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A new construction of subgroups of big mapping class groups

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ABSTRACT. We explicitly construct new subgroups of the mapping class groups of an uncountable collection of infinite-type surfaces, including, but not limited to, free groups, Baumslag-Solitar groups, mapping class groups of other surfaces, and a large collection of wreath products. For each such subgroup H and surface S, we show that there are countably many non-conjugate embeddings of H into $\mathrm{Map}(S)$; in certain cases, there are uncountably many such embeddings. The images of each of these embeddings cannot lie in the isometry group of S for any hyperbolic metric and are not contained in the closure of the compactly supported subgroup of $\mathrm{Map}(S)$. In this sense, our construction is new and does not rely on previously known techniques for constructing subgroups of mapping class groups. Notably, our embeddings of $\mathrm{Map}(S')$ into $\mathrm{Map}(S)$ are not induced by embeddings of S' into S. Our main tool for all of these constructions is the utilization of special homeomorphisms of S called shift maps, and more generally, multipush maps.

CONTENTS

1. Introduction

A fundamental question in low-dimensional topology asks which groups can arise as subgroups of the diffeomorphism group, homeomorphism group, and mapping class group of a surface S, denoted by $\mathrm{Homeo}(S)$, $\mathrm{Diffeo}(S)$, and $\mathrm{Map}(S)$, respectively. One approach to producing such subgroups is to consider embeddings of finite-type subsurfaces S' into an infinite-type surface S that induce injections

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of $\mathrm{Map}(S')$ into $\mathrm{Map}(S)$. In this case, every subgroup of $\mathrm{Map}(S')$ is a subgroup of $\mathrm{Map}(S)$. Another approach to this problem is to show that a particular group G acts by orientation-preserving isometries on a surface S, which implies that G can be realized as a subgroup of $\mathrm{Homeo}(S)$, $\mathrm{Diffeo}(S)$, and $\mathrm{Map}(S)$. However, these two classical approaches have limitations. For example, the strong Tits alternative holds for finite-type mapping class groups, meaning every subgroup of $\mathrm{Map}(S')$ is either virtually abelian or contains a free subgroup [Iva84, McC85]. In addition, Aougab, Patel, and Vlamis show that only finite groups can arise as the isometry group of a hyperbolic metric on S whenever S contains a *non-displaceable subsurface* (see [APV21, Lemma 4.2]). They also show that no uncountable group can be obtained as the isometry group of a hyperbolic metric on any infinite-type surface. These observations indicate that in order to fully understand the algebraic structure of big mapping class groups, we need other constructions of subgroups in $\mathrm{Map}(S)$, and we also need to understand the many different ways that a particular subgroup can embed in $\mathrm{Map}(S)$. This is precisely the goal of this paper.

To streamline the statements of our results below, we construct an uncountable collection of surfaces for which particular results hold. The precise definitions will appear in Section 3.2. When Π is a *distinguished surface*, we denote by $\mathcal{C}(\Pi)$ the collection of surfaces that admit a map which acts as a shift along a countable collection of copies of Π; see Definition 3.5. For example, in the case where Π is a torus with one boundary component, $\mathcal{C}(\Pi)$ includes the ladder surface, the connect sum of a ladder surface and any surface of genus 0, and the Cantor tree surface with exactly two ends accumulated by genus.

Our first construction produces embeddings between big mapping class groups that are not induced by embeddings of the underlying surfaces and that do not preserve the property of being compactly supported. The result is summarized as Theorem 1.1 below and is a consequence of Theorem 5.2. Recall that a group is indicable if it admits a surjection onto $\mathbb{Z}$.

Theorem 1.1. *Let Π be a distinguished surface. If $\mathrm{Map}(\Pi)$ is indicable, then for any surface $S \in \mathcal{C}(\Pi)$, there exist countably many non-conjugate embeddings of $\mathrm{Map}(\Pi)$ into $\mathrm{Map}(S)$ that are not induced by an embedding of Π into S.*

The above theorem is in line with a body of work dedicated to understanding and constructing homomorphisms between mapping class groups; see, for example, [ArLS09, AS13, ArLM24]. There are uncountably many distinguished surfaces Π for which $\mathrm{Map}(\Pi)$ is indicable; see Examples 5.4. When Π has at least two nonplanar ends, Theorem 1.1 also holds with $\mathrm{PMap}(\Pi)$ in place of $\mathrm{Map}(\Pi)$; see Corollary 5.3.

Theorem 1.1 also answers Question 4.75 from the AIM problem list on surfaces of infinite type [AIM] which asks, "Given a homomorphism $f\colon \mathrm{Map}(S) \to \mathrm{Map}(S')$, does f preserve the notion of being compactly supported?" Bavard, Dowdall and Rafi [BDR20] show that the answer is yes for surjective homomorphisms, and Aramayona, Leininger, and McLeay [ArLM24] give an example of two surfaces and self-maps for which the answer is no. Our results show that there is an uncountable family of surfaces and maps for which the answer is also no.

The second main result of our paper is Theorem 6.3, which is a combination theorem for indicable subgroups of $\mathrm{Map}(S)$. We summarize its statement as Theorem 1.2 below. The $\star$-product used in the statement is an interpolation between free products and direct products. The $\star$-product of two groups G_i with subgroups H_i is defined as

$$(G_1, H_1) \star (G_2, H_2) := G_1 * G_2 / \langle\!\langle [G_1, H_2], [H_1, G_2] \rangle\!\rangle.$$

More generally, given $G_1, \dots, G_n$ with subgroups $H_1, \dots, H_n$, let

$$(G_1, H_1) \star \cdots \star (G_n, H_n) := G_1 * \cdots * G_n / \langle\!\langle [G_i, H_j] : i \neq j \rangle\!\rangle.$$

Theorem 1.2. *Let G_i be an indicable group that embeds in $\mathrm{Map}(S_i)$ for $i = 1, \dots, n$, where S_i is a surface with exactly one boundary component. For each i, fix a surjective map $f_i : G_i \to \mathbb{Z}$, and let H_i be the kernel of f_i. The indicable group $(G_1, H_1) \star \cdots \star (G_n, H_n)$ embeds in $\mathrm{Map}(S)$ for $S = S_\Gamma(\Pi)$, where Π is obtained from $\#_n S_i$ by capping off $n - 1$ boundary components.*

We direct the reader to Section 3 for the definition of the surface $S_\Gamma(\Pi)$, the construction of which was inspired by work of Allcock [All06]. Importantly, the support of the homeomorphisms defined in our construction is not all of $S_\Gamma(\Pi)$. Consequently, we may change the topology outside the support of the homeomorphisms in any way we choose. In this way, Theorem 1.2 actually shows that $(G_1, H_1) \star \cdots \star (G_n, H_n)$ embeds in the mapping class group of a wide class of infinite-type surfaces. For instance, we can arrange for the edited surface to have a non-displaceable subsurface so that the subgroups we construct cannot arise from a construction using isometries.

A key aspect of the proof of Theorem 1.2 is a set of criteria on a collection of *shift maps* (or *multipush maps*) that guarantees they generate a free group (see Theorem 4.2). Shift maps are generalizations of *handleshifts*, homeomorphisms introduced by Patel and Vlamis in [PaV18] that have become integral to the theory of infinite-type surfaces. In particular, we augment the generators of the groups G_i in the statement of the theorem with these shift maps (or multipush maps). The fact that shift maps do not lie in $\overline{\mathrm{Map}_c(S)}$ implies that the subgroups we construct are also not completely contained in $\overline{\mathrm{Map}_c(S)}$. The only exceptions to this are when S is finite type or the Loch Ness Monster surface, in which case $\overline{\mathrm{Map}_c(S)} = \mathrm{Map}(S)$. We avoid the technical statement of Theorem 4.2 here and direct the reader to Section 4.1.

There are a variety of indicable groups that can play the role of G_i in the statement of Theorem 1.2 (or the role of G in the statement of Theorem 1.1). In particular, one can let G_i be any indicable subgroup of the mapping class group of a finite-type surface with exactly one boundary component, for example, free groups, braid groups, and right-angled Artin groups.

In the case of right-angled Artin groups, Clay, Leininger, Mangahas [CLM12] and Koberda [Kob12] show that every right-angled Artin group embeds as a subgroup of the mapping class group of some finite-type surface of sufficient complexity. The map $f_i : A_{\Gamma_i} \to \mathbb{Z}$ which sends each generator of the right-angled Artin group A_{Γ_i} with defining graph Γ_i to 1 is a homomorphism, so A_{Γ_i} is indicable. The kernel H_i of this map is called the *Bestvina-Brady group* defined on Γ_i and is denoted by BB_{Γ_i}. The pairs $(A_{\Gamma_i}, BB_{\Gamma_i})$ form a rich class of examples that can be used as the input for Theorem 1.2 and are discussed in detail in Section 6.1.

We also produce new examples of indicable subgroups of big mapping class groups that can be used as input for these theorems, including solvable Baumslag-Solitar groups $BS(1,n)$ and a large class of wreath products $G \wr H$. The following theorem is a particular case of Proposition 4.3 and Theorem 4.6, which both hold for a more general class of surfaces. We make the statement below to avoid technicalities.

Theorem 1.3. *If S is a Cantor tree surface, then solvable Baumslag-Solitar groups $BS(1,n)$ and wreath products $\mathbb{Z}^n \wr \mathbb{Z}$ for any $n \geq 1$ arise as subgroups of* $\mathrm{Map}(S)$.

Note that solvable Baumslag-Solitar and $\mathbb{Z}^n \wr \mathbb{Z}$ cannot embed in the mapping class group of any finite-type surfaces. Our theorem gives the first construction of these groups (for $n > 1$) in the mapping class groups of infinite-type surfaces. Lanier–Loving construct $\mathbb{Z} \wr \mathbb{Z}$ as a subgroup of the mapping class group of any infinite-type surface without boundary [LL20].

Outline. Section 2 contains preliminaries on infinite-type surfaces, mapping class groups, and shift and multipush maps. Section 3 gives a construction of surfaces based on Schreier graphs and describes how to obtain non-conjugate embeddings of subgroups generated by either shift or multipush maps. Our constructions of specific subgroups of big mapping class groups begins in Section 4, where we build embeddings of free groups, wreath products, and solvable Baumslag–Solitar groups into big mapping class groups. In Section 5, we prove Theorem 1.1. Finally, in Section 6, we prove and discuss applications of the combination theorem (Theorem 1.2).

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2. Preliminaries

2.1. Ends of surfaces. Essential to the classification of infinite-type surfaces is the notion of an end of a surface and the space of ends for an infinite-type surface S.

Definition 2.1. An *exiting sequence* in S is a sequence $\{U_n\}_{n\in\mathbb{N}}$ of connected open subsets of S satisfying:

(1) $U_n \subset U_m$ whenever $m < n$;
(2) U_n is not relatively compact for any $n \in \mathbb{N}$, that is, the closure of U_n in S is not compact;
(3) the boundary of U_n is compact for each $n \in \mathbb{N}$; and
(4) any relatively compact subset of S is disjoint from all but finitely many of the U_n's.

Two exiting sequences $\{U_n\}_{n\in\mathbb{N}}$ and $\{V_n\}_{n\in\mathbb{N}}$ are equivalent if for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $U_m \subset V_n$ and $V_m \subset U_n$. An *end* of S is an equivalence class of exiting sequences.

The *space of ends* of S, denoted by $E(S)$, is the set of ends of S equipped with a natural topology for which it is totally disconnected, Hausdorff, second countable, and compact. In particular, $E(S)$ is homeomorphic to a closed subset of a Cantor set. The definition of the topology on the space of ends is not relevant to this paper and so is omitted.

Ends of S can be isolated or not and can be *planar*, if there exists an i such that U_i is homeomorphic to an open subset of the plane $\mathbb{R}^2$, or *nonplanar*, if every U_i has infinite genus. The set of nonplanar ends of S is a closed subspace of $E(S)$ and will be denoted by $E^g(S)$; these are frequently called the *ends accumulated by genus*. We have the following classification theorem of Kerékjártó [Ker23] and Richards [Ric63]:

Theorem 2.2 (Classification of infinite-type surfaces)**.** *The homeomorphism type of an orientable, infinite-type surface S is determined by the quadruple*

$$(g, b, E^g(S), E(S))$$

where $g \in \mathbb{Z}_{\geq 0} \cup \infty$ is the genus of S and $b \in \mathbb{Z}_{\geq 0}$ is the number of (compact) boundary components of S.

There is a more complicated classification of infinite-type surfaces allowing for non-compact boundary components due to Prishlyak–Mischenko [PM07]. We use this classification once in Section 3.2, but in our setting, the surfaces we are comparing have precisely the same boundary, so the classification reduces to considering the triple $(g, E^g(S), E(S))$.

2.2. Mapping class group. The *mapping class group* of S, denoted $\mathrm{Map}(S)$, is the set of orientation-preserving homeomorphisms of S up to isotopy that fix the boundary pointwise. The natural topology on the set of homeomorphisms of S is the compact-open topology, and $\mathrm{Map}(S)$ is endowed with the induced quotient topology. Equipped with this topology, $\mathrm{Map}(S)$ is a topological group. When S is a finite-type surface, this topology on $\mathrm{Map}(S)$ agrees with the discrete topology, but when S is of infinite type, the two topologies are distinct. The *pure mapping class group*, denoted $\mathrm{PMap}(S)$, is the subgroup of $\mathrm{Map}(S)$ that fixes the set of ends of S pointwise, and $\overline{\mathrm{Map}_c(S)}$ is the subgroup of compactly supported mapping classes. Note that $\overline{\mathrm{Map}_c(S)} \leq \mathrm{PMap}(S)$. When S has at most one nonplanar end, $\overline{\mathrm{Map}_c(S)}$ is actually equal to $\mathrm{PMap}(S)$ [ArPV20].

Definition 2.3. A mapping class $f \in \mathrm{Map}(S)$ is of *intrinsically infinite type* if $f \notin \overline{\mathrm{Map}_c(S)}$. A subgroup $H \leq \mathrm{Map}(S)$ is of intrinsically infinite type if H is not completely contained in $\overline{\mathrm{Map}_c(S)}$.

In this paper, all of the subgroups of $\mathrm{Map}(S)$ that we construct contain many intrinsically infinite-type homeomorphisms and, therefore, cannot be completely contained in $\overline{\mathrm{Map}_c(S)}$, except when S is finite-type or the Loch Ness Monster, in which case $\overline{\mathrm{Map}_c(S)} = \mathrm{Map}(S)$. Recall that the Loch Ness Monster surface is the unique infinite-genus surface with one end (up to homeomorphism).

