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Dirichlet-type spaces of the unit bidisc and toral completely hyperexpansive operators

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ABSTRACT. We discuss a notion, originally introduced by Aleman in one variable, of Dirichlet-type space $\mathcal{D}(\mu_1, \mu_2)$ on the unit bidisc $\mathbb{D}^2$, with superharmonic weights related to finite positive Borel measures μ_1, μ_2 on $\overline{\mathbb{D}}$. The multiplication operators $\mathscr{M}_{z_1}$ and $\mathscr{M}_{z_2}$ by the coordinate functions z_1 and z_2, respectively, are bounded on $\mathcal{D}(\mu_1, \mu_2)$ and the set of polynomials is dense in $\mathcal{D}(\mu_1, \mu_2)$. We show that the commuting pair $\mathscr{M}_z = (\mathscr{M}_{z_1}, \mathscr{M}_{z_2})$ is a cyclic analytic toral completely hyperexpansive 2-tuple on $\mathcal{D}(\mu_1, \mu_2)$. Unlike the one variable case, not all cyclic analytic toral completely hyperexpansive pairs arise as multiplication 2-tuple $\mathscr{M}_z$ on these spaces. In particular, we establish that a cyclic analytic toral completely hyperexpansive operator 2-tuple $T = (T_1, T_2)$ satisfying $I - T_1^*T_1 - T_2^*T_2 + T_1^*T_2^*T_1T_2 = 0$ and having a cyclic vector f_0 is unitarily equivalent to $\mathscr{M}_z$ on $\mathcal{D}(\mu_1, \mu_2)$ for some finite positive Borel measures μ_1 and μ_2 on $\overline{\mathbb{D}}$ if and only if $\ker T^*$, spanned by f_0, is a wandering subspace for T.

CONTENTS

1. Introduction & preliminaries

Let $\mathbb{D}$ and $\mathbb{T}$ denote the open unit disc and the unit circle in the complex plane $\mathbb{C}$. For a non-empty subset S of $\mathbb{C}$, $M_+(S)$ denotes the set of finite positive Borel measures on S. Let $\mathcal{H}$ denote a complex separable Hilbert space and $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of bounded linear operators on $\mathcal{H}$. For an operator $S \in \mathcal{B}(\mathcal{H})$, S^* denotes the Hilbert space adjoint of S. A pair $T = (T_1, T_2)$ is called a *commuting*

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pair on $\mathcal{H}$ if $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ and $T_iT_j = T_jT_i$ for $1 \le i, j \le 2$. Following [16, p. 56], we say that a commuting pair $T = (T_1, T_2)$ on $\mathcal{H}$ is *analytic* if

$$\bigcap_{k=0}^{\infty} \sum_{\substack{\alpha_1,\alpha_2\ge 0\\ \alpha_1+\alpha_2=k}} T_1^{\alpha_1}T_2^{\alpha_2}\mathcal{H} = \{0\}.$$

A commuting pair $T = (T_1, T_2)$ is called *cyclic* with cyclic vector $f_0 \in \mathcal{H}$ if the closed linear span of $\{T_1^{\alpha_1}T_2^{\alpha_2}f_0, \alpha_1, \alpha_2 \in \mathbb{Z}_+\}$ equals to $\mathcal{H}$. Let $\alpha, \beta \in \mathbb{Z}_+^2$, we say $\alpha \le \beta$ if $\alpha_i \le \beta_i$ for $i = 1, 2$. For $\alpha \le \beta$ denote $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2}$. A commuting pair $T = (T_1, T_2)$ on $\mathcal{H}$ is said to be *toral completely hyperexpansive* (refer to [6, Definition 1], cf. [1, 5]) if for each $\alpha \in \mathbb{Z}_+^2 \setminus \{0\}$,

$$\beta_\alpha(T) := \sum_{\substack{\beta\in\mathbb{Z}_+^2\\ 0\le\beta\le\alpha}} (-1)^{|\beta|}\binom{\alpha}{\beta}T^{*\beta}T^{\beta} \le 0. \tag{1}$$

Using binomial expansion it is easy to check that for each $\alpha \in \mathbb{Z}_+^2$,

$$\beta_{\alpha+\epsilon_j}(T) = \beta_\alpha(T) - T_j^*\beta_\alpha(T)T_j, \quad j = 1, 2, \tag{2}$$

where $\epsilon_1 = (1, 0)$ and $\epsilon_2 = (0, 1)$. For a commuting pair $T = (T_1, T_2)$ on $\mathcal{H}$ we define the defect operator as

$$\beta_{(1,1)}(T) = I - T_1^*T_1 - T_2^*T_2 + T_1^*T_2^*T_1T_2. \tag{3}$$

From the identity (2) it is clear that whenever the defect operator of T is zero i.e. $\beta_{(1,1)}(T) = 0$, $\beta_\alpha(T) = 0$ for all $\alpha \in \mathbb{Z}_+^2$ with $\alpha_1\alpha_2 \ne 0$. It is evident that whenever two completely hyperexpansive operators T_1 and T_2 commute and have zero defect operator, the commuting 2-tuple $T = (T_1, T_2)$ is toral completely hyperexpansive. Not every toral completely hyperexpansive 2-tuple has zero defect operator. Indeed, if we take $d\nu$ to be Lebesgue area measure on $[0, 1]^2$ in [6, Eq. H] then the defect operator is non zero. In particular,

$$\begin{aligned} & \|e_0\|^2 - \|T_1e_0\|^2 - \|T_2e_0\|^2 + \|T_1T_2e_0\|^2 \\ = \; & 1 - (1 + b_1 + \frac{1}{2}) - (1 + b_2 + \frac{1}{2}) + (1 + b_1 + b_2 + \frac{3}{4}) = -\frac{1}{4} < 0. \end{aligned}$$

Richter [25] introduced the notion of *Dirichlet-type space* on the unit disc $\mathbb{D}$ with harmonic weight and proved that these spaces are model spaces for cyclic analytic 2-isometries (see [25, Theorem 5.1]). Later, in [4, Chapter IV] Aleman generalized this notion by considering superharmonic weights as follows: Let μ be a finite positive Borel measure on $\overline{\mathbb{D}}$. For a holomorphic function f on $\mathbb{D}$ consider

$$\mathcal{D}_\mu(f) = \int_{\mathbb{D}} |f'(z)|^2 U_\mu(z)dA(z), \tag{4}$$

where dA denotes the normalized Lebesgue area measure on $\mathbb{D}$ and U_μ is the superharmonic function on $\mathbb{D}$ given by

$$U_\mu(w) = \int_{\mathbb{D}} \log\left|\frac{1-\bar{\zeta}w}{w-\zeta}\right|^2 \frac{d\mu(\zeta)}{1-|\zeta|^2} + \int_{\mathbb{T}} \frac{1-|w|^2}{|\zeta-w|^2} d\mu(\zeta),\ w \in \mathbb{D}.$$

Note that any positive superharmonic function on $\mathbb{D}$ is of this form (see [21, Theorem 4.5.1]). The Dirichlet type space $\mathcal{D}(\mu)$ is the collection of holomorphic function f on $\mathbb{D}$ such that $\mathcal{D}_\mu(f) < \infty$. These spaces are subspaces of the Hardy space $H^2(\mathbb{D})$. Concerning the following norm

$$\|f\|^2_{\mathcal{D}(\mu)} = \|f\|^2_{H^2(\mathbb{D})} + \mathcal{D}_\mu(f), \quad f \in \mathcal{D}(\mu), \tag{5}$$

$\mathcal{D}(\mu)$ is a Hilbert space and the multiplication operator $\mathscr{M}_z$ by the coordinate function z is a cyclic analytic completely hyperexpansive (see [5, Eq D] and [4, Theorem 1.10(i), p 76]). Moreover, any cyclic analytic completely hyperexpansive operator is unitarily equivalent to $\mathscr{M}_z$ on $\mathcal{D}(\mu)$ for some $\mu \in M_+(\overline{\mathbb{D}})$ (see [4, Theorem 2.5, p. 79]). For further details on these spaces, please refer to [4, 15, 8, 19].

In [9] (see also [10]) a notion of Dirichlet-type spaces on unit bidisc with harmonic weights has been introduced and observed that the multiplication tuple $\mathscr{M}_z = (\mathscr{M}_{z_1}, \mathscr{M}_{z_2})$ is a toral 2-isometry, i.e. $\beta_\alpha(\mathscr{M}_z) = 0$ for each $\alpha \in \{(2,0),(0,2),(1,1)\}$. In this present paper we generalize the notion of Dirichlet-type space introduced in [9], by replacing the harmonic weights with superharmonic weights. Motivated by [4, Definition 1.8], [9, Definition 1.1] and [25, Eq 3.1] we define the following:

Definition 1.1. For $\mu_1, \mu_2 \in M_+(\overline{\mathbb{D}})$ and a holomorphic function f on the unit bidisc $\mathbb{D}^2$, *the Dirichlet integral* $\mathcal{D}_{\mu_1,\mu_2}(f)$ of f is given by

$$\begin{aligned}\mathcal{D}_{\mu_1,\mu_2}(f) := \sup_{0<r<1} \int_0^{2\pi} \int_{\mathbb{D}} |\partial_1 f(z_1, re^{i\theta})|^2 U_{\mu_1}(z_1)\, dA(z_1) \frac{d\theta}{2\pi} \\ + \sup_{0<r<1} \int_0^{2\pi} \int_{\mathbb{D}} |\partial_2 f(re^{i\theta}, z_2)|^2 U_{\mu_2}(z_2)\, dA(z_2) \frac{d\theta}{2\pi}.\end{aligned} \tag{6}$$

Consider the *Dirichlet-type space*

$$\mathcal{D}(\mu_1,\mu_2) := \{f \in H^2(\mathbb{D}^2) : \mathcal{D}_{\mu_1,\mu_2}(f) < \infty\},$$

where $H^2(\mathbb{D}^2)$ denotes the *Hardy space* on the unit bidisc $\mathbb{D}^2$ (see [27]).

