{P}^1_{\mathbb{Z}},\infty)$ given by Cazanave:

$$\begin{array}{ccccc}
(\mathbb{P}^1_{\mathbb{Z}},\infty) & \longrightarrow & (\mathbb{P}^1_{\mathbb{Z}},\mathbb{G}_m) & \xleftarrow{\ \sim\ } & (\mathbb{A}^1_{\mathbb{Z}},\mathbb{G}_m)_{t_\infty} \vee (\mathbb{A}^1_{\mathbb{Z}},\mathbb{G}_m)_{t_0} \\
&&&& \text{induced by } j_\infty \vee j_0 \Big\downarrow \sim \\
&&&& (\mathbb{P}^1_{\mathbb{Z}},\mathbb{A}^1_{\mathbb{Z}})_{j_0} \vee (\mathbb{P}^1_{\mathbb{Z}},\mathbb{A}^1_{\mathbb{Z}})_{j_\infty} \\
&&&& \Big\uparrow \sim \\
&&&& (\mathbb{P}^1_{\mathbb{Z}},0) \vee (\mathbb{P}^1_{\mathbb{Z}},\infty) \\
&&&& \sim \Big\downarrow \theta_0 \vee \mathrm{id} \\
&&&& \mathbb{P}^1_{\mathbb{Z}}/\mathbb{A}^1_{\mathbb{Z}} \vee (\mathbb{P}^1_{\mathbb{Z}},\infty) \\
&&&& \sim \Big\uparrow \theta_\infty \vee \mathrm{id} \\
&&&& (\mathbb{P}^1_{\mathbb{Z}},\infty) \vee (\mathbb{P}^1_{\mathbb{Z}},\infty) .
\end{array}$$

The weak equivalences in the diagram are indicated by $\sim$. We emphasize that we equip here $(\mathbb{P}^1_{\mathbb{Z}},\mathbb{G}_m)$ with the base point ∞ coming from the corresponding base point of $\mathbb{P}^1_{\mathbb{Z}}$. Analogously, we equip $(\mathbb{A}^1_{\mathbb{Z}},\mathbb{G}_m)_{t_\infty} \vee (\mathbb{A}^1_{\mathbb{Z}},\mathbb{G}_m)_{t_0}$ with the base point

$$\operatorname{Spec}\mathbb{Z} \xrightarrow{\ 0\ } \mathbb{A}^1_{\mathbb{Z}} \longrightarrow (\mathbb{A}^1_{\mathbb{Z}},\mathbb{G}_m)_{t_\infty} \hookrightarrow (\mathbb{A}^1_{\mathbb{Z}},\mathbb{G}_m)_{t_\infty} \vee (\mathbb{A}^1_{\mathbb{Z}},\mathbb{G}_m)_{t_0}$$

and we also call this base point 0. Furthermore, we equip $(\mathbb{P}^1_{\mathbb{Z}},\mathbb{A}^1_{\mathbb{Z}})_{j_0} \vee (\mathbb{P}^1_{\mathbb{Z}},\mathbb{A}^1_{\mathbb{Z}})_{j_\infty}$ with the base point

$$\operatorname{Spec}\mathbb{Z} \xrightarrow{\ \infty\ } \mathbb{P}^1_{\mathbb{Z}} \longrightarrow (\mathbb{P}^1_{\mathbb{Z}},\mathbb{A}^1_{\mathbb{Z}})_{j_0} \hookrightarrow (\mathbb{P}^1_{\mathbb{Z}},\mathbb{A}^1_{\mathbb{Z}})_{j_0} \vee (\mathbb{P}^1_{\mathbb{Z}},\mathbb{A}^1_{\mathbb{Z}})_{j_\infty}$$

and we call it ∞, too. Then we also equip $(\mathbb{P}^1_{\mathbb{Z}},0) \vee (\mathbb{P}^1_{\mathbb{Z}},\infty)$ with the base point

$$\operatorname{Spec}\mathbb{Z} \xrightarrow{\ \infty\ } (\mathbb{P}^1_{\mathbb{Z}},0) \hookrightarrow (\mathbb{P}^1_{\mathbb{Z}},0) \vee (\mathbb{P}^1_{\mathbb{Z}},\infty)$$

and also denote it by ∞. The motivic spaces $\mathbb{P}^1_{\mathbb{Z}}/\mathbb{A}^1_{\mathbb{Z}} \vee (\mathbb{P}^1_{\mathbb{Z}},\infty)$ and $(\mathbb{P}^1_{\mathbb{Z}},\infty) \vee (\mathbb{P}^1_{\mathbb{Z}},\infty)$ are equipped with the canonical base points. Using these base points and the diagram above, we get a pointed co-diagonal morphism $\tilde{\nabla} : (\mathbb{P}^1_{\mathbb{Z}}, \infty) \to (\mathbb{P}^1_{\mathbb{Z}}, \infty) \vee (\mathbb{P}^1_{\mathbb{Z}}, \infty)$ in $\mathcal{H}_{\bullet}(\mathbb{Z})$.

On the other hand, we also obtain a pointed co-diagonal morphism ∇ for $(\mathbb{P}^1_{\mathbb{Z}}, \infty)$ via the isomorphism $\alpha : (\mathbb{P}^1_{\mathbb{Z}}, \infty) \cong S^1 \wedge \mathbb{G}_m$. Now, we also write down this morphism explicitly. We recall that there is weak equivalence $I \to \Delta^1$ where I is $(\Delta^1, 1) \vee (\Delta^1, 0)$ and this weak equivalence is induced by projecting $(\Delta^1, 1)$ to the point $0 \in \Delta^1$. We denote the "mid-point" of I by $\frac{1}{2}$. Let ∂I denote the boundary of I, then this weak equivalence induces a weak equivalence $\mu : I/\partial I \to \Delta^1/\partial\Delta^1$. Particularly, the weak equivalence $\mu \wedge \mathrm{id} : I/\partial I \wedge \mathbb{G}_m \to S^1 \wedge \mathbb{G}_m$ is an isomorphism of cogroup objects in $\mathcal{H}_{\bullet}(\mathbb{Z})$. Moreover there is also a morphism from $I/\partial I$ to $S^1 \vee S^1$ by sending $(\Delta^1, 1)$ to the first copy of S^1 and $(\Delta^1, 0)$ to the second copy of S^1. Now, we have the diagram

$$\begin{array}{ccccc}
(\mathbb{P}^1_{\mathbb{Z}}, \infty) & \xleftarrow{\sim} & (\mathcal{X}, \infty) & \xrightarrow{\sim} & I/\partial I \wedge \mathbb{G}_m \\
 & & & & \downarrow \\
 & & & & (S^1 \vee S^1) \wedge \mathbb{G}_m \\
 & & & & \sim \uparrow (\mu\vee\mu)\wedge\mathrm{id} \\
 & & & & (I/\partial I \vee I/\partial I) \wedge \mathbb{G}_m \\
 & & & & \downarrow = \\
 & & & & (I/\partial I \wedge \mathbb{G}_m) \vee (I/\partial I \wedge \mathbb{G}_m) \\
 & & & & \uparrow \sim \\
 & & & & (\mathcal{X}, \infty) \vee (\mathcal{X}, \infty) \\
 & & & & \downarrow \sim \\
 & & & & (\mathbb{P}^1_{\mathbb{Z}}, \infty) \vee (\mathbb{P}^1_{\mathbb{Z}}, \infty) .
\end{array}$$

The weak equivalences are indicated again by $\sim$. Therefore, we get from this diagram a co-diagonal morphism $\nabla : (\mathbb{P}^1_{\mathbb{Z}}, \infty) \to (\mathbb{P}^1_{\mathbb{Z}}, \infty) \vee (\mathbb{P}^1_{\mathbb{Z}}, \infty)$ in $\mathcal{H}_{\bullet}(\mathbb{Z})$. In [Caz12], Cazanave did not show that his co-diagonal $\tilde{\nabla}$ coincides with ∇, so we prove it in this paper.

Proposition 3.1. *The two co-diagonal morphisms $\tilde{\nabla}$ and ∇ are the same.*

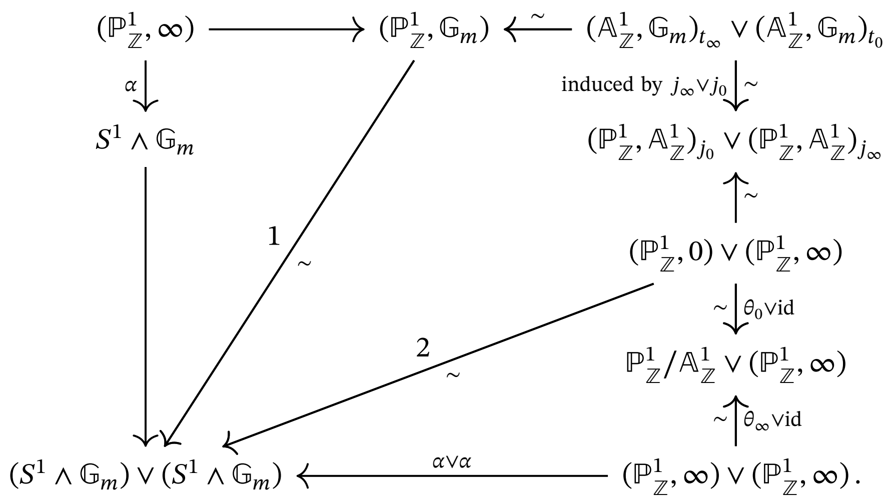
Proof. We have to show that the diagram

$$\begin{array}{ccc}
(\mathbb{P}^1_{\mathbb{Z}}, \infty) & \xrightarrow{\tilde{\nabla}} & (\mathbb{P}^1_{\mathbb{Z}}, \infty) \vee (\mathbb{P}^1_{\mathbb{Z}}, \infty) \\
\alpha \downarrow & & \downarrow \alpha\vee\alpha \\
S^1 \wedge \mathbb{G}_m & \longrightarrow & (S^1 \wedge \mathbb{G}_m) \vee (S^1 \wedge \mathbb{G}_m)
\end{array}$$

commutes in $\mathcal{H}_\bullet(\mathbb{Z})$, where $S^1 \wedge \mathbb{G}_m \to (S^1 \wedge \mathbb{G}_m) \vee (S^1 \wedge \mathbb{G}_m)$ is the morphism in $\mathcal{H}_\bullet(\mathbb{Z})$ induced by the sequence

$$S^1 \wedge \mathbb{G}_m \xleftarrow{\mu\wedge \mathrm{id}} I/\partial I \wedge \mathbb{G}_m \longrightarrow (S^1 \vee S^1) \wedge \mathbb{G}_m .$$

We can rewrite the square as follows:



We first explain what the weak equivalence indicated by 1 is. For this, we have to consider the diagram below again

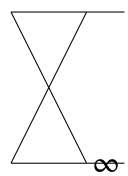
$$\begin{array}{ccccc}
\mathbb{G}_m \vee \mathbb{G}_m \xrightarrow{(\mathrm{id},\mathrm{id})} \mathbb{G}_m & & 0_+ \wedge \mathbb{G}_m \vee 1_+ \wedge \mathbb{G}_m & \xrightarrow{(l_0,l_1)} & I_+ \wedge \mathbb{G}_m \\
\downarrow{\scriptstyle t_\infty \vee t_0} & & \downarrow{\scriptstyle t_\infty \vee t_0} & & \\
\mathbb{A}^1_{\mathbb{Z}} \vee \mathbb{A}^1_{\mathbb{Z}} & \longleftarrow & 0_+ \wedge \mathbb{A}^1_{\mathbb{Z}} \vee 1_+ \wedge \mathbb{A}^1_{\mathbb{Z}} & & \\
 & & \downarrow & & \\
 & & 0_+ \wedge \mathbb{G}_m \vee 1_+ \wedge \mathbb{G}_m & \xrightarrow{(l_0,l_1)} & I_+ \wedge \mathbb{G}_m \\
 & & \downarrow & & \\
 & & * & & .
\end{array}$$

Recall that we denoted the pushout of the middle diagram by $\mathcal{X}$. We have here the inclusions $\{\frac{1}{2}\}_+ \wedge \mathbb{G}_m \hookrightarrow \mathcal{X}$ and $\{\frac{1}{2}\}_+ \wedge \mathbb{G}_m \hookrightarrow I/\partial I \wedge \mathbb{G}_m$. Via the first comparison map, the inclusion $\{\frac{1}{2}\}_+ \wedge \mathbb{G}_m \hookrightarrow \mathcal{X}$ corresponds to the inclusion $\mathbb{G}_m \hookrightarrow \mathbb{P}^1_{\mathbb{Z}}$. Via the second comparison map, the inclusion $\{\frac{1}{2}\}_+ \wedge \mathbb{G}_m \hookrightarrow \mathcal{X}$

corresponds to $\{\frac{1}{2}\}_+ \wedge \mathbb{G}_m \hookrightarrow I/\partial I \wedge \mathbb{G}_m$. Thus, we get a sequence of pointed motivic weak equivalences

$$((\mathbb{P}^1_{\mathbb{Z}}, \mathbb{G}_m), \infty) \xleftarrow{\sim} ((\mathcal{X}, \{\tfrac{1}{2}\}_+ \wedge \mathbb{G}_m), \infty) \xrightarrow{\sim} ((I/\partial I \wedge \mathbb{G}_m, \{\tfrac{1}{2}\}_+ \wedge \mathbb{G}_m), *)\,,$$

where $(\mathbb{P}^1_{\mathbb{Z}}, \mathbb{G}_m)$, $(\mathcal{X}, \{\frac{1}{2}\}_+ \wedge \mathbb{G}_m)$ and $(I/\partial I \wedge \mathbb{G}_m, \{\frac{1}{2}\}_+ \wedge \mathbb{G}_m)$ are the cofibers of the corresponding inclusions and $*$ is the canonical base point of $(I/\partial I \wedge \mathbb{G}_m, \{\frac{1}{2}\}_+ \wedge \mathbb{G}_m)$. Note that $(I/\partial I \wedge \mathbb{G}_m, \{\frac{1}{2}\}_+ \wedge \mathbb{G}_m)$ is just $(S^1 \wedge \mathbb{G}_m) \vee (S^1 \wedge \mathbb{G}_m)$. We can illustrate the pointed motivic space $((\mathcal{X}, \{\frac{1}{2}\}_+ \wedge \mathbb{G}_m), \infty)$ as follows:



The space $\mathcal{X}$ is obtained by glueing the two copies of $\mathbb{G}_m$ in $I_+ \wedge \mathbb{G}_m$ with $0_+ \wedge \mathbb{A}^1_{\mathbb{Z}}$ and $1_+ \wedge \mathbb{A}^1_{\mathbb{Z}}$, respectively. Therefore, in the illustration the top line demonstrates the glueing of $0_+ \wedge \mathbb{G}_m$ with $0_+ \wedge \mathbb{A}^1_{\mathbb{Z}}$; the bottom line means the glueing of $1_+ \wedge \mathbb{G}_m$ with $1_+ \wedge \mathbb{A}^1_{\mathbb{Z}}$. The point in the middle of the illustration represents the collapse of the subspace $\{\frac{1}{2}\}_+ \wedge \mathbb{G}_m$. The base point ∞ comes from $0_+ \wedge \mathbb{A}^1_{\mathbb{Z}}$.

Via this zig-zag of weak equivalences, we get the morphism indicated by 1 which is an isomorphism in $\mathcal{H}_\bullet(\mathbb{Z})$.

Next, we explain what the weak equivalence indicated by 2 is. We can equip $\mathcal{X}$ with the base point

$$\operatorname{Spec} \mathbb{Z} \xrightarrow{0} 1_+ \wedge \mathbb{A}^1_{\mathbb{Z}} \longrightarrow \mathcal{X}$$

and we also denote this base point by 0. Then the first comparison map induces a pointed weak equivalence $(\mathcal{X}, 0) \to (\mathbb{P}^1_{\mathbb{Z}}, 0)$ and the second comparison map induces a pointed weak equivalence $(\mathcal{X}, 0) \to S^1 \wedge \mathbb{G}_m$. Therefore, we also get an isomorphism $\tilde{\alpha} : (\mathbb{P}^1_{\mathbb{Z}}, 0) \cong S^1 \wedge \mathbb{G}_m$. Hence, the morphism indicated by 2 is $\tilde{\alpha} \vee \alpha$.

