New York Journal of Mathematics

New York J. Math. **32** (2026) 1010–1016.

Factorization of the Misiurewicz-Thurston polynomials

Nicolae Mihalache and François Vigneron

ABSTRACT. This note provides the complete factorization of the Misiurewicz-Thurston polynomial $q_{\ell,n}$ over $\mathbb{C}$, which plays a central role in the study of the Mandelbrot set and Multibrot sets. The roots can be classified into two categories. First, there are hyperbolic points $\mathrm{Hyp}(k)$ for any divisor k of n, which are parameters whose critical orbits are of exact period k. Those are roots of $q_{\ell,n}$ with multiplicity $(d-1)\left\lfloor \frac{\ell-1}{k} \right\rfloor + 2$. Next are the points $\mathrm{Mis}(j,k)$ for $2 \leq j \leq \ell$ whose critical orbits are pre-periodic of exact period k with an exact pre-period j. Those are simple roots of $q_{\ell,n}$.

CONTENTS

1. Introduction

In this article, we are interested in the factorization of two important families of polynomials. For any integer $d \geq 2$, the first family is defined recursively by

$$p_0(z) = 0, \qquad p_{n+1}(z) = p_n(z)^d + z. \tag{1}$$

For $n \geq 1$, one has $\deg p_n = d^{n-1}$. The second family is a two parameters family defined for $\ell, n \in \mathbb{N}$ by

$$q_{\ell,n}(z) = p_{\ell+n}(z) - p_\ell(z). \tag{2}$$

The factorisation of p_n over $\mathbb{C}$ is well known [DH82] (at least for $d = 2$) and is recalled first to fix essential notations. The main result of this article is the proof of Theorem 2 below, which gives the complete factorization of $q_{\ell,n}$ in terms of simple factors whose dynamical significance is explained briefly in the next

Received June 13, 2025.
2020 *Mathematics Subject Classification.* 37F10, 26A18, 12D05.
Key words and phrases. Misiurewicz-Thurston polynomials; Mandelbrot set; Multibrot set; Factorization.
Research supported by ANR-25-CE40-4532 (Smash).

ISSN 1076-9803/2026

section. This result plays a role in the systematic computation of hyperbolic centers and of pre-periodic parameters in the Mandelbrot set, up to period $n = 41$ for p_n and all types $\ell + n \leq 35$ for $q_{\ell,n}$, as detailed in [MihV25], yet remains of general interest.

2. About the Mandelbrot set

The polynomials p_n and $q_{\ell,n}$ are closely related to the *Mandelbrot* set $\mathcal{M}$ (when $d = 2$, Fig. 1) for $d > 2$ the set $\mathcal{M}_d$ is called the Multibrot of degree d. This set is composed of the parameters $c \in \mathbb{C}$ for which the sequence $(p_n(c))_{n\in\mathbb{N}}$ remains bounded. This sequence is the orbit by iterated compositions of the only critical point $z = 0$ of the map $f_c(z) = f_{d,c}(z) = z^d + c$, *i.e.*

$$\forall n \in \mathbb{N}, \qquad f_c^n(0) = \underbrace{f_c \circ \ldots \circ f_c}_{n \text{ times}}(0) = p_n(c). \tag{3}$$

For $c \in \mathbb{C}$, the dynamics of f_c splits $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ in two complementary sets. The *Fatou* set $\mathcal{F}_c$ is the open subset composed by the points $z \in \overline{\mathbb{C}}$ in the neighborhood of which, the sequence $(f_c^n)_{n\in\mathbb{N}}$ is a normal family *i.e.* precompact in the topology of local uniform convergence. On the contrary, on the *Julia* set $\mathcal{J}_c = \overline{\mathbb{C}} \setminus \mathcal{F}_c$, the dynamics are chaotic. Both $\mathcal{F}_c$ and $\mathcal{J}_c$ are fully invariant (*i.e.* invariant sets of the forward and backwards dynamics) and that $c \in \mathcal{M}$ if and only if $\mathcal{J}_c$ is connected. For a review of the properties of Fatou and Julia sets, see *e.g.* [CG93], [Mil90] for polynomial and rational maps and [Ber93], [MaPRW25] for entire and meromorphic functions.

