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Links on incompressible surfaces and volumes

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ABSTRACT. We consider volumes of two families of links that have been the focus of recent results on geometry, namely weakly generalised alternating (WGA) links and fully augmented links (FAL). Both have known lower bounds on hyperbolic volume in terms of their diagram combinatorics, but less is known about upper bounds. In fact, Kalfagianni and Purcell recently found a family of WGA knots on a compressible surface for which there can be no upper bounds on volume in terms of twist number. They asked if upper volume bounds always exist on incompressible surfaces. We show the answer is no: we find infinite families of WGA and FALs on incompressible surfaces with no upper bound on volume in terms of twist number.

CONTENTS

1. Introduction

Given a link, the hyperbolic volume of its complement can be a powerful invariant, but is difficult to calculate in general. A useful approach is to bound the volume of a knot or link complement from above or below in terms of tangible features of its diagram; consider for example [14], which gives universal upper and lower volume bounds on a hyperbolic link in terms of its twist number. This result applies specifically to planar links; that is, links in S^3 with a diagram on S^2. Similar results on upper and lower volume bounds have been achieved for non-planar links in certain other manifolds, such as thickened tori [31] and for general thickened surfaces [1], but have not been studied in general.

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Such two-sided bounds are not guaranteed in this new setting. Howie and Purcell [7] have previously given a lower bound on the volume of weakly generalised alternating links, but in [3], Kalfagianni and Purcell show there exist such links with genus-two diagrams in S^3 with fixed twist number but arbitrarily high hyperbolic volume; that is, there can not exist an upper volume bound derived from twist number. Notably, bounding the volume from above is typically the *easier* direction — consider, for example, that any triangulation of a manifold has volume bounded above by a triangulation composed of regular tetrahedra, and any known-volume hyperbolic Dehn surgery parent of a manifold provides an upper volume bound. In contrast, lower bounds typically require difficult arguments such as specific angled structures, or applying Ricci flow to the guts of a manifold, e.g. [28, 15, 2].

This paper demonstrates that this is not always the case for links in more general manifolds. The primary result is as follows:

Theorem 3.14. *Let Σ be a closed surface of genus at least two. Let M be either the doubled thickened surface $\Sigma \times S^1$ or the mapping torus $(\Sigma \times I)/\phi$ of a map ϕ that acts nontrivially on the isotopy class of at least one essential curve in Σ. There exist families of hyperbolic fully augmented links $\{J_n\}_{n\in\mathbb{N}}$ in M such that:*

1. *each link J_n in the family $\{J_n\}_{n\in\mathbb{N}}$ projects to an incompressible embedding of Σ in M,*
2. *the number of crossing circles is a fixed natural number c for all $J_n \in \{J_n\}_{n\in\mathbb{N}}$, and*
3. *the sequence of volumes $\{\mathrm{Vol}(M \setminus J_n)\}_{n\in\mathbb{N}}$ approaches infinity as n approaches infinity.*

In particular, this shows that there does *not* exist an analogous upper bound on volume to that of [14] for links in the above manifolds. A partial converse is also discussed; Theorem 3.22 details an upper volume bound for links in mapping tori of the identity in terms of the number of crossing circles.

An additional result relates this to the weakly generalised alternating links in the construction of [3], in which Kalfagianni-Purcell detail a family of links in S^3 with no upper volume bound in terms of twist number that are projected to a compressible surface. Question 1.5 of that paper asks if linear volume bounds on twist number always exist for incompressible projection surfaces, and this result answers that question in the negative — that is, there exist weakly generalised alternating links that project to surfaces that are incompressible in the ambient manifold, but which nevertheless do not admit a linear, or indeed any, upper bound on hyperbolic volume in terms of twist number.

Corollary 3.15. *Let Σ be a closed surface of genus at least two. There exist manifolds M and families of weakly generalised alternating knots and/or links $\{K_n\}_{n\in\mathbb{N}}$ in M such that:*

1. *each $K_n \in \{K_n\}_{n\in\mathbb{N}}$ projects to an incompressible embedding of Σ in M,*
2. *the twist number is a fixed natural number c for all $K_n \in \{K_n\}_{n\in\mathbb{N}}$, and*

(3) *the sequence of volumes* $\{\mathrm{Vol}(M \setminus K_n)\}_{n\in\mathbb{N}}$ *approaches infinity as n approaches infinity.*

Fully augmented links have been studied extensively as planar diagrams in S^3, for example in [4, 14, 17, 5, 18]. The particular generalisation of fully augmented links in this paper — projecting to closed oriented surfaces with nonzero genus — enables applications to 3-manifolds beyond S^3, and similar forms have appeared previously in [1, 31, 13, 12, 16].

The geometry of links with nonplanar diagrams and in various manifolds beyond S^3 is a growing area of research. In addition to the previously mentioned weakly generalised alternating links [7], which lie on surfaces with genus in general compact 3-manifolds, various classes of links have been defined that naturally fit in this setting for various applications. Twisted torus links [20] have a natural diagram on a torus in S^3, and are among the smallest-volume hyperbolic knots. Biperiodic links [19] lie in a thickened torus, and have been used to investigate density spectra of knots for various quantities [26, 25] and spanning tree entropy [22]. Fundamental shadow links [27] lie in connect sums of $S^1 \times S^2$ and are related to the Turaev-Viro volume conjecture in quantum topology.

Of particular note are virtual links [29] in thickened surfaces of nontrivial genus, which form the basis for the constructions in this paper. Also related to the setting of this paper, links in mapping tori have been considered in the context of quantum topology [23] and branched covers of orbifolds [24], though in both cases these links are of a different form, arising as orbits of marked points in the mapping rather than lying in the neighbourhood of the fibre surface as in this paper.

2. Fully augmented links in general surfaces

This section establishes the machinery that will be used throughout the paper. The first is the class of diagrams that will be used to investigate links in more general settings, adapted from [7].

Definition 2.1. Let a link L lie in the neighbourhood $\Sigma \times I$ of a (possibly disconnected) closed surface Σ. The *generalised projection* or *generalised diagram* $\pi(L)$ of L is the image of L under the projection $\pi : \Sigma \times I \longrightarrow \Sigma$, with crossing information.

The two common conditions for working with generalised projections below will be used throughout this paper.

Definition 2.2. The projection of a link is *cellular* in a projection surface if its complementary regions in that surface are homeomorphic to discs.

Definition 2.3. A link projection to a surface Σ is *weakly prime* if, given any disc D in Σ where ∂D meets the projection exactly twice, then D (or possibly $\Sigma \setminus D$, if Σ is a sphere) contains only an embedded arc.

Two main classes of links will be considered in this paper.

Definition 2.4. A *fully augmented link* is a link that admits a projection such that each component of L is one of two types:

(1) unknotted *crossing circles* that lie perpendicular to the projection surface, each bounding a disc called a *crossing disc*, and
(2) *projection components* that collectively intersect each crossing disc exactly twice.

In this definitive projection, projection components lie entirely in the projection surface save for at most one crossing next to each crossing disc (see Figure 1). The number of crossing circles is denoted c and the number of projection components, l.

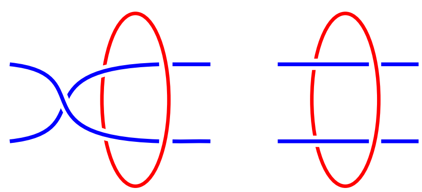

FIGURE 1. Crossing circles (red) shown with and without a half-twist in the projection components (blue).

In general, diagrams of fully augmented links are assumed to be of this definitive type unless otherwise specified. All projections of fully augmented links considered in this paper are of this definitive type. Note that if a fully augmented link projection is cellular, every projection component meets at least one crossing circle — if a component meets no crossing circles it has no crossings, and so meets a region of the projection surface that is not a disc on at least one side. Also note that this definition allows parallel crossing circles, but in practice this cannot occur for hyperbolic fully augmented links such as the ones in this paper.