We are particularly interested in indicable groups and various ways of embedding them in mapping class groups of infinite-type surfaces. A group G is *indicable* if there exists a surjective homomorphism $f\colon G \to \mathbb{Z}$. We show in Lemma 5.1 that a group G is indicable if and only if there is a presentation for G where the relators all have total exponent sum zero in the generators of G. Importantly, many of our constructions require an indicable subgroup G of $\mathrm{Map}(S)$ as an input, where S is a surface with exactly one boundary component. There are many examples of such groups that were mentioned in the introduction, but there are also some restrictions on what groups G can arise as subgroups of mapping class groups, as is evidenced by the following lemma, which generalizes the same result from the finite-type setting [FM12, Corollary 7.3].

Lemma 2.4 ([ACCL21, Corollary 3])**.** *If S is an orientable surface with nonempty compact boundary, the mapping class group fixing the boundary pointwise is torsion-free.*

2.3. Push and shift maps. In this section, we define shift maps and push maps, which are central to all of our constructions. A particular type of shift maps, called handle shifts, were first studied by Patel and Vlamis in [PaV18]. This inspired the following definition of Abbott, Miller, and Patel [AbMP25]. A similar definition of shift maps appears in [MaR23] and [LL20].

Definition 2.5. Let D_Π be the surface defined by taking the strip $\mathbb{R} \times [-1, 1]$, removing an open disk of radius $\frac{1}{4}$ with center $(n, 0)$ for $n \in \mathbb{Z}$, and attaching any fixed topologically nontrivial surface Π with exactly one boundary component to the boundary of each such disk. A *shift* on D_Π is the homeomorphism that acts like a translation, sending (x, y) to $(x + 1, y)$ for $y \in [-1 + \epsilon, 1 - \epsilon]$ and which tapers to the identity on ∂D_Π.

Given a surface S with a proper embedding of D_Π into S so that the two ends of the strip correspond to two different ends of S (see Figure 1), the shift on D_Π

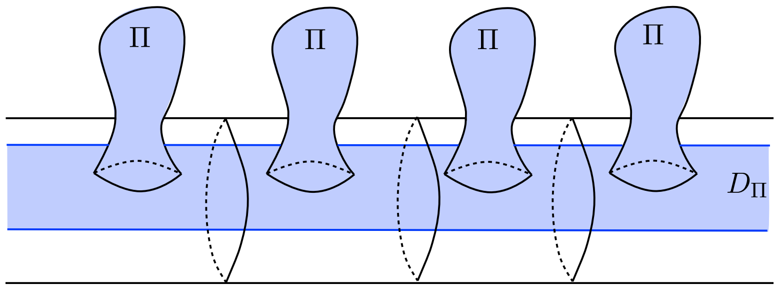

FIGURE 1. A surface S that admits a shift whose domain is an embedded copy of D_Π.

induces a *shift* on S, where the homeomorphism acts as the identity on the complement of D_Π. If instead, we have a proper embedding of D_Π into S where the two ends of the strip correspond to the same end, we call the resulting homeomorphism on S a *one-ended shift*. Given a shift or one-ended shift h on S, the embedded copy of D_Π in S is called the *domain* of h. By abuse of notation, we will sometimes say that the domain of the shift or one-ended shift h is D_Π rather than referring to it as an embedded copy of D_Π in S (when it is clear from context to which embedded copy of D_Π we are referring).

Remark 2.6. If the surface Π in Definition 2.5 has a nontrivial end space, then a shift or one-ended shift h on S with domain D_Π is not in $\mathrm{PMap}(S)$ since there are ends of S that are not fixed by h. Thus, $h \notin \overline{\mathrm{Map}_c(S)}$ and is of intrinsically infinite type. On the other hand, if h is a shift map and if Π is a finite-genus surface with no planar ends, then h is a power of a handle shift on S, and the proof of [PaV18, Proposition 6.3] again tells us that $h \notin \overline{\mathrm{Map}_c(S)}$. However, the second conclusion does not hold when h is a power of a one-ended handle shift since, in that case, it follows from work in [PaV18] that $h \in \overline{\mathrm{Map}_c(S)}$.

We now use the construction of shift maps to introduce *finite shifts*, which will be used in Section 4.3 to construct certain wreath products. These are constructed in a completely analogous way, starting with an annulus instead of a biinfinite strip.

Definition 2.7. Let A_Π be a surface defined by taking the annulus

$$([0,n]/0 \sim n) \times [-1,1],$$

removing an open disk of radius $\frac{1}{4}$ centered at the integer points, and attaching any fixed topologically nontrivial surface Π with exactly one boundary component to the boundary of each disk. A *finite shift* on A_Π is the homeomorphism that acts like a translation, sending (x,y) to $(x+1,y)$ (modulo n) for $y \in [-1+\epsilon, 1-\epsilon]$ and which tapers to the identity on ∂A_Π. Given a surface S with a proper embedding of A_Π into S, the finite shift on A_Π induces a *finite shift* on S, where the homeomorphism acts as the identity on the complement A_Π. We call the embedded copy of A_Π the *domain* of the finite shift.

Definition 2.8. A *push* is any map that is a finite shift, a one-ended shift, or a shift map.

In Section 3, we will introduce the notion of a *multipush*, which is roughly a collection of push maps with disjoint supports, once we have developed some further notation and language.

3. Surfaces from graphs and non-conjugate embeddings

In this section, we begin by constructing a broad class of surfaces using an underlying graph. We then introduce a type of homeomorphism called a *multipush* and show that these maps can be utilized to produce infinitely many non-conjugate embeddings of certain groups into mapping class groups.

3.1. A construction of surfaces. The basic building block for this construction is a d–holed sphere. The following definition of seams restricts to the normal notion of seams for a 3–holed sphere, i.e., a pair of pants.

Definition 3.1. A set of *seams* on a d–holed sphere is a collection of d disjointly embedded arcs such that each boundary component of the sphere intersects exactly two components of the seams at two distinct points and such that the collection of seams divides the sphere into two components. Call one component the *front side* and the other component the *back side*. These conditions imply that each component is homeomorphic to a disk.

Starting from any graph Γ with a countable vertex set and any surface Π with exactly one boundary component, we describe a procedure for building a surface $S_\Gamma(\Pi)$. This mirrors a construction of Allcock using the Cayley graph of a given group G [All06].

For each vertex v of valence d, start with a d–holed sphere. Remove a disk on the interior of the front side, and attach the surface Π along the boundary component. Call the resulting surface the *vertex surface for* v, which we denote by V_v, and let Π_v be the copy of Π on V_v. For each edge of the graph, define the edge surface E to be the 2–holed sphere with seams; topologically this is an annulus.

Whenever u and v are vertices of Γ connected by an edge, connect the vertex surfaces V_u and V_v with an edge surface $E(u, v)$ by gluing one boundary component of the edge surface to a boundary component of V_u and the other boundary component of the edge surface to a boundary component of V_v so that the gluing is *compatible* in the following sense: the union of the seams separates S_Γ into two disjoint connected components, the *front* and the *back*, containing the front and, respectively, the back of each vertex and edge surface. Call the resulting surface $S_\Gamma(\Pi)$. See Figure 2 for an example. Notice that the assumption that the vertex set $V(\Gamma)$ of Γ is countable is necessary for this construction to yield a surface. In particular, if $V(\Gamma)$ is uncountable, then $S_\Gamma(\Pi)$ is not second countable and therefore cannot be a surface.

We also define a more general class of surfaces constructed by editing the back of $S_\Gamma(\Pi)$ as follows. As above, fix a graph Γ with a countable vertex set and a

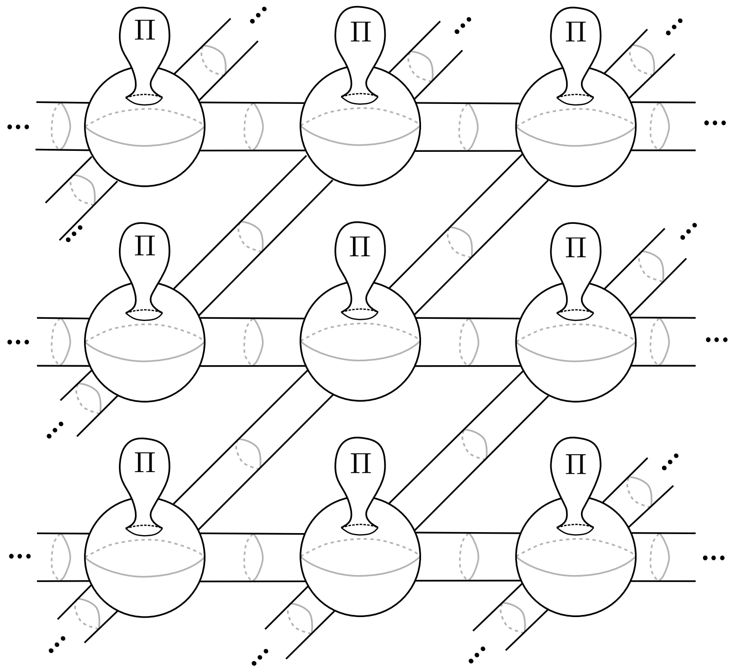

FIGURE 2. An example of the surface $S_\Gamma(\Pi)$ where the graph Γ is the Cayley graph of the group $\mathbb{Z}^2 = \langle a, b : [a,b] \rangle$.

surface with one boundary component Π, and let $S = S_\Gamma(\Pi)$. Given any collection of surfaces $\{\Omega_v\}_{v \in V(\Gamma)}$, only finitely many of which have boundary, we form the surface $S \underset{v \in V(\Gamma)}{\#} \Omega_v$ as follows. For each $v \in V(\Gamma)$, take the connect sum of V_v and the corresponding Ω_v. It is helpful to assume that the connect sum is done on the back of V_v, since we will perform certain homeomorphisms on the front of S later in the paper. We note that if every Ω_v is a sphere, then $S \underset{v \in V(\Gamma)}{\#} \Omega_v$ is homeomorphic to S. On the other hand, by choosing the Ω_v to be more topologically complex, the homeomorphism type of the resulting surface differs from S in either the genus or the space of ends. Thus, even for a fixed surface Π, this construction yields a large family of surfaces, formed by varying the topological type of the Ω_v.

The underlying graph Γ used to build $S_\Gamma(\Pi)$ throughout this paper will often be a Schreier graph, which is defined as follows. Let G be a finitely generated group, H a subgroup of G, and T a finite generating set for G. The Schreier graph $\Gamma(G, T, H)$ is the graph whose vertices are the right cosets of H and in which, for each coset Hg and each $s \in T$, there is an edge from Hg to Hgs labeled by s. If $Hg = Hgs$, there is a loop labeled by s at the vertex corresponding to Hg. Our assumption on the finiteness of T ensures that $\Gamma(G, T, H)$ has a countable vertex set. When Γ is a Schreier graph, let Π_{Hg} be the copy of Π on the vertex surface corresponding to the coset Hg. In the special case when $H = \{1\}$, the Schreier

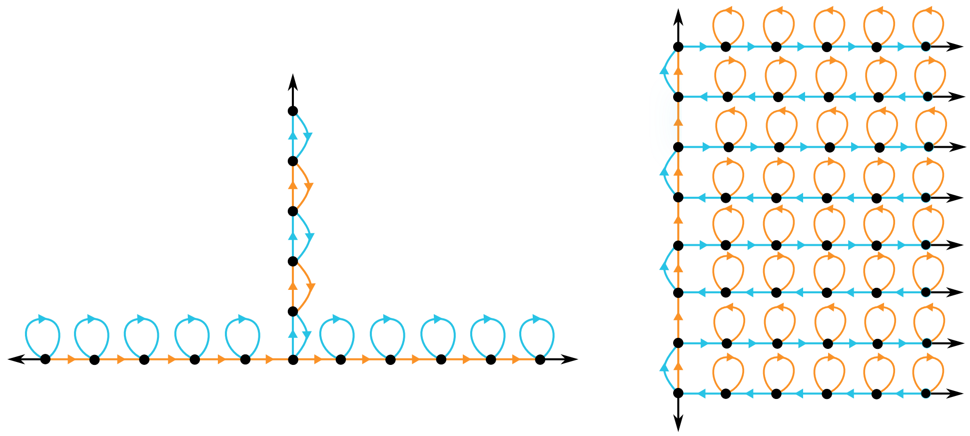

FIGURE 3. Each graph can be realized as a Schreier graph but not a Cayley graph. The graph on the left has 3 ends, and the graph on the right has end space homeomorphic to the 2-point compactification of $\mathbb{Z}$.

graph $\Gamma(G, T, \{1\})$ is simply the Cayley graph of G with respect to the generating set T, which we denote by $\Gamma(G, T)$.

Definition 3.2. Let Γ be a Schreier graph for a triple (G, T, H). A *Schreier surface associated to* (G, T, H) is a surface $S = S_\Gamma(\Pi) \underset{v \in V(\Gamma)}{\#} \Omega_v$ where Π has exactly one boundary component and is not a disk, and $\{\Omega_v\}$ is any collection of surfaces, only finitely many of which have boundary.

Using the more general class of Schreier graphs, rather than just Cayley graphs, to construct surfaces in this fashion broadens the class of surfaces our results apply to. For example, when Π is compact, the surface $S_\Gamma(\Pi)$ will have the same end space as the graph Γ. A Cayley graph for a finitely generated group has 1, 2 or a Cantor set of ends. On the other hand, there are many more possibilities for a Schreier graph; any regular graph with even degree can be realized as a Schreier graph [Gro77, Lub95]. Thus, there are Schreier graphs with any finite number of ends, or end spaces isomorphic to $\mathbb{N} \cup \{\infty\}$ or $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$. This implies that there are Schreier surfaces with these end spaces as well. See Figure 3 for two examples of Schreier graphs that cannot be realized as Cayley graphs.

We now move to defining a homeomorphism called a *multipush* on a Schreier surface.

Definition 3.3. Let $\Gamma = \Gamma(G, T, H)$ be a Schreier graph. Fix a surface Π with exactly one boundary component. Let $S = S_\Gamma(\Pi) \underset{v \in V(\Gamma)}{\#} \Omega_v$ be a Schreier surface.