It is clear from the definition that $\mathcal{D}_{\mu_1,\mu_2}(f)$ defines a seminorm on the space $\mathcal{D}(\mu_1,\mu_2)$. So we consider the following norm on $\mathcal{D}(\mu_1,\mu_2)$

$$\|f\|^2 := \|f\|^2_{H^2(\mathbb{D}^2)} + \mathcal{D}_{\mu_1,\mu_2}(f), \quad f \in \mathcal{D}(\mu_1,\mu_2). \tag{7}$$

With this norm $\mathcal{D}(\mu_1,\mu_2)$ is a reproducing kernel Hilbert space (see Lemma 2.2). If we assume $\mu_j(\mathbb{D}) = 0$ for $j = 1,2$, $\mathcal{D}(\mu_1,\mu_2)$ coincides with the notion of Dirichlet-type spaces appeared in [9].

1.1. Statement of the main theorem. Before stating the main theorem let us recall that a subspace $\mathcal{W}$ of $\mathcal{H}$ is said to be *wandering* (see [9, Definition 1.5], cf. [17, p. 103]) for a commuting pair $T = (T_1, T_2)$ on $\mathcal{H}$ if for $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{Z}_+$,

$$T_1^{\alpha_1}\mathcal{W} \quad \perp \quad T_1^{\beta_1}T_2^{\beta_2}\mathcal{W}, \text{ whenever } \beta_2 \neq 0,$$
$$T_2^{\alpha_2}\mathcal{W} \quad \perp \quad T_1^{\beta_1}T_2^{\beta_2}\mathcal{W}, \text{ whenever } \beta_1 \neq 0.$$

Theorem 1.2. *Let $T = (T_1, T_2)$ be a commuting pair on a complex separable Hilbert space $\mathcal{H}$. Then the following statements are equivalent:*

- (A) *T is a cyclic analytic toral completely hyperexpansive 2-tuple such that $\beta_{(1,1)}(T) = 0$, and T possesses a cyclic vector $f_0 \in \ker T^*$, where $\ker T^*$ is a wandering subspace of T,*
- (B) *there exist $\mu_1, \mu_2 \in M_+(\overline{\mathbb{D}})$ such that T is unitarily equivalent to $\mathscr{M}_z$ on $\mathcal{D}(\mu_1, \mu_2)$.*

Remark 1.3. From [23, Theorem 1] we know that any cyclic completely hyperexpansive operator on a complex separable Hilbert space has the wandering subspace property. But this fact fails in two-variable. For details one is refer to [9, Remark 2.5] (cf. [11, Example 6.8]).

Theorem 1.2 presents an analogue of [4, Theorem 2.5], (cf. [9, Theorem 2.4], [25, Theorem 5.1] and [10, Theorem 5.1]). In Section 2 we discuss the polynomial density, Gleason's problem and boundedness of the multiplication 2-tuple $\mathscr{M}_z = (\mathscr{M}_{z_1}, \mathscr{M}_{z_2})$ on $\mathcal{D}(\mu_1, \mu_2)$. A proof of Theorem 1.2 is presented in Section 3 along with some of its consequences.

2. Polynomial density and Gleason's problem

Let $\mathscr{H}$ be a space of holomorphic functions on a domain Ω in $\mathbb{C}^d$ $(d \geq 1)$. We say that the *Gleason's problem can be solved for $\mathscr{H}$ at $\lambda = (\lambda_1, \dots, \lambda_d) \in \Omega$*, if for every $f \in \mathscr{H}$, there exist functions $f_1, \dots, f_d$ in $\mathscr{H}$ such that

$$f(z) = f(\lambda) + \sum_{j=1}^{d}(z_j - \lambda_j)f_j(z), \quad z = (z_1, \dots, z_d) \in \Omega.$$

We say that $\mathscr{H}$ has the *Gleason property* if the Gleason's problem can be solved for $\mathscr{H}$ for each $\lambda \in \Omega$. The Hardy space of the bidisc $H^2(\mathbb{D}^d)$ has the Gleason property (see [9, Remark 5.2]). Kehe Zhu [29] showed that the Bergman space and Bloch space of the unit ball have the Gleason property. For further examples of the Gleason's problem on function spaces, see [12, 28].

We say that $\mathscr{H}$ has the *j-division property*, $j = 1, \dots, d$, if $\frac{f(z)}{z_j - \lambda_j}$ defines a function in $\mathscr{H}$ whenever $\lambda \in \Omega$, $f \in \mathscr{H}$ and $\{z \in \Omega : z_j = \lambda_j\}$ is contained in $Z(f)$, the zero set of f. If $\mathscr{H}$ has j-division property for every $j = 1, \dots, d$, then we say that $\mathscr{H}$ has the *division property*. Note that the Hardy space of bidisc $H^2(\mathbb{D}^2)$ ([9, Lemma 4.1]) and the Dirichlet-type spaces of the bidisc with harmonic weights ([9, Theorem 2.2]) have the division property.

Let $g \in H^2(\mathbb{D})$ and $\zeta \in \overline{\mathbb{D}}$. If $\zeta \in \mathbb{D}$, recall that the local Dirichlet integral of g at ζ (see [4, p. 74], [15, Theorem 2.1]) is defined as

$$D_\zeta(g) = \left\| \frac{g - g(\zeta)}{z - \zeta} \right\|^2_{H^2(\mathbb{D})} = \int_0^{2\pi} \left| \frac{g(e^{it}) - g(\zeta)}{e^{it} - \zeta} \right|^2 \frac{dt}{2\pi}. \tag{8}$$

If $\zeta \in \mathbb{T}$ and $g(\zeta) := \lim_{r\to 1^-} g(r\zeta)$ exists, we use the same formula (8) to denote the local Dirichlet integral $D_\zeta(g)$ of g at ζ. Otherwise, we set $D_\zeta(g) = \infty$ (see [24, p. 356]). For general $\mu \in M_+(\overline{\mathbb{D}})$, [4, Theorem 1.9, p. 74] (cf. [24, Proposition 2.2]) gives

$$\mathcal{D}_\mu(g) = \int_{\overline{\mathbb{D}}} \mathcal{D}_\zeta(g) d\mu(\zeta), \quad g \in \mathcal{D}(\mu). \tag{9}$$

Here is a two-variable analog of the above equation.

Proposition 2.1. *For $f \in \mathcal{D}(\mu_1, \mu_2)$,*

$$\mathcal{D}_{\mu_1,\mu_2}(f) = \sup_{0<r<1} \int_0^{2\pi} \int_{\overline{\mathbb{D}}} D_{\zeta_1}(f(\cdot, re^{i\theta})) d\mu_1(\zeta_1) \frac{d\theta}{2\pi} + \sup_{0<r<1} \int_0^{2\pi} \int_{\overline{\mathbb{D}}} D_{\zeta_2}(f(re^{i\theta}, \cdot)) d\mu_2(\zeta_2) \frac{d\theta}{2\pi}.$$

Proof. Let $f \in \mathcal{D}(\mu_1, \mu_2)$ so it belongs to $H^2(\mathbb{D}^2)$. By [9, Lemma 3.2], for each $r \in (0,1)$ and $\theta \in [0, 2\pi]$ the slice functions $f(\cdot, re^{i\theta})$ and $f(re^{i\theta}, \cdot)$ belong to $H^2(\mathbb{D})$. From (6) we get that for each $r \in (0,1)$ and almost every $\theta \in [0, 2\pi]$,

$$\int_{\mathbb{D}} |\partial_1 f(z_1, re^{i\theta})|^2 U_{\mu_1}(z_1) dA(z_1), \int_{\mathbb{D}} |\partial_2 f(re^{i\theta}, z_2)|^2 U_{\mu_2}(z_2) dA(z_2) < \infty.$$

In other words for each $r \in (0,1)$,

$$\begin{gathered}\text{there exists a measure zero subset } \Omega_r \subseteq [0, 2\pi] \text{ such that the slices} \\ f(\cdot, re^{i\theta}) \in \mathcal{D}(\mu_1) \text{ and } f(re^{i\theta}, \cdot) \in \mathcal{D}(\mu_2) \text{ for } \theta \in [0, 2\pi] \setminus \Omega_r.\end{gathered} \tag{10}$$

Thus (6) becomes

$$\mathcal{D}_{\mu_1,\mu_2}(f) = \sup_{0<r<1} \int_0^{2\pi} \mathcal{D}_{\mu_1}(f(\cdot, re^{i\theta})) \frac{d\theta}{2\pi} + \sup_{0<r<1} \int_0^{2\pi} \mathcal{D}_{\mu_2}(f(re^{i\theta}, \cdot)) \frac{d\theta}{2\pi}. \tag{11}$$