The definition of the second class of link below is reproduced from Howie and Purcell [7].

Definition 2.5. A *weakly generalised alternating* link projection on a (possibly disconnected) surface Σ:

(1) is weakly prime;
(2) has link components that project to each component of Σ;
(3) has at least one crossing in the projection of each link component;
(4) divides Σ into checkerboard-colourable faces;
(5) has representativity ≥ 4, i.e. each compression disc of Σ has at least four transverse intersections with the link projection;
(6) alternates about each complementary region of Σ, i.e. every strand about the edge of each complementary region of Σ passes from an over- to an understrand.

Fully augmented links are connected to general links by the following feature:

Definition 2.6. A *twist region* of a diagram consists of either a string of bigons arranged end-to-end or a single crossing adjacent to no bigons. The minimum number of twist regions in a diagram of a given link is the *twist number* of the link.

We require diagrams to be alternating in twist regions; if not, the number of crossings in the diagram may be reduced in an obvious way. The connection is as follows:

Definition 2.7. A link diagram is *augmented* by adding an unknotted component transversely about a twist region such that it bounds a disc through the twist region. For one-crossing twist regions, there are two choices of augmentation.

The complement of the link obtained by augmenting each twist region of a twist reduced link is homeomorphic to that obtained by removing all *full twists* from each twist region, leaving either only a crossing circle or a crossing circle with a *half-twist* (single crossing). This new link with full twists removed is a fully augmented link of Definition 2.4.

Lemma 2.8. *Let L be a fully augmented link lying in the neighbourhood of an incompressible surface Σ in a 3-manifold M. Suppose $M \setminus L$ is hyperbolic. Then there exists a projection $\pi(L)$ of L onto Σ that is weakly prime.*

Proof. This proof will show that given any generalised projection of a link L to a surface Σ where L is hyperbolic in the ambient manifold M, it is possible to construct a weakly prime projection. A key tool is Thurston's hyperbolisation Theorem [8] which, among other consequences, states that any annulus or sphere in a hyperbolic manifold is inessential.

Suppose there exists a disc $D \subset \Sigma$ such that $\pi(L)$ intersects ∂D exactly twice and D contains something other than a trivial arc, i.e. $\pi(L)$ is not weakly prime. First note that, analogous to the way split links in S^3 cannot be prime, if D contains a subdisc D' with $\pi(L) \cap D' \neq \emptyset$ and $\pi(L) \cap \partial D' = \emptyset$, then one can find an essential sphere in $M \setminus L$ by taking a neighbourhood $N(D')$ of D' in M such that $\partial N(D)$ does not intersect L. This is a contradiction, as $M \setminus L$ is hyperbolic and so contains no essential spheres.

Take a neighbourhood $N(D)$ of D in M such that $\partial N(D)$ intersects L exactly twice, i.e. $\partial N(D)$ is an annulus. Call this annulus A. Given $M \setminus L$ is hyperbolic, A cannot be essential. Suppose A is compressible; then the compression disc has boundary that is isotopic to the core curve of A, dividing $N(D)$ into two balls, the boundary of each of which meets L exactly once. As L is composed of closed curves, this is impossible. Then A is boundary compressible — in particular, there exists an ambient isotopy in M of $L \cap D$ onto an arc of ∂D. As D is a disc on Σ, ∂D is a simple closed curve. Thus, this isotopy removes all crossings from $L \cap D$. Repeat this for every $D \subset \Sigma$ where ∂D meets $\pi(L)$ exactly twice and $D \cap \pi(L)$ is not a trivial arc; after finitely many such isotopies, $\pi(L)$ is weakly prime. □

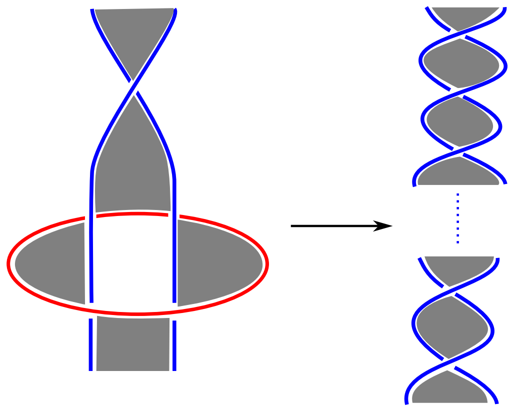

FIGURE 2. Converting a crossing circle to a twist region preserves checkerboard-colourability. It may be seen that the colouring is consistent with or without the half-twist.

Theorem 2.9. *Let $\pi(L)$ be a cellular, weakly prime projection of a fully augmented link L to an incompressible surface Σ in a 3-manifold M, where $\Sigma \setminus \pi(L)$ is checkerboard-colourable. Then an integer $t_k \neq 0$ may be chosen for each crossing circle of L such that performing $1/t_k$ Dehn filling on the k-th crossing circle of L for all k produces a weakly generalised alternating link.*

Proof. Recall the conditions of Definition 2.5. We will show each is satisfied by the link obtained by Dehn filling the crossing circles of a fully augmented link with the stated properties.

As Σ has no compression discs, the representativity of $\pi(L)$ is infinite. A hyperbolic fully augmented link admits a weakly prime projection by Lemma 2.8; it follows that the projection generated by the Dehn filling is weakly prime.

A cellular diagram $\pi(L)$ necessarily projects to all components of Σ, as Σ is composed of closed surfaces. Further, performing $1/t$ Dehn filling on a crossing circle with $t \neq 0$ creates a twist region with ≥ 1 crossing, ensuring at least one crossing is projected to each component of Σ.

Figure 2 shows that Dehn filling a crossing circle of a checkerboard-colourable fully augmented link diagram produces a twist region that is checkerboard colourable in the same manner. By Definition 2.7 of [7], choosing a checkerboard colourable $\pi(L)$ ensures that there exists a choice of sign for t_k in $1/t_k$ Dehn filling that produces a link projection that alternates around each complementary region.

Thus, by performing $1/t_k$ Dehn filling on the crossing circles of a fully augmented link with checkerboard colourable diagram, one can obtain a weakly generalised alternating knot or link. □

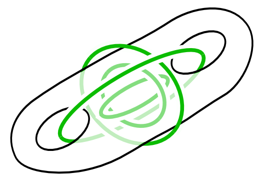

FIGURE 3. Adding curves about a surface such that they bound annuli through the surface.

3. Constructions

Construction 3.1 (Virtual link with layered curves)**.** Let L be a fully augmented link with projection $\pi(L)$ that is cellular in a surface Σ with genus $g \geq 2$ lying in the thickened surface $\Sigma \times I$. We will refer to L as the *base link*.

Fix two essential simple closed curves $\gamma_{odd}, \gamma_{even}$ on Σ, such that γ_{odd} intersects γ_{even} transversely; that is, the intersection number $\iota(\gamma_{odd}, \gamma_{even}) \geq 1$. Isotope γ_{odd} and γ_{even} such that they do not intersect any crossing circle of L. Fix an integer m; add m pairs of unknotted, unlinked curves $C_1, C_{-1}, ..., C_m, C_{-m}$ in a neighbourhood of Σ as in Figure 3, where:

(1) Each C_i is isotopic to γ_{odd} for i odd and γ_{even} for i even;
(2) For j, i of the same sign where $|j| > |i|$, C_i lies between C_j and Σ.

Collectively denote these *layered curves* by $\mathcal{C}_m = \bigcup_{i=1}^{m} C_{\pm i}$. Each pair $C_{\pm i}$ forms the boundaries of an annulus A_i. Each A_i intersects Σ in a curve isotopic to either γ_{odd} or γ_{even} based on the parity of i.

