

Factorization of the Misiurewicz-Thurston polynomials

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ABSTRACT. This note provides the complete factorization of the Misiurewicz-Thurston polynomial $q_{\ell,n}$ over \mathbb{C} , which plays a central role in the study of the Mandelbrot set and Multibrot sets. The roots can be classified into two categories. First, there are hyperbolic points $\text{Hyp}(k)$ for any divisor k of n , which are parameters whose critical orbits are of exact period k . Those are roots of $q_{\ell,n}$ with multiplicity $(d-1)\left\lfloor \frac{\ell-1}{k} \right\rfloor + 2$. Next are the points $\text{Mis}(j,k)$ for $2 \leq j \leq \ell$ whose critical orbits are pre-periodic of exact period k with an exact pre-period j . Those are simple roots of $q_{\ell,n}$.

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1. Introduction

In this article, we are interested in the factorization of two important families of polynomials. For any integer $d \geq 2$, the first family is defined recursively by

$$p_0(z) = 0, \quad p_{n+1}(z) = p_n(z)^d + z. \quad (1)$$

For $n \geq 1$, one has $\deg p_n = d^{n-1}$. The second family is a two parameters family defined for $\ell, n \in \mathbb{N}$ by

$$q_{\ell,n}(z) = p_{\ell+n}(z) - p_\ell(z). \quad (2)$$

The factorisation of p_n over \mathbb{C} is well known [DH82] (at least for $d = 2$) and is recalled first to fix essential notations. The main result of this article is the proof of Theorem 2 below, which gives the complete factorization of $q_{\ell,n}$ in terms of simple factors whose dynamical significance is explained briefly in the next

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section. This result plays a role in the systematic computation of hyperbolic centers and of pre-periodic parameters in the Mandelbrot set, up to period $n = 41$ for p_n and all types $\ell + n \leq 35$ for $q_{\ell,n}$, as detailed in [MihV25], yet remains of general interest.

2. About the Mandelbrot set

The polynomials p_n and $q_{\ell,n}$ are closely related to the *Mandelbrot set* \mathcal{M} (when $d = 2$, Fig. 1) for $d > 2$ the set \mathcal{M}_d is called the *Multibrot* of degree d . This set is composed of the parameters $c \in \mathbb{C}$ for which the sequence $(p_n(c))_{n \in \mathbb{N}}$ remains bounded. This sequence is the orbit by iterated compositions of the only critical point $z = 0$ of the map $f_c(z) = f_{d,c}(z) = z^d + c$, *i.e.*

$$\forall n \in \mathbb{N}, \quad f_c^n(0) = \underbrace{f_c \circ \dots \circ f_c}_{n \text{ times}}(0) = p_n(c). \tag{3}$$

For $c \in \mathbb{C}$, the dynamics of f_c splits $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ in two complementary sets. The *Fatou set* \mathcal{F}_c is the open subset composed by the points $z \in \overline{\mathbb{C}}$ in the neighborhood of which, the sequence $(f_c^n)_{n \in \mathbb{N}}$ is a normal family *i.e.* pre-compact in the topology of local uniform convergence. On the contrary, on the *Julia set* $\mathcal{J}_c = \overline{\mathbb{C}} \setminus \mathcal{F}_c$, the dynamics are chaotic. Both \mathcal{F}_c and \mathcal{J}_c are fully invariant (*i.e.* invariant sets of the forward and backwards dynamics) and that $c \in \mathcal{M}$ if and only if \mathcal{J}_c is connected. For a review of the properties of Fatou and Julia sets, see *e.g.* [CG93], [Mil90] for polynomial and rational maps and [Ber93], [MaPRW25] for entire and meromorphic functions.

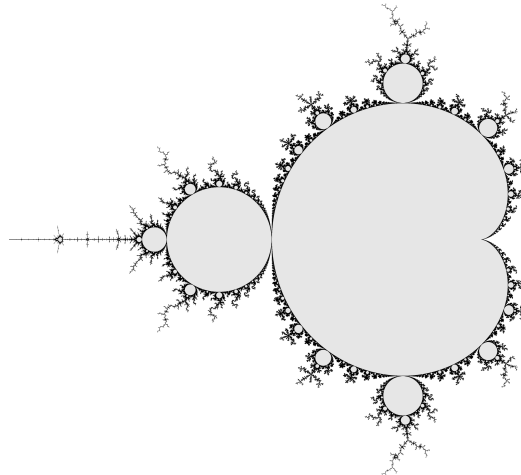


FIGURE 1. *The Mandelbrot set \mathcal{M} with $\partial\mathcal{M}$ in black and in gray, the interior of \mathcal{M} .*