In order to show that the square in the beginning is commutative, we have to show that the following three diagrams

$$\begin{array}{ccc}
(\mathbb{P}^1_{\mathbb{Z}}, \infty) & \longrightarrow & (\mathbb{P}^1_{\mathbb{Z}}, \mathbb{G}_m) \\
{\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle 1} \\
S^1 \wedge \mathbb{G}_m & \longrightarrow & (S^1 \wedge \mathbb{G}_m) \vee (S^1 \wedge \mathbb{G}_m)\,,
\end{array}$$

$$
\begin{array}{ccc}
(\mathbb{P}^1_{\mathbb{Z}}, \mathbb{G}_m) & \xleftarrow{\sim} & (\mathbb{A}^1_{\mathbb{Z}}, \mathbb{G}_m)_{t_\infty} \vee (\mathbb{A}^1_{\mathbb{Z}}, \mathbb{G}_m)_{t_0} \\
 & & \sim \Big\downarrow \text{induced by } j_\infty \vee j_0 \\
1 \Big\downarrow \sim & & (\mathbb{P}^1_{\mathbb{Z}}, \mathbb{A}^1_{\mathbb{Z}})_{j_0} \vee (\mathbb{P}^1_{\mathbb{Z}}, \mathbb{A}^1_{\mathbb{Z}})_{j_\infty} \\
 & & \Big\uparrow \sim \\
(S^1 \wedge \mathbb{G}_m) \vee (S^1 \wedge \mathbb{G}_m) & \xleftarrow[\tilde{\alpha} \vee \alpha]{} & (\mathbb{P}^1_{\mathbb{Z}}, 0) \vee (\mathbb{P}^1_{\mathbb{Z}}, \infty)
\end{array}
$$

and

$$
\begin{array}{ccc}
(\mathbb{P}^1_{\mathbb{Z}}, 0) \vee (\mathbb{P}^1_{\mathbb{Z}}, \infty) & \xrightarrow[\sim]{\theta_0 \vee \mathrm{id}} & \mathbb{P}^1_{\mathbb{Z}} / \mathbb{A}^1_{\mathbb{Z}} \vee (\mathbb{P}^1_{\mathbb{Z}}, \infty) \\
\tilde{\alpha} \vee \alpha \Big\downarrow & & \sim \Big\uparrow \theta_\infty \vee \mathrm{id} \\
(S^1 \wedge \mathbb{G}_m) \vee (S^1 \wedge \mathbb{G}_m) & \xleftarrow[\alpha \vee \alpha]{} & (\mathbb{P}^1_{\mathbb{Z}}, \infty) \vee (\mathbb{P}^1_{\mathbb{Z}}, \infty)
\end{array}
$$

commute in the pointed homotopy category. The commutativity of the first two diagrams holds by inspection.

In the next step, we show the commutativity of the third diagram. Recall that $\mathbb{P}^1_{\mathbb{Z}} / \mathbb{A}^1_{\mathbb{Z}}$ is the cofiber of the path $j : \mathbb{A}^1_{\mathbb{Z}} \to \mathbb{P}^1_{\mathbb{Z}}$ from ∞ to 0. The path j is the composition $\psi \circ j_\infty$ where ψ is induced by the ring isomorphism

$$\mathbb{Z}[T_0, T_1] \to \mathbb{Z}[T_0, T_1]; T_0 \mapsto T_0 - T_1, T_1 \mapsto T_1.$$

In particular, j induces an isomorphism from $\mathbb{A}^1_{\mathbb{Z}}$ to the open subset $\mathrm{D}_+(T_0 + T_1)$ of $\mathbb{P}^1_{\mathbb{Z}}$. Therefore, the cofiber $\mathbb{P}^1_{\mathbb{Z}} / \mathbb{A}^1_{\mathbb{Z}}$ is just the cofiber of the inclusion $\mathrm{D}_+(T_0 + T_1) \hookrightarrow \mathbb{P}^1_{\mathbb{Z}}$. Hence, we can replace in the third diagram $\mathbb{P}^1_{\mathbb{Z}} / \mathbb{A}^1_{\mathbb{Z}}$ by $\mathbb{P}^1_{\mathbb{Z}} / \mathrm{D}_+(T_0 + T_1)$.

Next, we consider the following pushout diagram

$$
\begin{array}{ccc}
0 \times \mathrm{D}_+((T_0 + T_1) T_0 T_1) \sqcup 1 \times \mathrm{D}_+((T_0 + T_1) T_0 T_1) & \hookrightarrow & I \times \mathrm{D}_+((T_0 + T_1) T_0 T_1) \\
\text{inclusion} \sqcup \text{inclusion} \Big\downarrow & & \\
0 \times \mathrm{D}_+((T_0 + T_1) T_0) \sqcup 1 \times \mathrm{D}_+((T_0 + T_1) T_1) & &
\end{array}
$$

where we denote the pushout by A. It is clear that there is a canonical weak equivalence from A to $\mathrm{D}_+(T_0 + T_1)$. Then there is a comparison map of diagrams

$$
\begin{array}{ccc}
0 \times \mathrm{D}_+((T_0 + T_1) T_0 T_1) \sqcup 1 \times \mathrm{D}_+((T_0 + T_1) T_0 T_1) & \hookrightarrow & I \times \mathrm{D}_+((T_0 + T_1) T_0 T_1) \\
\text{inclusion} \sqcup \text{inclusion} \Big\downarrow & & \\
0 \times \mathrm{D}_+((T_0 + T_1) T_0) \sqcup 1 \times \mathrm{D}_+((T_0 + T_1) T_1) & &
\end{array}
$$

$$\downarrow$$

$$\begin{array}{ccc} 0\times\mathbb{G}_m \sqcup 1\times\mathbb{G}_m & \longrightarrow & I\times\mathbb{G}_m \\ {\scriptstyle t_\infty\sqcup t_0}\downarrow & & \\ 0\times\mathbb{A}^1_{\mathbb{Z}} \sqcup 1\times\mathbb{A}^1_{\mathbb{Z}} & & \end{array}$$

where we denote the pushout of the second diagram by $\tilde{\mathcal{X}}$. Note that $\tilde{\mathcal{X}}/I\times\{1\}$ is the motivic space $\mathcal{X}$, where 1 is the base point of $\mathbb{G}_m$. Naturally, there is a weak equivalence from $\tilde{\mathcal{X}}$ to $\mathbb{P}^1_{\mathbb{Z}}$ just as for $\mathcal{X}$. Furthermore, there is also a canonical weak equivalence from $\tilde{\mathcal{X}}$ to $S^1\wedge\mathbb{G}_m$. It is the composition $\tilde{\mathcal{X}}\to\mathcal{X}\to S^1\wedge\mathbb{G}_m$.

As for $\mathcal{X}$ we can equip $\tilde{\mathcal{X}}$ with the base points 0 and ∞. Moreover, we have the following commutative diagrams

$$\begin{array}{ccc} & & (\mathbb{P}^1_{\mathbb{Z}},0) \\ & \nearrow & \uparrow \\ (\tilde{\mathcal{X}},0) & \xrightarrow{\sim} & (\mathcal{X},0) \\ & \searrow & \downarrow \\ & & S^1\wedge\mathbb{G}_m \end{array}$$

and

$$\begin{array}{ccc} & & (\mathbb{P}^1_{\mathbb{Z}},\infty) \\ & \nearrow & \uparrow \\ (\tilde{\mathcal{X}},\infty) & \xrightarrow{\sim} & (\mathcal{X},\infty) \\ & \searrow & \downarrow \\ & & S^1\wedge\mathbb{G}_m\,. \end{array}$$

Thus, we only have to show that the diagram

$$
\begin{array}{ccccc}
(\mathbb{P}^1_{\mathbb{Z}},0)\vee(\mathbb{P}^1_{\mathbb{Z}},\infty) & & \xrightarrow[\sim]{\theta_0\vee\mathrm{id}} & & \mathbb{P}^1_{\mathbb{Z}}/\mathbb{A}^1_{\mathbb{Z}}\vee(\mathbb{P}^1_{\mathbb{Z}},\infty) \\
\uparrow{\scriptstyle\sim} & & & & \\
(\tilde{\mathcal{X}},0)\vee(\tilde{\mathcal{X}},\infty) & & & & \uparrow{\scriptstyle\sim\ \theta_\infty\vee\mathrm{id}} \\
\downarrow{\scriptstyle\sim} & & & & \\
(S^1\wedge\mathbb{G}_m)\vee(S^1\wedge\mathbb{G}_m) & \xleftarrow[\sim]{} & (\tilde{\mathcal{X}},\infty)\vee(\tilde{\mathcal{X}},\infty) & \xrightarrow[\sim]{} & (\mathbb{P}^1_{\mathbb{Z}},\infty)\vee(\mathbb{P}^1_{\mathbb{Z}},\infty)
\end{array}
$$

is commutative.

The comparison map

$$
\begin{array}{ccc}
0\times \mathrm{D}_+((T_0+T_1)T_0T_1)\sqcup 1\times \mathrm{D}_+((T_0+T_1)T_0T_1) & \hookrightarrow & I\times \mathrm{D}_+((T_0+T_1)T_0T_1) \\
{\scriptstyle \mathrm{inclusion}\sqcup\mathrm{inclusion}}\big\downarrow & & \\
0\times \mathrm{D}_+((T_0+T_1)T_0)\sqcup 1\times \mathrm{D}_+((T_0+T_1)T_1) & &
\end{array}
$$

$$\downarrow$$

$$
\begin{array}{ccc}
0\times \mathbb{G}_m\sqcup 1\times\mathbb{G}_m & \longrightarrow & I\times\mathbb{G}_m \\
{\scriptstyle t_\infty\sqcup t_0}\big\downarrow & & \\
0\times\mathbb{A}^1_{\mathbb{Z}}\sqcup 1\times\mathbb{A}^1_{\mathbb{Z}} & &
\end{array}
$$

induces an inclusion $A\hookrightarrow\tilde{\mathcal{X}}$. Hence, we get a canonical weak equivalence $\tilde{\mathcal{X}}/A\to\mathbb{P}^1_{\mathbb{Z}}/\mathrm{D}_+(T_0+T_1)$. The canonical projection $\tilde{\mathcal{X}}\to\tilde{\mathcal{X}}/A$ induces two pointed weak equivalences $\tilde{\theta}_\infty:(\tilde{\mathcal{X}},\infty)\to\tilde{\mathcal{X}}/A$ and $\tilde{\theta}_0:(\tilde{\mathcal{X}},0)\to\tilde{\mathcal{X}}/A$.

The canonical morphism $\tilde{\mathcal{X}} \to \mathbb{P}^1_{\mathbb{Z}}$ induces pointed weak equivalences $\nu_\infty : (\tilde{\mathcal{X}}, \infty) \to (\mathbb{P}^1_{\mathbb{Z}}, \infty)$ and $\nu_0 : (\tilde{\mathcal{X}}, 0) \to (\mathbb{P}^1_{\mathbb{Z}}, 0)$. Now we can consider the following diagram

$$
\begin{array}{ccccc}
(\mathbb{P}^1_{\mathbb{Z}}, 0) \vee (\mathbb{P}^1_{\mathbb{Z}}, \infty) & & \xrightarrow[\sim]{\theta_0 \vee \mathrm{id}} & & \mathbb{P}^1_{\mathbb{Z}}/\mathrm{D}_+(T_0 + T_1) \vee (\mathbb{P}^1_{\mathbb{Z}}, \infty) \\
\sim \uparrow {\scriptstyle \nu_0 \vee \nu_\infty} & & & \nearrow \sim & \\
(\tilde{\mathcal{X}}, 0) \vee (\tilde{\mathcal{X}}, \infty) & \xrightarrow[\tilde{\theta}_0 \vee \mathrm{id}]{\sim} & \tilde{\mathcal{X}}/A \vee (\tilde{\mathcal{X}}, \infty) & & \uparrow {\scriptstyle \theta_\infty \vee \mathrm{id}} \sim \\
\sim \downarrow & & {\scriptstyle \tilde{\theta}_\infty \vee \mathrm{id}} \uparrow \sim & & \\
(S^1 \wedge \mathbb{G}_m) \vee (S^1 \wedge \mathbb{G}_m) & \xleftarrow{\sim} & (\tilde{\mathcal{X}}, \infty) \vee (\tilde{\mathcal{X}}, \infty) & \xrightarrow[\nu_\infty \vee \nu_\infty]{\sim} & (\mathbb{P}^1_{\mathbb{Z}}, \infty) \vee (\mathbb{P}^1_{\mathbb{Z}}, \infty)
\end{array}
$$

where the outer diagram is our third diagram.

Now we only have to show that the diagram

$$
\begin{array}{ccc}
(\tilde{\mathcal{X}}, 0) \vee (\tilde{\mathcal{X}}, \infty) & \xrightarrow[\tilde{\theta}_0 \vee \mathrm{id}]{\sim} & \tilde{\mathcal{X}}/A \vee (\tilde{\mathcal{X}}, \infty) \\
\sim \downarrow & & {\scriptstyle \tilde{\theta}_\infty \vee \mathrm{id}} \uparrow \sim \\
(S^1 \wedge \mathbb{G}_m) \vee (S^1 \wedge \mathbb{G}_m) & \xleftarrow{\sim} & (\tilde{\mathcal{X}}, \infty) \vee (\tilde{\mathcal{X}}, \infty)
\end{array}
$$

commutes as the other two inner diagrams commute already by construction. In order to show the commutativity, we first want to construct a morphism from $\tilde{\mathcal{X}}/A$ to $S^1 \wedge \mathbb{G}_m$ in the pointed homotopy category.

Taking the composition

$$
I \times \{1\} \hookrightarrow \tilde{\mathcal{X}} \longrightarrow \tilde{\mathcal{X}}/A
$$

where 1 is the canonical base point of $\mathbb{G}_m$, we can form the cofiber $(\tilde{\mathcal{X}}/A)/(I \times \{1\})$ and the projection

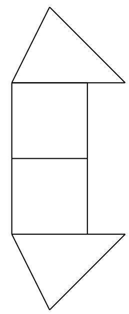
$$
\tilde{\mathcal{X}}/A \longrightarrow (\tilde{\mathcal{X}}/A)/(I \times \{1\})
$$

is a sectionwise weak equivalence of pointed spaces.

Let $\mathcal{C}(\mathbb{A}^1_{\mathbb{Z}})$ be $\Delta^1 \wedge \mathbb{A}^1_{\mathbb{Z}}$ where Δ^1 is based at 1 and $\mathbb{A}^1_{\mathbb{Z}}$ is based at 1. Let $\mathcal{C}'(\mathbb{A}^1_{\mathbb{Z}})$ be $\Delta^1 \wedge \mathbb{A}^1_{\mathbb{Z}}$ where Δ^1 is based at 0 and $\mathbb{A}^1_{\mathbb{Z}}$ is based at 1. Then we consider the following diagram

$$
\begin{array}{ccc}
0_+ \wedge \mathbb{A}^1_{\mathbb{Z}} \vee 1_+ \wedge \mathbb{A}^1_{\mathbb{Z}} & \longrightarrow & (\tilde{\mathcal{X}}/A)/(I \times \{1\}) \\
\big\downarrow & & \\
\mathcal{C}'(\mathbb{A}^1_{\mathbb{Z}}) \vee \mathcal{C}(\mathbb{A}^1_{\mathbb{Z}}) & &
\end{array}
$$

where we denote the pushout of this diagram by $\hat{\mathcal{X}}$. The canonical inclusion $\tilde{\mathcal{X}} \to \hat{\mathcal{X}}$ is a weak equivalence. We can illustrate $\hat{\mathcal{X}}$ as follows:



In the illustration, the bottom cone represents $\mathcal{C}'(\mathbb{A}^1_{\mathbb{Z}})$ and the top cone symbolizes $\mathcal{C}(\mathbb{A}^1_{\mathbb{Z}})$. The lines stretching out demonstrate the glueing of $0 \times \mathbb{G}_m$ with $0 \times \mathbb{A}^1_{\mathbb{Z}}$ and the glueing of $1 \times \mathbb{G}_m$ with $1 \times \mathbb{A}^1_{\mathbb{Z}}$, respectively.