For each $s \in T$, we construct a collection of push maps whose support corresponds to connected components of the subgraph of Γ which includes only edges labeled by s (see Figure 4). Fix a transversal $\mathcal{T}$ for the set of double cosets

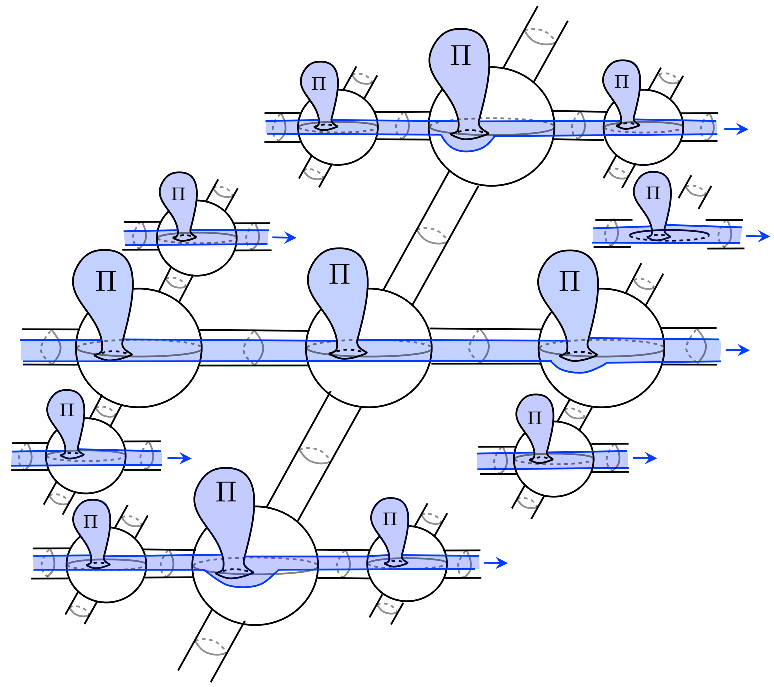

FIGURE 4. A portion of the domain D_a (in blue) of the multipush x_a on the surface $S_\Gamma(\Pi)$, where Γ is the Cayley graph $\Gamma = \Gamma(\mathbb{F}_2, \{a, b\})$ for $\mathbb{F}_2 = \langle a, b\rangle$.

$\{Hg\langle s\rangle \mid g \in G\}$, so that $\mathcal{T}$ contains exactly one element from each double coset in the set. In the case $H = \{1\}$, that is, when Γ is a Cayley graph, the set $\mathcal{T}$ is simply a transversal for (left cosets of) $\langle s\rangle$. For each element t in the transversal, we define a push $h_{t\langle s\rangle}$ which maps Π_{Hts^i} to $\Pi_{Hts^{i+1}}$. The support of $h_{t\langle s\rangle}$ is contained in the front of

$$\left(\bigcup_{i\in\mathbb{Z}} V_{Hts^i}\right) \cup \left(\bigcup_{i\in\mathbb{Z}} E(Hts^i, Hts^{i+1})\right).$$

Recall that, for each $i \in \mathbb{Z}$, V_{Hts^i} is the vertex surface associated to the vertex Hts^i and $E(Hts^i, Hts^{i+1})$ is the edge surface associated to the edge (Hts^i, Hts^{i+1}). This support corresponds to a connected component of Γ with all edges labeled by s; see Figure 4. The *multipush* x_s associated to s is the element of $\mathrm{Map}(S)$ that acts simultaneously as the pushes $h_{t\langle s\rangle}$ for each $t \in \mathcal{T}$. We let D_s denote the domain of the multipush x_s. If $h_{t\langle s\rangle}$ is not a finite shift for any $t \in T$, we say x_s is an *infinite* multipush.

Since supports of the pushes $h_{t\langle s\rangle}$ are disjoint, the multipush x_s is a well-defined homeomorphism of the surface. Note that if $Hg = Hgs$, then the edge $E(Hgs^i, Hgs^{i+1})$ in the support of $h_{g\langle s\rangle}$ is a loop. In particular, finite pushes can occur as part of a multipush. In Figure 4, all pushes are infinite, but the multipush with the orange domain shown in Figure 5 has both a finite and infinite push.

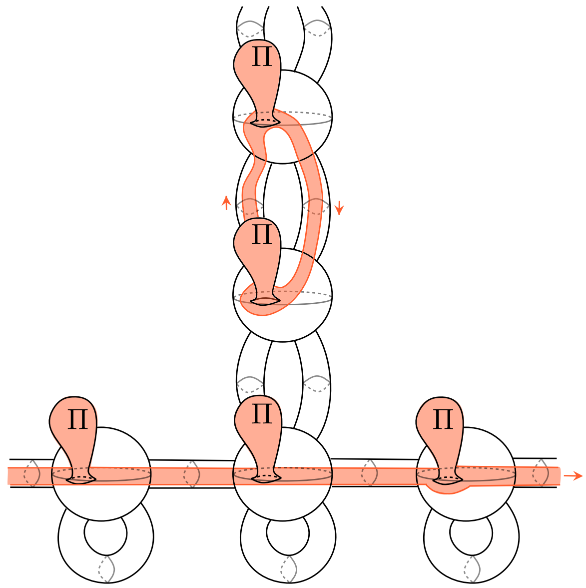

FIGURE 5. A multipush on a surface corresponding to the Schreier graph on the left in Figure 3 that contains both a finite and infinite push. The domain of the multipush is highlighted in orange.

Remark 3.4. We emphasize that multipushes are *not* induced by an action on the graph Γ, even when Γ is a Cayley graph. The construction simply uses the labeling of the vertices of Γ to define the homeomorphism x_s.

3.2. Non-conjugate embeddings. Given a shift map h corresponding to an embedding of D_Π into a surface S, one can define a new and distinct shift map h' on S by omitting some of the surfaces Π_i from the domain of h, so long as infinitely many remain. This gives another embedding of D_Π into S. See Figure 6. Since there are uncountably many infinite subsets of $\mathbb{Z}$, we can construct uncountably many distinct embeddings of D_Π into S, and thus uncountably many distinct shift maps on S, in this way. The same argument goes through for one-ended shifts as well. Similarly, one infinite multipush x_s associated to a generator $s \in T$ on a surface $S = S_\Gamma(\Pi)$ can be used to produce uncountably many distinct domains for multipushes associated to s by simply omitting some of the copies of Π from the domains of x_s for all $s \in T$.

In many cases, these distinct domains give rise to isomorphic but non-conjugate subgroups of $\mathrm{Map}(S)$. First, consider the case of shift maps. Given a shift map h on S, we define a new shift map h' on S by removing some copies of Π from the domain of the shift. The groups $\langle h \rangle$ and $\langle h' \rangle$ are isomorphic subgroups of $\mathrm{Map}(S)$. If they were conjugate, not only would $\mathrm{supp}(h)$ and $\mathrm{supp}(h')$ be homeomorphic, but their complements $S \setminus \mathrm{supp}(h)$ and $S \setminus \mathrm{supp}(h')$ would also be homeomorphic. There are many surfaces for which this latter condition fails. For example, let Π

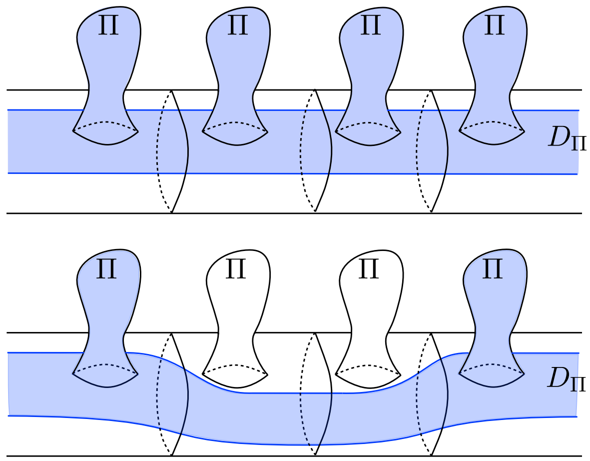

FIGURE 6. Two different embeddings of D_Π. The figure at the top corresponds to a shift h and the embedding shown at the bottom corresponds to a new and distinct shift h' obtained by leaving two copies of Π out of the domain of h.

be a handle, i.e. a torus with one boundary component, and let $S = S_\Gamma(\Pi)$ for $\Gamma(\mathbb{Z}, \{1\})$, i.e. the ladder surface (see Figure 6). In this case, $h = x_1$ is a shift map and $S \setminus \operatorname{supp}(h)$ has genus zero, while $S \setminus \operatorname{supp}(h')$ has nonzero genus coming from the copies of Π that were removed from the domain of h. Therefore, the embeddings of $\mathbb{Z}$ as $\langle h \rangle$ and $\langle h' \rangle$ are non-conjugate in $\operatorname{Map}(S)$. This example can be generalized by letting Π be any surface with a countable end space and one boundary component, so long as removing copies of Π from the domain of the shift map produces non-homeomorphic subsurfaces $S \setminus \operatorname{supp}(h)$ and $S \setminus \operatorname{supp}(h')$. This motivates the following definition.

Definition 3.5. A *distinguished surface* is a surface Π with exactly one boundary component, satisfying at least one of the following:

(1) Π has finite genus,
(2) $E(\Pi)$ consists of finitely many planar ends, or
(3) $E(\Pi)$ consists of finitely many nonplanar ends.

For each distinguished surface Π, let $\mathcal{C}(\Pi)$ be the collection of surfaces S that admit an embedding of D_Π such that the following holds. If Π satisfies (1), then $S \setminus D_\Pi$ has finite (possibly zero) genus. If Π satisfies (2) or (3), then $S \setminus D_\Pi$ has finitely many planar or nonplanar ends, respectively. If Π falls into more than one of the above categories, then $\mathcal{C}(\Pi)$ should consist of surfaces that satisfy either of the conditions on $S \setminus D_\Pi$.

If Π is a distinguished surface and $S \in \mathcal{C}(\Pi)$, then S admits a shift h with domain D_Π. If h' is another shift on S whose domain is embedded by omitting finitely many copies of Π from D_Π, then each of the three conditions on Π ensures that $S \setminus \operatorname{supp}(h)$ and $S \setminus \operatorname{supp}(h')$ are not homeomorphic. In particular, $S \setminus \operatorname{supp}(h)$

and $S \setminus \mathrm{supp}(h')$ will have different genus or will contain a different number of planar or nonplanar ends. Similarly, if h' and h'' are obtained from h by omitting different (finite) numbers of copies of Π from D_Π, then the complements of their supports are not homeomorphic.

The collection $\mathcal{C}(\Pi)$ for a distinguished surface Π is uncountable. To see this, suppose Π has finite genus or $E(\Pi)$ consists of finitely many nonplanar ends. Then S can be any surface such that $S \setminus D_\Pi$ has only planar ends. On the other hand, if $E(\Pi)$ consists of finitely many planar ends, then S can be any surface so that $S \setminus D_\Pi$ has no planar ends. In either case, there are uncountably many such S.

The definition of a distinguished surface Π and the collection of surfaces $\mathcal{C}(\Pi)$ ensure that we can "count" the number of copies of Π that have been removed from the domain of a shift, thus producing non-conjugate embeddings. We could expand the definition of a distinguished surface and the collection $\mathcal{C}(\Pi)$ to encompass a larger family of surfaces for which this is possible, but we choose the streamlined definition above for simplicity, while still demonstrating that our results hold for a broad class of surfaces.

We have shown that there are countably many non-conjugate infinite cyclic subgroups in $\mathrm{Map}(S)$. In Section 5, we will use these different embeddings of $\mathbb{Z}$ to construct non-conjugate embeddings of indicable subgroups into $\mathrm{Map}(S)$ (Theorem 1.1). The following lemma summarizes the discussion above.

Lemma 3.6. *Let S be any surface in the uncountable collection $\mathcal{C}(\Pi)$ for a distinguished surface Π. There exist countably many non-conjugate embeddings of the subgroup generated by the shift map on S with domain D_Π into $\mathrm{Map}(S)$.*

We now turn our attention to constructing non-conjugate embeddings of subgroups generated by multipushes. Let $S = S_\Gamma(\Pi)$ be infinite-type, and let x_s be the multipush defined by $s \in T$. In the same way as for a shift map, by omitting copies of Π from the domain of x_s so that the complements of the supports are not homeomorphic, we obtain a non-conjugate embedding of $\langle x_s \rangle$ in $\mathrm{Map}(S)$.

If several multipushes x_s for $s \in T$ have common copies of Π in their supports, such as in Figure 14, more care needs to be taken. It is possible to remove copies of Π from the domains of all the multipushes to obtain new multipushes x'_s in such a way that $\langle x_s \mid s \in T \rangle \cong \langle x'_s \mid s \in T \rangle$ and so that the complements of the supports of the subgroups are not homeomorphic. One way to formalize this is to consider the surface $S_m = S \mathop{\#}_{v \in V(\Gamma)} \Omega_v$ where exactly m of the Ω_v are homeomorphic to Π with the boundary component capped off, and the remainder of the Ω_v are spheres. By the classification of surfaces, the surfaces S and S_m are homeomorphic, and this homeomorphism induces an isomorphism of mapping class groups $\mathrm{Map}(S) \cong \mathrm{Map}(S_m)$. Let $x_s^{(m)}$ be the multipush on the surface S_m defined by $s \in T$. Notice that $G = \langle x_s \mid s \in T \rangle$ is isomorphic to $\langle x_s^{(m)} \mid s \in T \rangle$ because they are generated by multipushes with the same supports π_1-embedded into different surfaces. Let $G_m \leq \mathrm{Map}(S)$ be the image of $\langle x_s^{(m)} \mid s \in T \rangle \leq \mathrm{Map}(S_m)$ under the isomorphism of mapping class groups, so that $G \cong G_m$. By construction, there are m copies of

Π that are not in the support of G_m, while all copies of Π are in the support of G, and so G and G_m are not conjugate. Similarly, whenever $m \neq n$, the groups G_m and G_n are isomorphic and non-conjugate.

For the remainder of the paper, when we say that we remove copies of Π from the supports of multipushes, we will mean that we do so in the above manner, so that the resulting groups are isomorphic.

If the end space of Γ contains a Cantor set, then there are uncountably many non-conjugate copies of G in $\mathrm{Map}(S)$. To see this, use the procedure above to edit the domains of the multipush maps by removing a collection of copies of Π that accumulate onto a closed subset of the Cantor set of ends of $S = S_\Gamma(\Pi)$. By removing copies of Π that accumulate onto non-homeomorphic closed subsets of the Cantor set, we obtain a non-conjugate embedding of G into $\mathrm{Map}(S)$.