From this, we can now perform two separate constructions to “close" the manifold $(\Sigma \times I) \setminus L_m$; see Figure 4. The base link is assumed to be hyperbolic in the ambient manifold in both cases.

Construction 3.2 (Doubled thickened surface)**.** Begin with $L \cup \mathcal{C}_m \subset \Sigma \times I$, as in Construction 3.1. Take a second base link L' with projection $\pi(L')$ that is cellular in a copy Σ' of Σ lying in thickened surface $\Sigma' \times I$. We will refer to $L \cup L'$ as the base links for this manifold.

Glue via the identity $\Sigma \times \{0\}$ to $\Sigma' \times \{0\}$ and $\Sigma \times \{1\}$ to $\Sigma' \times \{1\}$. The result is an ambient manifold isomorphic to $\Sigma \times S^1$, the *doubled thickened surface*, where the base links $L \cup L'$ lie in neighbourhoods of Σ and Σ', and the layered curves $\mathcal{C}_m$ in lie in copies of the surface Σ between the projection surfaces Σ and Σ'.

Construction 3.3 (Mapping torus)**.** Begin with the virtual link with layered curves of Construction 3.1. Identify the boundaries of $\Sigma \times I$ by a nontrivial element ϕ of the mapping class group of Σ. The result is an ambient manifold isotopic to $(\Sigma \times I)/\phi$, the *mapping torus of* ϕ, with the base link L in a neighbourhood of Σ and the layered curves $\mathcal{C}_m$ in copies of Σ.

In this construction, we require that the isotopy classes of γ_{odd} and γ_{even} are nontrivially acted on by ϕ, denoted $\phi(\gamma) \nsim \pm\gamma$. This may restrict the possible choices of γ_{odd} and γ_{even} for some ϕ.

Let L_0 be the base links of either construction, and denote the family of links $L_0 \cup \mathcal{C}_m$ by $\{L_m\}_{m\in\mathbb{N}}$.

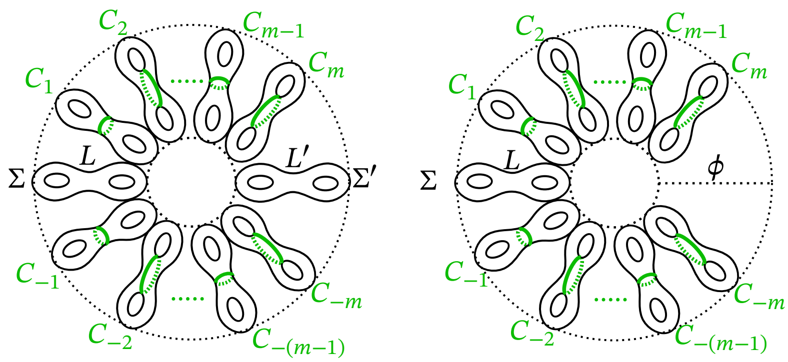

FIGURE 4. The two constructions of M, represented schematically; L and L' in $\Sigma \times S^1$ at left, and L in $(\Sigma \times I)/\phi$ at right.

Lemma 3.4. *Suppose an element ϕ of the mapping class group of a closed surface Σ acts nontrivially on the isotopy class of a nontrivial simple closed curve γ, i.e. $\phi(\gamma) \nsim \pm\gamma$. Then there exists a second such curve γ' such that $\iota(\gamma,\gamma') \geq 1$ and $\phi(\gamma') \nsim \pm\gamma'$.*

Proof. Given γ as above, take a second curve α such that $\iota(\gamma,\alpha) \geq 1$. If $\phi(\alpha) \nsim \pm\alpha$, take $\gamma' = \alpha$; done. Suppose instead that $\phi(\alpha) \sim \pm\alpha$. We will construct a curve γ' that satisfies the proposition.

Take the Dehn twist of γ about α, $T_\alpha(\gamma)$. This has intersection number $\iota\big(\gamma, T_\alpha(\gamma)\big) = \iota(\gamma,\alpha)^2 \geq 1$. As ϕ is a homeomorphism, it preserves intersections; thus, $\iota\Big(\phi(\gamma), \phi\big(T_\alpha(\gamma)\big)\Big) = \iota\big(\gamma, T_\alpha(\gamma)\big) \geq 1$. We now only require that $\phi\big(T_\alpha(\gamma)\big) \nsim T_\alpha(\gamma)$. We have that $\phi\big(T_\alpha(\gamma)\big) = T_{\phi(\alpha)}\big(\phi(\gamma)\big) = T_{\pm\alpha}\big(\phi(\gamma)\big)$, as $\phi(\alpha) \sim \pm\alpha$.

If $\phi(\alpha) \sim \alpha$, then we have that $T_\alpha(\gamma) = T_\alpha\big(\phi(\gamma)\big)$; this implies that either $\gamma \sim \alpha$, $\iota(\gamma,\alpha) = 0$, or $\gamma \sim \phi(\gamma)$, all of which are false by assumption.

If $\phi(\alpha) \sim -\alpha$, then $T_\alpha(\gamma) \sim \phi\big(T_\alpha(\gamma)\big) = T_\alpha^{-1}\big(\phi(\gamma)\big)$, which implies that $T_\alpha^2\gamma \sim \phi(\gamma)$; that is, the map ϕ acts on γ as two Dehn twists about α. This, in turn, implies that T_α is order two, as $T_\alpha(\gamma) \sim \phi\big(T_\alpha(\gamma)\big) = T_\alpha^3(\gamma)$; this is a contradiction, as Dehn twists have infinite order. Note also that this implies $\gamma \sim T_\alpha^2(\gamma) = \phi(\gamma)$, which is again false by assumption. □

Remark 3.5. To perform Construction 3.3 in the mapping torus of a map ϕ, it suffices to find one curve γ such that $\phi(\gamma) \nsim \gamma$, and construct a second as in Lemma 3.4.

Thus the only mapping tori $(\Sigma\times I)/\phi$ for which Construction 3.3 is not possible are those for which $\phi(\gamma) \sim \pm\gamma$ for all nontrivial simple closed curves $\gamma \subset \Sigma$. See Theorem 3.22 and Remark 3.23.

3.1. Hyperbolicity. We require some preliminary observations to prove Constructions 3.3 and 3.2 are hyperbolic.

Lemma 3.6. *Let γ be an essential closed curve lying in an incompressible surface Σ in a 3-manifold M. Suppose there exists an essential annulus A in $M \setminus N(\gamma)$ with one boundary on $\partial N(\gamma)$, and the other outside a neighbourhood of Σ; then $\partial A \cap N(\gamma)$ is parallel to $\partial N(\gamma) \cap \Sigma$.*

Proof. Let $A \cap N(\gamma) \subset \partial A$ form a nontrivial curve on $N(\gamma)$. We will consider arcs of intersection of A with the surface Σ. Let α be such an arc in $A \cap \Sigma$. As $\partial A \setminus N(\gamma)$ is disjoint from a neighbourhood of Σ, both ends of α are on $N(\gamma)$; thus, α co-bounds a disc on A with an arc of ∂A. Take an edgemost disc D on A such that the arc of ∂A is a single connected arc β. As β is contained in $N(\gamma) \setminus \Sigma$, it can be isotoped to lie on $\gamma \subset \Sigma$. Then ∂D bounds a disc E on Σ, as Σ is incompressible. Gluing D and E through ∂D forms a sphere in $N(\Sigma) = \Sigma \times I$. As $\Sigma \times I$ is irreducible, this sphere bounds a ball B. As D is assumed to be edgemost, it can be removed by isotopy of $A \cap B$ through B. After finitely many such isotopies, all arcs of intersection in $A \cap \Sigma$ are removed.