Now we can apply the geometric realization functor defined in the author's doctoral thesis [Don24]. Let $\mathrm{Pre}_\Delta(\mathbb{Z})_*$ the category of presheaves on $\mathcal{S}\mathrm{m}_{\mathbb{Z}}$ with values in pointed Δ-generated topological spaces (see Section 6.1). By Proposition 6.2, the geometric realization of a simplicial set is Δ-generated. Therefore, by applying the usual geometric realization functor sectionwise, we get a functor

$$|\cdot| : \mathrm{sPre}(\mathbb{Z})_* \to \mathrm{Pre}_\Delta(\mathbb{Z})_*.$$

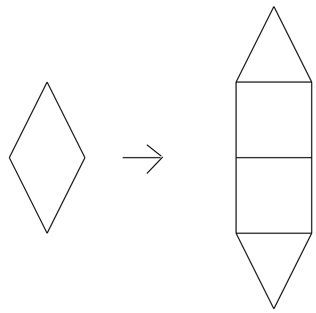
The basic properties of this functor can be found in Section 6.2. Furthermore, we can equip $\mathrm{Pre}_\Delta(\mathbb{Z})_*$ with an $\mathbb{A}^1$-local injective model structure (see Remark 6.7) and the corresponding homotopy category is denoted by $\mathcal{H}o_\Delta(\mathbb{Z})$. In particular, we can construct a pointed weak equivalence $|S^1 \wedge \mathbb{G}_m| \to |\hat{\mathcal{X}}|$. First note that we can consider the following pushout

$$\begin{array}{ccc} 0_+ \wedge \mathbb{G}_m \vee 1_+ \wedge \mathbb{G}_m & \longrightarrow & I_+ \wedge \mathbb{G}_m \\ \big\downarrow & & \\ \mathcal{C}'(\mathbb{G}_m) \vee \mathcal{C}(\mathbb{G}_m) & & \end{array}$$

where the reduced cones $\mathcal{C}(\mathbb{G}_m)$ and $\mathcal{C}'(\mathbb{G}_m)$ are just defined as for $\mathbb{A}^1_{\mathbb{Z}}$. We denote this pushout by $\mathcal{S}$. Moreover, we also have the comparison map

$$\begin{array}{ccccc} \mathcal{C}'(\mathbb{G}_m) \vee \mathcal{C}(\mathbb{G}_m) & \longleftarrow & 0_+ \wedge \mathbb{G}_m \vee 1_+ \wedge \mathbb{G}_m & \longrightarrow & I_+ \wedge \mathbb{G}_m \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathcal{C}'(\mathbb{A}^1_{\mathbb{Z}}) \vee \mathcal{C}(\mathbb{A}^1_{\mathbb{Z}}) & \longleftarrow & 0_+ \wedge \mathbb{A}^1_{\mathbb{Z}} \vee 1_+ \wedge \mathbb{A}^1_{\mathbb{Z}} & \longrightarrow & (\tilde{\mathcal{X}}/A)/I \times \{1\} \end{array}$$

which induces a pointed morphism $\rho : \mathcal{S} \to \hat{\mathcal{X}}$. Now it is easy to see that there is a pointed weak equivalence $\phi : |S^1 \wedge \mathbb{G}_m| \to |\mathcal{S}|$ by stretching $|S^1 \wedge \mathbb{G}_m|$. We illustrate this morphism as follows:



The pointed continuous morphism ϕ arises from stretching out the unit interval by a factor of 4.

Altogether we get a pointed morphism $|\rho|\circ\phi : |S^1\wedge\mathbb{G}_m| \to |\hat{\mathcal{X}}|$. We consider now the following diagram

$$\begin{array}{ccccc} |(\tilde{\mathcal{X}},0)| & & \xrightarrow[\sim]{|\tilde{\theta}_0|} & & |\tilde{\mathcal{X}}/A| \\ \downarrow{\scriptstyle\sim} & & & & {\scriptstyle\sim}\downarrow{\scriptstyle\text{canonical morphism}} \\ |S^1\wedge\mathbb{G}_m| & \xrightarrow[\phi]{} & |\mathcal{S}| & \xrightarrow[|\rho|]{} & |\hat{\mathcal{X}}| \end{array}$$

which commutes in the pointed homotopy category $\mathcal{H}o_{\Delta}(\mathbb{Z})$. We will now explain why this diagram commutes. For this we first look at the morphism

$$(0,1]\times\{0\} \longrightarrow |\mathcal{C}(\mathbb{A}^1_{\mathbb{Z}})| \longrightarrow |\hat{\mathcal{X}}|$$

where $\{0\}$ is the base point of $\mathbb{A}^1_{\mathbb{Z}}$. Let $|\hat{\mathcal{X}}|/|((0,1]\times\{0\}|)$ be the cofiber of this morphism. The projection $|\hat{\mathcal{X}}| \to |\hat{\mathcal{X}}/((0,1]\times\{0\})|$ is a sectionwise weak equivalence. Then we also can look at the morphism

$$[0,1)\times\{0\} \longrightarrow |\mathcal{C}'(\mathbb{A}^1_{\mathbb{Z}})| \longrightarrow |\hat{\mathcal{X}}/((0,1]\times\{0\})|\,.$$

In particular, the projection of $|\hat{\mathcal{X}}|$ to the cofiber of this morphism is a pointed sectionwise weak equivalence. Now we denote the cofiber by $\mathcal{Z}$.

We can deform the composition

$$|(\tilde{\mathcal{X}},0)| \longrightarrow |S^1\wedge\mathbb{G}_m| \xrightarrow{\phi} |\mathcal{S}| \xrightarrow{|\rho|} |\hat{\mathcal{X}}| \longrightarrow \mathcal{Z}$$

to

$$|(\tilde{\mathcal{X}},0)| \xrightarrow{|\tilde{\theta}_0|} |\tilde{\mathcal{X}}/A| \xrightarrow{\text{canonical morphism}} |\hat{\mathcal{X}}| \longrightarrow \mathcal{Z}$$

by stretching, too. In particular, the stretching gives us a pointed homotopy. In addition, it also follows that $|\rho|$ is a weak equivalence. Analogously, the diagram

$$
\begin{array}{ccccc}
|(\tilde{\mathcal{X}},\infty)| & & \xrightarrow[\sim]{|\tilde{\theta}_\infty|} & & |\tilde{\mathcal{X}}/A| \\
\downarrow{\scriptstyle\sim} & & & & {\scriptstyle\sim}\downarrow{\scriptstyle\text{canonical morphism}} \\
|S^1 \wedge \mathbb{G}_m| & \xrightarrow[\phi]{} & |\mathcal{S}| & \xrightarrow[|\rho|]{} & |\hat{\mathcal{X}}|
\end{array}
$$

commutes in the pointed homotopy category $\mathcal{H}o_\Delta(\mathbb{Z})$ by the same arguments. Since the derived geometric realization functor is an equivalence of categories (see Proposition 6.6), there is a unique isomorphism $\epsilon : S^1 \wedge \mathbb{G}_m \to \hat{\mathcal{X}}$ such that $|\epsilon|$ is $|\rho|\circ\phi$. Then the diagram

$$
\begin{array}{ccccc}
(\tilde{\mathcal{X}},0)\vee(\tilde{\mathcal{X}},\infty) & & \xrightarrow{\tilde{\theta}_0 \vee \mathrm{id}} & & \tilde{\mathcal{X}}/A \vee (\tilde{\mathcal{X}},\infty) \\
 & & & \swarrow{\scriptstyle\sim} & \\
\downarrow{\scriptstyle\sim} & & \hat{\mathcal{X}} \vee (\tilde{\mathcal{X}},\infty) & & \uparrow{\scriptstyle\tilde{\theta}_\infty \vee \mathrm{id}} \\
 & \swarrow{\scriptstyle\epsilon^{-1}\vee \mathrm{id}} & & & \\
(S^1 \wedge \mathbb{G}_m)\vee(S^1 \wedge \mathbb{G}_m) & & \xleftarrow{\sim} & & (\tilde{\mathcal{X}},\infty)\vee(\tilde{\mathcal{X}},\infty)
\end{array}
$$

commutes in $\mathcal{H}_\bullet(\mathbb{Z})$. Now we see that for

$$
\begin{array}{ccccc}
(\mathbb{P}^1_{\mathbb{Z}},0)\vee(\mathbb{P}^1_{\mathbb{Z}},\infty) & & \xrightarrow[\sim]{\theta_0 \vee \mathrm{id}} & & \mathbb{P}^1_{\mathbb{Z}}/\mathrm{D}_+(T_0+T_1)\vee(\mathbb{P}^1_{\mathbb{Z}},\infty) \\
{\scriptstyle\sim}\uparrow{\scriptstyle\nu_0\vee\nu_\infty} & & & \nearrow{\scriptstyle\sim} & \\
(\tilde{\mathcal{X}},0)\vee(\tilde{\mathcal{X}},\infty) & \xrightarrow[\tilde{\theta}_0\vee\mathrm{id}]{\sim} & \tilde{\mathcal{X}}/A\vee(\tilde{\mathcal{X}},\infty) & & {\scriptstyle\theta_\infty\vee\mathrm{id}}\uparrow{\scriptstyle\sim} \\
\downarrow{\scriptstyle\sim} & & {\scriptstyle\tilde{\theta}_\infty\vee\mathrm{id}}\uparrow{\scriptstyle\sim} & & \\
(S^1\wedge\mathbb{G}_m)\vee(S^1\wedge\mathbb{G}_m) & \xleftarrow{\sim} & (\tilde{\mathcal{X}},\infty)\vee(\tilde{\mathcal{X}},\infty) & \xrightarrow[\nu_\infty\vee\nu_\infty]{\sim} & (\mathbb{P}^1_{\mathbb{Z}},\infty)\vee(\mathbb{P}^1_{\mathbb{Z}},\infty)
\end{array}
$$

all three inner diagrams commute. Since all involved morphisms are isomorphisms in $\mathcal{H}_\bullet(\mathbb{Z})$, we can conclude that the outer diagram is commutative. □

4. Change of base points

There is a unique automorphism Φ of $\mathbb{P}^1_{\mathbb{Z}}$ which interchanges the base points 1 and ∞ and sends 0 to itself. It is induced by the ring isomorphism

$$\mathbb{Z}[T_0,T_1] \to \mathbb{Z}[T_0,T_1];\ T_0 \mapsto T_0, T_1 \mapsto T_0 - T_1.$$

Note that we have $\Phi\circ\Phi = \mathrm{id}$. We need the results in this section for Proposition 5.2.

Recall that we equip $(\mathbb{P}^1_{\mathbb{Z}}, \infty)$ with a cogroup structure via the following zig-zag of pointed weak equivalences

$$(\mathbb{P}^1_{\mathbb{Z}}, \infty) \xleftarrow{\sim} (\tilde{\mathcal{X}}, \infty) \xrightarrow{\sim} S^1 \wedge \mathbb{G}_m\,.$$

Now we would like to construct a similar zig-zag of pointed weak equivalences for $(\mathbb{P}^1_{\mathbb{Z}}, 1)$. It follows from the definition of Φ that we have $\Phi(\mathrm{D}_+(T_0)) = \mathrm{D}_+(T_0)$, $\Phi(\mathrm{D}_+(T_1)) = \mathrm{D}_+(T_0 - T_1)$ and $\Phi(\mathrm{D}_+(T_0T_1)) = \mathrm{D}_+((T_0 - T_1)T_0)$. The morphism $\infty : \mathrm{Spec}\mathbb{Z} \to \mathrm{D}_+(T_0)$ factors through $\mathrm{D}_+((T_0 - T_1)T_0)$:

$$\begin{array}{ccc}
\mathrm{Spec}\,\mathbb{Z} & \xrightarrow{\quad\infty\quad} & \mathrm{D}_+(T_0) \\
& \searrow \qquad \nearrow & \\
& \mathrm{D}_+((T_0 - T_1)T_0) &
\end{array}$$

Hence, we also denote the morphism $\mathrm{Spec}\,\mathbb{Z} \to \mathrm{D}_+((T_0 - T_1)T_0)$ by ∞.

We consider now the pushout diagram

$$\begin{array}{ccc}
0 \times \mathrm{D}_+((T_0 - T_1)T_0) \sqcup 1 \times \mathrm{D}_+((T_0 - T_1)T_0) & \rightarrow & I \times \mathrm{D}_+((T_0 - T_1)T_0) \\
\big\downarrow {\scriptstyle \text{inclusion}\sqcup\text{inclusion}} & & \\
0 \times \mathrm{D}_+(T_0) \sqcup 1 \times \mathrm{D}_+(T_0 - T_1) & & .
\end{array}$$

We denote the pushout of the diagram by $\mathcal{Y}$. We also can equip $\mathcal{Y}$ with the base point

$$\mathrm{Spec}\,\mathbb{Z} \xrightarrow{1} \mathrm{D}_+(T_0T_1) \hookrightarrow 0 \times \mathrm{D}_+(T_0)$$

which we also denote by 1.

In the next step, we can consider the comparison maps between pushout diagrams

$$\begin{array}{ccc}
0 \times \mathrm{D}_+((T_0 - T_1)T_0) \sqcup 1 \times \mathrm{D}_+((T_0 - T_1)T_0) & \rightarrow & I \times \mathrm{D}_+((T_0 - T_1)T_0) \\
\big\downarrow {\scriptstyle \text{inclusion}\sqcup\text{inclusion}} & & \\
0 \times \mathrm{D}_+(T_0) \sqcup 1 \times \mathrm{D}_+(T_0 - T_1) & &
\end{array}$$

$$\Big\downarrow$$

$$\begin{array}{ccc}
\mathrm{D}_+((T_0 - T_1)T_0) \sqcup \mathrm{D}_+((T_0 - T_1)T_0) & \longrightarrow & \mathrm{D}_+((T_0 - T_1)T_0) \\
\big\downarrow {\scriptstyle \text{inclusion}\sqcup\text{inclusion}} & & \\
\mathrm{D}_+(T_0) \sqcup \mathrm{D}_+(T_0 - T_1) & &
\end{array}$$

which induces a pointed weak equivalence $(\mathcal{Y}, 1) \to (\mathbb{P}^1_{\mathbb{Z}}, 1)$. Since the spaces $\mathrm{D}_+(T_0)$, $\mathrm{D}_+(T_0 - T_1)$ and $\mathrm{D}_+((T_0 - T_1)T_0)$ all contain the base point ∞, there is an inclusion $I \times \{\infty\} \hookrightarrow I \times \mathrm{D}_+((T_0 - T_1)T_0) \hookrightarrow \mathcal{Y}$. The induced projection $\mathcal{Y} \to \mathcal{Y}/I \times \{\infty\}$ is a weak equivalence. By collapsing $\mathrm{D}_+(T_0)$ and $\mathrm{D}_+(T_0 - T_1)$ to a point we get furthermore a weak equivalence

$$\mathcal{Y}/I \times \{\infty\} \to S^1 \wedge \mathrm{D}_+((T_0 - T_1)T_0)$$

such that the composition $(\mathcal{Y}, 1) \to \mathcal{Y}/I \times \{\infty\} \to S^1 \wedge \mathrm{D}_+((T_0 - T_1)T_0)$ is a pointed weak equivalence.