Combining (9) and (11) yields the result. □

In view of [9, Lemma 1.1], for any holomorphic function f on $\mathbb{D}^2$ and $\nu \in M_+(\mathbb{D})$, the function $r \to \int_{\mathbb{T}} \int_{\mathbb{D}} |f(z, re^{i\theta})|^2 d\nu(z) d\theta$ is increasing. So we can replace $\sup_{0<r<1}$ in (6) by $\lim_{r\to 1^-}$. Thus for each $f \in \mathcal{D}(\mu_1, \mu_2)$, $\mathcal{D}_{\mu_1,\mu_2}(f)$ breaks into two parts as $\mathcal{D}_{\mu_1,\mu_2}(f) = I_{\mu_1,\mu_2}(f) + B_{\mu_1,\mu_2}(f)$, where $I_{\mu_1,\mu_2}(f)$ and $B_{\mu_1,\mu_2}(f)$ are the integrals correspond to $\mu_j|_{\mathbb{D}}$ and $\mu_j|_{\mathbb{T}}$, respectively, for $j = 1, 2$ and given by

$$I_{\mu_1,\mu_2}(f) = \sup_{0<r<1} \int_0^{2\pi} \int_{\mathbb{D}} D_{\zeta_1}(f(\cdot, re^{i\theta})) d\mu_1(\zeta_1) \frac{d\theta}{2\pi} + \sup_{0<r<1} \int_0^{2\pi} \int_{\mathbb{D}} D_{\zeta_2}(f(re^{i\theta}, \cdot)) d\mu_2(\zeta_2) \frac{d\theta}{2\pi}, \tag{12}$$

$$B_{\mu_1,\mu_2}(f) = \sup_{0<r<1} \int_0^{2\pi} \int_{\mathbb{T}} D_{\zeta_1}(f(\cdot, re^{i\theta})) d\mu_1(\zeta_1) \frac{d\theta}{2\pi} + \sup_{0<r<1} \int_0^{2\pi} \int_{\mathbb{T}} D_{\zeta_2}(f(re^{i\theta}, \cdot)) d\mu_2(\zeta_2) \frac{d\theta}{2\pi}.$$

The following lemmas are fundamental to prove the polynomial density and boundedness of the multiplication 2-tuple $\mathscr{M}_z$ on $\mathcal{D}(\mu_1, \mu_2)$.

Lemma 2.2. *The Dirichlet-type space $\mathcal{D}(\mu_1, \mu_2)$ is a reproducing kernel Hilbert space. If $\kappa : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ is the reproducing kernel of $\mathcal{D}(\mu_1, \mu_2)$, then for any $r \in (0,1)$, $\bigvee\{\kappa(\cdot, w) : |w| < r\} = \mathcal{D}(\mu_1, \mu_2)$ and $\kappa(\cdot, 0) = 1$.*

Proof. By replacing P_{μ_1} by U_{μ_1} and P_{μ_2} by U_{μ_2} in [9, Lemma 3.1] and arguing similarly we get required result. □

Let $\mu \in M_+(\overline{\mathbb{D}})$ and $g \in \mathcal{D}(\mu)$. For each $r \in (0,1)$ define the function g_r on $\mathbb{D}$ as $g_r(w) := g(rw)$, $w \in \mathbb{D}$. Combining [4, Lemma 4.1, p. 87] and [4, Theorem 1.9, p. 74] gives $D_\mu(g_r) \leq \frac{5}{2} D_\mu(g)$. Later, in [15, Theorem 4.2] this inequality is improved to

$$D_\mu(g_r) \leq D_\mu(g). \tag{13}$$

The following lemma provides a similar estimate as of (13) for $\mathcal{D}(\mu_1, \mu_2)$. For $R = (R_1, R_2) \in (0,1)^2$ and $f \in \mathcal{O}(\mathbb{D}^2)$ let $f_R(z) = f(R_1 z_1, R_2 z_2)$ for $z = (z_1, z_2) \in \mathbb{D}^2$.

Lemma 2.3. *For any $R = (R_1, R_2) \in (0,1)^2$ and $f \in \mathcal{D}(\mu_1, \mu_2)$,*

$$\mathcal{D}_{\mu_1,\mu_2}(f_R) \leq \mathcal{D}_{\mu_1,\mu_2}(f).$$

Proof. Let $f \in \mathcal{D}(\mu_1, \mu_2)$. Fix $R = (R_1, R_2) \in (0,1)^2$. By (11),

$$\begin{aligned} &\mathcal{D}_{\mu_1,\mu_2}(f_R) \\ = \;& \sup_{0<r<1} \int_0^{2\pi} \mathcal{D}_{\mu_1}(f_R(\cdot, re^{i\theta})) \frac{d\theta}{2\pi} + \sup_{0<r<1} \int_0^{2\pi} \mathcal{D}_{\mu_2}(f_R(re^{i\theta}, \cdot)) \frac{d\theta}{2\pi} \\ \overset{(13)}{\leq} \;& \sup_{0<r<1} \int_0^{2\pi} \mathcal{D}_{\mu_1}(f(\cdot, R_2 re^{i\theta})) \frac{d\theta}{2\pi} + \sup_{0<r<1} \int_0^{2\pi} \mathcal{D}_{\mu_2}(f(R_1 re^{i\theta}, \cdot)) \frac{d\theta}{2\pi}. \end{aligned}$$

Finally applying [9, Lemma 1.1] yields the result. □

The next lemma is a prototype of [9, Lemma 3.7] (cf. [14, Theorem 7.3.1]) and the proof is similar so left to the reader.

Lemma 2.4. *Let $f \in \mathcal{D}(\mu_1, \mu_2)$ and $R = (R_1, R_2) \in (0,1)^2$. Then*

$$\lim_{R_1, R_2 \to 1^-} \mathcal{D}_{\mu_1,\mu_2}(f - f_R) = 0.$$

As an application of Lemma 2.4 we show that the set of polynomial is dense in $\mathcal{D}(\mu_1, \mu_2)$.

Lemma 2.5. *Polynomials are dense in $\mathcal{D}(\mu_1, \mu_2)$.*

Proof. Let $f \in \mathcal{D}(\mu_1, \mu_2)$ and choose $\varepsilon > 0$. It is enough to show that there exists a polynomial p such that $\|f - p\|_{\mathcal{D}(\mu_1,\mu_2)} < \varepsilon$. By Lemma 2.4 there exists $R = (R_1, R_2) \in (0,1)^2$ such that $\|f - f_R\|_{\mathcal{D}(\mu_1,\mu_2)} < \varepsilon/2$. Since f_R is holomorphic in a neighborhood of $\overline{\mathbb{D}}^2$, there exists a polynomial p such that

$$\|f_R - p\|_{\infty, \overline{\mathbb{D}}^2}, \left\| \frac{\partial f_R}{\partial z_j} - \frac{\partial p}{\partial z_j} \right\|_{\infty, \overline{\mathbb{D}}^2} < \frac{\sqrt{\varepsilon}}{4\sqrt{M}}, \quad j = 1, 2,$$

where $M = \max\{\int_{\mathbb{D}} U_{\mu_j}(w) dA(w) : j = 1, 2\} + 1$. This together with the fact that the norm on $H^2(\mathbb{D}^2)$ is dominated by the norm $\|\cdot\|_{\infty, \overline{\mathbb{D}}^2}$ shows that $\|f_R - p\|_{\mathcal{D}(\mu_1,\mu_2)} < \varepsilon/2$. Thus using triangle inequality we get that $\|f - p\|_{\mathcal{D}(\mu_1,\mu_2)} < \varepsilon$. Hence the proof. □

This next lemma is very crucial to prove the boundedness of the multiplication tuple $\mathscr{M}_z = (\mathscr{M}_{z_1}, \mathscr{M}_{z_2})$.

Lemma 2.6. *Let $f \in H^2(\mathbb{D}^2)$. Then*

$$\begin{aligned} I_{\mu_1,\mu_2}(z_1 f) &= \sup_{0<r<1} \int_0^{2\pi} \int_{\mathbb{D}} |f(\zeta_1, re^{i\theta})|^2 d\mu_1(\zeta_1) \frac{d\theta}{2\pi} + I_{\mu_1,\mu_2}(f), \\ I_{\mu_1,\mu_2}(z_2 f) &= \sup_{0<r<1} \int_0^{2\pi} \int_{\mathbb{D}} |f(re^{i\theta}, \zeta_2)|^2 d\mu_2(\zeta_2) \frac{d\theta}{2\pi} + I_{\mu_1,\mu_2}(f). \end{aligned}$$

Proof. For each $r \in (0,1)$ and $\theta \in [0, 2\pi]$ define $f_{r,\theta}(w) := f(w, re^{i\theta})$, $w \in \mathbb{D}$, then $f_{r,\theta} \in H^2(\mathbb{D})$ (see [9, Lemma 3.2]). Since $H^2(\mathbb{D})$ is closed under the multiplication of the coordinate function w so $w f_{r,\theta} \in H^2(\mathbb{D})$. Fixing $\zeta \in \mathbb{D}$ we know that for each $g \in H^2(\mathbb{D})$, $\frac{g - g(\zeta)}{w - \zeta} \in H^2(\mathbb{D})$. In particular, $g = w f_{r,\theta}$ gives $\frac{w f_{r,\theta} - (w f_{r,\theta})(\zeta)}{w - \zeta} \in H^2(\mathbb{D})$ and

$$\left\| \frac{w f_{r,\theta} - (w f_{r,\theta})(\zeta)}{w - \zeta} \right\|_{H^2(\mathbb{D})}^2 = \left\| f_{r,\theta}(\zeta) + w \frac{f_{r,\theta} - f_{r,\theta}(\zeta)}{w - \zeta} \right\|_{H^2(\mathbb{D})}^2.$$

As the constant functions are orthogonal to $wH^2(\mathbb{D})$ in $H^2(\mathbb{D})$ so the above equation becomes

$$\begin{aligned}\left\|\frac{wf_{r,\theta}-(wf_{r,\theta})(\zeta)}{w-\zeta}\right\|_{H^2}^2 &= |f_{r,\theta}(\zeta)|^2+\left\|\frac{f_{r,\theta}-f_{r,\theta}(\zeta)}{w-\zeta}\right\|_{H^2}^2 \\ &= |f(\zeta,re^{i\theta})|^2+\left\|\frac{f(\cdot,re^{i\theta})-f(\zeta,re^{i\theta})}{w-\zeta}\right\|_{H^2}^2.\end{aligned} \tag{14}$$