Above, we assumed that $S = S_\Gamma(\Pi)$. However, the argument applies more broadly. For example, if $S = S_\Gamma(\Pi) \mathop{\#}_{v \in V(\Gamma)} \Omega_v$ and each Ω_v has only planar ends and Π has nonzero finite genus, then adding two finite collections of handles of differing sizes to some Ω_v still results in the complements of the domains being non-homeomorphic subspaces. More generally, we could let Π be any surface with a countable end space and one boundary component (of which there are uncountably many), so long as removing two finite collections of Π of differing cardinalities still results in the complement subsurfaces being non-homeomorphic. This observation leads to the definition of the following family of surfaces.

Definition 3.7. Let $\mathcal{B}$ be the collection of Schreier surfaces $S = S_\Gamma(\Pi) \mathop{\#}_{v \in V(\Gamma)} \Omega_v$ such that $S_\Gamma(\Pi)$ is infinite-type, Π has a countable end space, and the surfaces Ω_v are compatible with Π in following sense: Let $Y = \bigcup_{s \in T} \mathrm{supp}(x_s)$, and $Y' = \bigcup_{s \in T} \mathrm{supp}(x'_s)$, $Y'' = \bigcup_{s \in T} \mathrm{supp}(x''_s)$, where x'_s is obtained by moving m copies of Π out of the domain of x_s and x''_s is obtained by moving n copies of Π out of the domain of x_s, with $m \neq n$. Then $S \setminus Y'$ and $S \setminus Y''$ are non-homeomorphic in S.

Let $\mathcal{B}_\infty$ be the subset of $\mathcal{B}$ consisting of those Schreier surfaces built from infinite-type surfaces Π.

The above discussion demonstrates that $\mathcal{B}$ is uncountable and proves the following lemma.

Lemma 3.8. *Let S be any surface in the uncountable collection $\mathcal{B}$. Letting G be the subgroup of $\mathrm{Map}(S)$ generated by the multipush maps x_s on S for $s \in T$, there exist countably many non-conjugate copies of G in $\mathrm{Map}(S)$. If the end space of Γ contains a Cantor set, then there are uncountably many non-conjugate copies of G in $\mathrm{Map}(S)$.*

3.3. Non-isometric embeddings. Throughout the paper, all constructions of subgroups will utilize push and multipush maps. If the complement of the domain of a (multi)push is not simply connected, then the map cannot act as an isometry for any hyperbolic metric on S. We can use the collection of subsurfaces $\{\Omega_v\}$ from the construction of a Schreier surface S to ensure this condition holds, and so all of our constructions can produce subgroups that are not contained in the isometry group of S for any hyperbolic metric on S.

For many surfaces, this is not simply an artifact of our particular construction. By choosing the collection $\{\Omega_v\}$ carefully, we can often ensure that the resulting surface S has a non-displaceable subsurface, and hence its isometry group (with respect to any hyperbolic metric) contains only finite groups [APV21, Lemma 4.2]. In particular, the groups we construct could not arise from a construction using isometries for any such surface.

4. Free groups, wreath products, and Baumslag-Solitar groups

In this section, we use shift maps and multipushes to construct free groups, certain wreath products, and solvable Baumslag-Solitar groups as subgroups of big mapping class groups.

4.1. Free groups. The construction of Schreier surfaces from Section 3.1 was motivated by the following construction of a free subgroup of intrinsically infinite type.

Example 4.1. Let Γ be the Cayley graph of the free group $\mathbb{F}_2 = \langle a, b\rangle$, which is the Schreier graph $\Gamma(\mathbb{F}_2, \{a, b\}, \{\mathrm{id}\})$, and build the Schreier surface $S = S_\Gamma(\Pi)$ with Π a torus with one boundary component. See Figure 4. This Schreier surface is homeomorphic to the blooming Cantor tree, that is, the surface with no boundary components, no planar ends, and a Cantor set of nonplanar ends. The multipushes x_a and x_b generate a copy of $\mathbb{F}_2$ in $\mathrm{PMap}(S)$. To see this, observe that for any $g \in \langle a, b\rangle$, the multipush x_a maps Π_g to Π_{ga}, and similarly for x_b. Thus, the only way for a word $w \in \langle x_a, x_b\rangle$ to act trivially on the surface is if the corresponding word in $\langle a, b\rangle$ is trivial. Moreover, Remark 2.6 shows that this copy of $\mathbb{F}_2$ in $\mathrm{PMap}(S)$ is not contained in $\overline{\mathrm{Map}_c(S)}$.

In this example, it is straightforward to prove that a non-trivial word $w \in \langle x_a, x_b\rangle$ acts non-trivially on the surface because $\mathbb{F}_2$ has no relations and Γ is a tree, so we only need to track where w sends Π_{id}. With a more nuanced analysis of the action of w, however, we can show that multipushes generate a free group in a much more general setting. Recall that the collection $\mathcal{B}$ of Schreier surfaces was defined in Definition 3.7.

Theorem 4.2. *Let Γ be a Schreier graph for a triple (G, T, H) and S any associated Schreier surface. The set $\{x_\alpha \mid \alpha \in T\}$ generates a free group of rank $|T|$ in $\mathrm{Map}(S)$. If $|T| = 1$ and Γ is finite, then we require that at least one Ω_v is not a sphere.*

Moreover, when $S \in \mathcal{B}$, there exist countably many non-conjugate embeddings of such a free group in $\mathrm{Map}(S)$, none of which can lie entirely in the isometry group for any hyperbolic metric on S. If S is not finite type and not the Loch Ness monster surface, these free groups cannot be completely contained in $\overline{\mathrm{Map}_c(S)}$.

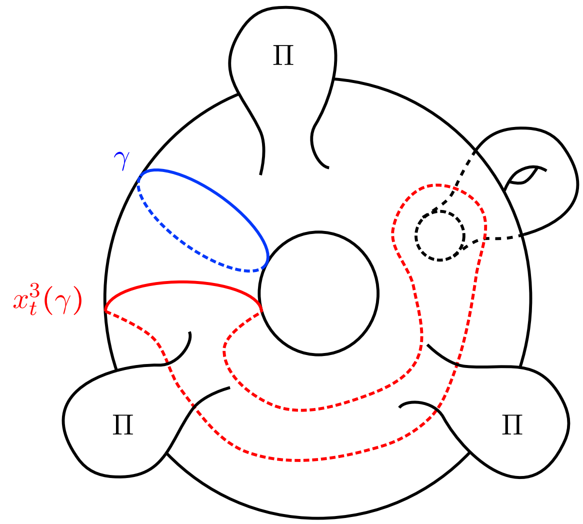

FIGURE 7. A surface S built from the Cayley graph of $\mathbb{Z}/3\mathbb{Z} = \langle t \rangle$. The curve γ is not homotopic to its image under x_t^3 due to the handle on the back of S.

Proof. Let $w = t_1 \ldots t_k$ be a nontrivial, freely reduced word in the free group generated by the set T, and let $x_w := x_{t_k} \cdots x_{t_1}$ be the product of multipushes. We aim to show that x_w is nontrivial in $\mathrm{Map}(S)$. We first observe that if $x_w(\Pi_{Hg}) = \Pi_{Hgw} \neq \Pi_{Hg}$ for any coset Hg, then x_w is nontrivial in $\mathrm{Map}(S)$. We may therefore assume that x_w returns each Π_{Hg} to itself. In particular, this implies that $Hgw = Hg$ for all $g \in G$, and so the edge path given by labels $(t_1, \ldots, t_k)$ in $\Gamma(G, T, H)$ based at any vertex describes a cycle.

First consider a one-generated group G. If Γ is infinite, then we must have $H = \{\mathrm{id}\}$, in which case $\Gamma(G, T, H) = \Gamma(\mathbb{Z}, \{1\}, \{\mathrm{id}\})$ is the Cayley graph of $\mathbb{Z}$ with its standard generator. Since this graph has no cycles, each element x_w with $w \in G$ is non-trivial in $\mathrm{Map}(S)$. On the other hand, suppose Γ is a finite cycle of order k, and consider the multipush x_t, where t is the generator of G. Then, x_t^k represents a cycle in Γ, but the requirement that some Ω_i is not a sphere guarantees that the curve γ and $x_t^k(\gamma)$ cobound a surface with non-trivial topology. See Figure 7 for the case $k = 3$. Thus γ and $x_t^k(\gamma)$ are not homotopic, so x_t^k is non-trivial and $\langle x_t \rangle \cong \mathbb{Z}$.

Now assume $|T| = n \geq 2$, so that every vertex of $\Gamma(G, T, H)$ has degree $2n \geq 4$. Let $p: \tilde{\Gamma} \to \Gamma$ be the universal cover of the labelled graph Γ, which is a tree of valency $2n$ with edge labels in the set T. Construct the Schreier surface $\tilde{S} = S_{\tilde{\Gamma}}(\Pi) \mathop{\#}_{\tilde{v} \in V(\tilde{\Gamma})} \Omega_{\tilde{v}}$, where $\Omega_{\tilde{v}} = \Omega_v$ whenever $p(\tilde{v}) = v$. By construction, $\tilde{S}$ is a cover of $S = S_\Gamma(\Pi) \mathop{\#}_{v \in V(\Gamma)} \Omega_v$. See Figure 8 for an example.

For each $t \in T$, let $\tilde{x}_t$ be the multipush on $\tilde{S}$ obtained by identifying $\tilde{\Gamma}$ with the Cayley graph of the free group with basis T. The covering map $P: \tilde{S} \to S$ induces a homomorphism from the group generated by the multipushes on $\tilde{S}$ to the group generated by the multipushes on S by mapping $\tilde{x}_t \mapsto x_t$. Recall that, by assumption, $w = t_1 \dots t_k$ is a non-trivial reduced word in the free generating set T and $x_w = x_{t_k} \dots x_{t_1}$. Let $\tilde{x}_w = \tilde{x}_{t_k} \dots \tilde{x}_{t_1}$.

Suppose towards a contradiction that x_w is trivial in $\mathrm{Map}(S)$. Then the following commutative diagram of homeomorphisms shows that $\tilde{x}_w$ is a deck transformation.

$$\begin{array}{ccc} S_{\tilde{\Gamma}}(\Pi) & \xrightarrow{\tilde{x}_w} & S_{\tilde{\Gamma}}(\Pi) \\ P\big\downarrow & & \big\downarrow P \\ S_{\Gamma}(\Pi) & \xrightarrow{x_w = \mathrm{id}} & S_{\Gamma}(\Pi) \end{array}$$

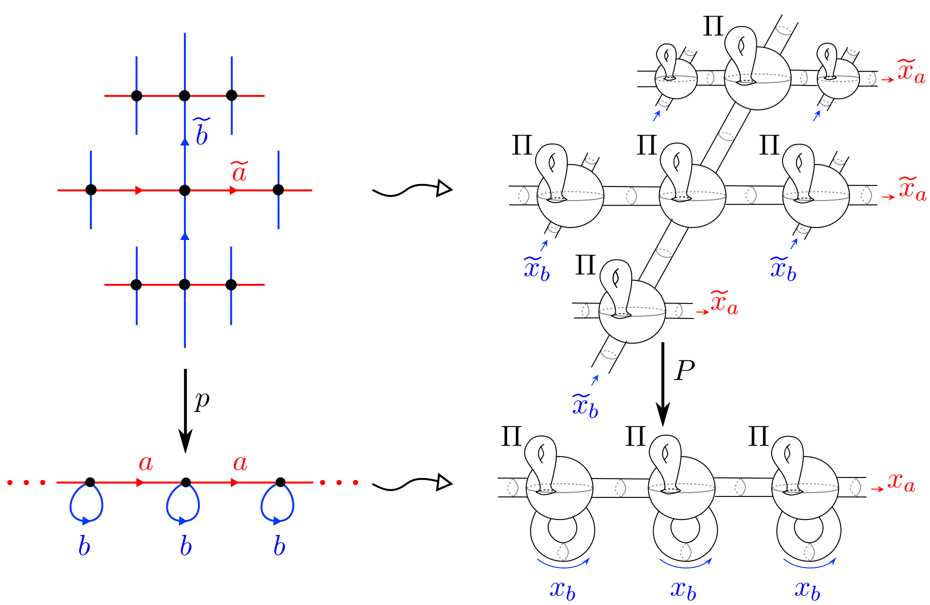

FIGURE 8. An example of the surface $\tilde{S} = S_{\tilde{\Gamma}}(\Pi)$ and lifts of multipushes x_a, x_b.

On the other hand, since $\tilde{x}_w$ is a multipush, it moves every vertex surface of $\tilde{S}$ at most k steps away from itself, a bounded distance. We claim this is a contradiction. Indeed, as the covering map sends vertex surfaces to vertex surfaces and edge surfaces to edge surfaces, respecting the edge labels in T, we see that any deck transformation of $P: \tilde{S} \to S$ is determined by a deck transformation of the covering $p: \tilde{\Gamma} \to \Gamma$. One readily checks that for any such nontrivial deck transformation and for all $j \geq 1$, there exists a vertex v in the tree $\tilde{\Gamma}$ such that the distance from v to its image is larger than j, and we have obtained our contradiction.

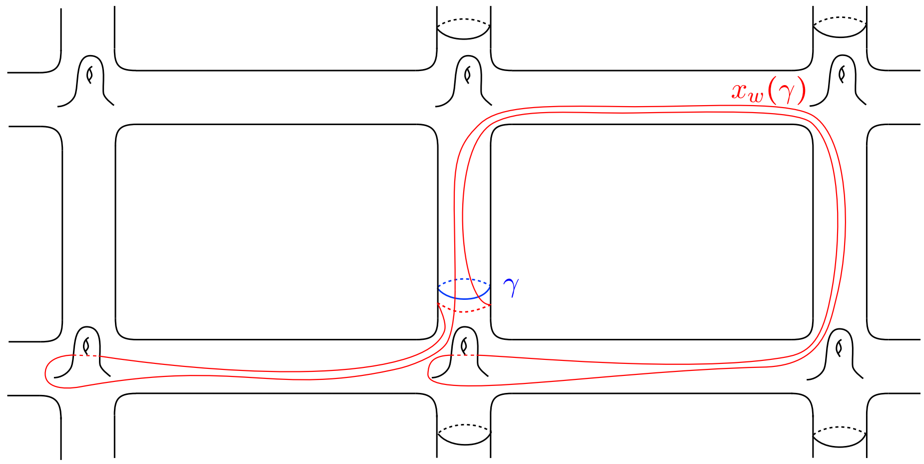

FIGURE 9. A portion of the Schreier surface for $(\mathbb{Z}^2, \{a, b\}, \{1\})$ and the image of the curve γ under the element $x_{bab^{-1}a^{-1}}$.