2.1. Hyperbolic centers. For $n \geq 1$, the roots of p_n are parameters $c \in \mathcal{M}$, called *hyperbolic centers*, whose critical orbits are periodic of period n . The roots of p_n are simple [DH82, Eps12] and the polynomial $p_n(z)$ is divisible by $p_k(z)$ for any divisor k of n . A. Douady and J.H. Hubbard [DH82, DH84, DH85] have shown that the hyperbolic centers are interior points of \mathcal{M} , with at most one hyperbolic center per connected component of the interior. The set of all hyperbolic centers is also dense in the boundary of \mathcal{M} in the sense that its closure in \mathbb{C} contains $\partial\mathcal{M}$.

Let us define the set of hyperbolic centers of *order* $n \geq 1$ as the subset of $p_n^{-1}(0)$ whose minimal (or fundamental) period is exactly n ; in other terms:

$$\text{Hyp}(n) = \{z \in p_n^{-1}(0) \mid \forall k \in \text{Div}(n)^*, p_k(z) \neq 0\} \quad (4)$$

where $\text{Div}(n) = \{k \in \mathbb{N}^* ; \exists k' \in \mathbb{N}, kk' = n\}$ and $\text{Div}(n)^* = \text{Div}(n) \setminus \{n\}$ is the set of strict divisors of n . For example, $\text{Hyp}(1) = \{0\}$ and $\text{Hyp}(2) = \{z \in \mathbb{C} ; z^{d-1} = -1\}$. The reduced polynomial, also known as Gleason's polynomial, is:

$$h_n(z) = \prod_{r \in \text{Hyp}(n)} (z - r). \quad (5)$$

The polynomials p_n and h_n have integer coefficients even for $d = 2$. While expected, the irreducibility of h_n over $\mathbb{Z}[z]$ remains conjectural; see [HT15, last remark of §3] and [SS17, p.155].

Theorem 1 ([DH82]). *The complete factorization of p_n is*

$$p_n(z) = \prod_{k|n} h_k(z). \quad (6)$$

Moreover, the cardinal of $\text{Hyp}(n)$ is given by

$$|\text{Hyp}(n)| = \sum_{k|n} \mu(n/k) d^{k-1} \quad (7)$$

where μ is the Möbius function, i.e.

$$\mu(n) = \begin{cases} (-1)^v & \text{if } n \text{ is square free and has } v \text{ distinct prime factors,} \\ 0 & \text{if } n \text{ is not square free.} \end{cases}$$

The notation $k|n$ means $k \in \text{Div}(n)$ and $|S|$ denotes the cardinal of a finite set. The identity (6) is well known and is key in the count of hyperbolic centers. The Online Encyclopedia of Integer Sequences [OEIS] attributes (7) to Warren D. Smith and Robert Munafo (2000), with no published reference. For the variant $d \geq 3$, see [HT15].

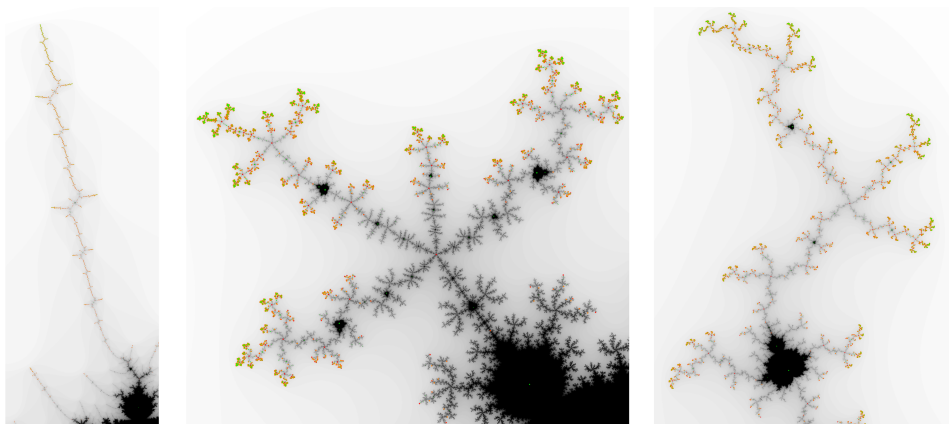


FIGURE 2. The hyperbolic points $\text{Hyp}(n)$ for $d = 2$ and $n \leq 18$ in green and the Misiurewicz-Thurston parameters $M_{\ell,n} = \text{Mis}(\ell, n)$ with $\ell + n \leq 16$ in red in different parts of the Mandelbrot set \mathcal{M} , with $\Re c$ increasing from left to right between images.