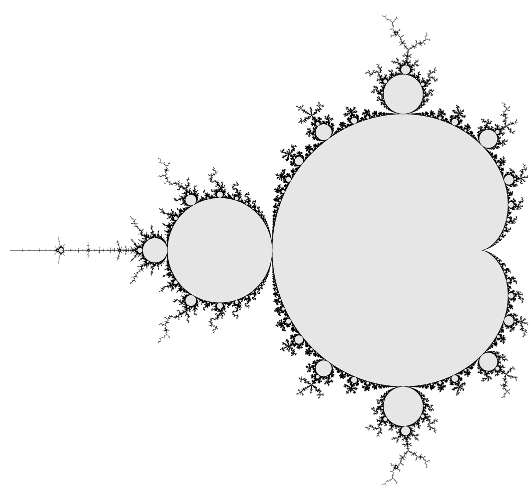

FIGURE 1. *The Mandelbrot set $\mathcal{M}$ with $\partial\mathcal{M}$ in black and in gray, the interior of $\mathcal{M}$.*

2.1. Hyperbolic centers. For $n \geq 1$, the roots of p_n are parameters $c \in \mathcal{M}$, called *hyperbolic centers*, whose critical orbits are periodic of period n. The roots of p_n are simple [DH82, Eps12] and the polynomial $p_n(z)$ is divisible by $p_k(z)$ for any divisor k of n. A. Douady and J.H. Hubbard [DH82, DH84, DH85] have shown that the hyperbolic centers are interior points of $\mathcal{M}$, with at most one hyperbolic center per connected component of the interior. The set of all hyperbolic centers is also dense in the boundary of $\mathcal{M}$ in the sense that its closure in $\mathbb{C}$ contains $\partial\mathcal{M}$.

Let us define the set of hyperbolic centers of *order* $n \geq 1$ as the subset of $p_n^{-1}(0)$ whose minimal (or fundamental) period is exactly n; in other terms:

$$\mathrm{Hyp}(n) = \left\{z \in p_n^{-1}(0) \,\middle|\, \forall k \in \mathrm{Div}(n)^*,\ p_k(z) \neq 0\right\} \tag{4}$$

where $\mathrm{Div}(n) = \{k \in \mathbb{N}^* ; \exists k' \in \mathbb{N}, \quad kk' = n\}$ and $\mathrm{Div}(n)^* = \mathrm{Div}(n)\backslash\{n\}$ is the set of strict divisors of n. For example, $\mathrm{Hyp}(1) = \{0\}$ and $\mathrm{Hyp}(2) = \{z \in \mathbb{C} ; z^{d-1} = -1\}$. The reduced polynomial, also known as Gleason's polynomial, is:

$$h_n(z) = \prod_{r \in \mathrm{Hyp}(n)} (z - r). \tag{5}$$

The polynomials p_n and h_n have integer coefficients even for $d = 2$. While expected, the irreducibility of h_n over $\mathbb{Z}[z]$ remains conjectural; see [HT15, last remark of §3] and [SS17, p.155].

Theorem 1 ([DH82]). *The complete factorization of p_n is*

$$p_n(z) = \prod_{k|n} h_k(z). \tag{6}$$

Moreover, the cardinal of Hyp(n) *is given by*

$$|\mathrm{Hyp}(n)| = \sum_{k|n} \mu(n/k) d^{k-1} \tag{7}$$

where μ is the Möbius function, i.e.

$$\mu(n) = \begin{cases} (-1)^\nu & \text{if } n \text{ is square free and has } \nu \text{ distinct prime factors,} \\ 0 & \text{if } n \text{ is not square free.} \end{cases}$$

The notation $k|n$ means $k \in \mathrm{Div}(n)$ and $|S|$ denotes the cardinal of a finite set. The identity (6) is well known and is key in the count of hyperbolic centers. The Online Encyclopedia of Integer Sequences [OEIS] attributes (7) to Warren D. Smith and Robert Munafo (2000), with no published reference. For the variant $d \geq 3$, see [HT15].

