New York Journal of Mathematics

New York J. Math. **32** (2026) 35–43.

Coxeter groups without infinite labels and the proper joint spectra of their faithful representations

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Abstract. We analyze faithful representations of dihedral groups and prove that Coxeter groups without infinite labels can be determined by the proper joint spectra of their faithful representations.

Contents

1. Introduction

The study of Coxeter groups is a classical topic in Lie theory and representation theory, with connections to many areas of mathematics. The notion of the projective spectrum for noncommuting operators was first introduced by Yang in [13] and has since become a powerful tool in functional analysis, group representation theory, Lie algebras, and spectral dynamical systems. Substantial related research has been carried out in works such as [1], [2], [4], [5], [6], [7], [8], and [10]. The proper joint spectrum is a special case of the projective spectrum, providing a bridge between operator theory and geometry. Some results concerning Coxeter groups and the proper joint spectra of their generators can be found in [3] and [12]. In particular, [3, Theorem 1.1] shows that a finite Coxeter group W can be recovered from the joint spectrum associated with its left regular representation. The proof relies largely on geometric and analytic techniques. In this paper, we establish a similar, more general result for all Coxeter groups whose Dynkin diagrams contain only finite bonds, using the proper joint

Received November 20, 2025.
2020 *Mathematics Subject Classification.* Primary 17B05; Secondary 17B10.
Key words and phrases. Proper joint spectrum, Coxeter group, dihedral group.
The first author is funded by the NSFC (Grant Numbers 12271298 and 11871308).

ISSN 1076-9803/2026

spectra attached to arbitrary faithful representations of Coxeter groups. Our approach is purely algebraic and is based on analyzing the structure of the faithful representation.

The paper is organized as follows. In Section 2, we recall some necessary background. In Section 3, we compute the characteristic polynomials and proper joint spectra for irreducible representations of dihedral groups, and summarize the results in three tables. In Section 4, we give an equivalent condition for a representation ρ of $W(\mathrm{I}_2(n))$ to be faithful, expressed in terms of its decomposition into irreducible representations. Finally, in Section 5, we prove our main theorem: a Coxeter group with finite bonds is determined by the proper joint spectrum of any faithful representation.

2. Some basic notions

We first recall the definition of Coxeter groups.

Definition 2.1. Let $M = (m_{ij})_{1\le i,j\le n}$ be a symmetric $n \times n$ matrix with entries from $\mathbb{N} \cup \infty$ such that $m_{ii} = 1$ for all $i \in [n]$ and $m_{ij} > 1$ whenever $i \neq j$. The Coxeter group of type M is the group

$$W(M) = \langle s_1, \dots, s_n | (s_i s_j)^{m_{ij}} = 1, i, j \in [n], m_{ij} < \infty\rangle.$$

We often write S instead of $s_1, \dots, s_n$ and if no confusion is imminent, W instead of $W(M)$. The pair (W, S) is called the Coxeter system of type M.

In this paper, we only consider Coxeter groups with all bonds m_{ij} being finite. We also recall some conceptions from [13].

Definition 2.2. Suppose $A_1, \dots, A_n$ are bounded linear operators on a Hilbert space V. The **projective joint spectrum** of $A_1, \dots, A_n$ is the set

$$\begin{aligned} &\sigma(A_1, \dots, A_n) \\ = \ &\{[x_1 : \cdots : x_n] \in \mathbb{CP}^n : x_1A_1 + \cdots + x_nA_n \text{ is not invertible}\}. \end{aligned}$$

The **proper joint spectrum** of $A_1, \cdots, A_n$ is the set

$$\begin{aligned} &\sigma_p(A_1, \cdots, A_n) \\ = \ &\{(x_1, \dots, x_n) \in \mathbb{C}^n : x_1A_1 + \cdots + x_nA_n - I \text{ is not invertible}\}. \end{aligned}$$

Let $S = \{s_1, \dots, s_n\}$ be a set of generators of a Coxeter group W associated to the Coxeter diagram of W, and let

$$\rho : W \longrightarrow \mathrm{GL}(V)$$

be a representation of W, with V being a complex linear space of finite dimension. Then

$$\begin{aligned} &\sigma_p(\rho(s_1), \dots, \rho(s_n)) = \\ &\{(x_1, \dots, x_n) \in \mathbb{C}^n | -I + x_1\rho(s_1) + \cdots + x_n\rho(s_n) \text{ is not invertible}\} \end{aligned}$$

is called the proper joint spectrum of (W, ρ).