Now the automorphism Φ induces a comparison map

$$\begin{array}{ccc} 0 \times \mathbb{G}_m \sqcup 1 \times \mathbb{G}_m & \longrightarrow & I \times \mathbb{G}_m \\ \downarrow & & \\ 0 \times \mathbb{A}^1_{\mathbb{Z}} \sqcup 1 \times \mathbb{A}^1_{\mathbb{Z}} & & \end{array}$$

$$\downarrow$$

$$\begin{array}{ccc} 0 \times \mathrm{D}_+((T_0 - T_1)T_0) \sqcup 1 \times \mathrm{D}_+((T_0 - T_1)T_0) & \longrightarrow & I \times \mathrm{D}_+((T_0 - T_1)T_0) \\ \text{inclusion}\sqcup\text{inclusion} \big\downarrow & & \\ 0 \times \mathrm{D}_+(T_0) \sqcup 1 \times \mathrm{D}_+(T_0 - T_1) & & \end{array}$$

which in turn induces a pointed weak equivalence $\tilde{\Phi} : (\tilde{\mathcal{X}}, \infty) \to (\mathcal{Y}, 1)$. Similarly, the automorphism Φ also induces a comparison map

$$\begin{array}{ccc} \mathbb{G}_m \sqcup \mathbb{G}_m & \longrightarrow & \mathbb{G}_m \\ \downarrow & & \\ \mathbb{A}^1_{\mathbb{Z}} \sqcup \mathbb{A}^1_{\mathbb{Z}} & & \end{array}$$

$$\downarrow$$

$$\begin{array}{ccc} \mathrm{D}_+((T_0 - T_1)T_0) \sqcup \mathrm{D}_+((T_0 - T_1)T_0) & \longrightarrow & \mathrm{D}_+((T_0 - T_1)T_0) \\ \text{inclusion}\sqcup\text{inclusion} \big\downarrow & & \\ \mathrm{D}_+(T_0) \sqcup \mathrm{D}_+(T_0 - T_1) & & \end{array}$$

The induced morphism between the pushouts is just Φ. Altogether we obtain the commutative diagram

$$\begin{array}{ccccc} S^1 \wedge \mathbb{G}_m & \xleftarrow{\sim} & (\tilde{\mathcal{X}}, \infty) & \xrightarrow{\sim} & (\mathbb{P}^1_{\mathbb{Z}}, \infty) \\ \downarrow{\scriptstyle \mathrm{id} \wedge \Phi|_{\mathbb{G}_m}} & & \downarrow{\scriptstyle \tilde{\Phi}} & & \downarrow{\scriptstyle \Phi} \\ S^1 \wedge \mathrm{D}_+((T_0 - T_1)T_0) & \xleftarrow{\sim} & (\mathcal{Y}, 1) & \xrightarrow{\sim} & (\mathbb{P}^1_{\mathbb{Z}}, 1)\,. \end{array}$$

In addition, we also can consider the following comparison map which is induced by inclusions

$$\begin{array}{ccc} 0 \times \mathrm{D}_+((T_0 - T_1)T_1T_0) \sqcup 1 \times \mathrm{D}_+((T_0 - T_1)T_1T_0) & \rightarrow & I \times \mathrm{D}_+((T_0 - T_1)T_1T_0) \\ \downarrow & & \\ 0 \times \mathrm{D}_+(T_0T_1) \sqcup 1 \times \mathrm{D}_+((T_0 - T_1)T_1) & & \\ & & \\ \downarrow & & \\ & & \\ 0 \times \mathrm{D}_+((T_0 - T_1)T_0) \sqcup 1 \times \mathrm{D}_+((T_0 - T_1)T_0) & \longrightarrow & I \times \mathrm{D}_+((T_0 - T_1)T_0) \\ \downarrow{\scriptstyle \text{inclusion} \sqcup \text{inclusion}} & & \\ 0 \times \mathrm{D}_+(T_0) \sqcup 1 \times \mathrm{D}_+(T_0 - T_1) & & \end{array}$$

where we denote the pushout of the first diagram by B and B is canonically weakly equivalent to $\mathrm{D}_+(T_1)$. We equip the cofiber $\mathcal{Y}/B$ with the canonical base point. In particular, the projection $(\mathcal{Y}, 1) \to \mathcal{Y}/B$ is a pointed weak equivalence. Moreover, there is a canonical a weak equivalence $\mathcal{Y}/B \to \mathbb{P}^1_{\mathbb{Z}}/\mathrm{D}_+(T_1)$ of pointed spaces such that the diagram

$$\begin{array}{ccc} (\mathcal{Y}, 1) & \xrightarrow{\sim} & (\mathbb{P}^1_{\mathbb{Z}}, 1) \\ {\scriptstyle \sim}\downarrow & & \downarrow{\scriptstyle \sim} \\ \mathcal{Y}/B & \xrightarrow[\sim]{} & \mathbb{P}^1_{\mathbb{Z}}/\mathrm{D}_+(T_1) \end{array}$$

commutes. We denote the composition

$$(\mathbb{P}^1_{\mathbb{Z}}, \infty) \xrightarrow{\Phi} (\mathbb{P}^1_{\mathbb{Z}}, 1) \longrightarrow \mathbb{P}^1_{\mathbb{Z}}/\mathrm{D}_+(T_1)$$

by $\bar{\Phi}$. Altogether we get the commutative diagram

$$\begin{array}{ccccc} S^1 \wedge \mathbb{G}_m & \xleftarrow{\sim} & (\tilde{\mathcal{X}}, \infty) & \xrightarrow{\sim} & (\mathbb{P}^1_{\mathbb{Z}}, \infty) \\ \downarrow{\scriptstyle \mathrm{id}\wedge\Phi|_{\mathbb{G}_m}} & & \downarrow{\scriptstyle \tilde{\Phi}} & & \downarrow{\scriptstyle \bar{\Phi}} \\ S^1 \wedge \mathrm{D}_+((T_0 - T_1)T_0) & \xleftarrow{\sim} & (\mathcal{Y}, 1) & \xrightarrow{\sim} & \mathbb{P}^1_{\mathbb{Z}}/\mathrm{D}_+(T_1) . \end{array}$$

5. An unstable null-Hopf relation

In this section, we work entirely over the base Spec $\mathbb{Z}$ and would like to prove the desired unstable Hopf relation. We recall here the definition of motivic spheres. Let $s, w \geq 0$ be integers. We define $S^{s+(w)}$ to be the pointed simplicial presheaf $S^s \wedge \mathbb{G}_m^w$ where S^s is the smash product $\underbrace{S^1 \wedge \ldots \wedge S^1}_{s\ times}$ of the simplicial circle $S^1 = \Delta^1/\partial\Delta^1$ and $\mathbb{G}_m$ is based at 1. We call s the degree and w the weight of $S^{s+(w)}$. Suspension from the right with $\mathbb{G}_m$ increases the weight (w) by 1. Suspension from the left by the simplicial circle S^1 increases the degree s by 1. Let $\mathcal{E}$ be an arbitrary pointed motivic space in sPre$(\mathbb{Z})_*$. Then we set $\pi_{s+(w)}\mathcal{E}$ to be the group $\mathcal{H}_\bullet(\mathbb{Z})(S^{s+(w)}, \mathcal{E})$ for $s > 0$ and $w \geq 0$.

Next, we recall the definition of the Hopf map $\eta : \mathbb{A}^2_{\mathbb{Z}} - \{0\} \to \mathbb{P}^1_{\mathbb{Z}}$. It is the canonical map $(T_0, T_1) \mapsto [T_0 : T_1]$. The reduced join $\mathbb{G}_m * \mathbb{G}_m$ is defined to be the quotient of $\Delta^1 \times \mathbb{G}_m \times \mathbb{G}_m$ by the relations $(0, x, y) = (0, x, y'), (1, x, y) = (1, x', y)$ and $(t, 1, 1) = (s, 1, 1)$ for any $t, s \in \Delta^1$. The motivic space $\mathbb{A}^2_{\mathbb{Z}} - \{0\}$ is canonically $\mathbb{A}^1$-weakly equivalent to the join $\mathbb{G}_m * \mathbb{G}_m$ via the classical covering of $\mathbb{A}^2_{\mathbb{Z}} - \{0\}$ by $\mathbb{G}_m \times \mathbb{A}^1_{\mathbb{Z}}$ and $\mathbb{A}^1_{\mathbb{Z}} \times \mathbb{G}_m$ with intersection $\mathbb{G}_m \times \mathbb{G}_m$.

Note that $\mathbb{G}_m$ is a sheaf of abelian groups. In particular, we can consider the pointed map

$$\mu'_{\mathbb{G}_m} : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m,\ (g, h) \mapsto g^{-1}h.$$

This morphism induces a pointed morphism

$$\eta_{\mathbb{G}_m} : \mathbb{G}_m * \mathbb{G}_m \to S^1 \wedge \mathbb{G}_m$$

which is called the algebraic Hopf map. Via the weak equivalence between $\mathbb{A}^2_{\mathbb{Z}} - \{0\}$ and $\mathbb{G}_m * \mathbb{G}_m$, we can show that the Hopf map η is $\mathbb{A}^1$-weakly equivalent to $\eta_{\mathbb{G}_m}$ (cf.[Mor04, Lemma 6.2.3], [DuI13, Proposition 4.10]). The canonical projection from $\mathbb{G}_m * \mathbb{G}_m$ to $S^1 \wedge \mathbb{G}_m \wedge \mathbb{G}_m$ is a motivic weak equivalence. We also call the composition in $\mathcal{H}_\bullet(\mathbb{Z})$

$$S^1 \wedge \mathbb{G}_m \wedge \mathbb{G}_m \longrightarrow \mathbb{G}_m * \mathbb{G}_m \xrightarrow{\eta_{\mathbb{G}_m}} S^1 \wedge \mathbb{G}_m$$

the Hopf map.

Lemma 5.1. *Let $w \geq 0$ be a natural number. The group $\pi_{1+(w)}S^{1+(2)}$ is commutative.*

Proof. The motivic sphere $S^{1+(2)}$ is $\mathbb{A}^1$-weakly equivalent to $\mathbb{A}^2_{\mathbb{Z}} - \{0\}$. Let SL_2 be the special linear group scheme Spec $\mathbb{Z}[T_{11}, T_{12}, T_{21}, T_{22}]/(\det - 1)$. The projection onto the last column $\mathrm{SL}_2 \to \mathbb{A}^2_{\mathbb{Z}} - \{0\}$ is an $\mathbb{A}^1$-weak equivalence [DuI13, Example 2.12(3)]. Therefore, we can equip $S^{1+(2)}$ with a group structure in $\mathcal{H}_\bullet(\mathbb{Z})$. Using the Eckmann-Hilton argument, we can show that $\pi_{1+(w)}S^{1+(2)}$ is commutative. □

Morphisms from motivic spheres to motivic spheres are indexed by the bidegree of the target. For example, if we have a morphism $\phi_{s_2+(w_2)} : S^{s_1+(w_1)} \to S^{s_2+(w_2)}$, then suspension yields suspended morphisms

$$\phi_{s_2+s+(w_2+w)} : S^{s_1+s+(w_1+w)} \to S^{s_2+s+(w_2+w)}$$

for $s > 0$ and $w > 0$. The Hopf map might be denoted by $\eta_{1+(1)}$. Suspension yields suspended Hopf maps $\eta_{s+(w)}$ for all $s > 0$ and $w > 0$. Let n be an arbitrary integer. We define the power map

$$P_n : \mathbb{G}_m \to \mathbb{G}_m,\ x \mapsto x^n.$$

For $n = -1$, we set $\epsilon_{(1)} := P_{-1}$. For $n = 2$ we set $q_{(1)} := P_2$. Furthermore, we define the hyperbolic plane $h_{1+(1)}$ to be $1_{1+(1)} - \epsilon_{1+(1)}$ where $1_{1+(1)}$ is just the identity morphism for $S^{1+(1)}$. We would like to study the relation between $q_{1+(1)}$ and $1_{1+(1)} - \epsilon_{1+(1)}$.

Via the zig-zag of pointed weak equivalences,

$$(\mathbb{P}^1_{\mathbb{Z}}, \infty) \xleftarrow{\sim} (\tilde{\mathcal{X}}, \infty) \xrightarrow{\sim} S^1 \wedge \mathbb{G}_m$$

the morphism $q_{1+(1)}$ corresponds to the pointed endomorphism

$$(\mathbb{P}^1_{\mathbb{Z}}, \infty) \to (\mathbb{P}^1_{\mathbb{Z}}, \infty), [T_0 : T_1] \to [T_0^2 : T_1^2].$$

Proposition 5.2. *Let $\tau : \mathbb{P}^1_{\mathbb{Z}} \to \mathbb{P}^1_{\mathbb{Z}}$ be the automorphism induced by $[T_0 : T_1] \mapsto [T_1 : T_0]$. Then under the zig-zag of pointed weak equivalences*

$$(\mathbb{P}^1_{\mathbb{Z}}, \infty) \xleftarrow{\sim} (\tilde{\mathcal{X}}, \infty) \xrightarrow{\sim} S^1 \wedge \mathbb{G}_m$$

the morphism $-\epsilon_{1+(1)}$ corresponds to the pointed automorphism $\Phi \circ \tau \circ \Phi$ of $(\mathbb{P}^1_{\mathbb{Z}}, \infty)$.

Proof. At the end of Section 4 we have the commutative diagram

$$\begin{array}{ccccc} S^1 \wedge \mathbb{G}_m & \xleftarrow{\sim} & (\tilde{\mathcal{X}}, \infty) & \xrightarrow{\sim} & (\mathbb{P}^1_{\mathbb{Z}}, \infty) \\ \downarrow{\scriptstyle (\Phi|_{\mathbb{G}_m})_{1+(1)}} & & \downarrow{\scriptstyle \tilde{\Phi}} & & \downarrow{\scriptstyle \bar{\Phi}} \\ S^1 \wedge \mathrm{D}_+((T_0 - T_1)T_0) & \xleftarrow{\sim} & (\mathcal{Y}, 1) & \xrightarrow{\sim} & \mathbb{P}^1_{\mathbb{Z}}/\mathrm{D}_+(T_1)\,. \end{array}$$

Therefore, via the isomorphism $(\Phi|_{\mathbb{G}_m})_{1+(1)}$, the morphism $\epsilon_{1+(1)}$ corresponds to $(\Phi|_{\mathbb{G}_m} \circ \epsilon \circ (\Phi|_{\mathbb{G}_m})^{-1})_{1+(1)}$. The morphism $\Phi|_{\mathbb{G}_m} \circ \epsilon \circ (\Phi|_{\mathbb{G}_m})^{-1}$ is induced by the ring homomorphism

$$\mathbb{Z}[T_0, T_1]_{((T_0 - T_1)T_0)} \to \mathbb{Z}[T_0, T_1]_{((T_0 - T_1)T_0)}$$

which interchanges $\dfrac{(T_0-T_1)^2}{(T_0-T_1)T_0}$ with $\dfrac{T_0^2}{(T_0-T_1)T_0}$.