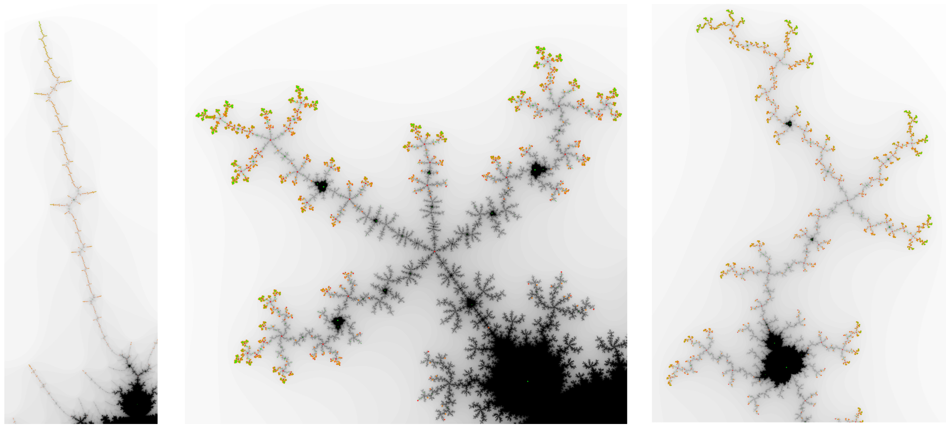

FIGURE 2. The hyperbolic points $\mathrm{Hyp}(n)$ for $d = 2$ and $n \leq 18$ in green and the Misiurewicz-Thurston parameters $M_{\ell,n} = \mathrm{Mis}(\ell, n)$ with $\ell + n \leq 16$ in red in different parts of the Mandelbrot set $\mathcal{M}$, with $\Re\mathrm{e}\, c$ increasing from left to right between images.

2.2. Pre-periodic or Misiurewicz-Thurston parameters. For integers $n \geq 1$ and $\ell \geq 2$, the roots of $q_{\ell,n}$ (defined in (2)) are parameters $c \in \partial\mathcal{M}_d$, called *pre-periodic parameters* or *Misiurewicz-Thurston* points, whose critical orbits are pre-periodic, that is it becomes periodic of period n after the first ℓ steps. For $\ell \in \{0, 1\}$, one can check immediately that

$$q_{0,n}(z) = p_n(z) \qquad \text{and} \qquad q_{1,n}(z) = p_n^d(z). \tag{8}$$

The dynamical reason for the second identity in (8) is that 0 is the only pre-image of c under f_c so a pre-periodic orbit of type $(1, n)$ starting at zero actually loops back to zero, so is periodic of period n. The polynomial $q_{\ell,n}(z)$ is divisible by $q_{\ell,k}(z)$ for any divisor k of n and by $q_{\ell',n}(z)$ for any $\ell' \leq \ell$. Some of the roots of $q_{\ell,n}(z)$ have multiplicity; however, the polynomial

$$s_{\ell,n}(z) = \frac{q_{\ell,n}(z)}{q_{\ell-1,n}(z)} \in \mathbb{Z}[z] \tag{9}$$

has simple roots [Eps12, Theorem 2], [HT15, Lemma 3.1] and [Buf18]. Douady–Hubbard [DH82, DH84, DH85] have shown that the set of all pre-periodic points is dense in the boundary of $\mathcal{M}$. Visually, those points are either branch tips, centers of spirals or points where branches meet (see Fig. 2).