3. Proper joint spectrum of an irreducible representation of $W(\mathrm{I}_2(n))$

According to [11, Example 8.2.3], the non-linear irreducible representations of a finite dihedral group are explicitly described. Moreover, using its generators and defining relations via Definition 2.1, one can readily determine its linear representations. In this section, we focus on computing the characteristic polynomial and the proper joint spectrum associated with each irreducible representation of the dihedral group. The results are summarized in tables and will be utilized in subsequent sections.

Let $W(\mathrm{I}_2(n))$ denote the dihedral group of order $2n$. Following Definition 2.1, we set

$$\mathrm{I}_2(n) = \begin{pmatrix} 1 & n \\ n & 1 \end{pmatrix},$$

$$\mathrm{Dih}_{2n} = W(\mathrm{I}_2(n)) = \{s_1, s_2 | s_1^2 = 1, s_2^2 = 1, (s_1 s_2)^n = 1\}.$$

Let ρ be an irreducible representation of $W(\mathrm{I}_2(n))$. We first compute the proper joint spectrum of (W, ρ), defined as the determinant $\det(-I + x_1\rho(s_1) + x_2\rho(s_2))$.

It is known that the irreducible representations of $W(\mathrm{I}_2(n))$ have degree either 1 or 2. Suppose the group has m_1 irreducible representations of degree one and m_2 of degree two. Then

$$|W(\mathrm{I}_2(n))| = 2n = m_1 + 4m_2.$$

We denote these representations by $\rho^n_{i,j}$, where $i = 1$ or 2 indicates the dimension of the representation, and $j = 1, \dots, m_1$ for $i = 1$, or $j = 1, \dots, m_2$ for $i = 2$.

We need to discuss the parity of n.

When $n = 2$, namely $s_1 s_2 = s_2 s_1$, then $W(\mathrm{I}_2(2)) = \mathbb{Z}/(2\mathbb{Z}) \times \mathbb{Z}/(2\mathbb{Z})$. $W(\mathrm{I}_2(2))$ has 4 irreducible representations of dimension 1 in the following.

(i) $\rho^2_{1,1}(s_1) = \rho^2_{1,1}(s_2) = 1$;
(ii) $\rho^2_{1,2}(s_1) = \rho^2_{1,2}(s_2) = -1$;
(iii) $\rho^2_{1,3}(s_1) = 1, \rho^2_{1,3}(s_2) = -1$;
(iv) $\rho^2_{1,4}(s_1) = -1, \rho^2_{1,4}(s_2) = 1$.

When $2 \nmid n$, there are two representations of dimension 1 as follows, and $m_1 = 2$.

(i) $\rho^n_{1,1}(s_1) = \rho^n_{1,1}(s_2) = 1$;
(ii) $\rho^n_{1,2}(s_1) = \rho^n_{1,2}(s_2) = -1$.

When $2|n$, there are four representations of dimension 1, and $m_1 = 4$.

(i) $\rho^n_{1,1}(s_1) = \rho^n_{1,1}(s_2) = 1$;
(ii) $\rho^n_{1,2}(s_1) = \rho^n_{1,2}(s_2) = -1$;
(iii) $\rho^n_{1,3}(s_1) = 1, \rho^n_{1,3}(s_2) = -1$;
(iv) $\rho^n_{1,4}(s_1) = -1, \rho^n_{1,4}(s_2) = 1$.

When n is an odd number, we have

$$m_1 = 2, \quad m_2 = \frac{2n-2}{4} = \frac{n-1}{2}.$$

When n is an even number,

$$m_1 = 4, \quad m_2 = \frac{2n-4}{4} = \frac{n-2}{2}.$$

Next, let us calculate $D_{i,j} = \det[-I + x_1\rho^n_{i,j}(s_1) + x_2\rho^n_{i,j}(s_2)]$.