Now (12) together with (8) implies

$$\begin{aligned}&I_{\mu_1,\mu_2}(z_1f)\\ &= \sup_{0<r<1}\int_0^{2\pi}\int_{\mathbb{D}}\left\|\frac{(z_1f)(\cdot,re^{i\theta})-(z_1f)(\zeta_1,re^{i\theta})}{z_1-\zeta_1}\right\|_{H^2(\mathbb{D})}^2 d\mu_1(\zeta_1)\frac{d\theta}{2\pi}\\ &+ \sup_{0<r<1}\int_0^{2\pi}\int_{\mathbb{D}}\left\|\frac{(z_1f)(re^{i\theta},\cdot)-(z_1f)(re^{i\theta},\zeta_2)}{z_2-\zeta_2}\right\|_{H^2(\mathbb{D})}^2 d\mu_2(\zeta_2)\frac{d\theta}{2\pi}\\ &\overset{(14)}{=} \sup_{0<r<1}\int_0^{2\pi}\int_{\mathbb{D}}|f(\zeta_1,re^{i\theta})|^2 d\mu_1(\zeta_1)\frac{d\theta}{2\pi}\\ &+ \sup_{0<r<1}\int_0^{2\pi}\int_{\mathbb{D}}\left\|\frac{f(\cdot,re^{i\theta})-f(\zeta_1,re^{i\theta})}{z_1-\zeta_1}\right\|_{H^2(\mathbb{D})}^2 d\mu_1(\zeta_1)\frac{d\theta}{2\pi}\\ &+ \sup_{0<r<1}\int_0^{2\pi}\int_{\mathbb{D}}\left\|\frac{f(re^{i\theta},\cdot)-f(re^{i\theta},\zeta_2)}{z_2-\zeta_2}\right\|_{H^2(\mathbb{D})}^2 d\mu_2(\zeta_2)\frac{d\theta}{2\pi}\\ &\overset{(12)}{=} \sup_{0<r<1}\int_0^{2\pi}\int_{\mathbb{D}}|f(\zeta_1,re^{i\theta})|^2 d\mu_1(\zeta_1)\frac{d\theta}{2\pi}+I_{\mu_1,\mu_2}(f).\end{aligned} \tag{15}$$

Similarly on can derive the expression involving $I_{\mu_1,\mu_2}(z_2f)$. ☐

Here we show that the coordinate functions z_1 and z_2 are multipliers of $\mathcal{D}(\mu_1,\mu_2)$.

Lemma 2.7. *Let $\mu_1,\mu_2\in M_+(\overline{\mathbb{D}})$. Then $\mathscr{M}_{z_1}$ and $\mathscr{M}_{z_2}$ are bounded linear operators on $\mathcal{D}(\mu_1,\mu_2)$.*

Proof. Let $f\in\mathcal{D}(\mu_1,\mu_2)$. So both $I_{\mu_1,\mu_2}(f)$ and $B_{\mu_1,\mu_2}(f)$ are finite. From the proof of [9, Lemma 3.4], there exists constant $C\geq 1$ such that

$$B_{\mu_1,\mu_2}(z_1f)\leq C\Big(\|f\|_{H^2(\mathbb{D}^2)}^2+B_{\mu_1,\mu_2}(f)\Big)<\infty.$$

Now we show that $I_{\mu_1,\mu_2}(z_1f)<\infty$. In view of Lemma 2.6 it is enough to show that

$$\sup_{0<r<1}\int_0^{2\pi}\int_{\mathbb{D}}|f(\zeta_1,re^{i\theta})|^2 d\mu_1(\zeta_1)\frac{d\theta}{2\pi}<\infty. \tag{16}$$

The following idea is motivated from [4, Proposition 1.6]. Let $h\in H^2(\mathbb{D}^2)$ and assume that $h(0,z_2)=0$ for all $z_2\in\mathbb{D}$. By the division property of $H^2(\mathbb{D}^2)$ (see

[9, Remark 4.2]) there exists $g \in H^2(\mathbb{D}^2)$ such that $h(z_1,z_2) = z_1 g(z_1,z_2)$ for all $(z_1,z_2) \in \mathbb{D}^2$. Then for each $r \in (0,1)$, $\theta \in [0,2\pi]$ and $\zeta_1 \in \mathbb{D}$, using (14) we get

$$\begin{aligned}\left\|\frac{h(\cdot,re^{i\theta}) - h(\zeta_1,re^{i\theta})}{z_1-\zeta_1}\right\|_{H^2}^2 &= |g(\zeta_1,re^{i\theta})|^2 + \left\|\frac{g(\cdot,re^{i\theta}) - g(\zeta_1,re^{i\theta})}{z_1-\zeta_1}\right\|_{H^2}^2 \\ &\geq |g(\zeta_1,re^{i\theta})|^2 \\ &\geq |\zeta_1|^2|g(\zeta_1,re^{i\theta})|^2 \\ &= |h(\zeta_1,re^{i\theta})|^2. \end{aligned} \tag{17}$$

Thus,

$$\begin{aligned}&\sup_{0<r<1}\int_0^{2\pi}\int_{\mathbb{D}} |f(\zeta_1,re^{i\theta})|^2 d\mu_1(\zeta_1)\frac{d\theta}{2\pi} \\ &= \sup_{0<r<1}\int_0^{2\pi}\int_{\mathbb{D}} |f(\zeta_1,re^{i\theta}) - f(0,re^{i\theta}) + f(0,re^{i\theta})|^2 d\mu_1(\zeta_1)\frac{d\theta}{2\pi} \\ &\leq 2\sup_{0<r<1}\int_0^{2\pi}\int_{\mathbb{D}} |f(\zeta_1,re^{i\theta}) - f(0,re^{i\theta})|^2 d\mu_1(\zeta_1)\frac{d\theta}{2\pi} \\ &+ 2\sup_{0<r<1}\int_0^{2\pi}\int_{\mathbb{D}} |f(0,re^{i\theta})|^2 d\mu_1(\zeta_1)\frac{d\theta}{2\pi} \\ &\overset{(17)}{\leq} 2\sup_{0<r<1}\int_0^{2\pi}\int_{\mathbb{D}} \left\|\frac{f(\cdot,re^{i\theta}) - f(\zeta_1,re^{i\theta})}{z_1-\zeta_1}\right\|_{H^2(\mathbb{D})}^2 d\mu_1(\zeta_1)\frac{d\theta}{2\pi} \\ &+ 2\mu_1(\mathbb{D})\sup_{0<r<1}\int_0^{2\pi} |f(0,re^{i\theta})|^2\frac{d\theta}{2\pi}. \end{aligned} \tag{18}$$

Let us assume that $f(z_1,z_2) = \sum_{m,n\geq 0} a_{m,n} z_1^m z_2^n$. As $f \in H^2(\mathbb{D}^2)$ by using dominated convergence theorem (see [26, p. 88])

$$\sup_{0<r<1}\int_0^{2\pi} |f(0,re^{i\theta})|^2\frac{d\theta}{2\pi} = \sum_{n\geq 0}|a_{0,n}|^2 \leq \|f\|_{H^2(\mathbb{D}^2)}^2. \tag{19}$$

Combining (18) with (12) and (19) yields (16).

Hence we conclude that $z_1 f \in \mathcal{D}(\mu_1,\mu_2)$. Similarly, one can show show that $z_2 f \in \mathcal{D}(\mu_1,\mu_2)$. Since $\mathcal{D}(\mu_1,\mu_2)$ is a reproducing kernel Hilbert space so using the closed graph theorem we conclude the result. □

Let $\kappa : \mathbb{D}\times\mathbb{D} \to \mathbb{C}$ denote the reproducing kernel of $\mathcal{D}(\mu_1,\mu_2)$.

Corollary 2.8. *For any $w \in \mathbb{D}^2$, $\ker(\mathscr{M}_z - w) = \{0\}$ and $\ker(\mathscr{M}_z^* - w)$ is the one-dimensional space spanned by $\kappa(\cdot,\overline{w})$.*

Proof. The proof goes the same as that of [9, Corollary 3.9]. □

The following lemma recovers a counterpart of [22, Lemma 2.1] for $\mathcal{D}(\mu)$.

Lemma 2.9. *For any $\mu \in M_+(\overline{\mathbb{D}})$, $\mathcal{D}(\mu)$ has the Gleason property.*

Proof. Here $\mathscr{M}_z$ is cyclic on $\mathcal{D}(\mu)$ then for each $\lambda \in \mathbb{D}$ the dim $\ker(\mathscr{M}_z^* - \overline{\lambda})$ is at most one (see [3, Proposition 1.1]). Let k denote the reproducing kernel of $\mathcal{D}(\mu)$. Then $k(\cdot, \lambda) \in \ker(\mathscr{M}_z^* - \overline{\lambda})$ so $\ker(\mathscr{M}_z^* - \overline{\lambda})$ is spanned by $k(\cdot, \lambda)$. For any $h \in \mathcal{D}(\mu)$ by the reproducing property we know that $h - h(\lambda)$ is orthogonal to $k(\cdot, \lambda)$. That means $h - h(\lambda)$ belongs the range closure of $(\mathscr{M}_z - \lambda)$. Since $\mathscr{M}_z$ is 2-concave, by [23, Lemma 1(a)] $\mathscr{M}_z$ is expansive on $\mathcal{D}(\mu)$ so $(\mathscr{M}_z - \lambda)$ is bounded below and hence range of $(\mathscr{M}_z - \lambda)$ is closed. Thus there exists $g \in \mathcal{D}(\mu)$ such that $h(z) - h(\lambda) = (z - \lambda)g(z)$ for $z \in \mathbb{D}$. □

The next proposition shows that $\mathcal{D}(\mu_1, \mu_2)$ has the division property.