When $S \in \mathcal{B}$, it follows from Lemma 3.8 that there are countably many non-conjugate embeddings of the free group $\mathbb{F}_{|T|}$ in $\mathrm{Map}(S)$. By the argument in Section 3.3, none of these embeddings lie in the isometry group for any hyperbolic metric on S. Finally, when S is not finite-type or the Loch Ness Monster (in which case $\overline{\mathrm{Map}_c(S)} = \mathrm{Map}(S)$), each multipush in the argument above is a collection of shift maps, so Remark 2.6 completes the proof. □

It follows from the proof of Theorem 4.2 that the support of every non-trivial element of $\mathbb{F}_{|T|}$ is not contained in the union of the vertex surfaces. This is clear if w does not fix every Π_{Hg}, because the shift domains are contained in the support of x_w. On the other hand, suppose w fixes each Π_{Hg}. Since x_w is a collection of pushes, it therefore restricts to the identity on each vertex surface. However, the proof of the theorem shows that x_w is a non-trivial homeomorphism, and so the support of x_w cannot be contained in the union of the vertex surfaces. See Figure 9 for an example of what the image of a loop γ might look like after the application of x_w when w is trivial in G. This will be a crucial ingredient in the proof of Theorem 6.3.

4.2. Shift maps that do not generate a free group. The construction above uses a countable collection of intersecting push maps to ensure the resulting group is free. The following example demonstrates why this is necessary by showing that the group generated by two shift maps with minimal intersection is not free. We use the convention that $[x, y] = xyx^{-1}y^{-1}$ and choose a right action.

Let Γ be the four-ended tree with a single vertex of valence four and all other vertices of valence two. Identify Γ with the coordinate axes in $\mathbb{R}^2$ to get a labeling of the vertices as integer coordinates. Let Π be any surface with one boundary component that is not a disk, and construct the surface $S = S_\Gamma(\Pi)$. There is

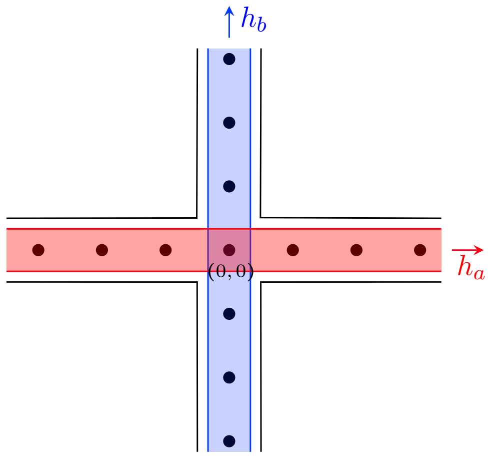

FIGURE 10. The shifts h_a and h_b do not generate a free group.

a horizontal shift h_a corresponding to the $+(1,0)$ map on the x–axis and a vertical shift h_b corresponding to the $+(0,1)$ map on the y–axis, as shown in Figure 10. The intersection of the supports of these shifts is contained in the front of $V_{(0,0)}$, the vertex surface at $(0,0)$. It can be checked that the support of $[h_a, h_b]$ is contained in the fronts of $V_{(-1,0)}$, $V_{(0,0)}$, and $V_{(0,-1)}$ and the adjoining edge surfaces. The word $w = h_a h_b h_a^2$ maps $\{\Pi_{(0,-1)}, \Pi_{(-1,0)}, \Pi_{(0,0)}\}$ to the collection $\{\Pi_{(1,0)}, \Pi_{(2,0)}, \Pi_{(3,0)}\}$. Thus, the elements $[h_a, h_b]$ and $w[h_a, h_b]w^{-1}$ have disjoint supports and so commute. More generally, the words $w_n = h_a^{3n+1} h_b h_a^2$ map $\{\Pi_{(0,-1)}, \Pi_{(-1,0)}, \Pi_{(0,0)}\}$ to $\{\Pi_{(1+3n,0)}, \Pi_{(2+3n,0)}, \Pi_{(3+3n,0)}\}$. From this, we see that $H := \langle h_a, h_b \rangle$ is not a free group and actually contains copies of $\mathbb{Z}^n$ for all n.

In fact, H is isomorphic to a 2–generated subgroup of an infinite strand braid group. To see this, note that the group structure of H is not dependent on the surface Π that we attach, so we may assume Π is a punctured disk. We can also realize each shift domain as a disk with countably many punctures with two distinct accumulation points on the boundary. Because braid groups are mapping class groups of punctured disks, this viewpoint allows us to realize H as a subgroup of the infinite strand braid group in which braids are allowed to have non-compact support. In particular, H is isomorphic to the subgroup of this braid group generated by the elements h_a and h_b, viewed as braids with non-compact support.

4.3. Wreath products. Recall that if H acts on a set Λ, then the (restricted) wreath product $G \wr_\Lambda H$ is defined as

$$G \wr_\Lambda H = G^\Lambda \rtimes_\gamma H,$$

that is, the semidirect product of H with the direct sum of copies of G indexed by Λ. Here, $G^\Lambda = \oplus_\Lambda G$ and is the set of $(g_\lambda)_{\lambda \in \Lambda}$. The automorphism $\gamma\colon H \to \mathrm{Aut}(G^\Lambda)$ is defined by $\gamma(h)(G_\lambda) = hG_\lambda h^{-1} = G_{h\lambda}$, so that H acts on G^Λ by permuting the coordinates according to the action on the indices. When it is clear from context, or when $\Lambda = H$, we may simply write $G \wr H$.

We now construct a collection of wreath products in big mapping class groups. The most straightforward example of this construction is when S is a surface which admits a shift whose domain is an embedded copy of D_Π for some surface Π with one boundary component. For any $G \leq \mathrm{Map}(\Pi)$, we generalize a construction of Lanier and Loving [LL20] to construct $G \wr \mathbb{Z}$ as a subgroup of $\mathrm{Map}(S)$. When G is chosen to be the infinite cyclic group generated by a single Dehn twist, we recover [LL20, Theorem 4].

Proposition 4.3. *Let $G \leq \mathrm{Map}(\Pi)$, where Π is a surface with a single boundary component. Let S be a surface and $H \leq \mathrm{Map}(S)$ be generated by a finite collection of pushes and multipushes, all of whose domains are (unions of) embedded copies of A_Π or D_Π. Index the copies of Π in these domains by Λ. The wreath product $G \wr_\Lambda H$ is a subgroup of $\mathrm{Map}(S)$.*

Proof. Let $h_1, \dots, h_n$ be the generators of H, so Λ is a set indexing the copies of Π contained in the union of the domains of the h_i. Each h_i permutes the copies of Π in its domain and so acts on Λ: if $\lambda \in \Lambda$, then $h_i(\lambda)$ is defined to be the index of $h_i(\Pi_\lambda)$. This induces an action of H on Λ.

Let $G \leq \mathrm{Map}(\Pi)$, and let $G_\lambda \cong G$ be the corresponding subgroup of $\mathrm{Map}(S)$ supported on Π_λ. Whenever $\lambda \neq \lambda'$, the subgroups G_λ and $G_{\lambda'}$ have disjoint supports and commute, so $\langle G_\lambda \mid \lambda \in \Lambda\rangle = G^\Lambda$. For any $h \in H$ and $\lambda \in \Lambda$, we have $hG_\lambda h^{-1} = G_{h(\lambda)}$ and $H \cap G_\lambda = \{1\}$. Therefore, the subgroup of $\mathrm{Map}(S)$ generated by $\langle H, G_\lambda \mid \lambda \in \Lambda\rangle$ is isomorphic to $G \wr_\Lambda H$. □

We illustrate this proposition with several examples.

Example 4.4. Proposition 4.3 applies whenever S and H are one of the following.

(1) Let S be a surface with an embedded copy of D_Π, and let H be generated by a (possibly one-ended) shift h, so that $H \cong \mathbb{Z}$. The index set Λ is simply $\mathbb{Z}$, and h acts on Λ as addition by 1.

(2) Let S be a Schreier surface for a triple (A, T, B) such that $t_1, \dots, t_n \in T$ correspond to biinfinite geodesics in $\Gamma(A, T, B)$. Let H be the subgroup of $\mathrm{Map}(S)$ generated by the multipushes $x_{t_1}, \dots, x_{t_n}$. By Theorem 4.2, $H \cong \mathbb{F}_n$. In this case, the index set Λ is the collection of right cosets $\{Ba \mid a \in A\}$. Each generator x_{t_i} acts on Λ as follows: if $Ba \in \Lambda$, then $x_{t_i} \cdot Ba = Bat_i$.

(3) Let $S = S_\Gamma(\Pi)$ be the surface described in Section 4.2, and let $H = \langle h_a, h_b\rangle$ be the subgroup of $\mathrm{Map}(S)$ constructed in that section. In this case, H is not free. The index set Λ is the set $\{(0, n), (n, 0) \mid n \in \mathbb{Z}\}$, and the generators h_a and h_b act on Λ as addition by $(1, 0)$ and $(0, 1)$, respectively.

When $S \in \mathcal{B}$, it follows from Lemma 3.8 that there are countably many non-conjugate embeddings of $G \wr H$ in $\mathrm{Map}(S)$ for G, H as in the statements of Proposition 4.3. Moreover, none of these embeddings lie in the isometry group for any hyperbolic metric on S by the discussion in Section 3.3 or in $\overline{\mathrm{Map}_c(S)}$ by Remark 2.6.

4.4. Solvable Baumslag-Solitar groups. For our third and final construction in this section, we focus on solvable Baumslag-Solitar groups. Fixing a positive integer n, recall that the Baumslag-Solitar group $BS(1,n)$ is the group with presentation

$$BS(1,n) = \langle a, t \mid tat^{-1}a^{-n} \rangle.$$

Given $n \in \mathbb{N}$, we first construct a shift map with n^kth roots for all $k \in \mathbb{N}$ on a certain class of surfaces. To do this, we borrow ideas from asymptotically rigid mapping class groups [ArF21]. We say that a mapping class group element is *rigid* with respect to some pants decomposition if the pants curves, seams, and fronts (as defined in Definition 3.1) are preserved setwise by the element.

Lemma 4.5. *Let S be a surface whose end space contains a clopen subset homeomorphic to a Cantor set of planar ends. For any $n \in \mathbb{N}$, there is a shift map in $\mathrm{Map}(S)$ with n^kth roots for all $k \in \mathbb{N}$.*

Proof. If the end space of S has a clopen subset homeomorphic to a Cantor set of planar ends, then there exists a subsurface $S' \subset S$ which is homeomorphic to a Cantor tree surface with a boundary component (see [AbMP25, Lemma 2.7] for a proof). That is, S' has exactly one compact boundary component, genus zero, and $E(S')$ is a Cantor set. We will construct an explicit realization of S' and a shift map that has roots. Then, because homeomorphisms induce isomorphisms of the respective mapping class groups, we see that any surface with a subsurface homeomorphic to S' also has a shift map with the desired property, proving the theorem in its full generality.

We start with a pair of pants with countably many cuffs, called B, which is constructed from countably many copies of a standard pair of pants (3-hold spheres) in the following way. Let P be a standard pair of pants with boundary components labeled by α, β_r, β_ℓ and equipped with seams so that P has a back side and a front side. Then β_r and β_ℓ are the right and left cuff of P, respectively. The surface B is constructed from countably many copies of P, labeled by $\{P_i\}_{i\in\mathbb{Z}}$, with the right cuff of P_i glued to the left cuff of P_{i+1} for all $i \in \mathbb{Z}$, such that the seams and fronts of the P_i are compatible. Here compatibility is the property described after Definition 3.1, i.e., the union of the seams separates B into two disjoint connected components, the front and the back, containing the front and, respectively, the back of each P_i. The boundary component of B corresponding to the cuff α of P_i will be labeled by α_i.

Next, let T be a Cantor tree surface with boundary that is built from $(n+1)$-holed spheres, called pants, equipped with compatible seams and fronts. Glue a copy of T, called T_0, to α_0 in B so that the seams and front line up with those of B. We will use finite strings in $\{1, \dots, n\}$ to label the curves in the pants decomposition of T_0 as follows. Start with the pants containing the boundary component of T_0 and, moving clockwise along the boundary of the polygon that forms the front of the pants, label the other boundary curves $\alpha_{0,1}, \dots \alpha_{0,n}$. Continue on adjacent pants, using the cyclic order to append the corresponding digit in $\{1, \dots, n\}$. See Figure 11 below for the case $n = 3$. Finally, glue a copy of T, called T_i, to each other curve α_i of B, so that the seams and fronts line up. Label the pants curves of T_i by $\alpha_{i,x}$ with $x \in \{1,\ldots,n\}^{<\mathbb{N}}$ analogous to the labeling of pants curves of T_0.

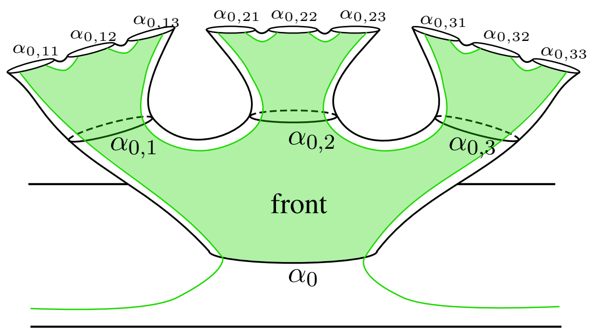

FIGURE 11. The labeling of curves in the pants decomposition for $n = 3$ on T_0.

The resulting surface R is homeomorphic to a Cantor tree surface since it has no boundary, genus zero, and its endspace is homeomorphic to a Cantor set of planar ends.

There is a rigid shift $\phi \in \mathrm{Map}(R)$ with shift domain a properly embedded copy of D_T, which shifts T_i to T_{i+1}, and which respects the seams and front of R. By construction, ϕ has an nth root which shifts $\alpha_{i,j}$ to $\alpha_{i,j+1}$, when $1 \le j < n$ and $\alpha_{i,n}$ to $\alpha_{i+1,1}$. See Figure 12 for an example in the case where $n = 3$.