When n is an odd number, for the irreducible representations of dimension 1, we have

$$D_{1,1} = -1 + x_1 + x_2,$$
$$D_{1,2} = -1 - x_1 - x_2;$$

and for the irreducible representations of dimension 2, it follows that

$$\rho^n_{2,k}(s_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho^n_{2,k}(s_2) = \begin{pmatrix} 0 & e^{\frac{2\pi ik}{n}} \\ e^{\frac{-2\pi ik}{n}} & 0 \end{pmatrix},$$

so

$$\begin{aligned} D_{i,j} &= \det[-I + x_1\rho^n_{i,j}(s_1) + x_2\rho^n_{i,j}(s_2)] \\ &= \begin{vmatrix} -1 & x_1 + x_2 e^{\frac{2\pi ik}{n}} \\ x_1 + x_2 e^{\frac{-2\pi ik}{n}} & -1 \end{vmatrix} \\ &= 1 - x_1^2 - x_2^2 + 2\cos\frac{2\pi k}{n} x_1 x_2, \end{aligned}$$

where $1 \le k \le \frac{n-1}{2}$.

Similarly, we can deal with the case when n is an even number, and it follows that

$$D_{1,1} = -1 + x_1 + x_2;$$
$$D_{1,2} = -1 - x_1 - x_2;$$
$$D_{1,3} = -1 + x_1 - x_2;$$
$$D_{1,4} = -1 - x_1 + x_2;$$
$$D_{2,k} = 1 - x_1^2 - x_2^2 + 2\cos\frac{2\pi k}{n} x_1 x_2, 1 \le k \le \frac{n-2}{2}.$$

For the general representation ρ for $W(\mathrm{I}_2(n))$, we use $F_\rho^{W(\mathrm{I}_2(n))}(x_1, x_2)$ to represent the equation defining the proper joint spectrum, which means that $\sigma_p(s_1, s_2)$ is defined by

$$F_\rho^{W(\mathrm{I}_2(n))}(x_1, x_2) = \det(-I + x_1\rho(s_1) + x_2\rho(s_2)) = 0.$$

For our aim in the next section, we also compute the kernel and image for each irreducible representation of $W(\mathrm{I}_2(n))$, which will be presented in the lemma below.

Now we summarize the above results in the below.

Lemma 3.1. *For the finite dihedral group $W(\mathrm{I}_2(n))$, its proper joint spectra for irreducible representations can be presented in the following three tables. Table* 1 *is for the case $n = 2$; Table* 2 *is for the case $n > 2$, $2 \nmid n$, and $1 \leq k \leq \frac{n-1}{2}$; Table* 3 *is for the case $n > 2$, $2|n$, and $1 \leq k \leq \frac{n-2}{2}$.*

ρ	*kernel*	*image*	$F_{\rho}^{W(\mathrm{I}_2(n))} = 0$
$\rho_{1,1}^2$	$W(\mathrm{I}_2(n))$	$\langle 1 \rangle$	$x_1 + x_2 - 1 = 0$
$\rho_{1,2}^2$	$\mathbb{Z}/(2\mathbb{Z}) = \langle s_1 s_2 \rangle$	$\mathbb{Z}/(2\mathbb{Z})$	$-x_1 - x_2 - 1 = 0$
$\rho_{1,3}^2$	$\mathbb{Z}/(2\mathbb{Z}) = \langle s_2 \rangle$	$\mathbb{Z}/(2\mathbb{Z})$	$-x_1 + x_2 - 1 = 0$
$\rho_{1,4}^2$	$\mathbb{Z}/(2\mathbb{Z}) = \langle s_1 \rangle$	$\mathbb{Z}/(2\mathbb{Z})$	$x_1 - x_2 - 1 = 0$

TABLE 1. The cases for $W(\mathrm{I}_2(2))$.

ρ	*kernel*	*image*	$F_{\rho}^{W(\mathrm{I}_2(n))} = 0$
$\rho_{1,1}^n$	$W(\mathrm{I}_2(n))$	$\langle 1 \rangle$	$x_1 + x_2 - 1 = 0$
$\rho_{1,2}^n$	$\mathbb{Z}/(2\mathbb{Z}) = \langle s_1 s_2 \rangle$	$\mathbb{Z}/(2\mathbb{Z})$	$-x_1 - x_2 - 1 = 0$
$\rho_{2,k}^n$	$\mathbb{Z}/(\gcd(n,k)\mathbb{Z}) = \left\langle (s_1 s_2)^{\frac{n}{\gcd(n,k)}} \right\rangle$	$W(\mathrm{I}_2(\frac{n}{\gcd(n,k)}))$	$x_1^2 + x_2^2 + 2\cos\frac{2\pi k}{n} x_1 x_2 - 1 = 0$

TABLE 2. The cases for $W(\mathrm{I}_2(n))$, n being odd.