As all arcs of intersection in $A \cap \Sigma$ have been removed, i.e. $A \cap \Sigma$ is disjoint from $\partial N(\gamma)$, $A \cap N(\gamma) \subset \partial A$ lies completely on one side of Σ, parallel to $\partial N(\gamma) \cap \Sigma$ as required. □

Lemma 3.7. *Let γ, γ' lie in distinct copies $\Sigma \times \{s\} = \Sigma_s$ and $\Sigma \times \{t\} = \Sigma_t$, respectively, of a surface Σ in the thickened surface $\Sigma \times I$. Suppose there exists an essential annulus A in $\Sigma \times I \setminus \big(N(\gamma) \cup N(\gamma')\big)$ with boundary on $\partial N(\gamma) \cup \partial N(\gamma')$; then γ and γ' are in the same isotopy class on Σ, and ∂A either:*

(1) *intersects both of $N(\gamma), N(\gamma')$, and is isotopic to $\gamma \times (s,t)$, or*
(2) *is contained in $\partial N(\gamma)$, and A co-bounds a solid torus containing $\gamma \times (s,t)$ that has core curve γ' with an annulus $A' \subset \partial N(\gamma)$, up to relabelling of γ, γ'.*

Proof. Suppose ∂A intersects both $N(\gamma)$ and $N(\gamma')$. Observe that $\partial A \cap N(\gamma)$ is disjoint from $N(\Sigma_t)$ and $\partial A \cap N(\gamma')$ is disjoint from $N(\Sigma_s)$. By Lemma 3.6, ∂A is composed of a longitude of each of $N(\gamma)$ and $N(\gamma')$, parallel to $\partial N(\gamma) \cap \Sigma_s$ and $\partial N(\gamma') \cap \Sigma_t$, respectively; thus, A describes an isotopy between γ and γ'. As the fundamental group $\pi_1(\Sigma \times I)$ of $\Sigma \times I$ is the same as $\pi_1(\Sigma) = \pi_1(\Sigma_s) = \pi_1(\Sigma_t)$, γ and γ' are in the same isotopy class on Σ.

Consider $A \cap \big(\gamma \times (s,t)\big)$. As ∂A is composed of longitudes of $N(\gamma), N(\gamma')$ as above, there are no arcs in $A \cap \big(\gamma \times (s,t)\big)$. If a component of $A \cap \big(\gamma \times (s,t)\big)$ bounds a disc D in $\gamma \times (s,t)$, it also bounds a disc E in A, as A is incompressible, and vice-versa; take an innermost such disc on A. Gluing D to E through $A \cap \big(\gamma \times (s,t)\big)$ forms a sphere in $\Sigma \times I$, which bounds a ball as $\Sigma \times I$ is irreducible. Isotoping E through this ball removes the trivial component of $A \cap \gamma \times (s,t)$, and after finitely many such isotopies, $A \cap \gamma \times (s,t)$ has no trivial components. Thus $A \cap \big(\gamma \times (s,t)\big)$ is composed of curves that are nontrivial on both A and $\gamma \times (s,t)$. Thus $A \cap \big(\gamma \times (s,t)\big)$ divides A and $\gamma \times (s,t)$ into parallel annuli. Take an innermost sub-annulus A' of A where $\gamma \subset \partial A'$. Then A' co-bounds a solid torus T disjoint from γ in $\Sigma \times I$ with a sub-annulus A'' of $\gamma \times (s,t)$. The sub-annulus A' may be isotoped past A'' through T to remove a component of $A \cap (\gamma \times (s,t))$, except if $\gamma' \subset T$. In that case, take a similar innermost annulus $\bar{A}'$ in A such that $\gamma' \subset \partial \bar{A}'$, co-bounding a solid torus T' with a sub-annulus $\bar{A}''$ of $\gamma \times (s,t)$; then $\gamma \not\subset T'$ and a similar isotopy of $\bar{A}'$ is possible. To see this, note $N(\gamma) \not\subset T$, so any solid torus containing γ has boundary that is not contained in T, but $\bar{A}'$ originates inside T, and so must exit T via $A'' \subset \gamma \times (s,t)$; then any $\bar{A}' \not\subset T$ is not innermost, so $\gamma \not\subset T'$ and isotoping $\bar{A}'$ past $\bar{A}''$ through T' removes components of $A \cap (\gamma \times (s,t))$. Thus, after finitely many isotopies, $A \cap (\gamma \times (s,t))$ is empty, and thus A co-bounds a solid torus in $(\Sigma \times I) \setminus (N(\gamma) \cup N(\gamma'))$ with $\gamma \times (s,t)$, through which A may be isotoped into the form of $\gamma \times (s,t)$.

Now consider the second case, $\partial A \subset \partial N(\gamma)$. In this case, ∂A is composed of two parallel closed curves on $N(\gamma)$, and thus A co-bounds a solid torus T in $\Sigma \times I$ with interior disjoint from γ with an annulus in $\partial N(\gamma)$. Note $\gamma' \subset T$, as A is essential; then there is a neighbourhood of Σ_s for which $A \cap N(\Sigma_s)$ is composed of two sub-annuli, and by Lemma 3.6, ∂A is composed of longitudes of $N(\gamma)$. Trivial curves of $A \cap \Sigma_t$ are removable as both surfaces are incompressible; isotope A such that $A \cap \Sigma_t$ is composed of nontrivial curves in A parallel to ∂A. As ∂A is composed of longitudes of $N(\gamma)$, any sub-annulus of A describes an isotopy of γ, and as such any nontrivial component of $A \cap \Sigma_t$ is isotopic to γ in Σ. As γ' is a nontrivial curve and $\gamma' \subset T \cap \Sigma_t$, γ' is contained in an annulus with boundaries isotopic to γ, and thus γ' is in the same isotopy class as γ on Σ.

As γ and γ' are in the same isotopy class, there exists an annulus $A' \subset T$ which describes the isotopy between them, of which T forms a neighbourhood. By the above, A' may be isotoped into the form $\gamma \times (s,t)$; performing a comparable isotopy on A gives the required result. ☐

Lemma 3.8. *Let $M = \Sigma \times I \setminus (N(\gamma) \cup N(\gamma'))$, where γ, γ' are two essential closed curves in distinct copies of Σ. Then M is toroidal if and only if γ and γ' are in the same isotopy class on Σ, in which case the only essential torus bounds a solid torus containing both γ and γ', and is isotopic to the boundary of a neighbourhood of $\gamma \times (s,t)$.*

Proof. As $\Sigma \times I$ is atoroidal, any essential torus in $(\Sigma \times I) \setminus (N(\gamma) \cup N(\gamma'))$ bounds a solid torus $T \subset \Sigma \times I$ that contains γ and/or γ'. Let $\gamma \subset T$; then γ is homotopic to the core curve of T. As γ is a nontrivial curve on an incompressible surface, the component of $T \cap \Sigma_s$ containing γ is an annulus up to isotopy, and thus nontrivial components of $\partial T \cap \Sigma_s$ are isotopic to γ in Σ_s. If $\gamma' \not\subset T$, ∂T is boundary parallel, which is a contradiction; thus $\gamma' \subset T$. Then, by identical argument, nontrivial components of $\partial T \cap \Sigma_t$ are isotopic to γ'. As Σ_s is disjoint from Σ_t, nontrivial curves in $\partial T \cap \Sigma_s$ are parallel in ∂T, and thus isotopic, to those to $\partial T \cap \Sigma_t$. This gives an isotopy between γ and γ' in $\Sigma \times I$, and thus γ and γ' are in the same isotopy class in Σ.