Similarly, the n^kth root shifts the curves at height k according to the lexicographic ordering on $\mathbb{N}\times\{1,\ldots,n\}^k$. Note that ϕ being rigid is part of what ensures these roots of ϕ exist. Finally, note that the domain of ϕ is not all of R so that we may remove an open disk, say on the back of R, to obtain a surface homeomorphic to S'. The corresponding mapping class of S', which we also call ϕ by an abuse of notation, is a shift map with n^kth roots for all $k \in \mathbb{N}$. □

Theorem 4.6. *Let S be a surface whose end space contains a clopen subset homeomorphic to a Cantor set of planar ends. Then $BS(1,n) \le \mathrm{Map}(S)$ for all $n > 0$.*

Proof. Let $S' \subset S$ be homeomorphic to the Cantor tree surface with a boundary component. Let $h \in \mathrm{Map}(S')$ be a rigid shift map with domain D_T, where T is also homeomorphic to a Cantor tree surface with a boundary component. Index the copies of T in D_T by $\mathbb{Z}$ according to the shift order.

Now, since T_0 is homeomorphic to a Cantor tree surface with one boundary component, we may choose a rigid shift ϕ on T_0 with n^kth roots by Lemma 4.5. For $k \ge 0$ let $\phi_k = \sqrt[n^k]{\phi}$, and for $k < 0$ let $\phi_k = \phi^{n^{-k}}$. Note that for all $k \in \mathbb{Z}$, we have $\phi_k^n = \phi_{k-1}$.

Next, let $\tilde{\phi_k} = h^k \phi_k h^{-k}$. Heuristically, $\tilde{\phi_k}$ performs ϕ_k on T_k instead of on T_0. Because the domains of $\tilde{\phi_k}$ are disjoint, the product

$$\Phi = \prod_{k\in Z} \tilde{\phi_k}$$

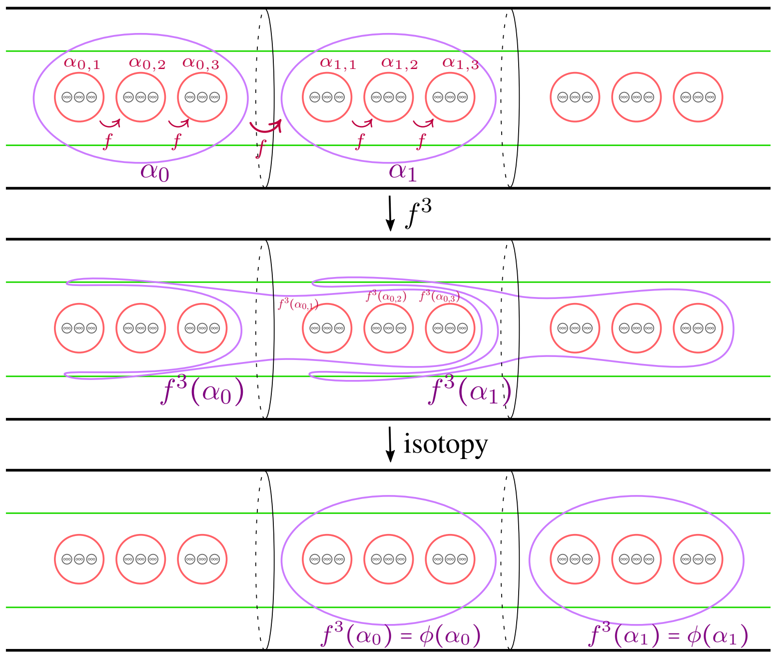

FIGURE 12. A flattened picture of the surface R for $n = 3$. Let f be the map that shifts $\alpha_{i,j}$ to $\alpha_{i,j+1}$, when $1 \leq j < 3$ and $\alpha_{i,3}$ to $\alpha_{i+1,1}$. The domain of f is shown in green. We see that f^3 is indeed equal to the rigid shift ϕ in $\mathrm{Map}(R)$ that takes the T_i to T_{i+1}.

is a well-defined element of $\mathrm{Map}(S)$. This map simultaneously performs ϕ_k on T_k for all $k \in \mathbb{Z}$. See Figure 13 for an example of the action of Φ on S when $n = 2$.

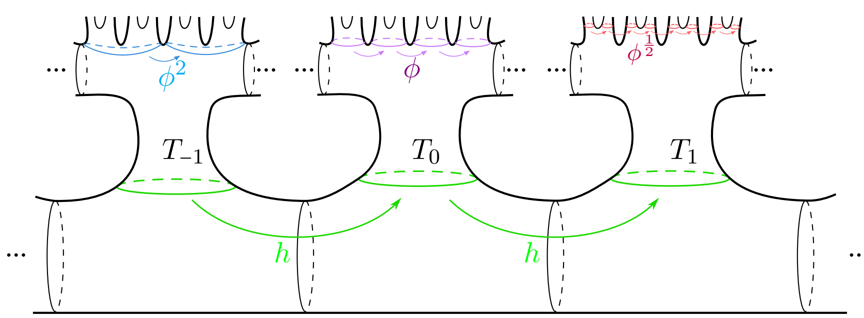

FIGURE 13. This figure shows the action of the shift h on S as well as the action of Φ on the subsurfaces T_{-1}, T_0, T_1 when $n = 2$.

Let $f\colon BS(1,n) \to \mathrm{Map}(S)$ be the map defined by $f(a) = \Phi$ and $f(t) = h$. We will show that f is an isomorphism onto its image, i.e., $\mathrm{Map}(S)$ contains a subgroup isomorphic to $BS(1,n)$.

For each $k \in \mathbb{Z}$, the mapping class $f(tat^{-1}) = h\Phi h^{-1}$ first shifts T_k to T_{k-1}, applies Φ, which now acts as $\phi_{k-1} = \phi_k^n$ on T_{k-1}, and then shifts T_{k-1} back to T_k. It follows that

$$f(tat^{-1}) = h\Phi h^{-1} = \Phi^n = f(a^n).$$

Therefore, f is a well-defined homomorphism. This can also be verified algebraically:

$$\begin{aligned}\Phi^n &= \prod_{k \in Z} \tilde{\phi_k}^n = \prod_{k \in Z} (h^k \phi_k h^{-k})^n = \prod_{k \in Z} h^k \phi_k^n h^{-k} = \prod_{k \in Z} h^k \phi_{k-1} h^{-k} \\ &= h \left(\prod_{k \in Z} h^{k-1} \phi_{k-1} h^{-(k-1)} \right) h^{-1} = h\Phi h^{-1}.\end{aligned}$$

Suppose there exists $g \in BS(1,n)$ such that $f(g)$ is the identity of $\mathrm{Map}(S)$. Using the relation in $BS(1,n)$, the element g can be written as $g = t^i a^k t^{-j}$ for some $k \in \mathbb{Z}$ and $i, j \in \mathbb{Z}_{\geq 0}$. Since $f(g) = h^i \Phi^k h^{-j}$ is the identity, it must fix each T_i, and so we must have $i = j$. Consider the surface T_0. Then, $f(g)$ first shifts T_0 to the left j times, applies Φ^k (which acts as ϕ_{-j}^k), and then shifts back to T_0. The result is that $f(g)$ acts by ϕ_{-j}^k on T_0. The only way that $f(g)$ can act as the identity on T_0 is if $k = 0$. Thus, $g = t^j a^0 t^{-j} = 1$, and f is injective, as desired. □

The construction above embeds solvable Baumslag-Solitar groups into mapping class groups of certain *infinite-type* surfaces. This is in contrast to the finite-type case, where $BS(1,n)$ is never a subgroup of the mapping class group due to the Tits alternative: every subgroup of such a mapping class group either contains a free subgroup or is virtually abelian [Iva84, McC85]. Since $BS(1,n)$ is solvable, it does not contain any free subgroups, but it is also not virtually abelian.

We expect that the techniques in Section 3.2 can be used to produce countably many non-conjugate embeddings of $BS(1,n)$ into $\mathrm{Map}(S)$ if the topology of the Cantor trees T_i are suitably edited. We also expect that a construction similar to the one above can be used to embed $BS(m,n)$ into $\mathrm{Map}(S)$ when $m \neq 1$. However, the lack of a normal form for elements in $BS(m,n)$ significantly increases the complexity of the proof.

5. Indicable groups

In this section, we give a general construction for embedding any indicable group which arises as a subgroup of a mapping class group of a surface with one boundary component into a big mapping class group in countably many non-conjugate and intrinsically infinite-type ways. We will need the following lemma in our construction.

Lemma 5.1. *A group G is indicable if and only if there exists a presentation $G = \langle T \mid R \rangle$ such that for each $r \in R$, the total exponent sum of r with respect to the generators T is zero.*

Before presenting the proof of the lemma, we give an example that motivates the argument. Consider the Baumslag-Solitar group $BS(1,n)$ with its standard presentation $BS(1,n) = \langle a,t \mid tat^{-1}a^{-n}\rangle$. This presentation does not have the desired property since the total exponent sum of the relator in the generators a and t is $1-n$. However, there exists a homomorphism $f\colon BS(1,n) \to \mathbb{Z}$ defined by letting $f(a) = 0$ and $f(t) = 1$, so Lemma 5.1 tells us that there must be a presentation of $BS(1,n)$ with the desired property. If we augment the generator a to be at instead, then

$$BS(1,n) = \left\langle at, t \,\middle|\, (t\cdot at\cdot t^{-1}\cdot t^{-1})\cdot \underbrace{t(at)^{-1}\cdots t(at)^{-1}}_{n \text{ times}} \right\rangle,$$

and the relator has zero total exponent sum in the generators at and t. In this presentation, the generators of $BS(1,n)$ both map to 1 under the homomorphism f, and we will use this property in the proof of the lemma.

Proof of Lemma 5.1. Given a group $G = \langle T \mid R\rangle$ with all relators having total exponent sum zero, there is a well-defined homomorphism $f\colon G \to \mathbb{Z}$ defined by sending each generator to $1 \in \mathbb{Z}$.

For the other direction, assume there exists a homomorphism $f\colon G \to \mathbb{Z}$, and let $N = \ker(f)$. Let $N = \langle V \mid W\rangle$ be a presentation for N, and let $a \in G$ be such that $f(a) = 1$. Then since $G/N \cong \mathbb{Z}$, the group G is generated by $T' = \{a\} \cup V$. If we augment the generators in $V \subseteq T'$ by a, then $T = \{a\} \cup \{av : v \in V\}$ is also a generating set for G. Importantly, the image of every one of these generators under f is $1 \in \mathbb{Z}$.

Let $G = \langle T \mid R\rangle$ be the presentation of G for the generating set T. If $r \in R$ is a relator, then r is a word in $\langle T\rangle$ that is the identity in G. Thus, $f(r) = 0$, and given that every element of T maps to $1 \in \mathbb{Z}$, the total exponent sum of r with respect to T must be zero. Therefore, $\langle T \mid R\rangle$ is one such desired presentation for G. □

We can now begin our construction. Take any indicable group G that arises as a subgroup of $\mathrm{Map}(\Pi)$, where Π is a surface with exactly one boundary component. Let h be a shift map on an infinite-type surface S whose domain is an embedded copy of D_Π in S. As discussed in Section 3, this includes a wide range of surfaces, including surfaces $S_\Gamma(\Pi)$ built from any graph with countable vertex set that contains a biinfinite path.

The most trivial way to embed G into $\mathrm{Map}(S)$ is to let G act on one copy of Π in S. Indexing the copies of Π in D_Π by $\mathbb{Z}$ and taking any subset of I of $\mathbb{Z}$, G can also act simultaneously on the subsurfaces Π_i of S for $i \in I$. Varying over all subsets of $\mathbb{Z}$ gives an uncountable collection of copies of G in $\mathrm{Map}(S)$. Unlike these embeddings, the construction in the next theorem produces an uncountable collection of copies of G which do not lie in the isometry group of S, even if G lies in the isometry group of Π, and do not lie in $\overline{\mathrm{Map}_c(S)}$, even if Π is compact. See Definition 3.5 for the definition of a distinguished surface and the family $\mathcal{C}(\Pi)$.

Theorem 5.2. *Let* Π *be a distinguished surface and* $G \leq \mathrm{Map}(\Pi)$ *an indicable group. Given a surface* $S \in \mathcal{C}(\Pi)$*, there are countably many non-conjugate embeddings of* G *in* $\mathrm{Map}(S)$ *such that no embedded copy is contained in* $\overline{\mathrm{Map}_c(S)}$ *and no embedded copy is contained in the isometry group for any hyperbolic metric on* S.

Proof. Let $h \in \mathrm{Map}(S)$ be a shift with domain D_Π, and let G be an indicable group. Fix a presentation $\langle T \mid R \rangle$ of G such that each $r \in R$ has total exponent sum zero with respect to T, which exists by Lemma 5.1. Since G is a subgroup of $\mathrm{Map}(\Pi)$, G acts by homeomorphisms on each Π_i in D_Π. For each $g \in G$, let $\bar{g} \in \mathrm{Map}(S)$ be the element that acts as g simultaneously on each Π_i in D_Π. We claim that the group generated by $\overline{T} = \{\bar{t}h : t \in T\}$ in $\mathrm{Map}(S)$ is isomorphic to G. Let $\phi: F_T \to \langle \overline{T} \rangle \leq \mathrm{Map}(S)$ be the surjective map defined by $t \mapsto \bar{t}h$ for all $t \in T$, where F_T is the free group on the generators T. We claim a word is in the kernel of this map if and only if it represents a trivial element in G.

Notice that h and $\bar{t}$ commute as elements of $\mathrm{Map}(S)$, so for any word $w \in F_T$ with total exponent sum $k \in \mathbb{Z}$, the image $\phi(w)$ can be written as $\bar{w}h^k$. Thus, $\phi(w)$ acts trivially on S if and only if $k = 0$ and $\bar{w}$ acts trivially on each copy of Π in S. The only elements w with this property are those that are trivial in G, and elements that are trivial in G have this property since products of conjugates of relators $r \in R$ have total exponent sum zero. Thus, the group G' generated by $\overline{T}$ in $\mathrm{Map}(S)$ is isomorphic to G.