Proposition 2.10. *Let $\mu_1, \mu_2 \in M_+(\overline{\mathbb{D}})$. Then $\mathcal{D}(\mu_1, \mu_2)$ has the division property.*

Proof. Let $f \in \mathcal{D}(\mu_1, \mu_2)$ and $\lambda \in \mathbb{D}$ such that $f(\lambda, z_2) = 0$ for each $z_2 \in \mathbb{D}$. We are required to show $\frac{f}{z_1 - \lambda}$ belongs $\mathcal{D}(\mu_1, \mu_2)$. Since $H^2(\mathbb{D}^2)$ has the division property (see [9, Lemma 4.1]), there exists $g \in H^2(\mathbb{D}^2)$ such that $f(z_1, z_2) = (z_1 - \lambda)g(z_1, z_2)$ for $z_1, z_2 \in \mathbb{D}$. Now it boils down to show $g \in \mathcal{D}(\mu_1, \mu_2)$. From (10) it is clear that $(z_1 - \lambda)g(\cdot, re^{i\theta}) \in \mathcal{D}(\mu_1)$ and $(re^{i\theta} - \lambda)g(re^{i\theta}, \cdot) \in \mathcal{D}(\mu_2)$ for every $r \in (0,1)$ and almost every $\theta \in [0, 2\pi]$. Clearly, for every $r \in (0,1)$ and almost every $\theta \in [0, 2\pi]$, (by Lemma 2.9) $g(\cdot, re^{i\theta}) \in \mathcal{D}(\mu_1)$ and $g(re^{i\theta}, \cdot) \in \mathcal{D}(\mu_2)$. The multiplication operator $\mathscr{M}_w$ by the coordinate function w is expansive on $\mathcal{D}(\mu_j)$, $j = 1, 2$. So

$$||g(\cdot, re^{i\theta})||_{\mathcal{D}(\mu_1)} \le ||wg(\cdot, re^{i\theta})||_{\mathcal{D}(\mu_1)}.$$

The rest of the proof is similar to [9, Proof of Theorem 2.2]. For the sake of completeness, we are providing the full argument. Note that

$$\begin{aligned} ||(w - \lambda)g(\cdot, re^{i\theta})||_{\mathcal{D}(\mu_1)} &\ge (1 - |\lambda|)^2 ||g(\cdot, re^{i\theta})||_{\mathcal{D}(\mu_1)} \\ &\ge (1 - |\lambda|)^2 \mathcal{D}_{\mu_1}(g(\cdot, re^{i\theta})). \end{aligned}$$

Integrating both sides with respect to θ

$$\begin{aligned} &(1 - |\lambda|)^2 \int_0^{2\pi} \mathcal{D}_{\mu_1}(g(\cdot, re^{i\theta})) \frac{d\theta}{2\pi} \\ &\le \int_0^{2\pi} ||(w - \lambda)g(\cdot, re^{i\theta})||^2_{H^2(\mathbb{D})} \frac{d\theta}{2\pi} + \int_0^{2\pi} \mathcal{D}_{\mu_1}((w - \lambda)g(\cdot, re^{i\theta})) \frac{d\theta}{2\pi}. \end{aligned}$$

By using [9, Lemma 3.2] and (11) we get that

$$(1 - |\lambda|)^2 \int_0^{2\pi} \mathcal{D}_{\mu_1}(g(\cdot, re^{i\theta})) \frac{d\theta}{2\pi} \le ||(z_1 - \lambda)g||^2_{H^2(\mathbb{D}^2)} + \mathcal{D}_{\mu_1, \mu_2}((z_1 - \lambda)g).$$

Taking supremum over $0 < r < 1$ on the above inequality gives

$$\sup_{0<r<1} \int_0^{2\pi} \mathcal{D}_{\mu_1}(g(\cdot, re^{i\theta})) \frac{d\theta}{2\pi} < \infty. \tag{20}$$

We already have

$$\sup_{0<r<1} \int_0^{2\pi} \mathcal{D}_{\mu_2}((re^{i\theta} - \lambda)g(re^{i\theta}, \cdot))\frac{d\theta}{2\pi} < \infty.$$

So for any $s \in (|\lambda|, 1)$,

$$\begin{aligned} &\sup_{0<r<1} \int_0^{2\pi} \mathcal{D}_{\mu_2}((re^{i\theta} - \lambda)g(re^{i\theta}, \cdot))\frac{d\theta}{2\pi} \\ \geq\ & \int_0^{2\pi} \mathcal{D}_{\mu_2}((se^{i\theta} - \lambda)g(se^{i\theta}, \cdot))\frac{d\theta}{2\pi} \\ \geq\ & (s - |\lambda|)^2 \int_0^{2\pi} \mathcal{D}_{\mu_2}(g(se^{i\theta}, \cdot))\frac{d\theta}{2\pi}. \end{aligned}$$

Now taking limit $s \to 1$ gives

$$\lim_{s\to 1} \int_0^{2\pi} \mathcal{D}_{\mu_2}(g(se^{i\theta}, \cdot))\frac{d\theta}{2\pi} < \infty. \tag{21}$$

Since [9, Lemma 1.1] suggest that we can replace the limit by supremum so combining (20) and (21) yields $g \in \mathcal{D}(\mu_1, \mu_2)$.

Similarly one can start with the assumption that $f(z_1, \lambda) = 0$ for all $z_1 \in \mathbb{D}$ and show that $\frac{f}{z_2-\lambda} \in \mathcal{D}(\mu_1, \mu_2)$. □

As an application of the Proposition 2.10, we have the following:

Lemma 2.11. *Gleason's problem for $\mathcal{D}(\mu_1, \mu_2)$ has solution over $\{(\lambda_1, \lambda_2) \in \mathbb{D}^2 : \lambda_1\lambda_2 = 0\}$.*

Proof. Let $f \in \mathcal{D}(\mu_1, \mu_2)$ and $\lambda \in \mathbb{D}$. It is clear from the Definition 1.1 that

$$\begin{aligned} \mathcal{D}_{\mu_1,\mu_2}(f(\cdot, 0)) &= \mathcal{D}_{\mu_1}(f(\cdot, 0)) \leq \mathcal{D}_{\mu_1,\mu_2}(f), \\ \mathcal{D}_{\mu_1,\mu_2}(f(0, \cdot)) &= \mathcal{D}_{\mu_2}(f(0, \cdot)) \leq \mathcal{D}_{\mu_1,\mu_2}(f). \end{aligned}$$

Consider the function $h(z_1, z_2) = f(z_1, z_2) - f(z_1, 0)$, $(z_1, z_2) \in \mathbb{D}^2$. Then $h \in \mathcal{D}(\mu_1, \mu_2)$. By Proposition 2.10, there exists $f_1 \in \mathcal{D}(\mu_1, \mu_2)$ such that

$$h(z_1, z_2) = f(z_1, z_2) - f(z_1, 0) = (z_2 - 0)f_1(z_1, z_2), \quad (z_1, z_2) \in \mathbb{D}^2. \tag{22}$$

Since $\mathcal{D}_{\mu_1}(f(\cdot, 0)) < \infty$ so $f(\cdot, 0) \in \mathcal{D}(\mu_1)$. Applying Lemma 2.9 to $\mathcal{D}(\mu_1)$ we get that for each $\lambda \in \mathbb{D}$, there exists $v \in \mathcal{D}(\mu_1)$ such that

$$f(z_1, 0) - f(\lambda, 0) = (z_1 - \lambda)v(z_1), \quad z_1 \in \mathbb{D}. \tag{23}$$

Now adding (22) and (23) gives us

$$f(z_1, z_2) - f(\lambda, 0) = (z_1 - \lambda)v(z_1) + z_2 f_1(z_1, z_2), \quad z_1, z_2 \in \mathbb{D}.$$

Defining $f_2(z_1, z_2) = v(z_1)$, $z_1, z_2 \in \mathbb{D}$ shows that $f_2 \in \mathcal{D}(\mu_1, \mu_2)$. Thus the Gleason's problem has solution at $(\lambda, 0)$ for every $\lambda \in \mathbb{D}$.

Similarly, starting with $H(z_1, z_2) = f(z_1, z_2) - f(0, z_2)$ for $z_1, z_2 \in \mathbb{D}$, one can show that the Gleason's problem can be solved on $\{(0, \lambda) : \lambda \in \mathbb{D}\}$. □

Theorem 2.12. *Let $\mu_1, \mu_2 \in M_+(\overline{\mathbb{D}})$. Then the followings hold:*

(*a*) *the commuting pair $\mathscr{M}_z = (\mathscr{M}_{z_1}, \mathscr{M}_{z_2})$ is cyclic with cyclic vector* 1,
(*b*) *Gleason's problem can be solved for $\mathcal{D}(\mu_1, \mu_2)$ on an open neighborhood of $\{(\lambda_1, \lambda_2) \in \mathbb{D}^2 : \lambda_1\lambda_2 = 0\}$.*

Proof. (a) Combining Lemmas 2.7, 2.5 yields the result.
(b) Lemma 2.11 suggests that Gleason's problem has solution over

$$A = \{(\lambda_1, \lambda_2) \in \mathbb{D}^2 : \lambda_1\lambda_2 = 0\}.$$

So for each $\lambda = (\lambda_1, \lambda_2) \in A$ the row operator $T_\lambda := [\mathscr{M}_{z_1} - \lambda_1 \ \mathscr{M}_{z_2} - \lambda_2]$ has closed range. With the help of Corollary 2.8

$$\begin{aligned} \dim\left(\mathcal{D}(\mu_1, \mu_2)/T_\lambda(\mathcal{D}(\mu_1, \mu_2) \oplus \mathcal{D}(\mu_1, \mu_2))\right) &= \dim\ker T_\lambda^* \\ &= \dim\ker(\mathscr{M}_z^* - \overline{\lambda}) = 1. \end{aligned}$$