By Lemma 3.7, there exists an essential annulus $\gamma \times (s,t)$ in M when γ and γ' are in the same isotopy class on Σ. As γ, γ' are isotopic to the core curve of T, T is isotopic to the boundary of a neighbourhood of $\gamma \times (s,t)$, as required. □

Lemma 3.9. *Let L be a fully augmented link with cellular projection $\pi(L)$ to a surface Σ in $\Sigma \times I$, i.e. L is a virtual fully augmented link. Let A be an embedded annulus in $(\Sigma \times I) \setminus L$ with $\partial A \subset \partial(\Sigma \times I)$ that intersects a crossing disc D of L. Either the intersection can be removed by isotopy, or least one of A and D is compressible in $(\Sigma \times I) \setminus L$.*

Proof. An annulus A that is embedded in $(\Sigma \times I) \setminus L$ is also embedded in $\Sigma \times I$. Both components of ∂A are isotopic curves in (copies of) Σ, as the projection $\pi(A)$ describes an isotopy in Σ. As both components of ∂A are outside a neighbourhood of D, all components of the intersection $A \cap D$ are simple closed curves. If there exist components of $A \cap D$ that are trivial in both A and D, the innermost is removable by isotopy of A. After finitely many such isotopies, there are no such mutually trivial intersections; if these were the only kind of intersection, A and D are then disjoint.

Suppose then that after removing all mutually trivial intersections, $A \cap D \neq \emptyset$. Then each component of $A \cap D$ is nontrivial in at least one of A or D. By definition, if there is a curve that bounds a disc E in one and is nontrivial in the other, then E is a compression disc. The remaining case is that each component of $A \cap D$ is nontrivial in both surfaces. As D lies in a simply connected region of $\Sigma \times I$, components of $A \cap D$ are trivial in $\Sigma \times I$. Each divides A into sub-annuli, as they are nontrivial in A, and are isotopic through A to ∂A. By the above argument, this isotopy implies that each component of ∂A is trivial in $\Sigma \times I$; as ∂A lies outside a neighbourhood of L, this implies that A is compressible in $(\Sigma \times I) \setminus L$. □

We return to $\{L_m\}_{m \in \mathbb{N}}$ as obtained by Constructions 3.2 and 3.3. Recall L_0 is the base link — L in Construction 3.3 or $L \cup L'$ in Construction 3.2 — and let Σ_0 refer to the projection surfaces of each, which may or may not be connected depending on the construction.

Proposition 3.10. *Assume the base link L_0 has a hyperbolic complement in the respective manifolds obtained in Constructions 3.2 and 3.3. Consider the families of links $\{L_m\}_{m \in \mathbb{N}} = L_0 \cup \mathcal{C}_m$ obtained in these constructions, i.e. the base links with layered curves added. Each link L_m also has a hyperbolic complement.*

Proof. Let M be either $\Sigma \times S^1$ or $(\Sigma \times I)/\phi$ containing links as specified in Constructions 3.2 and 3.3. The base curves ($m = 0$) of both constructions are hyperbolic by assumption. Let $m > 0$; that is, drill the layered curves $\mathcal{C}_m$ of Construction 3.1. This proof will proceed using Thurston's hyperbolization theorem [8], ruling out the existence of essential discs, spheres, annuli and tori. As $M \setminus L_0$ is hyperbolic, it contains none of these essential surfaces.

Suppose $M \setminus L_m$ is reducible. As $M \setminus L_0$ is irreducible, a reducing sphere in $M \setminus L_m$ bounds a ball in $M \setminus L$ that contains components of $\mathcal{C}_m$ in $M \setminus L_m$.

This is a contradiction, as all $C_i \subset \mathcal{C}_m$ are essential curves in incompressible surfaces and thus not contained in any simply connected region. Thus, $M \setminus L_m$ is irreducible.

Suppose $M \setminus L_m$ is boundary reducible. All components of $\mathcal{C}_m$ are essential curves on incompressible surfaces in M, and so do not bound discs; this rules out any disc with boundary a curve with longitudinal components on a neighbourhood of C_i. Any essential disc with boundary a meridian or trivial curve on a neighbourhood of C_i may be capped by a disc in M by trivially Dehn filling C_i to find a reducing sphere for $M \setminus L_0$, which is a contradiction. Any reducing disk then has boundary on L_0. Such a disc must co-bound a ball in $M \setminus L_0$ with a disc of $\partial N(L_0)$, as $M \setminus L_0$ is boundary irreducible; but as all C_i are essential curves in incompressible surfaces of M, no component of $\mathcal{C}$ is contained in a ball, so such a disc cannot be essential in $M \setminus L_m$.

As $M \setminus L_0$ is anannular, an essential annulus A in $M \setminus L_m$ either has at least one boundary on $\mathcal{C}_m$ or is inessential in $M \setminus L_0$. The argument is divided into several cases.

Case 1: The boundary of A lies on two distinct $C_i \in \mathcal{C}_m$.

By construction, all components of $\mathcal{C}_m$ sit in distinct copies of the surface Σ. By Lemma 3.7, there exists an annulus between two of these components only if they are isotopic to the same curve on Σ, i.e. of the same parity. Any annulus between two same-parity, same-sign components of $\mathcal{C}_m$ is punctured by at least one opposite-parity C_i between them.

Let A have boundary on C_1 and C_{-1} or (in Construction 3.2) C_m and C_{-m}, such that A intersects Σ_0. Any annulus that meets Σ_0 in any closed curve in a complementary region of $\pi(L_0)$ is compressible, as $\pi(L_0)$ is cellular. An annulus embedded in $M \setminus L_m$ that meets Σ_0 and intersects $\pi(L_0)$ intersects at least one crossing disc, as neighbourhoods of crossing discs are the only areas of Σ_0 where components of L_0 are not embedded in Σ_0. As $N(\Sigma_0)$ is a thickened surface containing L_0, Lemma 3.9 applies to $A \cap N(\Sigma_0)$: if all such intersections with crossing discs can be removed, the annulus is compressible as noted previously, and if the annulus is compressible as per the lemma it is not essential. Then the crossing disc is compressible, but in that case there is either an essential annulus or disc with boundary on L_0, contradicting that $M \setminus L_0$ is hyperbolic.

If $M = (\Sigma \times I)/\phi$, there is no annulus with boundary on C_m and C_{-m} as$\phi(\gamma_{odd}) \nsim \gamma_{odd}$ and $\phi(\gamma_{even}) \nsim \gamma_{even}$ by construction. There exists an annulus disjoint from Σ_0 between opposite-parity and -sign components of $\mathcal{C}_m$ if $\phi(\gamma_{even}) \sim \gamma_{odd}$, or vice-versa, but note that this annulus is punctured by the paired curve of the higher-magnitude index. For example, suppose there exists an annulus between C_{-m} and C_{m-1} with m even, i.e. $\phi(\gamma_{even}) \sim \gamma_{odd}$. Then the annulus is punctured by C_m, as $\gamma_{even} \nsim \gamma_{odd}$.

Case 2: Both boundaries of A lie on one C_i.

If $A \cap \Sigma_0$ is nonempty, Lemma 3.9 applies to a subannulus of A in $N(\Sigma_0)$. Then either the essentiality of A or the hyperbolicity of $M \setminus L_0$ is contradicted, or all components bound discs on Σ_0. As A is incompressible, $A \cap \Sigma_0$ must also bound a disc on A; as $M \setminus L_0$ is irreducible, these discs co-bound a sphere and so the innermost such discs are removable by isotopy of A. After finitely many isotopies, A is disjoint from $N(\Sigma_0)$. Then A is as described in Lemma 3.7(2), and bounds a solid torus that contains a C_j where i, j are of the same parity; it is then punctured by at least one C_k of the opposite parity.

Case 3: The annulus A has one boundary on a C_i and one on a component L^* of L_0.