Any element of G' that does not have total exponent sum zero with respect to $\overline{T}$ is not in $\overline{\mathrm{Map}_c(S)}$, since it must shift the surfaces Π_i. Remove finitely many copies of Π from the domain of h to obtain a new shift h'', and construct the group $G'' = \langle \bar{t}h'' \mid t \in T \rangle \leq \mathrm{Map}(S)$. This group G'' is isomorphic to G for the same reason that $G' \cong G$. By the same reasoning as in Section 3.2, the complements of the supports of G' and G'' are non-homeomorphic. In particular, G' and G'' are not conjugate. As in Lemma 3.8, this procedure produces countably many non-conjugate embeddings of G into $\mathrm{Map}(S)$. Finally, no such embedding is contained in $\overline{\mathrm{Map}_c(S)}$ by construction. □

It was suggested to the authors by Mladen Bestvina that one can get around constructing the presentation in Lemma 5.1 for the indicable group G by working instead with the wreath product construction in Proposition 4.3. More specifically, let $f: G \to \mathbb{Z}$ be a surjection from the indicable group to $\mathbb{Z}$. Let Π be a surface with exactly one boundary component such that G arises as a subgroup of $\mathrm{Map}(\Pi)$, and let S be a surface which admits a shift h with domain D_Π. For $g \in G$, let $\bar{g}$ be the element which acts as g on each Π_i. Then, for $g \in G$, define a new map $\psi: G \to G \wr \mathbb{Z} \leq \mathrm{Map}(S)$ via $g \mapsto \bar{g}h^{f(g)}$. One readily checks that this map is an injective homomorphism by observing that the restriction of the image of G to $\bigoplus_{-\infty}^{\infty} G$ is the diagonal subgroup, and so the action of $\mathbb{Z}$ is trivial. The embedding in the proof of Theorem 5.2 is exactly this map.

Theorem 5.2 applies to all subgroups constructed in Section 3. Another interesting class of examples produces embeddings of pure mapping class groups into a full mapping class group that are not induced by embeddings of the underlying surfaces.

The following corollary is immediate from Theorem 5.2 and work of Aramayona, Patel, and Vlamis [ArPV20, Corollary 6], which shows that the *pure* mapping class group of any surface with at least two nonplanar ends is indicable.

Corollary 5.3. *Let Π be an infinite-type surface with at least two nonplanar ends and exactly one boundary component. Given any surface S that admits a shift whose domain is D_Π, there exist uncountably many embeddings of* $\mathrm{PMap}(\Pi)$ *into* $\mathrm{Map}(S)$ *that are not induced by an embedding of* Π *into* S*. In addition, none of these embeddings preserve the notion of being compactly supported. When* Π *is a distinguished surface and* $S \in \mathcal{C}(\Pi)$*, countably many of these embeddings are non-conjugate.*

Corollary 5.3 is in line with a body of work aiming to find interesting homomorphisms between big mapping class groups. It also gives a natural set of examples of uncountable groups G to which one can apply Theorem 5.2. We note that determining which *full* mapping class groups are indicable is an important open question for both finite- and infinite-type surfaces. We now give a few examples of indicable big mapping class groups.

Examples 5.4. Mann and Rafi build continuous homomorphisms from finite-index subgroups of mapping class groups to $\mathbb{Z}^k$ and to $\mathbb{Z}$ in the proofs of [MaR23, Lemma 6.7 & Theorem 1.7], respectively. To find surfaces whose full mapping class groups are indicable, we focus on the cases where the subgroup has index 1, a few of which we list below. We will define the homomorphism to $\mathbb{Z}$ explicitly for example (1); the others are defined similarly.

(1) Let Π be the surface with infinite genus whose end space is homeomorphic to the two-point compactification of $\mathbb{Z}$, that is, $E(\Pi) = \{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$, where $E^g(\Pi) = \{\infty\}$. Let $A \subset E(\Pi)$ be the subset of ends corresponding to $-\mathbb{N}$, and let B be the subset of ends corresponding to $\{0\} \cup \mathbb{N}$. This surface is colloquially called the bi-infinite flute with one end accumulated by genus, and it admits a shift with domain D_Σ for a punctured disk Σ. A homomorphism $\ell\colon \mathrm{Map}(\Pi) \to \mathbb{Z}$ can be defined by

$$\ell(\phi) = |\{x \in E \mid x \in A,\ \phi(x) \in B\}| - |\{x \in E \mid x \in B,\ \phi(x) \in A\}|.$$

The map ℓ counts the difference in the number of punctures mapped from negative to positive and punctures mapped from positive to negative. Note that the shift map mentioned above evaluates to 1 under ℓ, so the map ℓ is surjective.

(2) Let Π be a surface of any genus whose end space consists of a Cantor set and $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$, equipped with the same topology as in part (1), where the end $\{\infty\}$ is identified with a point in the Cantor set. The ends corresponding to $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$ must all be planar or all nonplanar; the other Cantor set of ends can be planar or not. The homomorphism to $\mathbb{Z}$ is defined as above, with sets $A = -\mathbb{N}$ and $B = \{0\} \cup \mathbb{N}$.

(3) Let Π be the surface with infinite genus and end space $\mathbb{N} \cup \{\infty\}$, where only the ends corresponding to 1 and ∞ are nonplanar. This surface can be visualized as the ladder surface with punctures accumulating to one end. Here we can similarly define a homomorphism to $\mathbb{Z}$, which instead counts the number of genus that are moved across a simple closed curve separating the ends in E^G.

The common thread in the examples above is that the two ends of the shift map are of different topological types so that no element of $\mathrm{Map}(\Pi)$ can exchange the two ends. This is the key fact necessary to ensure that the map ℓ above is a well-defined homomorphism of $\mathrm{Map}(\Pi)$ and not of a proper subgroup of $\mathrm{Map}(\Pi)$.

Each of the examples above can be modified to have exactly one boundary component. The third example can be extended to uncountably many more examples by replacing one of the isolated planar ends with a disk punctured by any closed subset of the Cantor set, of which there are uncountably many.

Moreover, in each case, Π is like a distinguished surface in the sense that if $S - D_\Pi$ has finitely many nonplanar ends in Cases (1) and (3), then $\mathrm{Map}(S)$ can be used as the input for Theorem 5.2. In Case (2), if $S - D_\Pi$ has finitely many nonplanar (resp. planar) ends when the ends of $E(\Pi)$ corresponding to $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$ are nonplanar (resp. planar), then $\mathrm{Map}(S)$ can be used as the input for Theorem 5.2. Therefore, we can construct countably many non-conjugate embeddings of $\mathrm{Map}(\Pi)$ into $\mathrm{Map}(S)$ in all such cases.

6. Combination Theorem

In this section, we give a construction that takes as its input a set of indicable subgroups of mapping class groups of surfaces with one boundary component and outputs a new surface whose mapping class group contains a new indicable subgroup, called the $\star$-product, of intrinsically infinite type built from the original subgroups.

Definition 6.1. Given two subgroups H_1 and H_2 of groups G_1 and G_2, respectively, let

$$(G_1, H_1) \star (G_2, H_2) := (G_1 * G_2)/\langle\langle [G_1, H_2], [H_1, G_2] \rangle\rangle.$$

More generally, given $G_1, \dots, G_n$ with subgroups $H_1, \dots, H_n$, let

$$(G_1, H_1) \star \cdots \star (G_n, H_n) := G_1 * \cdots * G_n/\langle\langle [G_i, H_j] : i \neq j \rangle\rangle.$$

These groups are an interpolation between free products (where the H_i are trivial) and direct products (where $H_i = G_i$ for all i). Examples include some right-angled Artin groups and certain graph products. We are interested in the case where G_i are indicable groups and the H_i are the kernels of the surjections to $\mathbb{Z}$.

Lemma 6.2. *Let $G_1, \dots, G_n$ be indicable groups with surjective maps $f_i: G_i \to \mathbb{Z}$, and let $H_i = \ker(f_i)$. Then the group $(G_1, H_1) \star \cdots \star (G_n, H_n)$ is also indicable.*

Proof. Let T_i be a generating set for G_i. Then there is a map $\phi: (G_1, H_1) \star \cdots \star (G_n, H_n) \to G_1$ defined by $\phi(t) = 1$ for each $t \in T_i$ with $i \neq 1$, and $\phi(t') = t'$ for

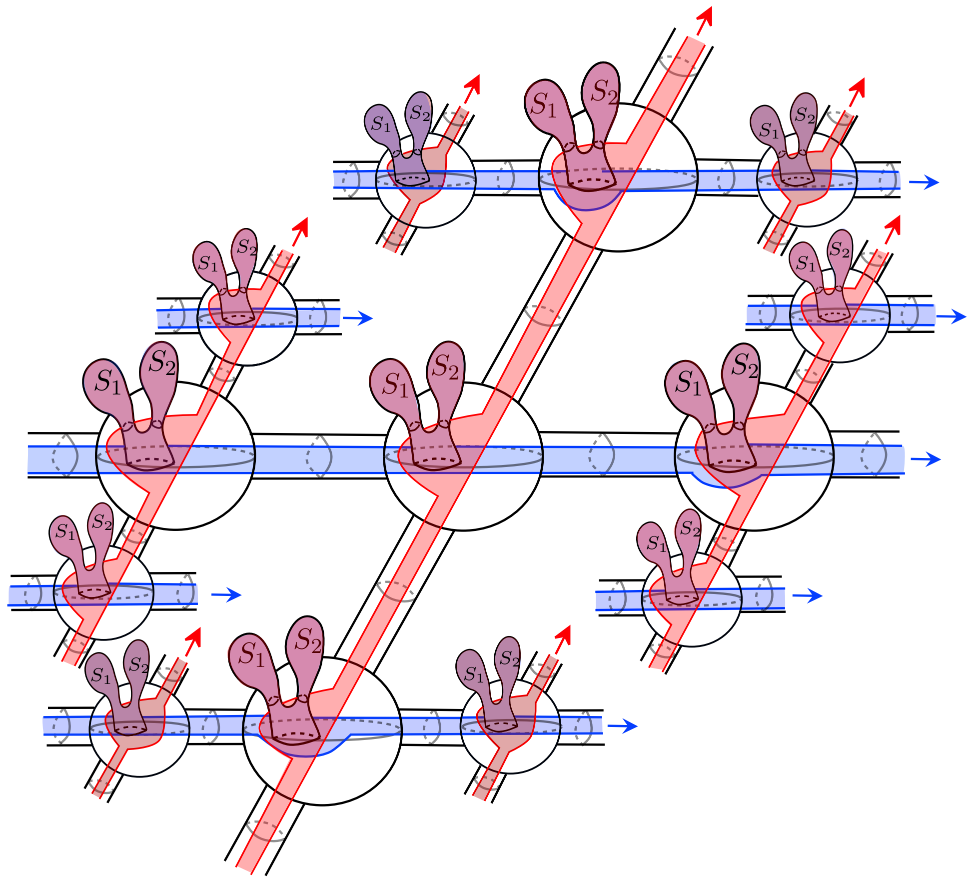

FIGURE 14. The domains of the two multipushes x_a (blue) and x_b (red) in the proof of Theorem 6.3 in the case that Γ is the Cayley graph of the free group generated by a and b.

each $t' \in T_1$. Here 1 is the identity element of G_1. This map ϕ is a homomorphism which restricts to the identity on G_1. By post-composing ϕ with f_1, we obtain the desired map $(G_1, H_1) \star \cdots \star (G_n, H_n) \to \mathbb{Z}$. □

We are now ready to prove our main combination theorem, of which Theorem 1.2 is a special case.

Theorem 6.3. *For $i = 1, \ldots, n$, let S_i be a distinguished surface, and let Π be obtained from $\#_n S_i$ by capping off $n-1$ boundary components. Let S be a Schreier surface in $\mathcal{C}(\Pi)$ for a triple (G, T, H) with $|T| = n$. For each i, let G_i be an indicable group that embeds in $\mathrm{Map}(S_i)$, fix a surjective map $f_i\colon G_i \to \mathbb{Z}$, and let $H_i = \ker f_i$.*

Then, there are countably many non-conjugate embeddings of the indicable group

$$G_1 * G_2 * \cdots * G_n / \langle\!\langle \bigcup_{i \neq j} [G_i, H_j] \rangle\!\rangle$$

into $\mathrm{Map}(S)$*, none of which lie in* $\overline{\mathrm{Map}_c(S)}$.

Proof. We prove the theorem for $n = 2$ for simplicity of notation, but the same proof works for all n. Let a and b be two distinct generators of the group G for the Schreier surface S. By construction, S admits two multipushes x_a and x_b, where each acts as simultaneous pushes, as in Definition 3.3. See Figure 14.

By Lemma 5.1, each surjection $f_i\colon G_i \to \mathbb{Z}$ gives rise to a presentation $G_i = \langle T_i \mid R_i \rangle$ such that every $r \in R_i$ has total exponent sum zero with respect to T_i for $i = 1, 2$. Note that, under this map to $\mathbb{Z}$, the image of each generator is 1. As this is a homomorphism, the map f_i sends a group element of G_i to its total exponent sum with respect to T_i, and this is an invariant of the group element.

Similarly to Theorem 5.2, for each $g \in G_i$, define an element $\bar{g} \in \mathrm{Map}(S)$, where $\bar{g}$ acts as g simultaneously on each copy of Π in the domains of x_a and x_b in S. Note that since the copies of S_1 and S_2 in each copy of Π are disjoint, the elements $\bar{g}_1$ and $\bar{g}_2$ of $\mathrm{Map}(S)$ commute for any $g_1 \in G_1$ and $g_2 \in G_2$. This key fact will be used several times throughout the proof. Letting $\widetilde{T_1} = \{\bar{t}x_a : t \in T_1\}$ and $\widetilde{T_2} = \{\bar{t}x_b : t \in T_2\}$, we claim that the group generated by $\widetilde{T_1} \cup \widetilde{T_2}$ in $\mathrm{Map}(S)$ is isomorphic to $G_1 * G_2 / \langle\!\langle [G_1, H_2] \cup [H_1, G_2] \rangle\!\rangle$.

First, we claim that the group generated by $\widetilde{T_1}$ in $\mathrm{Map}(S)$ is isomorphic to G_1 under the map $t \to \bar{t}x_a$. Observe that for any $t \in T_1$, the element $\bar{t}$ commutes with x_a, and in fact also with x_b, although we will only need the first fact now. Thus, the image of a word $w \in \langle T_1 \rangle$ under this map can be expressed as the product of the corresponding word in $\{\bar{t} : t \in T_1\}$ and x_a^n where n is the exponent sum of the word w with respect to T_1. For $n \neq 0$, the support of x_a^n is not contained in the vertex surfaces, so we observe that a word in $\widetilde{T_1}$ is the trivial element in $\mathrm{Map}(S)$ if and only if both the corresponding word in $\{\bar{t} : t \in T_1\}$ acts trivially on S and $n = 0$. However, since $\langle \bar{t} : t \in T_1 \rangle$ is canonically isomorphic to $G_1 = \langle t : t \in T_1 \rangle$ and the presentation for G_1 was chosen so that every relator has total exponent sum zero with respect to T_1, the second condition is redundant. Thus, a word on $\widetilde{T_1}$ is the trivial element in $\mathrm{Map}(S)$ iff the corresponding word in G_1 is trivial. Therefore, the group generated by $\widetilde{T_1}$ and G_1 are isomorphic, and a symmetric argument shows that the group generated by $\widetilde{T_2}$ is isomorphic to G_2. Setting up notation for the rest of the proof, denote by $\widetilde{G}_1$ and $\widetilde{G}_2$, respectively, these isomorphic copies of G_1 and G_2. That is to say $\widetilde{G}_i = \langle \widetilde{T_i} \rangle$ as subgroups of $\mathrm{Map}(S)$ for $i = 1, 2$. Moreover, let $\widetilde{H}_i$ be the subgroup of $\widetilde{G}_i$ that is canonically isomorphic to H_i, for $i = 1, 2$.