On the other hand, the automorphism $\Phi\circ\tau\circ\Phi$ of $\mathbb{P}^1_{\mathbb{Z}}$ is induced by the ring isomorphism

$$\mathbb{Z}[T_0,T_1]\to\mathbb{Z}[T_0,T_1]; T_0\mapsto T_0-T_1,\ T_1\mapsto -T_1.$$

The restriction $\Phi\circ\tau\circ\Phi|_{\mathrm{D}_+((T_0-T_1)T_0)}$ is $\Phi|_{\mathbb{G}_m}\circ\epsilon\circ(\Phi|_{\mathbb{G}_m})^{-1}$. Moreover, the automorphism $\Phi\circ\tau\circ\Phi$ interchanges $\mathrm{D}_+(T_0)$ with $\mathrm{D}_+(T_0-T_1)$ and sends $\mathrm{D}_+(T_1)$ to itself. Hence, $\Phi\circ\tau\circ\Phi$ also induces a morphism $\overline{\Phi\circ\tau\circ\Phi}:\mathbb{P}^1_{\mathbb{Z}}/\mathrm{D}_+(T_1)\to\mathbb{P}^1_{\mathbb{Z}}/\mathrm{D}_+(T_1)$. Now recall that in Section 4 we also construcuted the pointed space $\mathcal{Y}/B$. Analogously, we can consider the comparison map between pushout diagrams induced by inclusions

$$\begin{array}{ccc} 0\times\mathrm{D}_+((T_0-T_1)T_1T_0)\sqcup 1\times\mathrm{D}_+((T_0-T_1)T_1T_0) & \to & I\times\mathrm{D}_+((T_0-T_1)T_1T_0) \\ \downarrow & & \\ 0\times\mathrm{D}_+((T_0-T_1)T_1)\sqcup 1\times\mathrm{D}_+(T_0T_1) & & \\ & & \\ \downarrow & & \\ & & \\ 0\times\mathrm{D}_+((T_0-T_1)T_0)\sqcup 1\times\mathrm{D}_+((T_0-T_1)T_0) & \to & I\times\mathrm{D}_+((T_0-T_1)T_0) \\ {\scriptstyle \text{inclusion}\sqcup\text{inclusion}}\hookdownarrow & & \\ 0\times\mathrm{D}_+(T_0-T_1)\sqcup 1\times\mathrm{D}_+(T_0) & & \end{array}$$

where we denote the pushout of the first diagram by B' and the second by $\mathcal{Y}'$. Then we take the cofiber $\mathcal{Y}'/B'$. There is again a canonical weak equivalence $\mathcal{Y}'/B'\to\mathbb{P}^1_{\mathbb{Z}}/\mathrm{D}_+(T_1)$. Since the automorphism $\Phi\circ\tau\circ\Phi$ interchanges $\mathrm{D}_+(T_0)$ with $\mathrm{D}_+(T_0-T_1)$ and keeps $\mathrm{D}_+(T_1)$ invariant, it induces an isomorphism $\mathcal{Y}/B\cong\mathcal{Y}'/B'$. We have then the commutative diagram

$$\begin{array}{ccc} \mathcal{Y}/B & \xrightarrow{\sim} & \mathbb{P}^1_{\mathbb{Z}}/\mathrm{D}_+(T_1) \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\overline{\Phi\circ\tau\circ\Phi}} \\ \mathcal{Y}'/B' & \xrightarrow[\sim]{} & \mathbb{P}^1_{\mathbb{Z}}/\mathrm{D}_+(T_1). \end{array}$$

Next, we apply the geometric realization functor (see Section 6.2). There is an isomorphism s from $|\mathcal{Y}'/B'|$ to $|\mathcal{Y}/B|$ which is induced by the swap morphism

$|I| \to |I|$. In particular, the following diagram:

$$\begin{array}{ccc}
|\mathcal{Y}/B| & \xrightarrow{\sim} & |\mathbb{P}^1_{\mathbb{Z}}/\mathrm{D}_+(T_1)| \\
{\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle |\overline{\Phi\circ\tau\circ\Phi}|} \\
|\mathcal{Y}'/B'| & \xrightarrow[\sim]{} & |\mathbb{P}^1_{\mathbb{Z}}/\mathrm{D}_+(T_1)| \\
{\scriptstyle s}\downarrow & \nearrow_{\sim} & \\
|\mathcal{Y}/B| & &
\end{array}$$

is commutative. Moreover the swap morphism also induces an isomorphism $s' : |S^1 \wedge \mathrm{D}_+((T_0 - T_1)T_0)| \to |S^1 \wedge \mathrm{D}_+((T_0 - T_1)T_0)|$ which is the inverse morphism for the cogroup object $|S^1 \wedge \mathrm{D}_+((T_0 - T_1)T_0)|$. We can equip $\mathcal{Y}'$ with the base point

$$\operatorname{Spec}\mathbb{Z} \xrightarrow{0} 0 \times \mathrm{D}_+(T_0 - T_1) \hookrightarrow \mathcal{Y}'\,.$$

Then it follows from the construction of $\mathcal{Y}'$ that there is a pointed weak equivalence $(\mathcal{Y}', 0) \to S^1 \wedge \mathrm{D}_+((T_0 - T_1)T_0)$. At the end, we have now the diagram

$$\begin{array}{ccccc}
|\mathcal{Y}/B| & \longleftarrow & |(\mathcal{Y},1)| & \longrightarrow & |S^1 \wedge \mathrm{D}_+((T_0 - T_1)T_0)| \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle |(\Phi|_{\mathbb{G}_m}\circ\epsilon\circ(\Phi|_{\mathbb{G}_m})^{-1})_{1+(1)}|} \\
|\mathcal{Y}'/B'| & \longleftarrow & |(\mathcal{Y}',0)| & \longrightarrow & |S^1 \wedge \mathrm{D}_+((T_0 - T_1)T_0)| \\
{\scriptstyle s}\downarrow & & & & \downarrow{\scriptstyle s'} \\
|\mathcal{Y}/B| & \longleftarrow & |(\mathcal{Y},1)| & \longrightarrow & |S^1 \wedge \mathrm{D}_+((T_0 - T_1)T_0)|
\end{array}$$

where $|\mathcal{Y}/B| \to |\mathcal{Y}'/B'|$ and $|(\mathcal{Y},1)| \to |(\mathcal{Y}',0)|$ are induced by $\Phi\circ\tau\circ\Phi$. The first part of the previous diagram

$$\begin{array}{ccccc}
|\mathcal{Y}/B| & \longleftarrow & |(\mathcal{Y},1)| & \longrightarrow & |S^1 \wedge \mathrm{D}_+((T_0 - T_1)T_0)| \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle |(\Phi|_{\mathbb{G}_m}\circ\epsilon\circ(\Phi|_{\mathbb{G}_m})^{-1})_{1+(1)}|} \\
|\mathcal{Y}'/B'| & \longleftarrow & |(\mathcal{Y}',0)| & \longrightarrow & |S^1 \wedge \mathrm{D}_+((T_0 - T_1)T_0)|
\end{array}$$

is commutative. We would like to show that the second part commutes, too. First, we also can equip $\mathcal{Y}$ with the base point

$$\operatorname{Spec}\mathbb{Z} \xrightarrow{0} 1 \times \mathrm{D}_+(T_0 - T_1) \hookrightarrow \mathcal{Y}\,.$$

Then the swap morphism induces an isomorphism $s'' : |(\mathcal{Y}',0)| \to |(\mathcal{Y},0)|$. We also have pointed weak equivalences $|(\mathcal{Y},0)| \to |\mathcal{Y}/B|$ and $|(\mathcal{Y},0)| \to |S^1 \wedge$

$D_+((T_0 - T_1)T_0)|$. Now we have the diagram

$$\begin{array}{ccccc} |\mathcal{Y}'/B'| & \longleftarrow & |(\mathcal{Y}',0)| & \longrightarrow & |S^1 \wedge D_+((T_0 - T_1)T_0)| \\ \Big\downarrow s & & \Big\downarrow s'' & & \Big\downarrow s' \\ & & |(\mathcal{Y},0)| & \xrightarrow{\sim} & |S^1 \wedge D_+((T_0 - T_1)T_0)| \\ & \swarrow\sim & & & \Big\uparrow \\ |\mathcal{Y}/B| & & \longleftarrow & & |(\mathcal{Y},1)|\,. \end{array}$$

It follows from the definition of the morphisms s, s' and s'' that the diagrams

$$\begin{array}{ccc} |\mathcal{Y}'/B'| & \longleftarrow & |(\mathcal{Y}',0)| \\ \Big\downarrow s & & \Big\downarrow s'' \\ |\mathcal{Y}/B| & \longleftarrow & |(\mathcal{Y},0)| \end{array}$$

and

$$\begin{array}{ccc} |(\mathcal{Y}',0)| & \longrightarrow & |S^1 \wedge D_+((T_0 - T_1)T_0)| \\ \Big\downarrow s'' & & \Big\downarrow s' \\ |(\mathcal{Y},0)| & \longrightarrow & |S^1 \wedge D_+((T_0 - T_1)T_0)| \end{array}$$

are commutative.

In Proposition 3.1, we proved that the diagram

$$\begin{array}{ccc} (\tilde{\mathcal{X}},0) & \xrightarrow{\sim} & S^1 \wedge \mathbb{G}_m \\ \sim\Big\downarrow & & \Big\uparrow\sim \\ \tilde{\mathcal{X}}/A & \xleftarrow{\sim} & (\tilde{\mathcal{X}},\infty) \end{array}$$

commutes. Now we can use exactly the same methods to show that

$$\begin{array}{ccc} (\mathcal{Y},0) & \xrightarrow{\sim} & S^1 \wedge D_+((T_0 - T_1)T_0) \\ \sim\Big\downarrow & & \Big\uparrow\sim \\ \mathcal{Y}/B & \xleftarrow[\sim]{} & (\mathcal{Y},1) \end{array}$$

is commutative, too. Therefore, all three inner diagrams in

$$
\begin{array}{ccccc}
|\mathcal{Y}'/B'| & \longleftarrow & |(\mathcal{Y}',0)| & \longrightarrow & |S^1 \wedge \mathrm{D}_+((T_0-T_1)T_0)| \\
 & & \big\downarrow{\scriptstyle s''} & & \big\downarrow{\scriptstyle s'} \\
{\scriptstyle s}\big\downarrow & & |(\mathcal{Y},0)| & \xrightarrow{\sim} & |S^1 \wedge \mathrm{D}_+((T_0-T_1)T_0)| \\
 & \swarrow{\scriptstyle \sim} & & & \big\uparrow \\
|\mathcal{Y}/B| & \longleftarrow & & & |(\mathcal{Y},1)|
\end{array}
$$

commute. Since all involved morphisms are weak equivalences, it also follows that the outer diagram is commutative. In particular, we have the following commutative diagram

$$
\begin{array}{ccccccc}
|\mathbb{P}^1_{\mathbb{Z}}/\mathrm{D}_+(T_1)| & \leftarrow & |\mathcal{Y}/B| & \leftarrow & |(\mathcal{Y},1)| & \rightarrow & |S^1 \wedge \mathrm{D}_+((T_0-T_1)T_0)| \\
{\scriptstyle |\overline{\Phi\circ\tau\circ\Phi}|}\big\downarrow & & \big\downarrow & & & & \big\downarrow{\scriptstyle |(\Phi|_{\mathbb{G}_m}\circ\epsilon\circ(\Phi|_{\mathbb{G}_m})^{-1})_{1+(1)}|} \\
|\mathbb{P}^1_{\mathbb{Z}}/\mathrm{D}_+(T_1)| & \leftarrow & |\mathcal{Y}'/B'| & & & & |S^1 \wedge \mathrm{D}_+((T_0-T_1)T_0)| \\
 & \nwarrow & {\scriptstyle s}\big\downarrow & & & & \big\downarrow{\scriptstyle s'} \\
 & & |\mathcal{Y}/B| & \leftarrow & |(\mathcal{Y},1)| & \rightarrow & |S^1 \wedge \mathrm{D}_+((T_0-T_1)T_0)|
\end{array}
$$

which implies that via the zig-zag of weak equivalences

$$
\mathbb{P}^1_{\mathbb{Z}}/\mathrm{D}_+(T_1) \longleftarrow (\mathcal{Y},1) \longrightarrow S^1 \wedge \mathrm{D}_+((T_0-T_1)T_0)
$$

the morphism $-(\Phi|_{\mathbb{G}_m}\circ\epsilon\circ(\Phi|_{\mathbb{G}_m})^{-1})_{1+(1)}$ corresponds to $\overline{\Phi\circ\tau\circ\Phi}$.

In the next step, we consider again the commutative diagram:

$$
\begin{array}{ccccc}
S^1 \wedge \mathbb{G}_m & \xleftarrow{\sim} & (\tilde{\mathcal{X}},\infty) & \xrightarrow{\sim} & (\mathbb{P}^1_{\mathbb{Z}},\infty) \\
{\scriptstyle (\Phi|_{\mathbb{G}_m})_{1+(1)}}\big\downarrow & & {\scriptstyle \tilde{\Phi}}\big\downarrow & & \big\downarrow{\scriptstyle \bar{\Phi}} \\
S^1 \wedge \mathrm{D}_+((T_0-T_1)T_0) & \xleftarrow{\sim} & (\mathcal{Y},1) & \xrightarrow{\sim} & \mathbb{P}^1_{\mathbb{Z}}/\mathrm{D}_+(T_1)\,.
\end{array}
$$

Via the isomorphism $(\Phi|_{\mathbb{G}_m})_{1+(1)}$, the morphism $-\epsilon_{1+(1)}$ corresponds to $-(\Phi|_{\mathbb{G}_m}\circ\epsilon\circ(\Phi|_{\mathbb{G}_m})^{-1})_{1+(1)}$ and under the zig-zag of weak equivalences

$$
\mathbb{P}^1_{\mathbb{Z}}/\mathrm{D}_+(T_1) \xleftarrow{\sim} (\mathcal{Y},1) \xrightarrow{\sim} S^1 \wedge \mathrm{D}_+((T_0-T_1)T_0)
$$

the morphism $-(\Phi|_{\mathbb{G}_m}\circ\epsilon\circ(\Phi|_{\mathbb{G}_m})^{-1})_{1+(1)}$ corresponds to $\overline{\Phi\circ\tau\circ\Phi}$. Therefore, we only need to determine which pointed endomorphism of $(\mathbb{P}^1_{\mathbb{Z}},\infty)$ equals to $\bar{\Phi}^{-1}\circ(\overline{\Phi\circ\tau\circ\Phi})\circ\bar{\Phi}$ in $\mathcal{H}_\bullet(\mathbb{Z})$, because this is then the morphism which corresponds to $-\epsilon_{1+(1)}$ under the zig-zag of weak equivalences

$$
S^1 \wedge \mathbb{G}_m \xleftarrow{\sim} (\tilde{\mathcal{X}},\infty) \xrightarrow{\sim} (\mathbb{P}^1_{\mathbb{Z}},\infty)\,.
$$

We claim that the diagram

$$\begin{array}{ccc} (\mathbb{P}^1_{\mathbb{Z}}, \infty) & \xrightarrow{\Phi\circ\tau\circ\Phi} & (\mathbb{P}^1_{\mathbb{Z}}, \infty) \\ \bar{\Phi}\downarrow & & \downarrow\bar{\Phi} \\ \mathbb{P}^1_{\mathbb{Z}}/\mathrm{D}_+(T_1) & \xrightarrow[\overline{\Phi\circ\tau\circ\Phi}]{} & \mathbb{P}^1_{\mathbb{Z}}/\mathrm{D}_+(T_1) \end{array}$$

commutes in $\mathcal{H}_\bullet(\mathbb{Z})$. The composition $\overline{\Phi\circ\tau\circ\Phi}\circ\bar{\Phi}$ is just

$$(\mathbb{P}^1_{\mathbb{Z}}, \infty) \xrightarrow{\tau} (\mathbb{P}^1_{\mathbb{Z}}, 0) \xrightarrow{\Phi} (\mathbb{P}^1_{\mathbb{Z}}, 0) \longrightarrow \mathbb{P}^1_{\mathbb{Z}}/\mathrm{D}_+(T_1)$$

where $(\mathbb{P}^1_{\mathbb{Z}}, 0) \to \mathbb{P}^1_{\mathbb{Z}}/\mathrm{D}_+(T_1)$ is the canonical projection. Similarly, $\bar{\Phi}\circ\Phi\circ\tau\circ\Phi$ is

$$(\mathbb{P}^1_{\mathbb{Z}}, \infty) \xrightarrow{\Phi} (\mathbb{P}^1_{\mathbb{Z}}, 1) \xrightarrow{\tau} (\mathbb{P}^1_{\mathbb{Z}}, 1) \longrightarrow \mathbb{P}^1_{\mathbb{Z}}/\mathrm{D}_+(T_1)$$

where $(\mathbb{P}^1_{\mathbb{Z}}, 0) \to \mathbb{P}^1_{\mathbb{Z}}/\mathrm{D}_+(T_1)$ is the canonical projection. First, we would like to find a sequence of naive $\mathbb{A}^1$-homotopies $H : \mathbb{P}^1_{\mathbb{Z}} \times_{\mathrm{Spec}\ \mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}} \to \mathbb{P}^1_{\mathbb{Z}}$ from $\Phi\circ\tau$ to $\tau\circ\Phi$ such that every composition

$$\mathrm{Spec}\ \mathbb{Z} \times_{\mathrm{Spec}\ \mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}} \xrightarrow{\infty\times\mathrm{id}_{\mathbb{A}^1_{\mathbb{Z}}}} \mathbb{P}^1_{\mathbb{Z}} \times_{\mathrm{Spec}\ \mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}} \xrightarrow{H} \mathbb{P}^1_{\mathbb{Z}}$$

factors through $\mathrm{D}_+(T_1)$. This condition is equivalent to the condition that $H((T_1, \rho))$ are all contained in $\mathrm{D}_+(T_1)$ where (T_1, ρ) is the homogeneous prime ideal of $\mathbb{Z}[T][T_0, T_1]$ generated by ρ and T_1 and ρ runs over the prime ideals of $\mathbb{Z}[T]$. Then such a sequence of naive $\mathbb{A}^1$-homotopies induces a sequence of pointed naive $\mathbb{A}^1$-homotopies $\mathrm{Spec}\ \mathbb{Z} \wedge (\mathbb{A}^1_{\mathbb{Z}})_+ \to \mathbb{P}^1_{\mathbb{Z}}/\mathrm{D}_+(T_1)$ from $\overline{\Phi\circ\tau\circ\Phi}\circ\bar{\Phi}$ to $\bar{\Phi}\circ\Phi\circ\tau\circ\Phi$.