By Lemma 3.9, if $A \cap N(\Sigma_0)$ contains a subannulus of A with boundary contained in $\partial N(\Sigma_0)$ — that is, A nontrivially intersects Σ_0 — either A is not essential or $M \setminus L_0$ is not hyperbolic, each of which is a contradiction. Thus, A is contained in some $\Sigma \times (s,t)$ disjoint from Σ_0. Consider the natural embedding of A into $M \setminus (C_i \cup L^*)$, which contains $M \setminus L_m$ as a subset. Lemmas 3.6 and 3.7 imply that C_i, L^* are isotopic and A describes an isotopy between them. As A is embedded in $M \setminus L_m$ by assumption, A also describes an isotopy between C_i and L^* in $M \setminus L_m$. This implies L^* forms a nontrivial curve in Σ_0 and is thus a projection component of L_0, as crossing circles project to trivial curves in Σ_0. If L^* meets no crossing discs of L_0, by Definition 2.4 L^* has no crossings in $\pi(L_0)$ and meets a non-disc region of the projection surface, contradicting that $\pi(L_0)$ is cellular. If L^* does meet a crossing disc, A describes an isotopy of L^* away from Σ_0 that does not affect any other components of L_0, removing intersections of L^* with any crossing discs. As crossing discs are twice-punctured by definition, removing an intersection implies there exists an essential annulus or disc in $M \setminus L_0$, contradicting that $M \setminus L_0$ is hyperbolic.

Case 4: The boundaries of A lie on L_0.

Now suppose an essential annulus A in $M \setminus L_m$ has both boundaries on L_0; then A is either compressible or boundary parallel in $M \setminus L_0$. If A is compressible in $M \setminus L_0$, then surgering A along a compression disc produces two embedded discs which are also inessential in $M \setminus L_0$; thus each component of ∂A is trivial in $N(L_0)$. Such a compression disc cannot be punctured by any C_i to form an essential annulus in $M \setminus L_m$. Thus, A is boundary parallel in $M \setminus L_0$, and ∂A lies on a single component $L^* \subset L_0$. By similar argument to Case 3, A is in the form of Lemma 3.7(2) bounding a solid torus containing C_i. Then there is an annulus A' that describes an isotopy from L^* to C_i, which is a contradiction as in Case 3.

This concludes the argument for essential annuli.

Finally, consider essential tori. The intersection of an essential torus T in $M \setminus L_m$ with any given copy of Σ in M can be reduced by isotopy to simple closed curves that are mutually nontrivial in T and the copy of Σ, as mutually trivial intersections can be removed and curves that are trivial in only one surface give a compression disc, which is a contradiction in either case. Thus, the intersection of T with any thickened surface region of M that does not completely contain T is composed of annuli. If $T \cap \Sigma_0$ is nonempty, either T is contained in $N(\Sigma_0)$, contradicting that $M \setminus L_0$ is hyperbolic, or Lemma 3.9 applies to all components of $T \cap N(\Sigma_0)$, each of which is an annulus; as argued above, this contradicts either the essentiality of T or the hyperbolicity of $M \setminus L_0$. Thus, T is disjoint from $N(\Sigma_0)$. Then T is contained in some $\Sigma\times(s,t)$ and bounds a solid torus that contains at least one C_i. If this solid torus contains only one C_i, T is boundary parallel by similar argument to Lemma 3.8. If the solid torus bounded by T contains distinct C_i, C_j, Lemma 3.8 applies and i, j are of the same parity; T then punctured by at least one C_k of the opposite parity.

As $M \setminus L_m$ contains no essential discs, spheres, annuli, or tori for all m, it is hyperbolic. □

Remark 3.11. The condition that Σ is of genus at least two is required for Proposition 3.10. Performing an analogous construction when Σ is genus zero or one produces a manifold with essential spheres or tori, respectively, parallel to Σ, and by Thurston hyperbolization [8], links on Σ in these manifolds are not hyperbolic.

To end this section, note that requiring a hyperbolic base link in Constructions 3.3 and 3.2 is not a strenuous or unreasonable assumption.

Lemma 3.12. *There exists a cellular fully augmented link L_0 such that $(\Sigma\times I)/\phi\setminus L_0$ is hyperbolic, where ϕ is as in Construction 3.3. Similarly, there exist cellular fully augmented links L_0 and L_0' such that $\Sigma\times S^1\setminus(L_0\cup L_0')$ as in Construction 3.2 is hyperbolic.*

Proof. A set of sufficient conditions for hyperbolicity of fully augmented links with cellular projections to incompressible surfaces is given in Reid [30], along with hyperbolicity for much more general fully augmented links. The existence of hyperbolic examples of the base links obtained in Construction 3.1 — that is, fully augmented links in thickened surfaces, also known as virtual fully augmented links — is shown in [12] and [1]. Further, by [1], hyperbolicity of the base links of Construction 3.2 follows on observing $(\Sigma\times S^1)\setminus(L_0\cup L_0')$ is a doubled manifold; for example, this is immediate in the case L_0 and L_0' are fully augmented links without half twists.

To show there exist hyperbolic examples of the base links of Construction 3.3, start with hyperbolic $(\Sigma\times I)\setminus L_0$ and denote the image of $\partial(\Sigma\times I)$ in $(\Sigma\times I)/\phi$ by Σ_ϕ. The argument follows similarly to Proposition 3.10, using Thurston hyperbolization.

Suppose there exists an essential disc $D\subset(\Sigma\times I)/\phi\setminus L_0$. Then $\Sigma_\phi\cap D$ is composed of trivial curves on D; as Σ_ϕ is incompressible in $(\Sigma\times I)/\phi$, each component of $\Sigma_\phi\cap D$ also bounds a disc in Σ_ϕ. As each M is irreducible, D co-bounds a ball and the innermost such D is removable by isotopy of D. After finitely many such isotopies, $\Sigma_\phi\cap D$ is empty, i.e. D is an essential disc in $\Sigma\times I\setminus L_0$, contradicting the hyperbolicity of $(\Sigma\times I)\setminus L_0$. By identical argument, if there exists an essential sphere $S\subset(\Sigma\times I)/\phi$, all components of $\Sigma_\phi\cap S$ are removable by isotopy of S, implying S is essential in $(\Sigma\times I)\setminus L_0$, which is a contradiction.

Suppose there exists an essential annulus $A\subset(\Sigma\times I)/\phi\setminus L_0$. If $\Sigma_\phi\cap A$ is entirely removable by isotopy of A, then again the hyperbolicity of $(\Sigma\times I)/\phi\setminus L_0$ is contradicted; thus $\Sigma_\phi\cap A$ is composed of non-removable curves that are nontrivial in both A and Σ_ϕ. Cutting the manifold along Σ_ϕ then cuts A into essential annuli in $(\Sigma\times I)/\phi\setminus L_0$ with boundary components on $\partial(\Sigma\times I)$, which is again a contradiction. By identical argument, an essential torus $T\subset(\Sigma\times I)/\phi\setminus L_0$ is either isotopic to an essential torus disjoint from Σ_ϕ or is cut along Σ_ϕ into essential annuli in $(\Sigma\times I)\setminus L_0$, each of which is again a contradiction.

As $(\Sigma\times I)/\phi\setminus L_0$ contains no essential discs, spheres, annuli, or tori, it is hyperbolic; thus there exist hyperbolic base links obtained from Construction 3.3. Following an identical argument for Construction 3.2 starting with hyperbolic $(\Sigma\times I)\setminus L_0$ and $(\Sigma\times I)\setminus L_0'$ provides an alternate proof of the existence of hyperbolic base links for Construction 3.2. □

3.2. Main results. To obtain the desired result, we employ annular Dehn filling.

Definition 3.13. Given two link components that co-bound an annulus A in a manifold M, an *annular Dehn filling* or $1/t$*-annular Dehn filling* consists of performing a $1/t$ Dehn filling on one link component and a $-1/t$ Dehn filling on the other for some $t\in\mathbb{N}$. The effect is of a Dehn twist through a neighbourhood of A; i.e. for a transverse curve γ through A, $1/t$-annular filling of A acts as the identity outside a neighbourhood of A and spins the curve γ a total of t times about the core of A.

The annular Dehn filling is a homeomorphism between the manifold M and the filled manifold; see, for example, [21, Section 2.3]. In particular, an annular filling that intersects a surface in a nontrivial curve acts as a Dehn twist of the surface on that curve.