Now using [9, Lemma 4.4] and the fact that the joint kernel $\ker(\mathscr{M}_z - \lambda) = \{0\}$, we conclude that the pair $\mathscr{M}_z - \lambda$ is Fredholm. Thus A is in the complement of the essential spectrum $\sigma_e(\mathscr{M}_z)$ of $\mathscr{M}_z$. Since $\sigma_e(\mathscr{M}_z)$ is closed, there exists an open subset V of $\mathbb{D}^2 \setminus \sigma_e(\mathscr{M}_z)$ containing A. Applying [9, Lemma 5.1] completes the proof. □

3. A Representation theorem

Let μ be a finite positive Borel measure on $\overline{\mathbb{D}}$. For two non-negative integers i and j, the (i, j)*-th moment of* μ (see [7, 13, 18, 20]) is defined as

$$\hat{\mu}\{i, j\} = \int_{\overline{\mathbb{D}}} (\overline{\zeta})^i \zeta^j d\mu(\zeta).$$

Proposition 3.1. *Let i and j be two nonnegative integers and $i \leq j$. Then*

$$\langle z^i, z^j \rangle_{\mathcal{D}(\mu)} = \delta(i, j) + \sum_{k=0}^{i-1} \hat{\mu}\{j - k - 1, i - k - 1\},$$

where $\delta(\cdot, \cdot)$ denotes the two variable Kronecker delta function.

Proof. Substituting (9) in (5) gives

$$\|g\|^2_{\mathcal{D}(\mu)} = \|g\|^2_{H^2(\mathbb{D})} + \int_{\overline{\mathbb{D}}} \mathcal{D}_\zeta(g) d\mu(\zeta), \ g \in \mathcal{D}(\mu).$$

Using polarization identity in the above equation gives

$$\begin{aligned}\langle z^i,\, z^j\rangle_{\mathcal{D}(\mu)} &= \langle z^i,\, z^j\rangle_{H^2(\mathbb{D})} + \int_{\overline{\mathbb{D}}} \Big\langle \frac{z^i-\zeta^i}{z-\zeta}, \frac{z^j-\zeta^j}{z-\zeta}\Big\rangle_{H^2(\mathbb{D})} d\mu(\zeta)\\ &= \delta(i,j) + \int_{\overline{\mathbb{D}}} \sum_{k=0}^{i-1} \zeta^{i-1-k}(\overline{\zeta})^{j-1-k} d\mu(\zeta)\\ &= \delta(i,j) + \sum_{k=0}^{i-1} \int_{\overline{\mathbb{D}}} \zeta^{i-1-k}(\overline{\zeta})^{j-1-k} d\mu(\zeta)\\ &= \delta(i,j) + \sum_{k=0}^{i-1} \hat{\mu}\{j-k-1, i-k-1\}.\end{aligned}$$

Hence the result. □

Next, we derive a formula for the inner product of monomials in $\mathcal{D}(\mu_1,\mu_2)$.

Proposition 3.2. *Let $\mu_1,\mu_2 \in M_+(\overline{\mathbb{D}})$ and $m,n,p,q \in \mathbb{N}$ then*

$$\langle z_1^m z_2^n,\, z_1^p z_2^q\rangle_{\mathcal{D}(\mu_1,\mu_2)} = \begin{cases} 0 & \text{if } m \neq p,\ n \neq q,\\ \langle z_2^n,\, z_2^q\rangle_{\mathcal{D}(\mu_2)} & \text{if } m = p,\ n\neq q,\\ \langle z_1^m,\, z_1^p\rangle_{\mathcal{D}(\mu_1)} & \text{if } m\neq p,\ n = q,\\ \|z_1^m\|^2_{\mathcal{D}(\mu_1)} + \|z_2^n\|^2_{\mathcal{D}(\mu_2)} - 1 & \text{if } m = p,\ n = q.\end{cases}$$

Proof. Using the polarization identity on (6) gives

$$\begin{aligned}&\langle z_1^m z_2^n,\, z_1^p z_2^q\rangle_{\mathcal{D}(\mu_1,\mu_2)} = \delta(m,p)\delta(n,q)\\ &+ \lim_{r\to 1}\int_0^{2\pi}\int_{\mathbb{D}} r^{n+q}e^{i(n-q)\theta} mp z_1^{m-1}(\overline{z}_1)^{p-1} U_{\mu_1}(z_1) dA(z_1)\frac{d\theta}{2\pi}\\ &+ \lim_{r\to 1}\int_0^{2\pi}\int_{\mathbb{D}} r^{m+p}e^{i(m-p)\theta} nq z_2^{n-1}(\overline{z}_2)^{q-1} U_{\mu_2}(z_2) dA(z_2)\frac{d\theta}{2\pi}\\ &= \delta(n,q)\delta(m,p) + \delta(n,q)\int_{\mathbb{D}} m z_1^{m-1} p(\overline{z}_1)^{p-1} U_{\mu_1}(z_1) dA(z_1)\\ &+ \delta(m,p)\int_{\mathbb{D}} n z_2^{n-1} q(\overline{z}_2)^{q-1} U_{\mu_2}(z_2) dA(z_2).\end{aligned}$$

Rest follows from (4) and polarisation identity. □

An immediate corollary of the of the above proposition is the following:

Corollary 3.3. *For $\mu_1,\mu_2 \in M_+(\overline{\mathbb{D}})$, the subspace spanned by the constant vector 1 in $\mathcal{D}(\mu_1,\mu_2)$ is a wandering subspace for $\mathscr{M}_z$ on $\mathcal{D}(\mu_1,\mu_2)$.*

Recall that a commuting pair $T = (T_1, T_2)$ is called a *toral 2-isometry* (see [9, Eq (1.1)]) if it satisfies the equations $I - T_i^*T_i - T_j^*T_j + T_i^*T_j^*T_iT_j = 0$ for

$i, j = 1, 2$, i.e. $\beta_\alpha(T) = 0$ for $\alpha \in \{(2,0), (1,1), (0,2)\}$. For future references we state the following lemma concerning toral 2-isometry (see [9, Corollary 3.8]).

Lemma 3.4. *Let the supports of μ_1 and μ_2 be contained in the unit circle. Then the commuting pair $\mathscr{M}_z$ on $\mathcal{D}(\mu_1, \mu_2)$ is a cyclic toral 2-isometry with cyclic vector 1.*

The following is a noteworthy observation regarding the commuting pair $\mathscr{M}_z$ on $\mathcal{D}(\mu_1, \mu_2)$.

Lemma 3.5. *Let $\mu_1, \mu_2 \in M_+(\overline{\mathbb{D}})$. Then $\mathscr{M}_z = (\mathscr{M}_{z_1}, \mathscr{M}_{z_2})$ is a toral completely hyperexpansive 2-tuple with zero defect operator on $\mathcal{D}(\mu_1, \mu_2)$.*

Proof. Let $n \geq 2$ and $f \in \mathcal{D}(\mu_1, \mu_2)$.

$$\begin{aligned}
&\langle \beta_{(n,0)}(\mathscr{M}_z)(f),\, f\rangle \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \|z_1^k f\|^2 \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \Big(\|z_1^k f\|^2_{H^2(\mathbb{D}^2)} + I_{\mu_1,\mu_2}(z_1^k f) + B_{\mu_1,\mu_2}(z_1^k f) \Big) \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \Big(\|z_1^k f\|^2_{H^2(\mathbb{D}^2)} + B_{\mu_1,\mu_2}(z_1^k f) \Big) \\
&+ \sum_{k=0}^{n} (-1)^k \binom{n}{k} I_{\mu_1,\mu_2}(z_1^k f).
\end{aligned}$$

By Lemma 3.4 and the fact that every 2-isometry is automatically a k-isometry for each $k \geq 2$ (see [1, 2]), the first part of the above sum is zero. So we are left with

$$\langle \beta_{(n,0)}(\mathscr{M}_z)(f),\, f\rangle = \sum_{k=0}^{n} (-1)^k \binom{n}{k} I_{\mu_1,\mu_2}(z_1^k f). \tag{24}$$

Let $k \geq 1$. Now replacing f by $z_1^k f$ in (15) gives us

$$I_{\mu_1,\mu_2}(z_1^{k+1} f) - I_{\mu_1,\mu_2}(z_1^k f) = \lim_{r \to 1} \int_0^{2\pi} \int_{\mathbb{D}} |\zeta_1|^{2k} |f(\zeta_1, re^{i\theta})|^2 d\mu_1(\zeta_1) \frac{d\theta}{2\pi}.$$

Thus (24) becomes

$$
\begin{aligned}
&\langle \beta_{(n,0)}(\mathscr{M}_z)(f),\, f\rangle \\
&= \sum_{k=0}^{n}(-1)^k\binom{n}{k} I_{\mu_1,\mu_2}(z_1^k f) \\
&= \sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\left(I_{\mu_1,\mu_2}(z_1^k f) - I_{\mu_1,\mu_2}(z_1^{k+1} f)\right) \\
&= -\lim_{r\to 1}\int_0^{2\pi}\int_{\mathbb{D}}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}|\zeta_1|^{2k}|f(\zeta_1, re^{i\theta})|^2 d\mu_1(\zeta_1)\frac{d\theta}{2\pi} \\
&= -\lim_{r\to 1}\int_0^{2\pi}\int_{\mathbb{D}}(1-|\zeta_1|^2)^{n-1}|f(\zeta_1, re^{i\theta})|^2 d\mu_1(\zeta_1)\frac{d\theta}{2\pi} \le 0. \qquad (25)
\end{aligned}
$$