The scheme morphism $\tau\circ\Phi$ is induced by the ring isomorphism

$$\mathbb{Z}[T_0, T_1] \to \mathbb{Z}[T_0, T_1];\ T_0 \mapsto T_0 - T_1,\ T_1 \mapsto T_0$$

and $\Phi\circ\tau$ is induced by the ring isomorphism

$$\mathbb{Z}[T_0, T_1] \to \mathbb{Z}[T_0, T_1];\ T_0 \mapsto T_1,\ T_1 \mapsto T_1 - T_0.$$

We first have a naive $\mathbb{A}^1$-homotopy $H_1 : \mathbb{P}^1_{\mathbb{Z}} \times_{\mathrm{Spec}\ \mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}} \to \mathbb{P}^1_{\mathbb{Z}}$ induced by the ring homomorphism

$$\mathbb{Z}[T_0, T_1] \to \mathbb{Z}[T][T_0, T_1];\ T_0 \mapsto TT_0 - T_1,\ T_1 \mapsto T_0.$$

It follows from the definition of H_1 that $H_1((T_1, \rho))$ are all contained in $\mathrm{D}_+(T_1)$ where ρ runs over the prime ideals of $\mathbb{Z}[T]$. Hence, $\tau\circ\Phi$ is $\mathbb{A}^1$-homotopic to the scheme endomorphism of $\mathbb{P}^1_{\mathbb{Z}}$ defined by $[T_0 : T_1] \mapsto [-T_1 : T_0] = [T_1 : -T_0]$. Next, we can give an $\mathbb{A}^1$-homotopy H_2 from this morphism to $\Phi\circ\tau$. It is given by

$$\mathbb{Z}[T_0, T_1] \to \mathbb{Z}[T][T_0, T_1];\ T_0 \mapsto T_1,\ T_1 \mapsto TT_1 - T_0.$$

Again, we have $H_2((T_1, \rho)) \in D_+(T_1)$ for all prime ideals ρ of $\mathbb{Z}[T]$. Thus we obtain a sequence of pointed naive $\mathbb{A}^1$-homotopies $\operatorname{Spec}\mathbb{Z} \wedge (\mathbb{A}^1_{\mathbb{Z}})_+ \to \mathbb{P}^1_{\mathbb{Z}}/D_+(T_1)$ from $\overline{\Phi \circ \tau \circ \Phi} \circ \bar{\Phi}$ to $\bar{\Phi} \circ \Phi \circ \tau \circ \Phi$. □

Proposition 5.3. *The morphism $q_{1+(1)}$ is equal to $1_{1+(1)} - \epsilon_{1+(1)}$ in $\mathcal{H}_\bullet(\mathbb{Z})$.*

Proof. Recall that $q_{(1)}$ is the pointed morphism

$$\mathbb{G}_m \to \mathbb{G}_m,\ x \mapsto x^2.$$

It is easy to see that via the zig-zag of pointed weak equivalences

$$(\mathbb{P}^1_{\mathbb{Z}}, \infty) \xleftarrow{\sim} (\tilde{\mathcal{X}}, \infty) \xrightarrow{\sim} S^1 \wedge \mathbb{G}_m$$

the morphism $q_{1+(1)}$ corresponds to the pointed endomorphism of $(\mathbb{P}^1_{\mathbb{Z}}, \infty)$ which is given by $[T_0 : T_1] \mapsto [T_0^2 : T_1^2]$. By Proposition 5.2, the morphism $-\epsilon_{1+(1)}$ corresponds to $\Phi \circ \tau \circ \Phi$. Since we equipped $(\mathbb{P}^1_{\mathbb{Z}}, \infty)$ with a cogroup structure using the zig-zag above, $1_{1+(1)} - \epsilon_{1+(1)}$ corresponds to $\mathrm{id}_{\mathbb{P}^1_{\mathbb{Z}}} + \Phi \circ \tau \circ \Phi$.

By Proposition 2.5, the morphism given by $[T_0 : T_1] \mapsto [T_0^2 : T_1^2]$ is represented by the pair of polynomials $\frac{X^2}{1}$ where X is $\frac{T_0}{T_1}$. Simiarly, $\Phi \circ \tau \circ \Phi$ is represented by $\frac{X-1}{-1}$ and $\mathrm{id}_{\mathbb{P}^1_{\mathbb{Z}}}$ is represented by $\frac{X}{1}$.

Cazanave gives the set $[\mathbb{P}^1_k, \mathbb{P}^1_k]^{\mathrm{N}}$ of pointed naive $\mathbb{A}^1$-homotopy classes of scheme morphisms a monoid structure, where $\mathbb{P}^1_k$ is equipped with the base point ∞. We denote the addition for this monoid structure by $\oplus^{\mathrm{N}}$. Via the same co-diagonal $\tilde{\nabla}$ as for $(\mathbb{P}^1_{\mathbb{Z}}, \infty)$(see Proposition 3.1), we can equip $\mathbb{P}^1_k$ with a cogroup structure. Proposition 3.1 also holds for $\mathbb{P}^1_k$. Then we have a group structure on $\mathcal{H}_\bullet(k)(\mathbb{P}^1_k, \mathbb{P}^1_k)$. We denote addition for this group structure by $\oplus^{\mathbb{A}^1}$.

In [Caz12, Appendix B], Cazanave shows that $\frac{X}{a} \oplus^{\mathrm{N}} g$ is equal to $\frac{X}{a} \oplus^{\mathbb{A}^1} g$ for any units $a \in k$ and g a pair of polynomials which represents a pointed endomorphism of $(\mathbb{P}^1_k, \infty)$. Actually, his methods also work over $\operatorname{Spec}\mathbb{Z}$ for units $a \in \mathbb{Z}$ because his proof relies on the homotopy purity theorem (see [MV99, Theorem 2.23, page 115]) and does not use any specific facts about fields. Therefore, we also have $\frac{X}{1} \oplus^{\mathrm{N}} \frac{X-1}{-1} = \frac{X}{1} \oplus^{\mathbb{A}^1} \frac{X-1}{-1}$ over $\operatorname{Spec}\mathbb{Z}$. By Definition 2.7 the sum $\frac{X}{1} \oplus^{\mathrm{N}} \frac{X-1}{-1}$ is equal to $\frac{X^2-X+1}{X-1}$. In the following, we give a sequence of pointed naive $\mathbb{A}^1$-homotopies between $\frac{X^2}{1}$ and $\frac{X^2-X+1}{X-1}$. We characterized pointed naive $\mathbb{A}^1$-homotopies in Proposition 2.6.

At first we have the pointed $\mathbb{A}^1$-homotopy

$$\frac{X^2}{TX+1}$$

from $\frac{X^2}{1}$ to $\frac{X^2}{X+1}$. Then

$$\frac{X^2+2TX+2T}{X+1}$$

is a pointed $\mathbb{A}^1$-homotopy from $\frac{X^2}{X+1}$ to $\frac{X^2+2X+2}{X+1}$. Next,

$$\frac{X^2+2TX+2T}{X+(2T-1)}$$

is a pointed $\mathbb{A}^1$-homotopy from $\frac{X^2+2X+2}{X+1}$ to $\frac{X^2}{X-1}$. Finally

$$\frac{X^2-TX+T}{X-1}$$

is a pointed $\mathbb{A}^1$-homotopy from $\frac{X^2}{X-1}$ to $\frac{X^2-X+1}{X-1}$. □

If we suspend from the right with $\mathbb{G}_m$, we get the following corollary.

Corollary 5.4. *Let $w > 0$ be a natural number. Then the morphism $q_{1+(w)}$: $S^{1+(w)} \to S^{1+(w)}$ coincides with $(1-\epsilon)_{1+(w)}$ in $\mathcal{H}_\bullet(\mathbb{Z})$.*

Now we are interested in the case $w = 2$.

Proposition 5.5. *In $\mathcal{H}_\bullet(\mathbb{Z})$, the morphism $q_{1+(2)}$ is equal to $1_{1+(1)} \wedge q_{(1)}$.*

Proof. The morphism $q_{1+(2)}$ is given by

$$S^1 \wedge \mathbb{G}_m^2 \to S^1 \wedge \mathbb{G}_m^2;\ t \wedge x \wedge y \mapsto t \wedge x^2 \wedge y.$$

And, the morphism $1_{1+(1)} \wedge q_{(1)}$ is given by

$$S^1 \wedge \mathbb{G}_m^2 \to S^1 \wedge \mathbb{G}_m^2;\ t \wedge x \wedge y \mapsto t \wedge x \wedge y^2.$$

Next, we consider the space $\mathbb{A}^2_{\mathbb{Z}} - \{0\}$. It is the pushout of

$$\mathbb{A}^1_{\mathbb{Z}} \times \mathbb{G}_m \hookleftarrow \mathbb{G}_m \times \mathbb{G}_m \hookrightarrow \mathbb{G}_m \times \mathbb{A}^1_{\mathbb{Z}},$$

so we can equip it with the base point $(1,1)$ coming from $\mathbb{G}_m \times \mathbb{G}_m$. Via the open covering of $\mathbb{A}^2_{\mathbb{Z}} - \{0\}$ above, there is a zig-zag of pointed weak equivalences from $(\mathbb{A}^2_{\mathbb{Z}} - \{0\}, (1,1))$ to $\mathbb{G}_m * \mathbb{G}_m$ and the projection $\mathbb{G}_m * \mathbb{G}_m \to S^1 \wedge \mathbb{G}_m^2$ is a weak equivalence. Via this isomorphism, $q_{1+(2)}$ corresponds to a pointed morphism $(\mathbb{A}^2_{\mathbb{Z}} - \{0\}, (1,1)) \to (\mathbb{A}^2_{\mathbb{Z}} - \{0\}, (1,1))$ which is induced by the ring homomorphism

$$\mathbb{Z}[T_0, T_1] \to \mathbb{Z}[T_0, T_1];\ T_0 \mapsto T_0^2,\ T_1 \mapsto T_1\ \ (*).$$

Similarly, $1_{1+(1)} \wedge q_{(1)}$ corresponds to the morphism induced by

$$\mathbb{Z}[T_0, T_1] \to \mathbb{Z}[T_0, T_1];\ T_0 \mapsto T_0,\ T_1 \mapsto T_1^2\ \ (**).$$

We would like to show that these two pointed endomorphisms of $\mathbb{A}^2_{\mathbb{Z}} - \{0\}$ are pointed $\mathbb{A}^1$-homotopic. Actually, it is enough to show that they are $\mathbb{A}^1$-homotopic since $\mathbb{A}^2_{\mathbb{Z}} - \{0\}$ is $\mathbb{A}^1$-weakly equivalent to SL_2. As SL_2 is a group scheme, the classical arguments show that the canonical map $\mathcal{H}_{\bullet}(\mathbb{Z})((\mathbb{A}^2_{\mathbb{Z}} - \{0\}, (1,1)), (\mathbb{A}^2_{\mathbb{Z}} - \{0\}, (1,1))) \to \mathcal{H}(\mathbb{Z})(\mathbb{A}^2_{\mathbb{Z}} - \{0\}, \mathbb{A}^2_{\mathbb{Z}} - \{0\})$ is injective, where $\mathcal{H}(\mathbb{Z})$ is the unstable motivic homotopy category over $\mathbb{Z}$. Now we give an explicit sequence of $\mathbb{A}^1$-homotopies between the two morphisms $(*)$ and $(**)$. We need $\mathbb{A}^1$-homotopies of the form $\mathbb{A}^2_{\mathbb{Z}} - \{0\} \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}} \to \mathbb{A}^2_{\mathbb{Z}} - \{0\}$. In the following, we give ring homomorphisms $f : \mathbb{Z}[T_0, T_1] \to \mathbb{Z}[T_0, T_1, T]$ which satisfy the property: If $\rho \subset \mathbb{Z}[T_0, T_1, T]$ is a prime ideal which does not contain the prime ideal (T_0, T_1), then the preimage $f^{-1}(\rho)$ also does not contain (T_0, T_1). Such ring homomorphisms induce morphisms of the form $\mathbb{A}^2_{\mathbb{Z}} - \{0\} \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}} \to \mathbb{A}^2_{\mathbb{Z}} - \{0\}$.

The first one is given by the ring homomorphism

$$\mathbb{Z}[T_0, T_1] \to \mathbb{Z}[T_0, T_1, T];\ T_0 \mapsto (T_0 + TT_1)^2,\ T_1 \mapsto T_1.$$

This ring homomorphism induces a morphism $\mathbb{A}^2_{\mathbb{Z}} - \{0\} \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}} \to \mathbb{A}^2_{\mathbb{Z}} - \{0\}$ which is an $\mathbb{A}^1$-homotopy from $(*)$ to the morphism induced by

$$\mathbb{Z}[T_0, T_1] \to \mathbb{Z}[T_0, T_1];\ T_0 \mapsto (T_0 + T_1)^2,\ T_1 \mapsto T_1.$$

The second one is induced by

$$\mathbb{Z}[T_0, T_1] \to \mathbb{Z}[T_0, T_1, T];\ T_0 \mapsto (T_0 + T_1)^2,\ T_1 \mapsto TT_1 + (T-1)T_0.$$

The third one is given by

$$\mathbb{Z}[T_0, T_1] \to \mathbb{Z}[T_0, T_1, T];\ T_0 \mapsto (TT_0 + T_1)^2,\ T_1 \mapsto -T_0.$$

The fourth one is given by

$$\mathbb{Z}[T_0, T_1] \to \mathbb{Z}[T_0, T_1, T];\ T_0 \mapsto TT_0 + T_1^2,\ T_1 \mapsto -T_0.$$

The fifth one is given by

$$\mathbb{Z}[T_0, T_1] \to \mathbb{Z}[T_0, T_1, T];\ T_0 \mapsto T_0 + TT_1^2,\ T_1 \mapsto -T_0 + (1-T)T_1^2.$$

And the last one is given by

$$\mathbb{Z}[T_0, T_1] \to \mathbb{Z}[T_0, T_1, T];\ T_0 \mapsto T_0,\ T_1 \mapsto -TT_0 + T_1^2.$$

The last ring homomorphism induces an $\mathbb{A}^1$-homotopy between $(**)$ and the morphism induced by

$$\mathbb{Z}[T_0, T_1] \to \mathbb{Z}[T_0, T_1];\ T_0 \mapsto T_0,\ T_1 \mapsto -T_0 + T_1^2.$$

Therefore, the two morphisms $q_{1+(2)}$ and $1_{1+(1)} \wedge q_{(1)}$ coincide. □

Let $\tau' : \mathbb{G}_m \wedge \mathbb{G}_m \to \mathbb{G}_m \wedge \mathbb{G}_m$ be the morphism defined by $x \wedge y \mapsto y \wedge x$.