Theorem 3.14. *Let Σ be a closed surface of genus at least two. Let M be either the doubled thickened surface $\Sigma\times S^1$ or the mapping torus $(\Sigma\times I)/\phi$ of a map ϕ that acts nontrivially on the isotopy class of at least one essential curve in Σ. There exist families of hyperbolic fully augmented links $\{J_n\}_{n\in\mathbb{N}}$ in M such that:*

1. *each link J_n in the family $\{J_n\}_{n\in\mathbb{N}}$ projects to an incompressible embedding of Σ in M,*
2. *the number of crossing circles is a fixed natural number c for all $J_n\in\{J_n\}_{n\in\mathbb{N}}$, and*
3. *the sequence of volumes $\{\mathrm{Vol}(M\setminus J_n)\}_{n\in\mathbb{N}}$ approaches infinity as n approaches infinity.*

Proof. Consider a fully augmented link J in M obtained by annular Dehn filling, in order from A_1 to A_m, the $2m$ cusps $\mathcal{C}_m=C_1,C_{-1},...C_m,C_{-m}$ of a link $\{L_m\}_{m\in\mathbb{N}}$ in $\Sigma\times S^1$ or $(\Sigma\times I)/\phi$ as in Constructions 3.2 and 3.3. By Definition 3.13, this has the effect of full twists on L_0 along γ_{odd} or γ_{even} as appropriate, giving a natural projection of J to Σ. Further, by construction γ_{odd} and γ_{even} have no intersection with crossing circles, and Dehn twists on Σ introduce no new crossings on Σ; thus, the number of crossing circles is a constant c for all such J obtained from a given family $\{L_m\}_{m\in\mathbb{N}}$.

Fix $V>0$ and fix an integer $m>0$ such that $2mv_{tet}>V$, where v_{tet} is the volume of a regular ideal hyperbolic tetrahedron. The manifold $M\setminus L_m$, as defined above, is hyperbolic by Proposition 3.10, and has either $l + c + 2m$ or $l' + c' + l + c + 2m$ cusps when M is the mapping torus or doubled thickened surface, respectively. By Adams [9], the volume satisfies:

$$\mathrm{Vol}(M \setminus L_m) > (l + c + 2m)v_{tet} > 2mv_{tet} > V,$$

when $M = (\Sigma \times I)/\phi$, and a similar relation holds when $M = \Sigma \times S^1$.

For $i = 1, ..., m$, let $t_i > 0$ be such that J is obtained by $1/t_i$ annular Dehn filling $A_i, i \in \{1, ..., m\}$. As the Dehn filling coefficients t_i approach infinity, the volume of $M \setminus J$ approaches $\mathrm{Vol}(M \setminus L_m)$ from below, since Dehn filling is known to decrease volume from work of Jorgensen and Thurston [8]; see also work of Futer, Kalfagianni, and Purcell [5]. Thus, for infinitely many such links J, the volume of the complement in M is strictly greater than $2mv_{tet} > V$ for arbitrary positive V. In general, there is such a collection of links for any appropriate choice of L and either L' or ϕ (depending on the construction) with compatible choices of γ_{odd} and γ_{even} on Σ.

Pick a family of links $\{L_m\}_{m\in\mathbb{N}}$ in M, where the fully augmented link L_0 has c crossing circles. By the above, for any unbounded sequence $\{V_n\}_{n\in\mathbb{N}}$ with $V_n > 0$ for all n, there exists a sequence of links $\{J_n\}_n \in \mathbb{N}$ where each J_n is obtained by annular Dehn filling the cusps $\mathcal{C}_m$ of a link $L_m \in \{L_m\}_{m\in\mathbb{N}}$, such that $\mathrm{Vol}(M \setminus J_n) > V_n$ for all n; in particular, $\mathrm{Vol}(M \setminus J_n) \to \infty$ as n approaches infinity. □

We may relate this result to weakly generalised alternating links as below.

Corollary 3.15. *Let Σ be a closed surface of genus at least two. There exist manifolds M and families of weakly generalised alternating knots and/or links $\{K_n\}_{n\in\mathbb{N}}$ in M such that:*

1. *each $K_n \in \{K_n\}_{n\in\mathbb{N}}$ projects to an incompressible embedding of Σ in M,*
2. *the twist number is a fixed natural number c for all $K_n \in \{K_n\}_{n\in\mathbb{N}}$, and*
3. *the sequence of volumes $\{\mathrm{Vol}(M \setminus K_n)\}_{n\in\mathbb{N}}$ approaches infinity as n approaches infinity.*

Proof. By Theorem 2.9, each hyperbolic fully augmented link L with cellular projection $\pi(L)$ whose complementary regions are checkerboard-colourable on a surface Σ is associated to a weakly generalised alternating link K via $1/t_k$ Dehn filling of crossing circles. Perform these Dehn fillings on the crossing circles of appropriately chosen $L \subset M$. Suppose the resulting link is hyperbolic; again by [5], the volume of $M \setminus K$ is bounded above by the volume of $M \setminus L$ as each $t_k \longrightarrow \infty$. Add the layered curves $\mathcal{C}_m = C_1, C_{-1}, ..., C_m, C_{-m}$ as above. By identical argument to Theorem 3.14, generate a family of links $\{K_n\}_{n\in\mathbb{N}}$ such that the sequence $\mathrm{Vol}(M \setminus K_n) \longrightarrow \infty$ as n approaches infinity, each K_n is projected to Σ, and twist number remains constant (as no new crossings are introduced by annular Dehn filling). □

Remark 3.16. In the construction used in [3], Kalfagianni and Purcell drill m pairs of unknotted, unlinked curves $C_1, C_2, ..., C_{2m-1}, C_{2m}$, about a WGA link to produce a manifold with hyperbolic volume bounded below by a specified (arbitrarily large) volume. We observe that these drilled curves are ambient isotopic to curves in a neighbourhood of the projection surface of the WGA link in S^3 (Figure 5). In that case, the drilled curves bound compression discs, and thus the Dehn fillings of the original construction may then be realised as Dehn twists on the projection surface. In contrast, the setting of this paper features incompressible projection surfaces by construction, but a similar result is achieved by annular Dehn filling. This operation has been used for similar constructions of large-volume manifolds in, for example, [21] and [20].

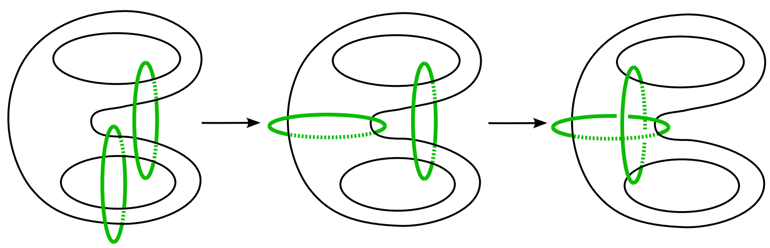

FIGURE 5. Ambient isotopy of a pair of curves about handles of an example surface of genus 2 to lie in a neighbourhood of the surface.

3.3. Bounded case and open questions. The complement of a hyperbolic fully augmented link in the trivial mapping torus (i.e. Construction 3.3 where ϕ is the identity and $m = 0$) *does* admit a linear upper bound on hyperbolic volume. To obtain this bound, we use a natural decomposition of fully augmented link projections.

Proposition 3.17. *Let L be a fully augmented link with a projection $\pi(L)$ onto a (possibly disconnected) closed, orientable surface Σ embedded in a manifold M. There exists a decomposition of the link complement $M \setminus L$ into a collection of manifolds with boundary where each such manifold is isotopic to a component of $M \backslash\backslash \Sigma = M \setminus N(\Sigma)$, where the boundary components are decorated with an ideal checkerboard-colourable graph that depends on $\pi(L)$.*

Proof. This decomposition is analogous to a standard decomposition for fully augmented links which appears in, for example, [4], but this decomposition allows generalised projection surfaces.