ρ	*kernel*	*image*	$F_{\rho}^{W(\mathrm{I}_2(n))} = 0$
$\rho_{1,1}^n$	$W(\mathrm{I}_2(n))$	$\langle 1 \rangle$	$x_1 + x_2 - 1 = 0$
$\rho_{1,2}^n$	$\mathbb{Z}/(n\mathbb{Z}) = \langle s_1 s_2 \rangle$	$\mathbb{Z}/(2\mathbb{Z})$	$-x_1 - x_2 - 1 = 0$
$\rho_{1,3}^n$	$W(\mathrm{I}_2(\frac{n}{2})) = \langle s_2, s_1 s_2 s_1 \rangle$	$\mathbb{Z}/(2\mathbb{Z})$	$-x_1 + x_2 - 1 = 0$
$\rho_{1,4}^2$	$W(\mathrm{I}_2(\frac{n}{2})) = \langle s_1, s_2 s_1 s_2 \rangle$	$\mathbb{Z}/(2\mathbb{Z})$	$x_1 - x_2 - 1 = 0$
$\rho_{2,k}^n$	$\mathbb{Z}/(\gcd(n,k)\mathbb{Z}) = \left\langle (s_1 s_2)^{\frac{n}{\gcd(n,k)}} \right\rangle$	$W(\mathrm{I}_2(\frac{n}{\gcd(n,k)}))$	$x_1^2 + x_2^2 + 2\cos\frac{2\pi k}{n} x_1 x_2 - 1 = 0$

TABLE 3. The cases for $W(\mathrm{I}_2(n))$, n being even.

4. The faithfulness for a representation ρ of $W(\mathrm{I}_2(n))$

Let ρ be a finite-dimensional representation of $W(\mathrm{I}_2(n))$. We establish a criterion for determining whether ρ is faithful by examining its decomposition into irreducible components.

Let $\mathrm{Irr}(W(\mathrm{I}_2(n)))$ denote the set of all irreducible representations of $W(\mathrm{I}_2(n))$. Suppose ρ decomposes as $\rho = \bigoplus (\rho_i)^{t_i}$, where each $\rho_i \in \mathrm{Irr}(W(\mathrm{I}_2(n)))$ and t_i is the multiplicity of ρ_i. The proper joint spectrum $V_{\rho}^{W(\mathrm{I}_2(n))}$ is defined by the equation $F_\rho^{W(\mathrm{I}_2(n))} = \det(-I + x_1\rho(s_1) + x_2\rho(s_2)) = 0$. The following theorem then holds.

Theorem 4.1. *For the representations of $W(\mathrm{I}_2(n))$, the following hold.*

(i) For each irreducible representation ρ of $W(\mathrm{I}_2(n))$, the set $V_\rho^{W(\mathrm{I}_2(n))}$ is a line or an ellipse, and $V_{\rho_1}^{W(\mathrm{I}_2(n))} \neq V_{\rho_2}^{W(\mathrm{I}_2(n))}$ if $\rho_1 \neq \rho_2 \in \mathrm{Irr}(W(\mathrm{I}_2(n)))$.

(ii) For any finite representation ρ of $W(\mathrm{I}_2(n))$, the irreducible components of $V_\rho^{W(\mathrm{I}_2(n))}$ correspond one to one with the irreducible representations of $W(\mathrm{I}_2(n))$ occurring in the decomposition of ρ.

Proof. The conclusion in (i) can be verified by the tables in Lemma 3.1.

For (ii), we have $\rho = \bigoplus \rho_i^{t_i}$ being its decomposition, $\rho_i \in \mathrm{Irr}(W(\mathrm{I}_2(n)))$, which implies the equation $F_\rho^{W(\mathrm{I}_2(n))}$ determining $V_\rho^{W(\mathrm{I}_2(n))}$ has the decomposition

$$F_\rho^{W(\mathrm{I}_2(n))} = \prod (F_{\rho_i}^{W(\mathrm{I}_2(n))})^{t_i}.$$

Therefore $V_\rho^{W(\mathrm{I}_2(n))}$ has irreducible components $V_{\rho_1}^{W(\mathrm{I}_2(n))}, \dots, V_{\rho_k}^{W(\mathrm{I}_2(n))}$ each being a line or an ellipse, having nothing to do with the multiplicities t_i. □

Then the following corollary holds for the Theorem 4.1.