Similarly, $\langle \beta_{(0,n)}(\mathscr{M}_z)(f),\, f\rangle \le 0$. We now show that the defect operator $\beta_{(1,1)}(\mathscr{M}_z)$ of $\mathscr{M}_z$ is zero. If we replace f by $z_2 f$ in (15),

$$
\begin{aligned}
I_{\mu_1,\mu_2}(z_1 z_2 f) - I_{\mu_1,\mu_2}(z_2 f) &= \lim_{r\to 1}\int_0^{2\pi}\int_{\mathbb{D}} r^2|f(\zeta_1, re^{i\theta})|^2 d\mu_1(\zeta_1)\frac{d\theta}{2\pi} \\
&= \lim_{r\to 1}\int_0^{2\pi}\int_{\mathbb{D}} |f(\zeta_1, re^{i\theta})|^2 d\mu_1(\zeta_1)\frac{d\theta}{2\pi}. \qquad (26)
\end{aligned}
$$

So applying Lemma 3.4 we have

$$
\begin{aligned}
&\|f\|^2 - \|\mathscr{M}_{z_1} f\|^2 - \|\mathscr{M}_{z_2} f\|^2 + \|\mathscr{M}_{z_1}\mathscr{M}_{z_2} f\|^2 \\
&= \left(I_{\mu_1,\mu_2}(f) - I_{\mu_1,\mu_2}(z_1 f)\right) + \left(I_{\mu_1,\mu_2}(z_1 z_2 f) - I_{\mu_1,\mu_2}(z_2 f)\right) \\
&\overset{(15)\&(26)}{=} 0.
\end{aligned}
$$

From the discussion after (1) it is clear that $\beta_\alpha(T) \le 0$ for all $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$. Hence the result. □

The next lemma is extracted from [9, Lemma 6.1] and very useful in the proof of the main theorem.

Lemma 3.6. *Let $T = (T_1, T_2)$ is a commuting pair on $\mathcal{H}$ such that the defect operator (see (3)) is zero. Assume that* $\ker T^*$ *is a wandering subspace of T. Then for each $f_0 \in \ker T^*$,*

$$
\langle T_1^m T_2^n f_0,\, T_1^p T_2^q f_0\rangle_{\mathcal{H}} = \begin{cases} 0 & \text{if } m \ne p,\ n \ne q, \\ \langle T_2^n f_0,\, T_2^q f_0\rangle_{\mathcal{H}} & \text{if } m = p,\ n \ne q, \\ \langle T_1^m f_0,\, T_1^p f_0\rangle_{\mathcal{H}} & \text{if } m \ne p,\ n = q, \\ \|T_1^m f_0\|_{\mathcal{H}}^2 + \|T_2^n f_0\|_{\mathcal{H}}^2 - \|f_0\|_{\mathcal{H}}^2 & \text{if } m = p,\ n = q. \end{cases}
$$

Proof. From the proof of [9, Lemma 6.1(i)] we get that whenever $I - T_1^*T_1 - T_2^*T_2 + T_1^*T_2^*T_1T_2 = 0$, for each $k, l \geq 0$,

$$T_1^{*k}T_2^{*l}T_1^kT_2^l = T_1^{*k}T_1^k + T_2^{*l}T_2^l - I. \tag{27}$$

Since $\ker T^*$ is a wandering subspace, using (27) and following the proof of [9, Lemma 6.1(ii)] one recovers the required result. □

Proof of Theorem 1.2. $(B) \implies (A)$ This follows from Corollaries 2.8, 3.3 and Lemma 3.5.

$(A) \implies (B)$ As T is analytic on $\mathcal{H}$ so is T_1 and T_2. Fix $j \in \{1, 2\}$. Consider the T_j invariant subspace $\mathcal{H}_j = \overline{span}\{T_j^k f_0 : k \geq 0\}$ of $\mathcal{H}$. Then $T_j|_{\mathcal{H}_j}$ is cyclic analytic completely hyperexpansive operator i.e. an operator of Dirichlet-type (refer to [4, Definition 1.2, p.70]). Hence, by [4, Theorem 2.5, p.79] there exists a unique measure $\mu_j \in M_+(\overline{\mathbb{D}})$ and a unitary operator $U_j : \mathcal{H}_j \to \mathcal{D}(\mu_j)$ such that

$$U_jf_0 = 1, \quad U_jT_j = \mathscr{M}_w^{(j)}U_j, \tag{28}$$

where $\mathscr{M}_w^{(j)}$ denotes the multiplication by coordinate function w on $\mathcal{D}(\mu_j)$. Now consider the map U as

$$U(T_1^mT_2^nf_0) = z_1^mz_2^n, \quad m, n \geq 0.$$

Here we have $\mathcal{H} = \overline{span}\{T_1^mT_2^nf_0 : m, n \geq 0\}$ and $\mathcal{D}(\mu_1, \mu_2) = \overline{span}\{z_1^mz_2^n : m, n \geq 0\}$. For any $m, p \geq 0$, by (28)

$$\begin{aligned} \langle T_j^mf_0, T_j^pf_0\rangle_{\mathcal{H}} &= \langle U_jT_j^mf_0, U_jT_j^pf_0\rangle_{\mathcal{D}(\mu_j)} \\ &= \langle (M_w^{(j)})^mU_jf_0, (M_w^{(j)})^pU_jf_0\rangle_{\mathcal{D}(\mu_j)} \\ &= \langle w^m, w^p\rangle_{\mathcal{D}(\mu_j)}. \end{aligned}$$

Now combining Lemma 3.6 and Proposition 3.2 yields

$$\langle T_1^mT_2^nf_0, T_1^pT_2^qf_0\rangle_{\mathcal{H}} = \langle z_1^mz_2^n, z_1^pz_2^q\rangle_{\mathcal{D}(\mu_1,\mu_2)}, \; m, n, p, q \geq 0.$$

So U extends as a unitary from $\mathcal{H}$ onto $\mathcal{D}(\mu_1, \mu_2)$. Hence the result. □

Corollary 3.7. *The commuting pair $\mathscr{M}_z = (\mathscr{M}_{z_1}, \mathscr{M}_{z_2})$ is a toral 2-isometry if and only if μ_1 and μ_2 are supported on $\partial\mathbb{D}$.*

Proof. As $\beta_{(2,0)}(\mathscr{M}_z) = 0$ putting $n = 2$ in (25) gives

$$0 = \langle \beta_{(2,0)}(\mathscr{M}_z)(f), f\rangle = -\lim_{r\to 1}\int_0^{2\pi}\int_{\mathbb{D}}(1 - |\zeta_1|^2)|f(\zeta_1, re^{i\theta})|^2d\mu_1(\zeta_1)\frac{d\theta}{2\pi}.$$

By substituting $f = 1$ into the equation above, we find that the support of μ_1 lies outside $\mathbb{D}$. Similarly, $\beta_{(0,2)}(\mathscr{M}_z) = 0$ implies the support of μ_2 is outside $\mathbb{D}$. Conversely, if you assume that μ_1 and μ_2 are supported on the unit circle $\mathbb{T}$ then by [9, Theorem 2.4] $\mathscr{M}_z = (\mathscr{M}_{z_1}, \mathscr{M}_{z_2})$ becomes a toral 2-isometry. □

The following theorem is an extended version of [9, Theorem 6.4](cf. [25, Theorem 5.2]).

Theorem 3.8. *For $i = 1,2$ consider $\mu_1^{(i)}, \mu_2^{(i)} \in M_+(\overline{\mathbb{D}})$. Then the multiplication 2-tuple $\mathscr{M}_z^{(1)}$ on $\mathcal{D}(\mu_1^{(1)}, \mu_2^{(1)})$ is unitarily equivalent to $\mathscr{M}_z^{(2)}$ on $\mathcal{D}(\mu_1^{(2)}, \mu_2^{(2)})$ if and only if $\mu_j^{(1)} = \mu_j^{(2)}$, $j = 1,2$.*

Proof. Let $\nu_1, \nu_2 \in M_+(\overline{\mathbb{D}})$ and p be a two variable polynomial. We have $\mathcal{D}_{\nu_1,\nu_2}(z_1 p) = I_{\nu_1,\nu_2}(z_1 p) + B_{\nu_1,\nu_2}(z_1 p)$. By (15),

$$I_{\nu_1,\nu_2}(z_1 p) = \int_0^{2\pi} \int_{\mathbb{D}} |p(\zeta_1, e^{i\theta})|^2 d\nu_1(\zeta_1) \frac{d\theta}{2\pi} + I_{\nu_1,\nu_2}(p). \tag{29}$$

From [9, Lemma 3.5] we get that

$$B_{\nu_1,\nu_2}(z_1 p) = \int_0^{2\pi} \int_{\mathbb{T}} |p(\zeta_1, e^{i\theta})|^2 d\nu_1(\zeta_1) \frac{d\theta}{2\pi} + B_{\nu_1,\nu_2}(p). \tag{30}$$

Now combining (29) and (30) together with (7) gives

$$\|z_1 p\|^2 = \|p\|^2 + \int_0^{2\pi} \int_{\overline{\mathbb{D}}} |p(\zeta_1, e^{i\theta})|^2 d\nu_1(\zeta_1) \frac{d\theta}{2\pi}. \tag{31}$$