The steps to decompose a fully augmented link projection $\pi(L)$ are as follows:

(1) Remove all half-twists from $\pi(L)$.
(2) Cut along Σ, bisecting each crossing circle and the twice-punctured disc it bounds.
(3) Cut along each half of each twice-punctured disc.
(4) Flatten each of the twice-punctured disc halves to the boundary.
(5) Collapse each link component to an ideal vertex.

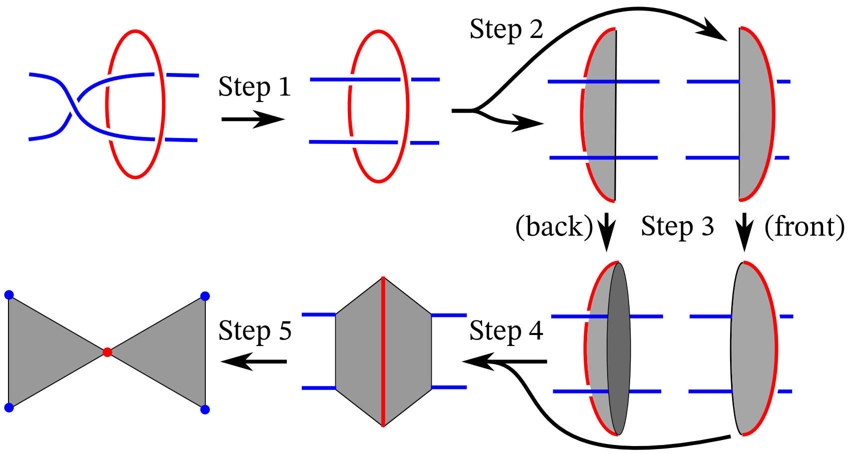

FIGURE 6. The steps of the bowtie decomposition of a fully augmented link with projection $\pi(L)$ on a surface Σ, displayed for a single crossing circle. Though the resulting image is identical for both components, note it describes two separate objects with interiors on opposite sides of the page.

These steps are shown for a single crossing circle in Figure 6.

In this process, the components of the projection surface become the white faces and the twice punctured discs become the shaded faces. All vertices of this decomposition are ideal. Each boundary arising from Σ is decorated with such a graph.

The manifold may be recovered from the above decomposition by gluing the faces in a way that reverses the decomposition; namely, gluing white faces that arose from the same surface by the identity and folding shaded faces to form crossing circles. Half-twists at crossings are recovered at this step by gluing shaded faces between components instead of within a single component; see Figure 7. □

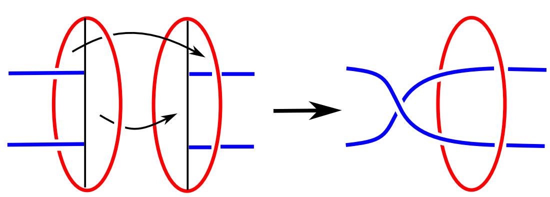

FIGURE 7. Gluing the shaded faces of the bowtie decomposition to form a half-twist at a crossing circle, reversing steps 1 and 3.

Definition 3.18. The decomposition of the complement of a fully augmented link described in Proposition 3.17 is the *bowtie decomposition* of $M \setminus L$, after the distinctive shapes formed by the shaded faces. In full specificity, it is referred to as the bowtie decomposition of $\pi(L)$ on Σ in M, but these are often omitted where the manifold and projection surface are clear.

Definition 3.19. Given a cellular fully augmented link L with projection $\pi(L)$ to a surface Σ, apply the bowtie decomposition. The *nerve* of $\pi(L)$, or of L when the projection surface is clear, is the graph on Σ which has a vertex for each white face and an edge between vertices wherever the associated white face(s) meet at an ideal vertex of the decomposition.

Lemma 3.20. *Let L be a fully augmented link with c crossing circles. The bowtie decomposition of a projection $\pi(L)$ that is cellular in a surface Σ of genus g has $c + 2 - 2g$ white faces.*

Proof. Proceed by argument of Euler characteristic.

Each face of the nerve of $\pi(L)$ corresponds to a shaded face of the bowtie decomposition, of which there are $2c = F$, as each crossing circle gives rise to exactly two shaded faces. Each shaded face of the bowtie decomposition is triangular, and thus each face of the nerve is also a triangle with one edge through each vertex of the shaded face. Each edge is shared by two faces, so the number of edges is $\frac{3}{2}2c = 3c = E$. The number of vertices V is the number of white faces in the decomposition. The Euler characteristic $\chi(\Sigma)$ of a genus g surface is $2 - 2g$.

Thus, we have:

$$\begin{aligned} \chi(\Sigma) &= V - E + F \\ 2 - 2g &= V - 3c + 2c \\ V &= c + 2 - 2g. \end{aligned}$$

□

Lemma 3.21. *The volume of an ideal hyperbolic triangular prism is bounded above by $3v_{tet}$, where $v_{tet} = 1.01494...$ is the volume of the regular ideal hyperbolic tetrahedron.*

Proof. Cut off one vertex of the triangular prism along the triangle through its three adjacent vertices; this forms one tetrahedron. What remains is a square-base pyramid, which may be decomposed into two further tetrahedra by splitting it along the triangle between the apex and a diagonal of the base. Each of these three ideal tetrahedra have volume bounded above by v_{tet}. □

Theorem 3.22. *Let Σ be a closed surface of genus at least two and M be the trivial mapping torus $\Sigma \times S^1$. Let L be a hyperbolic fully augmented link with projection $\pi(L)$ to $\Sigma \subset M$. The volume of the complement is bounded above by*

$$\mathrm{Vol}(M \setminus L) \leq 6(3c + 2g - 2)v_{tet}.$$

Proof. Apply the bowtie decomposition to $\pi(L)$. The result is a thickened surface $\Sigma \times I$ decorated identically on both boundaries with the decomposition of $\pi(L)$. Each white face is a (not necessarily geodesic) ideal n-gon that can be triangulated with $n-2$ ideal triangles. The bowtie decomposition has exactly $6c$ edges, as each crossing circle gives rise to exactly two triangular shaded faces, each of which is shared by a white face. By Lemma 3.20, there are $c+2-2g$ white faces; thus, triangulating all the white faces in this way produces $6c - 2(c+2-2g) = 4c+4g-4$ triangles. With the $2c$ shaded faces, each boundary is then decorated with an ideal triangulation with $6c+4g-4$ triangles.

Because both boundaries are decorated identically, every face of this triangulation matches exactly to a face on the opposite boundary of the thickened surface. Connect all matching vertices via an edge. Cutting along all the rectangles formed with the matching edges of the triangulation forms a triangular prism for every face of the triangulation. Each of these has volume bounded above by $3v_{tet}$ by Lemma 3.21. As there are $6c+4g-4$ triangles on each boundary, the result follows. □

Remark 3.23. By Theorem 3.14, there does not exist a linear bound on hyperbolic volume in terms of the number of crossing circles c of a fully augmented link in any mapping torus $(\Sigma \times I)/\phi$, provided that there exists a nontrivial curve $\gamma \subset \Sigma$ such that $\phi(\gamma) \nsim \pm\gamma$. By Theorem 3.22, links in $\Sigma \times S^1$, the trivial mapping torus, *do* admit such a bound.

There is an example of a map which is not covered by either of these two cases; that is, a map that is nontrivial, yet preserves the isotopy class of all simple closed curves — the *hyperelliptic involution* of a genus two surface. By [10], this is the only such map. Neither of the techniques employed in this paper are conclusive on whether a link in the mapping torus of the genus two hyperelliptic involution admits a linear, or indeed any, upper volume bound.

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