Lemma 5.6. *The relation* $\epsilon_{1+(2)} = 1_{1+(1)} \wedge \epsilon_{(1)}$ *holds.*

Proof. The morphism $(\mathrm{id}_{S^1}\wedge\tau')\circ(1_{1+(1)}\wedge q_{(1)})$ is given by $t\wedge x\wedge y\mapsto t\wedge y^2\wedge x$. Therefore, it is equal to $q_{1+(2)}\circ(\mathrm{id}_{S^1}\wedge\tau')$. Next, we have:

$$\begin{aligned} q_{1+(2)}\circ(\mathrm{id}_{S^1}\wedge\tau') &= (1_{1+(2)}-\epsilon_{1+(2)})\circ(\mathrm{id}_{S^1}\wedge\tau')\\ &= \mathrm{id}_{S^1}\wedge\tau'-\epsilon_{1+(2)}\circ(\mathrm{id}_{S^1}\wedge\tau')\\ &= \mathrm{id}_{S^1}\wedge\tau'-(\mathrm{id}_{S^1}\wedge\tau')\circ(1_{1+(1)}\wedge\epsilon_{(1)})\\ &= (\mathrm{id}_{S^1}\wedge\tau')\circ(1_{1+(2)}-1_{1+(1)}\wedge\epsilon_{(1)}).\end{aligned}$$

Since $\mathrm{id}_{S^1}\wedge\tau'$ is an isomorphism in the pointed homotopy category, we get $1_{1+(1)}\wedge q_{(1)}=1_{1+(2)}-1_{1+(1)}\wedge\epsilon_{(1)}$. By Lemma 3.4.5 we have that $1_{1+(1)}\wedge q_{(1)}$ is equal to $q_{1+(2)}$. Hence, we obtain $1_{1+(2)}-\epsilon_{1+(2)}=q_{1+(2)}=1_{1+(2)}-1_{1+(1)}\wedge\epsilon_{(1)}$. It follows that $\epsilon_{1+(2)}=1_{1+(1)}\wedge\epsilon_{(1)}$. ☐

Corollary 5.7. *Under the canonical isomorphism from $\mathbb{A}^2_{\mathbb{Z}}-\{0\}$ to $S^{1+(2)}$ the morphism $\mathbb{A}^2_{\mathbb{Z}}-\{0\}\to\mathbb{A}^2_{\mathbb{Z}}-\{0\},(T_0,T_1)\mapsto(T_0^2,T_1^2)$ corresponds to $q_{1+(2)}\circ(1_{1+(1)}\wedge q_{(1)})=q_{1+(2)}\circ q_{1+(2)}=(1-\epsilon)_{1+(2)}\circ(1-\epsilon)_{1+(2)}=2(1-\epsilon)_{1+(2)}$ in the commutative group $\pi_{1+(2)}S^{1+(2)}$.*

Proof. It is clear that the morphism $\mathbb{A}^2_{\mathbb{Z}}-\{0\}\to\mathbb{A}^2_{\mathbb{Z}}-\{0\},(T_0,T_1)\mapsto(T_0^2,T_1^2)$ corresponds to $q_{1+(2)}\circ(1_{1+(1)}\wedge q_{(1)})$. Furthermore, we have that

$$\begin{aligned}(1-\epsilon)_{1+(2)}\circ(1-\epsilon)_{1+(2)} &= (1-\epsilon)_{1+(2)}\circ(1_{1+(2)}-1_{1+(1)}\wedge\epsilon_{(1)})\\ &= 1_{1+(2)}-1_{1+(1)}\wedge\epsilon_{(1)}-\epsilon_{1+(2)}+\epsilon_{1+(2)}\circ(1_{1+(1)}\wedge\epsilon_{(1)})\\ &= 1_{1+(2)}-1_{1+(1)}\wedge\epsilon_{(1)}-\epsilon_{1+(2)}+\epsilon_{1+(2)}\circ\epsilon_{1+(2)}\\ &= 1_{1+(2)}-\epsilon_{1+(2)}-\epsilon_{1+(2)}+1_{1+(2)}\\ &= 2(1-\epsilon)_{1+(2)}.\end{aligned}$$ ☐

In the next proposition, we show that $\eta_{1+(2)}\circ h_{1+(3)}$ and $h_{1+(2)}\circ\eta_{1+(2)}$ are $\mathbb{A}^1$-nullhomotopic. Similar computations in the stable case can be found in [DuI13].

Proposition 5.8. *The elements*

$$\eta_{1+(2)}\circ h_{1+(3)}\quad\text{and}\quad h_{1+(2)}\circ\eta_{1+(2)}$$

are $\mathbb{A}^1$-nullhomotopic.

Proof. The smash product $\eta_{1+(1)}\wedge q_{(1)}$ can be expressed in two different ways. First we have that

$$\begin{aligned}\eta_{1+(1)}\wedge q_{(1)} &= (\eta_{1+(1)}\wedge\mathrm{id}_{\mathbb{G}_m})\circ(1_{1+(2)}\wedge q_{(1)})\\ &= \eta_{1+(2)}\circ(1_{1+(2)}\wedge q_{(1)}).\end{aligned}$$

By Proposition 5.5, we get $1_{1+(1)}\wedge q_{(1)}\wedge\mathrm{id}_{\mathbb{G}_m}=q_{1+(2)}\wedge\mathrm{id}_{\mathbb{G}_m}=q_{1+(3)}$. Moreover we have $(1_{1+(1)}\wedge\tau')\circ(1_{1+(2)}\wedge q_{(1)})\circ(1_{1+(1)}\wedge\tau')=1_{1+(1)}\wedge q_{(1)}\wedge\mathrm{id}_{\mathbb{G}_m}$. Therefore, the equation $1_{1+(2)}\wedge q_{(1)}=(1_{1+(1)}\wedge\tau')\circ q_{1+(3)}\circ(1_{1+(1)}\wedge\tau')=q_{1+(3)}$ holds.

On the other hand, we have that

$$\begin{aligned}\eta_{1+(1)}\wedge q_{(1)} &= (1_{1+(1)}\wedge q_{(1)})\circ(\eta_{1+(1)}\wedge\mathrm{id}_{\mathbb{G}_m})\\ &= q_{1+(2)}\wedge\eta_{1+(2)}.\end{aligned}$$

Hence, we get the equation $\eta_{1+(2)}\circ q_{1+(3)} = q_{1+(2)}\circ\eta_{1+(2)}$.

Furthermore, we have the commutative diagram

$$\begin{array}{ccc} \mathbb{A}^2_{\mathbb{Z}} - \{0\} & \xrightarrow{(T_0,T_1)\mapsto(T_0^2,T_1^2)} & \mathbb{A}^2_{\mathbb{Z}} - \{0\} \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\eta} \\ \mathbb{P}^1_{\mathbb{Z}} & \xrightarrow{[T_0:T_1]\mapsto[T_0^2:T_1^2]} & \mathbb{P}^1_{\mathbb{Z}} \end{array}$$

where η is the geometric Hopf map. This commutative diagram implies the equation $q_{1+(1)}\circ\eta_{1+(1)} = \eta_{1+(1)}\circ q_{1+(2)}\circ q_{1+(2)}$. Since $q_{1+(1)}$ is equal to $h_{1+(1)}$, we get the equations

$$\begin{aligned} 0 &= (\eta_{1+(1)}\circ q_{1+(2)}\circ q_{1+(2)}) \wedge \mathrm{id}_{\mathbb{G}_m} - (q_{1+(1)}\circ\eta_{1+(1)}) \wedge \mathrm{id}_{\mathbb{G}_m} \\ &= \eta_{1+(2)}\circ h_{1+(3)}\circ h_{1+(3)} - \eta_{1+(2)}\circ h_{1+(3)} \\ &= \eta_{1+(2)}\circ(h_{1+(3)}\circ h_{1+(3)} - h_{1+(3)}). \end{aligned}$$

It follows from Corollary 3.4.8 that $h_{1+(3)}\circ h_{1+(3)} - h_{1+(3)} = h_{1+(3)}$. Thus we get $\eta_{1+(2)}\circ h_{1+(3)} = 0$. Finally, we get $h_{1+(2)}\circ\eta_{1+(2)} = \eta_{1+(2)}\circ h_{1+(3)}\circ h_{1+(3)}$ from the equation $q_{1+(1)}\circ\eta_{1+(1)} = \eta_{1+(1)}\circ q_{1+(2)}\circ q_{1+(2)}$. Since $\eta_{1+(2)}\circ h_{1+(3)} = 0$, we obtain $h_{1+(2)}\circ\eta_{1+(2)} = 0$. □

Proposition 5.9. *The Toda brackets*

$$\{h_{1+(2)}, \eta_{1+(2)}, h_{1+(3)}\}$$

and

$$\{\eta_{1+(2)}, h_{1+(3)}, \eta_{1+(3)}\}$$

are not trivial, in the sense that they do not contain the homotopy class of the corresponding constant morphism.

Proof. The complex realizations of both Toda brackets are not trivial by [Tod63, Example 2, p.84] and [Tod63, Proposition 5.6], respectively. □

Let k be a field of characteristic zero. Then analogously, the Toda bracket $\{\eta_{1+(2)}, h_{1+(3)}, \eta_{1+(3)}\}$ is also defined over k. Using the same argument as for the base $\mathbb{Z}$ we can show that this Toda bracket is not trivial. Let ν' be an arbitrary element of $\{\eta_{1+(2)}, h_{1+(3)}, \eta_{1+(3)}\}$. The left derived complex realization functor sends this element to the element of the same name defined in [Tod63, Proposition 5.6]. By [AF14, Proposition 4.14], the group $\pi_{2+(4)}\mathrm{SL}_3$ is equal to $\mathbb{Z}/6\mathbb{Z}$ over k. Let j denote the inclusion $\mathrm{SL}_2 \hookrightarrow \mathrm{SL}_3$. Since $S^{1+(2)}$ is isomorphic to SL_2, ν' can be viewed as an element of $\pi_{2+(4)}\mathrm{SL}_2$. We claim that the element $j_*(\nu')$ generates the 2-primary component of $\pi_{2+(4)}\mathrm{SL}_3$.

Proposition 5.10. *The element $j_*(\nu')$ generates the 2-primary component of $\pi_{2+(4)}\mathrm{SL}_3$ over any field k of characteristic zero.*

Proof. Let i be the inclusion of the topological groups $\mathrm{SL}(2,\mathbb{C}) \hookrightarrow \mathrm{SL}(3,\mathbb{C})$. Using the complex realization, we get the following commutative diagram of abelian groups

$$\begin{array}{ccc} \pi_{2+(4)}\mathrm{SL}_2 & \xrightarrow{j_*} & \pi_{2+(4)}\mathrm{SL}_3 \\ \downarrow & & \downarrow \\ \pi_6\mathrm{SL}(2,\mathbb{C}) & \xrightarrow{i_*} & \pi_6\mathrm{SL}(3,\mathbb{C})\,. \end{array}$$

By [AF14, Theorem 5.5], the homomorphism $\pi_{2+(4)}\mathrm{SL}_3 \to \pi_6\mathrm{SL}(3,\mathbb{C})$ induced by the complex realization is an isomorphism over any field k of characteristic zero. Since $\mathrm{SL}(n,\mathbb{C})$ is homotopy equivalent to $\mathrm{SU}(n)$, we can rewrite the previous diagram in the following way

$$\begin{array}{ccc} \pi_{2+(4)}\mathrm{SL}_2 & \xrightarrow{j_*} & \pi_{2+(4)}\mathrm{SL}_3 \\ \downarrow & & \downarrow \cong \\ \pi_6\mathrm{SU}(2) & \xrightarrow{i_*} & \pi_6\mathrm{SU}(3) \end{array}$$

where we abuse the notation and also denote the inclusion $\mathrm{SU}(2) \hookrightarrow \mathrm{SU}(3)$ by i. By [MiT63, Theorem 4.1], the 2-primary component of $\pi_6\mathrm{SU}(3)$ is generated by $i_*(\nu')$. Therefore, $j_*(\nu')$ generates the 2-primary component of $\pi_{2+(4)}\mathrm{SL}_3$ over any field k of characteristic zero. □

At the end of the paper, we construct another motivic Toda bracket over Spec $\mathbb{Z}$ whose complex realization is trivial; nevertheless this Toda bracket itself is not trivial.

Let $\Delta_{(2)} : \mathbb{G}_m \to \mathbb{G}_m \wedge \mathbb{G}_m$ be the diagonal morphism $x \mapsto x \wedge x$.

Proposition 5.11. *The element* $\Delta_{1+(3)} \circ h_{1+(2)}$ *is* $\mathbb{A}^1$*-nullhomotopic.*

Proof. Since $h_{1+(2)}$ is by definition equal to $1_{1+(2)} - \epsilon_{1+(2)}$, we have that

$$\Delta_{1+(3)} \circ h_{1+(2)} = \Delta_{1+(3)} - \Delta_{1+(3)} \circ \epsilon_{1+(2)}.$$

In the following, we show that $\Delta_{1+(3)} \circ \epsilon_{1+(2)}$ is equal to $\Delta_{1+(3)}$. The morphism $\Delta_{1+(3)} \circ \epsilon_{1+(2)}$ is given by $t \wedge x \wedge y \mapsto t \wedge x^{-1} \wedge x^{-1} \wedge y$. Therefore, $\Delta_{1+(3)} \circ \epsilon_{1+(2)}$ equals to $\epsilon_{1+(3)} \circ 1_{1+(1)} \wedge \epsilon_{(1)} \wedge \mathrm{id}_{\mathbb{G}_m} \circ \Delta_{1+(3)}$. By Lemma 5.6, we get $\epsilon_{1+(3)} = 1_{1+(1)} \wedge \epsilon_{(1)} \wedge \mathrm{id}_{\mathbb{G}_m}$. Hence, we have that

$$\epsilon_{1+(3)} \circ 1_{1+(1)} \wedge \epsilon_{(1)} \wedge \mathrm{id}_{\mathbb{G}_m} = \epsilon_{1+(3)} \circ \epsilon_{1+(3)} = 1_{1+(3)}.$$

It follows that $\Delta_{1+(3)} \circ \epsilon_{1+(2)}$ is equal to $\Delta_{1+(3)}$. □

By Proposition 5.8, we also have that $h_{1+(2)} \circ \eta_{1+(2)}$ is $\mathbb{A}^1$-nullhomotopic. In particular, the Toda bracket $\{\Delta_{1+(3)}, h_{1+(2)}, \eta_{1+(2)}\}$ is defined. The complex realization of the morphism $\Delta_{1+(3)}$ is a pointed continuous map from S^3 to S^4; hence it is nullhomotopic. The left derived complex realization functor sends the Toda bracket $\{\Delta_{1+(3)}, h_{1+(2)}, \eta_{1+(2)}\}$ to $\{0, 2\mathrm{id}_{S^3}, \Sigma\eta_{top}\}$. The topological Toda bracket $\{0, 2\mathrm{id}_{S^3}, \Sigma\eta_{top}\}$ is trivial. So the complex realization of $\{\Delta_{1+(3)}, h_{1+(2)}, \eta_{1+(2)}\}$ is trivial, too. Although the complex realization of $\{\Delta_{1+(3)}, h_{1+(2)}, \eta_{1+(2)}\}$ is trivial, we can show that this Toda bracket itself is not trivial.