2.2. Pre-periodic or Misiurewicz-Thurston parameters. For integers $n \geq 1$ and $\ell \geq 2$, the roots of $q_{\ell,n}$ (defined in (2)) are parameters $c \in \partial\mathcal{M}_d$, called *pre-periodic parameters* or *Misiurewicz-Thurston points*, whose critical orbits are pre-periodic, that is it becomes periodic of period n after the first ℓ steps. For $\ell \in \{0, 1\}$, one can check immediately that

$$q_{0,n}(z) = p_n(z) \quad \text{and} \quad q_{1,n}(z) = p_n^d(z). \tag{8}$$

The dynamical reason for the second identity in (8) is that 0 is the only pre-image of c under f_c so a pre-periodic orbit of type $(1, n)$ starting at zero actually loops back to zero, so is periodic of period n . The polynomial $q_{\ell,n}(z)$ is divisible by $q_{\ell,k}(z)$ for any divisor k of n and by $q_{\ell',n}(z)$ for any $\ell' \leq \ell$. Some of the roots of $q_{\ell,n}(z)$ have multiplicity; however, the polynomial

$$s_{\ell,n}(z) = \frac{q_{\ell,n}(z)}{q_{\ell-1,n}(z)} \in \mathbb{Z}[z] \tag{9}$$

has simple roots [Eps12, Theorem 2], [HT15, Lemma 3.1] and [Buf18]. Douady-Hubbard [DH82, DH84, DH85] have shown that the set of all pre-periodic points is dense in the boundary of \mathcal{M} . Visually, those points are either branch tips, centers of spirals or points where branches meet (see Fig. 2).

By analogy with the hyperbolic case, let us define the set of *Misiurewicz points of type (ℓ, n)* as the subset of $q_{\ell,n}^{-1}(0)$ whose dynamical parameters are exactly ℓ and n ; in other terms:

$$\text{Mis}(\ell, n) = \left\{ z \in q_{\ell,n}^{-1}(0) \mid \begin{array}{l} q_{\ell-1,n}(z) \neq 0, \\ \forall k \in \text{Div}(n)^*, q_{\ell,k}(z) \neq 0 \end{array} \right\}. \tag{10}$$

We call $\ell + n$ the *order* of $\text{Mis}(\ell, n)$ because $q_{\ell, n}$ is a polynomial of degree $d^{\ell+n-1}$. The reduced polynomial whose roots are exactly $\text{Mis}(\ell, n)$ is denoted by

$$m_{\ell, n}(z) = \prod_{r \in \text{Mis}(\ell, n)} (z - r) \in \mathbb{Z}[z]. \quad (11)$$

Note that $\text{Mis}(0, n) = \text{Hyp}(n)$ and $\text{Mis}(1, n) = \emptyset$ because of (8). The count of the Misiurewicz points has been established by Benjamin Hutz and Adam Towsley [HT15, Cor. 3.3]:

$$|\text{Mis}(\ell, n)| = \Phi(\ell, n) |\text{Hyp}(n)| \quad (12)$$

where

$$\Phi(\ell, n) = \begin{cases} 1 & \text{if } \ell = 0, \\ d^\ell - d^{\ell-1} - d + 1 & \text{if } \ell \neq 0 \text{ and } n|\ell - 1, \\ d^\ell - d^{\ell-1} & \text{otherwise,} \end{cases}$$

and $|\text{Hyp}(n)|$ is given by (7).

To get a complete factorization of $q_{\ell, n}(z)$ in terms of (5) and (11), one needs to understand the multiplicity of the hyperbolic factors. We claim the following result.