Corollary 4.2. *Let ρ, ρ' be finite dimensional representations of $W(\mathrm{I}_2(n))$. Then $V_\rho^{W(\mathrm{I}_2(n))} = V_{\rho'}^{W(\mathrm{I}_2(n))}$ if and only if the irreducible representations in $W(\mathrm{I}_2(n))$ occurring in the decompositions of ρ and ρ' are the same.*

For the faithfulness of ρ for $W(\mathrm{I}_2(n))$, we have the following theorem.

Theorem 4.3. *The representation ρ of $W(\mathrm{I}_2(n))$ is a faithful representation if and only if the following conditions hold for different n.*

(i) The representation ρ has at least 2 of $\rho_{1,2}^2, \rho_{1,3}^2, \rho_{1,4}^2$ in its irreducible decomposition when $n = 2$.

(ii) The representation ρ has distinct $\rho_{2,k_i}^n, i = 1, \dots, t$ in its irreducible decomposition with $\gcd((n, k_1), \dots, (n, k_t)) = 1$ when $2 \nmid n$.

(iii) The representation ρ has neither $\rho_{1,3}^n$ or $\rho_{1,4}^n$ in its decomposition, and ρ has $\rho_{2,k_1}^n, \dots, \rho_{2,k_t}^n$ with $\gcd((n, k_1), \dots, (n, k_t)) = 1$ or ρ has either $\rho_{1,3}^n$ or $\rho_{1,4}^n$ in its decomposition, and ρ has $\rho_{2,k_1}^n, \dots, \rho_{2,k_t}^n$ with $\gcd(2, (n, k_1), \dots, (n, k_t)) = 1$ when $2 | n$.

Proof. Suppose $\rho = \rho_1^{t_1} \oplus \rho_2^{t_2} \oplus \cdots \oplus \rho_k^{t_k}, \rho_i \in \mathrm{Irr}(W(\mathrm{I}_2(n)))$, then ρ is faithful if and only if $\ker(\rho_1) \cap \ker(\rho_2) \cap \cdots \cap \ker(\rho_k) = 1$.

The case (i) can be verified from Table 1 in Lemma 3.1.

Since $W(\mathrm{I}_2(n)) = \langle s_1, s_2 | s_1^2 = 1, s_2^2 = 1, (s_1s_2)^n = 1 \rangle$, write $r = s_1s_2$. Hence the order of r is n.

Let us prove (ii). By Table 2 in Lemma 3.1, we see that $\ker\rho_{2,k}^n \subseteq \ker\rho_{1,2}^n \subseteq \ker\rho_{1,1}^n$, and $\ker\rho_{1,2}^n \neq \{1\}$. Hence, when ρ is faithful, ρ must have some $\rho_{2,k}^n$

in its decomposition. Suppose for those 2 dimensional representations, ρ has $\rho^n_{2,k_i}, i = 1, \dots, t$ in its decomposition. Hence $\ker\rho^n_{2,k_i} = \langle r^{\frac{n}{\gcd(n,k_i)}} \rangle$ by Table 2 in Lemma 3.1. Therefore, it follows that

$$\ker\rho = \ker\rho^n_{2,k_1} \cap \ker\rho^n_{2,k_2} \cap \cdots \cap \ker\rho^n_{2,k_t} = \bigcap \left\langle r^{\frac{n}{\gcd(n,k_i)}} \right\rangle.$$

Since $\left\langle r^{\frac{n}{\gcd(n,k_i)}} \right\rangle$ is a cyclic subgroup in $\langle r \rangle$ of order $\gcd(n, k_i)$, when ρ is faithful, it is equivalent to $\gcd((n, k_1), \dots, (n, k_t)) = 1$ or $\gcd(k_1, \dots, k_t, n) = 1$.