One can get a similar expression for $z_2 p$. Let U be a unitary map from $\mathcal{D}(\mu_1^{(1)}, \mu_2^{(1)})$ onto $\mathcal{D}(\mu_1^{(2)}, \mu_2^{(2)})$ which satisfies

$$U \mathscr{M}_{z_j}^{(1)} = \mathscr{M}_{z_j}^{(2)} U, \ j = 1,2. \tag{32}$$

Since the joint kernels $\ker \mathscr{M}_z^{(1)*}$ and $\ker \mathscr{M}_z^{(2)*}$ are spanned by the constant function 1 (see Crorllary 2.8) so (32) suggests that $U^*1 \in \ker \mathscr{M}_z^{(1)*}$. Hence U^*1 must be a unimodular constant. By multiplying suitable unimodular constant one can assume that $U1 = 1$. It now follows from (32) that U is identity on the polynomials. Thus (31) suggests that for any two-variable polynomial p,

$$\begin{aligned}
\int_0^{2\pi} \int_{\overline{\mathbb{D}}} |p(\zeta_1, e^{i\theta})|^2 d\mu_1^{(1)}(\zeta_1) \frac{d\theta}{2\pi} &= \int_0^{2\pi} \int_{\overline{\mathbb{D}}} |p(\zeta_1, e^{i\theta})|^2 d\mu_1^{(2)}(\zeta_1) \frac{d\theta}{2\pi}, \\
\int_0^{2\pi} \int_{\overline{\mathbb{D}}} |p(e^{i\theta}, \zeta_2)|^2 d\mu_2^{(1)}(\zeta_2) \frac{d\theta}{2\pi} &= \int_0^{2\pi} \int_{\overline{\mathbb{D}}} |p(e^{i\theta}, \zeta_2)|^2 d\mu_2^{(2)}(\zeta_2) \frac{d\theta}{2\pi}.
\end{aligned}$$

Thus for any one variable polynomial p,

$$\int_{\overline{\mathbb{D}}} |p(\zeta)|^2 d\mu_j^{(1)}(\zeta) = \int_{\overline{\mathbb{D}}} |p(\zeta)|^2 d\mu_j^{(2)}(\zeta), \quad j = 1,2.$$

Using polarization identity and the uniqueness of the two-variable moment problem on $\overline{\mathbb{D}}$ (see [7, Remark 1, p. 321]) we conclude the theorem. □

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References

[1] Agler, Jim. A disconjugacy theorem for Toeplitz operators. *Amer. J. Math.* **112** (1990), 1–14. MR1037599, Zbl 0708.47020, doi: 10.2307/2374849. 100, 112

[2] Agler, Jim; Stankus, Mark. *m*-isometric transformations of Hilbert space. I. *Integral Equations Operator Theory* **21** (1995), no. 4, 383–429. MR1321694, Zbl 0836.47008, doi: 10.1007/BF01222016. 112

[3] Albrecht, Ernst; Wirtz, Peter Maria. Multicyclic systems of commuting operators, in *Linear operators in function spaces (Timişoara, 1988)*, 39–61, *Oper. Theory Adv. Appl.* **43**, Birkhäuser, Basel. MR1090116, Zbl 0734.47009, doi: 10.1007/978-3-0348-7250-8_2. 108

[4] Aleman, Alexandru. *The multiplication operator on Hilbert spaces of analytic functions*. Habilitationsschrift, Hagen, 1993. 100, 101, 102, 103, 104, 106, 114

[5] Athavale, Ameer. On completely hyperexpansive operators. *Proc. Amer. Math. Soc.* **124** (1996), no. 12, 3745–3752. MR1353373, Zbl 0863.47017, doi: 10.1090/S0002-9939-96-03609-X. 100, 101

[6] Athavale, Ameer; Sholapurkar, Vinayak M. Completely hyperexpansive operator tuples. *Positivity* **3** (1999), no. 3, 245–257. MR1708652, Zbl 0985.47003, doi: 10.1023/A:1009719803199. 100

[7] Atzmon, Aharon. A moment problem for positive measures on the unit disc. *Pacific J. Math.* **59** (1975), 317–325. MR0402057, Zbl 0319.44009, doi: 10.2140/pjm.1975.59.317. 110, 115

[8] Bao, Guanlong; Göğüş, Nihat Gökhan; Pouliasis, Stamatis. On Dirichlet spaces with a class of superharmonic weights. *Canad. J. Math.* **70** (2018), no. 4, 721–741. MR3813510, Zbl 1410.30028, doi: 10.4153/CJM-2017-005-1. 101

[9] Bera, Santu; Chavan, Sameer; Ghara, Soumitra. Dirichlet-type spaces of the unit bidisc and toral 2-isometries. *Canad. J. Math.* **77** (2025), no. 4, 1271–1293. MR4921124, Zbl 08058266, doi: 10.4153/S0008414X24000300. 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115

[10] Bhattacharjee, Monojit; Gupta, Rajeev; Venugopal, Vidhya. Dirichlet-type spaces in the unit bidisc and wandering subspace property for operator tuples. *arXiv preprint* (2024). arXiv:2406.16541. 101, 102

[11] Bhattacharjee, Monojit; Eschmeier, Jörg; Keshari, Dinesh Kumar; Sarkar, Jaydeb. Dilations, wandering subspaces, and inner functions. *Linear Algebra Appl.* **523** (2017), 263–280. MR3624676, Zbl 06766346, doi: 10.1016/j.laa.2017.02.032. 102

[12] Choe, Boo Rim; Koo, Hyungwoon; Yi, Heungsu. Gleason's problem for harmonic Bergman and Bloch functions on half-spaces. *Integral Equations Operator Theory* **36** (2000), no. 3, 269–287. MR1753420 Zbl 0947.31002 doi: 10.1007/BF01213925 102

[13] Devinatz, Allen. Two parameter moment problems. *Duke Math. J.* **24** (1957), 481–498. MR0092019, Zbl 0081.10104, doi: 10.1215/S0012-7094-57-02455-9. 110

[14] El-Fallah, Omar; Kellay, Karim; Mashreghi, Javad; Ransford, Thomas. *A primer on the Dirichlet space*. Cambridge Tracts in Mathematics, 203, Cambridge University Press, Cambridge, 2014. MR3185375, Zbl 1304.30002, doi: 10.1017/CBO9781107239425. 105

[15] El-Fallah, Omar; Kellay, Karim; Klaja, Hubert; Mashreghi, Javad; Ransford, Thomas. Dirichlet spaces with superharmonic weights and de Branges–Rovnyak spaces. *Complex Anal. Oper. Theory* **10** (2016), no. 1, 97–107. MR3441469, Zbl 1341.30046, doi: 10.1007/s11785-015-0474-7. 101, 103, 104

[16] Eschmeier, Jörg; Langendörfer, Sebastian. Multivariable Bergman shifts and Wold decompositions. *Integral Equations Operator Theory* **90** (2018), Paper No. 56, 17 pp. MR3830213, Zbl 06965953, doi: 10.1007/s00020-018-2481-3. 100

[17] Halmos, Paul R. Shifts on Hilbert spaces. *J. Reine Angew. Math.* **208** (1961), 102–112. MR0152896, Zbl 0107.09802, doi: 10.1515/crll.1961.208.102. 102

[18] Haviland, E. K. On the momentum problem for distribution functions in more than one dimension. *Amer. J. Math.* **57** (1935), no. 3, 562–568. MR1507095, Zbl 0015.10901, doi: 10.2307/2371187. 110

[19] Pouliasis, Stamatis. Weighted Dirichlet spaces that are de Branges–Rovnyak spaces with equivalent norms. *J. Funct. Anal.* **288** (2025), no. 3, Paper No. 110717. MR4815908, Zbl 1555.30061, doi: 10.1016/j.jfa.2024.110717. 101

[20] Putinar, Mihai. A two-dimensional moment problem. *J. Funct. Anal.* **80** (1988), no. 1, 1–8. MR0960218, Zbl 0668.47024, doi: 10.1016/0022-1236(88)90060-2. 110

[21] Ransford, Thomas. *Potential theory in the complex plane*. London Math. Soc. Stud. Texts, 28, Cambridge University Press, Cambridge, 1995. MR1334766, Zbl 0828.31001, doi: 10.1017/CBO9780511623776. 101

[22] Richter, Stefan. Invariant subspaces in Banach spaces of analytic functions. *Trans. Amer. Math. Soc.* **304** (1987), 585–616. MR0911086, Zbl 0646.47023, doi: 10.1090/S0002-9947-1987-0911086-8. 107

[23] Richter, Stefan. Invariant subspaces of the Dirichlet shift. *J. Reine Angew. Math.* **386** (1988), 205–220. MR0936999, Zbl 0635.46021, doi: 10.1515/crll.1988.386.205. 102, 108

[24] Richter, Stefan; Sundberg, Carl. A formula for the local Dirichlet integral. *Michigan Math. J.* **38** (1991), no. 3, 355–379. MR1116495, Zbl 0768.30040, doi: 10.1307/mmj/1029004388. 103

[25] Richter, Stefan. A representation theorem for cyclic analytic two-isometries. *Trans. Amer. Math. Soc.* **328** (1991), 325–349. MR1013337, Zbl 0762.47009, doi: 10.1090/S0002-9947-1991-1013337-1. 100, 101, 102, 115

[26] Royden, Halsey L. *Real analysis*. Macmillan, New York, 1963. MR0151555, Zbl 0121.05501. 107

[27] Rudin, Walter. *Function theory in polydiscs*. W. A. Benjamin, New York-Amsterdam, 1969. MR0255841, Zbl 0177.34101, doi: 10.1002/zamm.19730530624. 101

[28] Ren, Guangbin; Kähler, Uwe. Weighted harmonic Bloch spaces and Gleason's problem. *Complex Var. Theory Appl.* **48** (2003), no. 3, 235–245. MR1968839 Zbl 1037.31004 doi: 10.1080/0278107031000073605 102

[29] Zhu, Ke He. The Bergman spaces, the Bloch space, and Gleason's problem. *Trans. Amer. Math. Soc.* **309** (1988), no. 1, 253–268. MR0931533 Zbl 0657.32002 doi: 10.2307/2001168. 102

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