Proposition 5.12. *The Toda bracket* $\{\Delta_{1+(3)}, h_{1+(2)}, \eta_{1+(2)}\}$ *does not contain 0.*

Proof. The Toda bracket $\{\Delta_{1+(3)}, h_{1+(2)}, \eta_{1+(2)}\}$ is again defined over any field k of characteristic zero. In the following, we first work over such a base k. The author showed in their doctoral thesis that motivic Toda brackets satisfy almost the same computational rules as topological Toda brackets [Don24, Section 2.3]. By [Don24, Proposition 2.3.3], we get the equation

$$\{\Delta_{1+(3)}, h_{1+(2)}, \eta_{1+(2)}\}\circ h_{2+(3)} = -(\Delta_{1+(3)}\circ\{h_{1+(2)}, \eta_{1+(2)}, h_{1+(3)}\}).$$

We want to study $\Delta_{1+(3)}\circ\{h_{1+(2)}, \eta_{1+(2)}, h_{1+(3)}\}$. We have to consider now the stable motivic catagory $\mathcal{SH}(k)$ (see [PPR09, A.5]). The category $\mathcal{SH}(k)$ is naturally a triangulated category, therefore Toda brackets are defined in this category. Furthermore, there is a suspension spectrum functor

$$\Sigma^\infty_{\mathbb{P}^1} : \mathcal{H}_\bullet(k) \to \mathcal{SH}(k)$$

We set $h := \Sigma^\infty_{\mathbb{P}^1}(h_{1+(1)})$ and $\eta := \Sigma^\infty_{\mathbb{P}^1}(\eta_{1+(1)})$. Then the functor $\Sigma^\infty_{\mathbb{P}^1}$ sends $\{h_{1+(2)}, \eta_{1+(2)}, h_{1+(3)}\}$ to the Toda bracket $< h, \eta, h >$ in $\mathcal{SH}(k)$.

Let $[-1] : S^0 \to \mathbb{G}_m$ be the pointed morphism defined by sending $* \in S^0 = \{*\}_+$ to $-1 \in \mathbb{G}_m$. In [DuI13, Proposition 3.5], Dugger and Isaksen show that $\rho := \Sigma^\infty_{\mathbb{P}^1}([-1])$ is equal to $\Sigma^\infty_{\mathbb{P}^1}(\Delta_{(2)})$. In the following, we will denote the suspension spectrum $\Sigma^\infty_{\mathbb{P}^1}(\mathcal{E})$ of a pointed motivic space simply by $\mathcal{E}$. The suspension spectrum of S^0 is called the sphere spectrum.

We can equip $\mathcal{SH}(k)$ with a smash product $\wedge$ which makes $\mathcal{SH}(k)$ into a tensor triangulated category (see [PPR09, Remark A.39]). Both S^1 and $\mathbb{G}_m$ are $\wedge$-invertible. We define the bigraded homotopy groups

$$\pi_{s+(w)}\mathbf{1} := \mathcal{SH}(k)((S^1)^m \wedge \mathbb{G}_m^w, S^0)$$

for all $s, w \in \mathbb{Z}$. In particular, we have that $h \in \pi_{0+(0)}\mathbf{1}, \eta \in \pi_{0+(1)}\mathbf{1}$ and $\rho \in \pi_{0+(-1)}\mathbf{1}$. Furthermore, the Toda bracket $< h, \eta, h >$ is contained in $\pi_{1+(1)}\mathbf{1}$.

By work of Morel [Mor04, Theorem 6.4.1], we have an isomorphism

$$K^{MW}_{-*}(k) \to \bigoplus_{w\in\mathbb{Z}} \pi_{0+(w)}\mathbf{1}$$

of graded rings, where $K^{MW}_*(k)$ is the Milnor-Witt K-theory of the field k (see [IØ19, Definition 6.4]). Under this isomorphism the morphisms h, η and ρ are sent to the elements in K^{MW}_{-*} of the same names. The Milnor K-theory $K^M_*(k)$ is defined to be $K^{MW}_*(k)/(\eta)$. Let $\eta_{top} : S^3 \to S^2$ be the topological Hopf map. It is an element of $\pi_{1+(0)}\mathbf{1}$. In [RSØ19, (1.1)], we can find a short exact sequence

$$0 \longrightarrow K^M_2(k)/24 \longrightarrow \pi_{1+(0)}\mathbf{1} \longrightarrow k^\times/2 \oplus \mathbb{Z}/2 \longrightarrow 0\,.$$

In this short exact sequence, the notation $k^\times/2$ means $k^\times$ modulo squares. The kernel $K^M_2(k)/24$ is generated by the second motivic Hopf map $\nu \in \pi_{1+(2)}\mathbf{1}$ (see [DuI13, Definition 4.7]), in the sense that its elements are of the form $\alpha\nu$ for

$\alpha \in \pi_{0+(-2)}\mathbf{1}$. The second factor of $k^\times/2 \oplus \mathbb{Z}/2$ is generated by the image of η_{top} and the first factor is generated by $\eta\eta_{top}$, in the sense that its elements are of the form $\alpha\eta\eta_{top}$ for $\alpha \in \pi_{0+(-1)}\mathbf{1}$. These generators are subject to the relations $24\nu = 0$ and $12\nu = \eta^2\eta_{top}$.

By [Rön20, Proposition 4.1], the Toda bracket $< h, \eta, h >$ is of the form $\eta\eta_{top} + 2K_1^M(k)/24$, where $2K_1^M(k)/24$ is the indeterminacy. The suspension functor $\Sigma^\infty_{\mathbb{P}^1}$ sends the set $\Delta_{1+(3)}\circ\{h_{1+(2)}, \eta_{1+(2)}, h_{1+(3)}\}$ to the set $\rho \cdot (\eta\eta_{top} + 2K_1^M(k)/24) \subseteq \pi_{1+(0)}\mathbf{1}$. It suffices to show that $\rho \cdot (\eta\eta_{top} + 2K_1^M(k)/24)$ does not contain 0. The surjection $\pi_{1+(0)}\mathbf{1} \to k^\times/2 \oplus \mathbb{Z}/2$ in the short exact sequence above sends the elements of $\rho \cdot (\eta\eta_{top} + 2K_1^M(k)/24)$ to $\rho \cdot \eta\eta_{top}$. Therefore, if -1 is not a quadratic root in k, then $\rho \cdot \eta\eta_{top}$ is not equal to 0. This is in particular the case if $k = \mathbb{Q}$. It follows that for $k = \mathbb{Q}$ the set $\Delta_{1+(3)}\circ\{h_{1+(2)}, \eta_{1+(2)}, h_{1+(3)}\}$ does not contain 0, hence the Toda bracket $\{\Delta_{1+(3)}, h_{1+(2)}, \eta_{1+(2)}\}$ is not trivial over $\mathbb{Q}$.

Finally, we would like to show that $\{\Delta_{1+(3)}, h_{1+(2)}, \eta_{1+(2)}\}$ is not trivial over $\mathbb{Z}$. For this, we use base change arguments. Let $f : \operatorname{Spec}\mathbb{Q} \to \operatorname{Spec}\mathbb{Z}$ be the canonical morphism. Then there is a functor

$$f_* : \mathrm{sPre}(\mathbb{Q}) \to \mathrm{sPre}(\mathbb{Z})$$

which is induced by

$$\mathcal{S}\mathrm{m}_{\mathbb{Z}} \to \mathcal{S}\mathrm{m}_{\mathbb{Q}};\ X \mapsto X \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Q}$$

for all $X \in \mathcal{S}\mathrm{m}_{\mathbb{Z}}$. The functor f_* admits a left adjoint $f^* : \mathrm{sPre}(\mathbb{Z}) \to \mathrm{sPre}(\mathbb{Q})$ with the property that it maps the sheaf represented by $X \in \mathcal{S}\mathrm{m}_{\mathbb{Z}}$ to the sheaf represented by $X \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Q}$. The explicit construction can be found in ([Jar15] p.108). By [Jar15, Corollary 5.24] and [Hir03, Proposition 3.3.18] the adjoint functors

$$f^* : \mathrm{sPre}(\mathbb{Z}) \rightleftarrows \mathrm{sPre}(\mathbb{Q}) : f_*$$

form a Quillen adjunction for the $\mathbb{A}^1$-local injective models. Since both functors f^* and f_* preserve terminal objects, we also get the pointed version of this Quillen adjunction

$$f^* : \mathrm{sPre}(\mathbb{Z})_* \rightleftarrows \mathrm{sPre}(\mathbb{Q})_* : f_*.$$

We denote motivic spheres over $\mathbb{Z}$ by $S_{\mathbb{Z}}^{s+(w)}$ and motivic spheres over $\mathbb{Q}$ by $S_{\mathbb{Q}}^{s+(w)}$. It follows from the construction of f^* that $f^*(S_{\mathbb{Z}}^{s+(w)}) = S_{\mathbb{Q}}^{s+(w)}$. Furthermore, the left derived functor $\mathbb{L}f^*$ sends the morphisms $\Delta_{1+(3)}, h_{1+(2)}$ and $\eta_{1+(2)}$ in $\mathcal{H}_\bullet(\mathbb{Z})$ to the morphisms of the same names in $\mathcal{H}_\bullet(\mathbb{Q})$. Therefore, $\mathbb{L}f^*$ sends the Toda bracket $\{\Delta_{1+(3)}, h_{1+(2)}, \eta_{1+(2)}\}$ over $\mathbb{Z}$ to the corresponding Toda bracket $\{\Delta_{1+(3)}, h_{1+(2)}, \eta_{1+(2)}\}$ over $\mathbb{Q}$. We already know that the Toda bracket $\{\Delta_{1+(3)}, h_{1+(2)}, \eta_{1+(2)}\}$ over $\mathbb{Q}$ does not contain 0, hence the Toda bracket $\{\Delta_{1+(3)}, h_{1+(2)}, \eta_{1+(2)}\}$ over $\mathbb{Z}$ is also not trivial. □

6. Appendix

6.1. Δ-generated spaces. This section contains some facts about Δ-generated spaces which are used in the paper. The notion of Δ-generated spaces was originally proposed by Jeff Smith as a nice category of spaces for homotopy theory. However, Jeff Smith never published his ideas and there are only few references on this notion. In the following, we will follow the unpublished notes by Daniel Dugger [Dug03].

Let $\mathcal{T}op$ denote the category of all topological spaces and continuous maps and Δ be the full subcategory of $\mathcal{T}op$ consisting of the topological simplices Δ^n.

Definition 6.1 ([Dug03, Definition 1.2]). A topological space X is called Δ-generated if it has the property that a subset $S \subseteq X$ is open if and only if $f^{-1}(S)$ is open for every continuous map $f : Z \to X$ with $Z \in \Delta$. Let $\mathcal{T}op_\Delta$ denote the full subcategory of Δ-generated spaces.

Proposition 6.2 ([Dug03, Proposition 1.3]). *Any object of Δ is Δ-generated. Any colimit of Δ-generated spaces is again Δ-generated.*

Therefore, $\mathcal{T}op_\Delta$ is a cocomplete category and the colimits are the same as those in $\mathcal{T}op$. Moreover, it also follows that $\mathcal{T}op_\Delta$ contains the geometric realization of every simplicial set. We now show that this category is also complete. Let X be a topological space and $(\Delta \downarrow X)$ be the overcategory. Then there is a canonical diagram $(\Delta \downarrow X) \to \mathcal{T}op$ sending every object $(f : Z \to X)$ to Z. The colimit of this diagram will be denoted by $k_\Delta(X)$. By the above proposition this colimit is again Δ-generated, and there is a canonical map $k_\Delta(X) \to X$.

Proposition 6.3 ([Dug03, Proposition 1.5]). *(a) $k_\Delta(X) \to X$ is a set-theoretic bijection.*
(b) X is Δ-generated if and only if $k_\Delta(X) \to X$ is a homeomorphism.
(c) A space is Δ-generated if and only if it is a colimit of some diagram whose objects belong to Δ.
(d) The functors $i : \mathcal{T}op_\Delta \rightleftarrows \mathcal{T}op : k_\Delta$ are an adjoint pair, where i is the inclusion.

Now by Proposition 6.3 (b) and (d), we see that $\mathcal{T}op_\Delta$ is also complete; limits are computed by first taking the limit in $\mathcal{T}op$ and then applying the functor $k_\Delta(-)$.

One of the most important properties of the category $\mathcal{T}op_\Delta$ is that it is locally presentable.

Proposition 6.4 ([FR07, Corollary 3.7]). *The category $\mathcal{T}op_\Delta$ is locally presentable.*

It follows that in particular every object of $\mathcal{T}op_\Delta$ is small, therefore $\mathcal{T}op_\Delta$ permits the small object argument. Furthermore, by [Wyl73, 3.3], the category $\mathcal{T}op_\Delta$ is even cartesian closed. For $X, Y \in \mathcal{T}op_\Delta$ we write $X \otimes Y$ for the product in $\mathcal{T}op_\Delta$ and $X \times Y$ for the usual cartesian product in $\mathcal{T}op$.

Proposition 6.5 ([Dug03, Proposition 1.14]). *The natural map $X \otimes Y \to X \times Y$ is a homeomorphism.*

At the end of this section we also mention that every open subset of a Δ-generated space is again Δ-generated (see [Dug03, Proposition 1.18]).

6.2. The geometric realization functor. We summarize the properties of the geometric realization functor which the author considered in her doctoral thesis [Don24, Section 2].

Let S be a noetherian base scheme of finite Krull dimension. We write $\mathcal{S}\mathrm{m}_S$ for the category of smooth schemes of finite type over S. The category $\mathrm{sPre}(S)$ is the category of simplicial presheaves on $\mathcal{S}\mathrm{m}_S$. The category of pointed simplicial presheaves on $\mathcal{S}\mathrm{m}_S$ is denoted by $\mathrm{sPre}(S)_*$. Let $\mathrm{Pre}_\Delta(S)$ denote the category of presheaves on $\mathcal{S}\mathrm{m}_S$ with values in Δ-generated topological spaces and $\mathrm{Pre}_\Delta(S)_*$ the category of presheaves on $\mathcal{S}\mathrm{m}_S$ with values in pointed Δ-generated topological spaces. By Proposition 6.2 the geometric realization of a simplicial set is Δ-generated. Therefore, by applying the usual geometric realization functor sectionwise, we get a functor

$$|\cdot| : \mathrm{sPre}(S)_* \to \mathrm{Pre}_\Delta(S)_*.$$

Recall that the geometric realization functor for simplicial sets has a right adjoint $Sing : \mathcal{T}op_\Delta \to \mathrm{s}\mathcal{S}\mathrm{et}$. Then if we apply this functor again sectionwise, we obtain a right adjoint for $|\cdot|$ and we denote the right adjoint still by $Sing$. Hence, we have the following adjoint pair

$$|\cdot| : \mathrm{sPre}(S)_* \rightleftarrows \mathrm{Pre}_\Delta(S)_* : Sing.$$

Proposition 6.6. [Don24, Remark 2.1.9] *There exists an $\mathbb{A}^1$-local injective model structure on* $\mathrm{Pre}_\Delta(S)_*$ *such that*

$$\mathrm{sPre}(S)_{*\ \mathbb{A}^1-local\ inj} \underset{Sing}{\overset{|\cdot|}{\rightleftarrows}} \mathrm{Pre}_\Delta(S)_{*\ \mathbb{A}^1-local\ inj}$$

is a Quillen equivalence.

Remark 6.7. The homotopy category associated to the $\mathbb{A}^1$-local injective model structure on $\mathrm{Pre}_\Delta(S)_*$ will be denoted by $\mathcal{H}o_\Delta(S)$.

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