We now have a surjective map

$$\phi\colon G_1 * G_2 \to \langle \widetilde{T_1} \cup \widetilde{T_2} \rangle,$$

where $\langle \widetilde{T_1} \cup \widetilde{T_2} \rangle = \langle \widetilde{G}_1 \cup \widetilde{G}_2 \rangle$, and it suffices to show that the kernel of ϕ is normally generated by the commutators $[G_1, H_2]$ and $[H_1, G_2]$.

For this, we first show that every element of $[G_1, H_2]$ and $[H_1, G_2]$ is in the kernel. Let $w = [g, h]$ where $g \in G_1$ and $h \in H_2$. As $h \in H_2$, it has exponent sum 0 and so $\phi(h) = \bar{h}$. Let m be the exponent sum of g. As noted above, for any $t \in T_1 \cup T_2$, the element $\bar{t}$ commutes with both x_a and x_b, and so it is also true that $\bar{h}$ and $\bar{g}$ commute with both x_a and x_b. Finally, observe that as $\bar{g}$ and $\bar{h}$ have disjoint supports, they also commute. We now evaluate $\phi(w)$:

$$\begin{aligned}\phi(w) &= \phi(g^{-1}h^{-1}gh)\\ &= \bar{g}^{-1}x_a^{-m}\bar{h}^{-1}\bar{g}x_a^m\bar{h}\\ &= x_a^{-m}x_a^m[\bar{g},\bar{h}]\\ &= 1.\end{aligned}$$

An analogous argument shows $[H_1, G_2]$ is in the kernel of ϕ. Therefore, the induced map

$$\bar{\phi} : G_1 * G_2/\langle\langle [G_1, H_2] \cup [H_1, G_2] \rangle\rangle \to \langle \widetilde{T}_1 \cup \widetilde{T}_2 \rangle,$$

is surjective.

To complete the proof, we must show that $\bar{\phi}$ is injective, as well. We view elements of $[G_1, H_2] \cup [H_1, G_2]$ as relators in $G_1 * G_2/\langle\langle [G_1, H_2] \cup [H_1, G_2] \rangle\rangle$ and note that applying a relator in $[G_1, H_2] \cup [H_1, G_2]$ to the word w in $G_1 * G_2$ amounts to applying the corresponding element of $[\widetilde{G}_1, \widetilde{H}_2] \cup [\widetilde{H}_1, \widetilde{G}_2]$ to the corresponding word $\widetilde{w}$ in $\widetilde{G}_1 * \widetilde{G}_2$. Thus, showing that $\bar{\phi}$ is injective is equivalent to showing that if $w \in G_1 * G_2$ maps to the trivial element in $\mathrm{Map}(S)$, then $\widetilde{w}$ can be reduced to the identity via free reductions and repeated applications of relators in the set $[\widetilde{G}_1, \widetilde{H}_2] \cup [\widetilde{H}_1, \widetilde{G}_2]$.

We now proceed with the proof, which will be via strong induction on the length of w as an alternating word in G_1 and G_2. Recall that an element $w = g_1 g_2 \ldots g_n \in G_1 * G_2$ is in normal form if each g_j is in either $G_1 \backslash \{1\}$ or $G_2 \backslash \{1\}$ and if $g_j \in G_i$ then $g_{j+1} \in G_{i+1}$, where the indices i are taken modulo 2.

We establish two base cases. Assume $n = 1$. Then w is in either G_1 or G_2. As ϕ restricted to either G_1 or G_2 is injective, w must be the trivial element, so no reductions are needed and the $n = 1$ case is vacuous.

Similarly, if $n = 2$, then w is of the form $g_1 g_2$, and, without loss of generality, we can assume $g_1 \in G_1$ and $g_2 \in G_2$. In this case, the $\widetilde{w}$ is $x_a^k \bar{g}_1 x_b^\ell \bar{g}_2$ for some values of k and ℓ, which is equal to $x_a^k x_b^\ell \bar{g}_1 \bar{g}_2$ in $\mathrm{Map}(S)$. Again, as x_a and x_b generate a free group for which nontrivial elements have support not contained in the vertex surfaces, it must be that $k = \ell = 0$. Moreover, as $\bar{g}_1$ and $\bar{g}_2$ act with disjoint supports, they too must be the identity. Thus, we conclude that $g_1 = g_2 = 1$, and this case is also vacuous.

Let, $n \geq 3$, and suppose $w = g_1 \cdots g_n$. Again, without loss of generality, we assume $g_1 \in G_1$. We will also assume n is odd, but one checks that if n is even, the argument differs only in notation. In this case,

$$\widetilde{w} = x_a^{k_1} \bar{g}_1 x_b^{k_2} \bar{g}_2 x_a^{k_3} \bar{g}_3 \cdots x_a^{k_n} \bar{g}_n$$

which equals

$$x_a^{k_1} x_b^{k_2} \cdots x_a^{k_n} \bar{g}_1 \bar{g}_2 \cdots \bar{g}_n$$

as an element in $\mathrm{Map}(S)$.

In order to act trivially, the product $x_a^{k_1} x_b^{k_2} \cdots x_a^{k_n}$ must freely reduce the to identity, and as the product alternates, this implies that there is some j such that $k_j = 0$. In turn, this implies that g_j is in H_1 or H_2.

Suppose $j = 1$ so that $x_a^{k_1} \bar{g}_1 = x_a^0 \bar{g}_1 \in \widetilde{H}_1$. Then an application of a relator in $[\widetilde{H}_1, \widetilde{G}_2]$ allows us to move the first term $x_a^{k_1} \bar{g}_1 = x_a^0 \bar{g}_1$ across the second term $x_b^{k_2} \bar{g}_2$, and we can shorten the original normal form of $\widetilde{w}$ to

$$\begin{aligned} & x_b^{k_2} \bar{g}_2 (x_a^{k_1} \bar{g}_1 x_a^{k_3} \bar{g}_3) \cdots x_a^{k_{n-1}} \bar{g}_{n-1} x_b^{k_n} \bar{g}_n \\ = & x_b^{k_2} \bar{g}_2 (x_a^{k_1 + k_3} \bar{g}_1 \bar{g}_3) \cdots x_a^{k_{n-1}} \bar{g}_{n-1} x_b^{k_n} \bar{g}_n . \end{aligned}$$

If this new word is in normal form in the free product $\widetilde{G}_1 * \widetilde{G}_2$, then by induction, $\widetilde{w}$ can be reduced to the identity and we are done. On the other hand, if it is not in normal form, which is precisely when $x_a^{k_1+k_3} \bar{g}_1 \bar{g}_3 = 1$, then we apply free reductions until it is. After there are no more free reductions to apply, the word is either already trivial or in normal form of shorter length. Once again, the inductive hypothesis implies that the word can be reduced to the identity via free reductions and repeated applications of relators in the set $[\widetilde{G}_1, \widetilde{H}_2] \cup [\widetilde{H}_1, \widetilde{G}_2]$. This completes the induction step for the case $j = 1$.

It remains to check the cases where $1 < j \leq n$. When $j = n$, the proof follows as above with minor notational differences. When $1 < j < n$, the proof again follows as above with the exception that when $n \geq 4$, the word length will drop by two immediately before free reductions. Combining all cases, we see that $\bar{\phi}$ is injective and the group generated by $\widetilde{T}_1 \cup \widetilde{T}_2$ in $\mathrm{Map}(S)$ is isomorphic to $G_1 * G_2 / \langle\langle [G_1, H_2] \cup [H_1, G_2] \rangle\rangle$.

We have shown that $(G_1, H_1) \star (G_2, H_2)$ embeds in $\mathrm{Map}(S)$. As in the proof of Theorem 1.1, by removing finitely many copies of Π from the domains of x_a and x_b, we obtain countably many non-conjugate embeddings of $(G_1, H_1) \star (G_2, H_2)$ into $\mathrm{Map}(S)$. By construction, no such embedding is contained in $\overline{\mathrm{Map}_c(S)}$. □

6.1. Applications of Theorem 1.2. Note that Theorem 1.2 produces embeddings of $\star$-products into $\mathrm{Map}(S)$. In general, $(G_1, H_1) \star (G_2, H_2)$ need not be finitely presented, even when the groups G_i are finitely presented. For example, consider the indicable group $\mathbb{F}_2 = \langle a, b \rangle$ with the map to $\mathbb{Z}$ defined by $a \mapsto 1$ and $b \mapsto 0$. It is an exercise to see that the kernel K of this map is not finitely generated; see Exercise 7 of Section 1.A in [Hat02]. Therefore, if $G_1 = G_2 = \mathbb{F}_2$ and $H_1 = H_2 = K$, then $(G_1, H_1) \star (G_2, H_2)$ is a finitely generated but infinitely presented group. However, there are instances where the $\star$-product is a recognizable finitely presented group.

Example 6.4. As a first example, consider a collection of groups $H_1, \ldots, H_n$ which are subgroups of $\mathrm{Map}(S_i)$ where S_i is a surface with one compact boundary component. Then after possibly increasing the complexity of S_i, the groups

$G_i = \mathbb{Z} \times H_i$ are also subgroups of $\mathrm{Map}(S_i)$. The groups H_i are the kernels of the projection maps of G_i onto the $\mathbb{Z}$ factor. As such, the following group is the associated $\star$-product, which embeds in $\mathrm{Map}(S)$ for $S = S_\Gamma(\Pi)$ by Theorem 1.2:

$$(G_1, H_1) \star \cdots \star (G_n, H_n) \simeq H_1 \times \cdots \times H_n \times F_n.$$

For a second class of examples, consider the case where G is a right-angled Artin group defined by the finite graph Γ. By work of Clay, Leininger, Mangahas [CLM12] and Koberda [Kob12], every right-angled Artin group embeds as a subgroup of the mapping class group of some finite-type surface of sufficient complexity.

The map $\chi: G \to \mathbb{Z}$ which sends each generator of G from Γ to 1 is a homomorphism, so G is indicable. The kernel of this map is called the *Bestvina-Brady group* defined on Γ, denoted by BB_Γ. Bestvina-Brady groups are a vast and varied class of groups. There is a large body of work connecting combinatorial conditions on Γ to algebraic properties of BB_Γ. First, the group BB_Γ is finitely generated exactly when Γ is connected. Further, BB_Γ is finitely presented exactly when the flag complex associated to Γ is simply connected. See [BeB97] for more details.

For finitely presented BB_Γ, Dicks and Leary compute an explicit presentation for BB_Γ in [DL99], which can be expressed in terms of the generators of A_Γ. In the particular case when Γ is any tree on n vertices, this presentation can be used to show that the Bestvina-Brady group is F_{n-1}. The following example makes use of this fact.

Example 6.5. Let P_4 represent the path graph on 4 vertices a, b, c, d as in Figure 15. Then the right-angled Artin group A_{P_4} has presentation

$$\langle a, b, c, d \mid [a, b], [b, c], [c, d] \rangle.$$

The associated group BB_Γ is the free group generated by $\langle ab^{-1}, bc^{-1}, cd^{-1} \rangle$.

FIGURE 15. The path graph

Letting $G_1 = G_2 = A_{P_4}$, with $H_1 \simeq H_2 \simeq F_3$, one can obtain an explicit presentation for $(A_{P_4}, F_3) \star (A_{P_4}, F_3)$ following the work of Dicks and Leary. Theorem 1.2 then implies that $(A_{P_4}, F_3) \star (A_{P_4}, F_3)$ embeds into $\mathrm{Map}(S)$ with $S = S_\Gamma(\Pi)$ for a sufficiently complicated Π. In particular, $(A_{P_4}, F_3) \star (A_{P_4}, F_3)$ embeds into $\mathrm{Map}(S)$ for $S \in \mathcal{B}_\infty$ (see Definition 3.7). An example of such a surface S is the blooming Cantor tree surface (the boundaryless surface whose end space is a Cantor set of nonplanar ends). More generally, for any right-angled Artin groups A_{Γ_1}, A_{Γ_2}, the group $(A_{\Gamma_1}, BB_{\Gamma_1}) \star (A_{\Gamma_2}, BB_{\Gamma_2})$ embeds in $\mathrm{Map}(S)$ for $S \in \mathcal{B}_\infty$.

In addition, Chang and Ruffoni [ChR24] solve the right-angled Artin group recognition problem for Bestvina-Brady groups, i.e., they give sufficient and necessary conditions on Γ that determine if BB_Γ is itself a right-angled Artin group. In particular, they show the following.

Theorem 6.6 (Chang-Ruffoni, [ChR24])**.** *If* Γ *admits a tree 2-spanner* T*, then* BB_Γ *is a right-angled Artin group. More precisely, the Dicks–Leary presentation can be simplified to the standard right-angled Artin group presentation with generating set* $E(T)$*. Moreover, we have* $BB_\Gamma = A_{T^*}$*.*

Here, $E(T)$ denotes the edge set of T, T^* is the dual graph to T, and A_{T^*} is the right-angled Artin group with defining graph T^*. A tree 2-spanner of Γ is a spanning tree T of Γ such that for all $x, y \in V(T)$, we have $d_T(x,y) \le 2d_\Gamma(x,y)$.

One consequence of their work is that the class of Bestvina-Brady groups includes all right-angled Artin groups. Therefore, letting A_1 and A_2 be any right-angled Artin groups, their work implies that Γ_1 and Γ_2 can be chosen so that the corresponding Bestvina-Brady groups are precisely A_1 and A_2. It follows from Theorem 1.2 that the group $(A_{\Gamma_1}, A_1) \star (A_{\Gamma_2}, A_2)$ embeds in $\mathrm{Map}(S)$ for $S \in \mathcal{B}_\infty$.

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