Theorem 2. *For $\ell \in \mathbb{N}$ and $n \in \mathbb{N}^*$, one has*

$$q_{\ell, n}(z) = p_n^d(z) \prod_{j=2}^{\ell} s_{j, n}(z) = \prod_{k|n} \left(h_k(z)^{\eta_\ell(k)} \prod_{j=2}^{\ell} m_{j, k}(z) \right) \quad (13)$$

where the multiplicity $\eta_\ell(k)$ is given by¹

$$\eta_\ell(k) = (d-1) \left\lfloor \frac{\ell-1}{k} \right\rfloor + d. \quad (14)$$

In other words, the roots of $q_{\ell, n}$ are composed of the points $\text{Hyp}(k)$ for any divisor k of n , which are roots with multiplicity $\eta_\ell(k)$, and of the points $\text{Mis}(j, k)$ for $2 \leq j \leq \ell$, which are simple roots.

We propose a direct proof in Section 3. In [MihV25], we use this result to compare the performance of a new splitting algorithm in the case of simple roots, like for p_n , or in the presence of high-multiplicity roots, like for $q_{\ell, n}$. Knowing the exact multiplicity is crucial for the interpretation of the numerical benchmarks.

¹In equation (14), the notation $\lfloor \cdot \rfloor$ represents the floor function, i.e. $\lfloor x \rfloor = n$ if $n \in \mathbb{Z}$ and $x \in [n, n+1)$.

3. Proof of Theorem 2

Let us give here a direct proof of the factorization theorem.

Proof. The first identity $q_{\ell,n}(z) = p_n^d \prod_{j=2}^{\ell} s_{j,n}$ is a direct consequence of definition (9) and formula (8). The term p_n^d and formula (6) explain the constant term in expression (14). We fix $\ell \geq 2$ and decompose $s_{\ell,n}$.

As observed in Theorem 2 in [Eps12],

$$s_{\ell,n} = \prod_{\substack{\omega^d=1 \\ \omega \neq 1}} \Delta_{\ell,n}^{\omega}, \tag{15}$$

where $\Delta_{\ell,n}^{\omega} = p_{n+\ell-1} - \omega p_{\ell-1}$, and all $\Delta_{\ell,n}^{\omega}$ ($\omega \neq 1$) have simple roots. Moreover, it is immediate that for $\omega_1 \neq \omega_2$ (of which one may be 1)

$$\gcd(\Delta_{\ell,n}^{\omega_1}, \Delta_{\ell,n}^{\omega_2}) = \gcd(p_{n+\ell-1}, p_{\ell-1}),$$

that is, a common root of $\Delta_{\ell,n}^{\omega_1}$ and $\Delta_{\ell,n}^{\omega_2}$ is also a common root of $p_{n+\ell-1}$ and of $p_{\ell-1}$ and thus of all $\Delta_{\ell,n}^{\omega}$. In turn, by formula (15), such a root becomes a root of order $d - 1$ of $s_{\ell,n}$ (see also Lemma 3.1 in [HT15]). In particular, all other roots of $s_{\ell,n}$ are simple and disjoint from the roots of $\Delta_{\ell,n}^1 = q_{\ell-1,n}$.

As by factorization (6),

$$\gcd(p_{n+\ell-1}, p_{\ell-1}) = p_{\gcd(n+\ell-1, \ell-1)} = p_{\gcd(n, \ell-1)},$$

we may conclude that

$$s_{\ell,n} = \left(\prod_{k | \gcd(n, \ell-1)} h_k^{d-1} \right) \left(\prod_{k | n} m_{\ell,k} \right), \tag{16}$$

which, combined with the first part of relation (13) and the fact that there are $\lfloor \frac{\ell-1}{k} \rfloor$ multiples of k in $\llbracket 1, \ell - 1 \rrbracket$, proves the result. □

Example 1. *The simplest case of a composite period for $d = 2$ is:*

$$q_{2,4}(z) = p_6(z) - p_2(z) = p_5^2(z) - p_1^2(z) = q_{1,4}(z)(p_5(z) + p_1(z)) = p_4^2(z)(p_4^2(z) + 2z).$$

As $p_4 = h_1 h_2 h_4$, the polynomial $q_{2,4}$ is divisible by h_4^2 but not by h_4^3 . The exponent of Hyp(4) is therefore exactly 2 and we are left with a single hyperbolic factor, namely Hyp(2). Using formulas (13) and (16),

$$q_{2,4}(z) = h_1^3(z) h_2^2(z) h_4^2(z) m_{2,1}(z) m_{2,2}(z) m_{2,4}(z)$$

and

$$s_{2,4}(z) = p_4^2(z) + 2z = h_1(z) m_{2,1}(z) m_{2,2}(z) m_{2,4}(z).$$

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