By analogy with the hyperbolic case, let us define the set of *Misiurewicz points of type* (ℓ, n) as the subset of $q_{\ell,n}^{-1}(0)$ whose dynamical parameters are exactly ℓ and n; in other terms:

$$\mathrm{Mis}(\ell, n) = \left\{ z \in q_{\ell,n}^{-1}(0) \,\middle|\, \begin{array}{l} q_{\ell-1,n}(z) \neq 0, \\ \forall k \in \mathrm{Div}(n)^*,\ q_{\ell,k}(z) \neq 0 \end{array} \right\}. \tag{10}$$

We call $\ell + n$ the *order* of $\operatorname{Mis}(\ell, n)$ because $q_{\ell,n}$ is a polynomial of degree $d^{\ell+n-1}$. The reduced polynomial whose roots are exactly $\operatorname{Mis}(\ell, n)$ is denoted by

$$m_{\ell,n}(z) = \prod_{r \in \operatorname{Mis}(\ell,n)} (z - r) \in \mathbb{Z}[z]. \tag{11}$$

Note that $\operatorname{Mis}(0, n) = \operatorname{Hyp}(n)$ and $\operatorname{Mis}(1, n) = \emptyset$ because of (8). The count of the Misiurewicz points has been established by Benjamin Hutz and Adam Towsley [HT15, Cor. 3.3]:

$$\left|\operatorname{Mis}(\ell, n)\right| = \Phi(\ell, n) \left|\operatorname{Hyp}(n)\right| \tag{12}$$

where

$$\Phi(\ell, n) = \begin{cases} 1 & \text{if } \ell = 0, \\ d^{\ell} - d^{\ell-1} - d + 1 & \text{if } \ell \neq 0 \text{ and } n | \ell - 1, \\ d^{\ell} - d^{\ell-1} & \text{otherwise,} \end{cases}$$

and $|\operatorname{Hyp}(n)|$ is given by (7).

To get a complete factorization of $q_{\ell,n}(z)$ in terms of (5) and (11), one needs to understand the multiplicity of the hyperbolic factors. We claim the following result.

Theorem 2. *For $\ell \in \mathbb{N}$ and $n \in \mathbb{N}^*$, one has*

$$q_{\ell,n}(z) = p_n^d(z) \prod_{j=2}^{\ell} s_{j,n}(z) = \prod_{k|n} \left(h_k(z)^{\eta_\ell(k)} \prod_{j=2}^{\ell} m_{j,k}(z) \right) \tag{13}$$

where the multiplicity $\eta_\ell(k)$ is given by[1]

$$\eta_\ell(k) = (d-1) \left\lfloor \frac{\ell - 1}{k} \right\rfloor + d. \tag{14}$$

In other words, the roots of $q_{\ell,n}$ are composed of the points $\operatorname{Hyp}(k)$ *for any divisor k of n, which are roots with multiplicity $\eta_\ell(k)$, and of the points* $\operatorname{Mis}(j, k)$ *for $2 \le j \le \ell$, which are simple roots.*

We propose a direct proof in Section 3. In [MihV25], we use this result to compare the performance of a new splitting algorithm in the case of simple roots, like for p_n, or in the presence of high-multiplicity roots, like for $q_{\ell,n}$. Knowing the exact multiplicity is crucial for the interpretation of the numerical benchmarks.

[1] In equation (14), the notation $\lfloor \cdot \rfloor$ represents the floor function, *i.e.* $\lfloor x \rfloor = n$ if $n \in \mathbb{Z}$ and $x \in [n, n+1)$.

3. Proof of Theorem 2

Let us give here a direct proof of the factorization theorem.

Proof. The first identity $q_{\ell,n}(z) = p_n^d \prod_{j=2}^{\ell} s_{j,n}$ is a direct consequence of definition (9) and formula (8). The term p_n^d and formula (6) explain the constant term in expression (14). We fix $\ell \geq 2$ and decompose $s_{\ell,n}$.