Now we prove (iii). For the case 1 of (iii), the argument is similar to the proof of (ii). For the case 2 of (iii), it is known that

$$\left\langle r^2 \right\rangle = \ker\rho^n_{1,3} \bigcap \langle r \rangle = \ker\rho^n_{1,4} \bigcap \langle r \rangle \subseteq \ker\rho^n_{1,2} \subseteq \ker\rho^n_{1,1}.$$

When ρ has exactly the 2-dimensional irreducible representations $\rho^n_{2,k_i}, i = 1, \dots, t$ in its decomposition, we see that

$$\ker\rho = \left\langle r^2 \right\rangle \bigcap (\bigcap \ker\rho^n_{2,k_i}) = \left\langle r^2 \right\rangle \bigcap \left(\bigcap \left\langle r^{\frac{n}{\gcd(n,k_i)}} \right\rangle \right).$$

Therefore, similarly to the argument in (ii), it follows that ρ is faithful if and only if $\gcd(\frac{n}{2}, (n, k_1), \dots, (n, k_t)) = 1$, namely $\gcd(\frac{n}{2}, k_1, \dots, k_t) = 1$. □

5. Main theorem

Compared with [3, Theorem 1.1], for general Coxeter groups without infinite bonds in their Coxeter diagrams, we will prove the new more general version of the theorem through a faithful representation.

Theorem 5.1. *Let W be a Coxeter group with generators $\{s_1, \dots, s_n\}$ associated to its Coxeter diagram without infinite bonds, and ρ be a faithful representation of W. If the proper joint spectrum U relative to $\{s_1, \dots, s_n\}$ of ρ is known, then the Coxeter group can be determined by the set U.*

Proof. Take 2 generators s_i, s_j of W. The theorem is equivalent to proving that the $m_{ij} = \mathrm{ord}(s_i s_j)$ are determined by the set U.

Since ρ is a faithful representation of W, we have ρ is also a faithful representation of the dihedral group generated by s_i and s_j.

Now, let $V_{ij} = \{(x_1, \dots, x_n) \in \mathbb{C}^n | x_k = 0 \text{ if } k \neq i \text{ or } j\}$ and $U_{ij} = U \bigcap V_{ij}$.

By Theorem 4.3, we divide our argument into 3 cases.

Case 1: When the set U_{ij} consists of lines only, then by Theorem 2, we must have $m_{ij} = 2$.

Case 2: Suppose the set U_{ij} consists of some ellipses $E_1, \dots, E_t$ and lines in $\{x_i + x_j - 1 = 0, -x_i - x_j - 1 = 0\}$. Say E_j is defined by the equation $x_1^2 + x_2^2 + 2\cos\frac{2\pi n_j}{m_j} x_1 x_2 - 1 = 0$ for $j = 1, \dots, t$ with $0 < \frac{n_j}{m_j} < \frac{1}{2}, \gcd(n_j, m_j) = 1$. By Table 2 in Lemma 3.1, we see E_j corresponds to an irreducible representation of $\langle s_i, s_j \rangle$ with kernel $\langle (s_i s_j)^{m_j} \rangle$. By (ii) of Theorem 4.3 and the first case of (iii) of Theorem 4.3, when ρ is faithful, we have

$$\gcd(\frac{m_{ij}}{m_1}, \dots, \frac{m_{ij}}{m_t}) = 1,$$

and then m_{ij} is the least common multiple of $m_1, \dots, m_t$.

Case 3: Suppose U_{ij} consists of some ellipses $E_1, \dots, E_t$ and lines in $\{x_i - x_j - 1 = 0, x_j - x_i - 1 = 0\}$. Therefore the order m_{ij} must be even.

We suppose E_j is in the form of case 2 in the above, by the second case of (iii) of Theorem 4.3, it follows that

$$\gcd\left(\frac{m_{ij}}{2}, \frac{m_{ij}}{m_1}, \dots, \frac{m_{ij}}{m_t}\right) = 1.$$

Therefore, we suppose θ is the least common multiple of $m_1, \dots, m_t$. When $2 \nmid \theta$, then we have $m_{ij} = 2\theta$; when $2|\theta$, it implies that $m_{ij} = \theta$. □

Remark 5.2. From this paper, we observe that for Coxeter groups, the generating relations of the group, its representations, and their characteristic polynomials mutually determine one another. This establishes a trinity of unification among Coxeter groups, their faithful representations, and their geometric realizations. A natural question arises: what is the relationship between the representations of Coxeter groups with infinite labels and the characteristic polynomials of these representations? Is there also a one-to-one correspondence? This remains an unresolved issue in our research. We think some results about representations of the infinite dihedral group should be helpful.

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This paper is available via http://nyjm.albany.edu/j/2026/32-2.html.