As observed in Theorem 2 in [Eps12],

$$s_{\ell,n} = \prod_{\substack{\omega^d=1 \\ \omega\neq 1}} \Delta^{\omega}_{\ell,n}, \tag{15}$$

where $\Delta^{\omega}_{\ell,n} = p_{n+\ell-1} - \omega p_{\ell-1}$, and all $\Delta^{\omega}_{\ell,n}$ $(\omega \neq 1)$ have simple roots. Moreover, it is immediate that for $w_1 \neq \omega_2$ (of which one may be 1)

$$\gcd\left(\Delta^{\omega_1}_{\ell,n}, \Delta^{\omega_2}_{\ell,n}\right) = \gcd(p_{n+\ell-1}, p_{\ell-1}),$$

that is, a common root of $\Delta^{\omega_1}_{\ell,n}$ and $\Delta^{\omega_2}_{\ell,n}$ is also a common root of $p_{n+\ell-1}$ and of $p_{\ell-1}$ and thus of all $\Delta^{\omega}_{\ell,n}$. In turn, by formula (15), such a root becomes a root of order $d-1$ of $s_{\ell,n}$ (see also Lemma 3.1 in [HT15]). In particular, all other roots of $s_{\ell,n}$ are simple and disjoint from the roots of $\Delta^{1}_{\ell,n} = q_{\ell-1,n}$.

As by factorization (6),

$$\gcd(p_{n+\ell-1}, p_{\ell-1}) = p_{\gcd(n+\ell-1,\ell-1)} = p_{\gcd(n,\ell-1)},$$

we may conclude that

$$s_{\ell,n} = \left(\prod_{k \mid \gcd(n,\ell-1)} h_k^{d-1}\right)\left(\prod_{k\mid n} m_{\ell,k}\right), \tag{16}$$

which, combined with the first part of relation (13) and the fact that there are $\left\lfloor \frac{\ell-1}{k} \right\rfloor$ multiples of k in $[\![1, \ell-1]\!]$, proves the result.

□

Example 1. *The simplest case of a composite period for $d = 2$ is:*

$$q_{2,4}(z) = p_6(z) - p_2(z) = p_5^2(z) - p_1^2(z) = q_{1,4}(z)(p_5(z)+p_1(z)) = p_4^2(z)(p_4^2(z)+2z).$$

As $p_4 = h_1 h_2 h_4$, the polynomial $q_{2,4}$ is divisible by h_4^2 but not by h_4^3. The exponent of Hyp(4) *is therefore exactly 2 and we are left with a single hyperbolic factor, namely* Hyp(2)*. Using formulas* (13) *and* (16)*,*

$$q_{2,4}(z) = h_1^3(z) h_2^2(z) h_4^2(z) m_{2,1}(z) m_{2,2}(z) m_{2,4}(z)$$

and

$$s_{2,4}(z) = p_4^2(z) + 2z = h_1(z) m_{2,1}(z) m_{2,2}(z) m_{2,4}(z).$$

References

[Ber93] BERGWEILER, WALTER. Iteration of meromorphic functions. *Bull. Amer. Math Soc. (N.S.)* **29** (1993), no. 2, 151–188, MR1216719, Zbl 0791.30018, arXiv:math/9310226, doi: 10.1090/S0273-0979-1993-00432-4. 1011

[Buf18] BUFF, XAVIER. On postcritically finite unicritical polynomials. *New York J. Math.* **24** (2018), 1111–1122, MR3890968, Zbl 1412.37084. 1013

[CG93] CARLESON, LENNART; GAMELIN, THEODORE W. Complex dynamics. Universitext: Tracts in Mathematics. *Springer-Verlag, New York*, 1993. x+175 pp. ISBN: 0-387-97942-5. MR1230383, Zbl 0782.30022, doi: 10.1007/978-1-4612-4364-9. 1011

[DH82] DOUADY, ADRIEN; HUBBARD, JOHN HAMAL. Itération des polynômes quadratiques complexes. *C.R. Acad. Sci. Paris, Sér. I Math.* **294** (1982), no. 3, 123–126, MR651802, Zbl 0483.30014. 1010, 1012, 1013

[DH84] DOUADY, ADRIEN; HUBBARD, JOHN HAMAL. Étude dynamique des polynômes complexes. Partie I. Publ. Math. Orsay, 84-2. *Université de Paris-Sud, Département de Mathématiques, Orsay*, 1984. 75 pp. MR0762431, Zbl 0552.30018. 1012, 1013

[DH85] DOUADY, ADRIEN; HUBBARD, JOHN HAMAL. Étude dynamique des polynômes complexes. Partie II. With the collaboration of P. Lavaurs, Tan Lei and P. Sentenac. Publ. Math. Orsay, 85-4. *Université de Paris-Sud, Département de Mathématiques, Orsay*, 1985. v+154 pp. MR0812271, Zbl 0571.30026. 1012, 1013

[Eps12] EPSTEIN, ADAM. Integrality and rigidity for postcritically finite polynomials. With an appendix by Epstein and Bjorn Poonen. *Bull. Lond. Math. Soc.* **44** (2012), no. 1, 39–46, MR2881322, Zbl 1291.37057, arXiv:1010.2780, doi: 10.1112/blms/bdr059. 1012, 1013, 1015

[HT15] HUTZ, BENJAMIN; TOWSLEY, ADAM. Misiurewicz points for polynomial maps and transversality. *New York J. Math.* **21** (2015), 297–319, MR3358544, Zbl 1391.37071, arXiv:1309.4048. 1012, 1013, 1014, 1015

[MaPRW25] MARTÍ-PETE, DAVID; REMPE, LASSE; WATERMAN, JAMES. Bounded Fatou and Julia components of meromorphic functions. *Math. Ann.*, **391** (2025), no. 1, 95–111, MR4846778, Zbl 1564.37046. 1011

[MihV25] MIHALACHE, NICOLAE; VIGNERON, FRANÇOIS. How to split a tera-polynomial. To appear in *SMAI J. Comput. Math.*, (2026). arXiv:2402.06083. 1011, 1014

[Mil90] MILNOR, JOHN. Dynamics in one complex variable. Third edition. Annals of Mathematics Studies, 160. *Princeton University Press, Princeton, NJ*, 2006. viii+304 pp. ISBN: 978-0-691-12488-9; 0-691-12488-4. MR2193309, Zbl 1085.30002, doi: 10.1515/9781400835539. 1011

[OEIS] SLOANE, NEIL J. A. The Online Encyclopedia of Integer Sequences. Sequence A000740. https://oeis.org/A000740. 1012

[SS17] SCHLEICHER, DIERK; STOLL, ROBIN. Newton's method in practice: Finding all roots of polynomials of degree one million efficiently. *Theoret. Comput. Sci.* **681** (2017), 146–166, MR3659421, Zbl 1375.65068, arXiv:1508.02935, doi: 10.1016/j.tcs.2017.03.025. 1012

(Nicolae Mihalache) UNIVERSITÉ PARIS-EST CRETEIL, UMR 8050 CNRS, LAMA, F-94010 CRETEIL, FRANCE & UNIVERSITÉ GUSTAVE EIFFEL, LAMA, F-77447 MARNE-LA-VALLÉE, FRANCE
`nicolae.mihalache@u-pec.fr`

(François Vigneron) UNIVERSITÉ DE REIMS CHAMPAGNE-ARDENNE, LABORATOIRE DE MATHÉMATIQUES DE REIMS, UMR 9008 CNRS, MOULIN DE LA HOUSSE, BP 1039, F-51687 REIMS, FRANCE
`francois.vigneron@univ-reims.fr`

This paper is available via `http://nyjm.albany.edu/j/2026/32-39.html`.