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Paschke duality and assembly maps

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Abstract. We construct a natural transformation between two versions of G-equivariant K-homology with coefficients in a G-C^*-category for a countable discrete group G. Its domain is a coarse geometric K-homology and its target is the usual analytic K-homology. Following classical terminology, we call this transformation the Paschke transformation. We show that under certain finiteness assumptions on a G-space X, the Paschke transformation is an equivalence on X. As an application, we provide a direct comparison of the homotopy theoretic Davis–Lück assembly map with Kasparov's analytic assembly map appearing in the Baum–Connes conjecture.

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1. Introduction and statements

The main result of the present paper is the construction of a natural transformation

$$K_{\mathbf{C}}^{G,\mathcal{X}} \to K_{\mathbf{C}}^{G,\mathrm{An}} \tag{1.1}$$

between two versions of spectrum-valued equivariant K-homology functors, where G is a countable discrete group. The evaluation of this transformation on G-finite G-simplicial complexes with finite stabilzers is an equivalence. Following the classical terminology, we call this transformation the Paschke transformation. The functor $K_{\mathbf{C}}^{G,\mathcal{X}}$ in the domain is called the equivariant local K-homology and is derived from an equivariant coarse K-homology functor using coarse geometric constructions, while the target $K_{\mathbf{C}}^{G,\mathrm{An}}$ is a spectrum-valued version of the classical equivariant analytic K-homology. In both versions, the subscript indicates a natural dependence on a coefficient G-C^*-category $\mathbf{C}$.

The Paschke transformation (1.1) will be used to compare the domains of the Davis–Lück type assembly map and the Baum–Connes type assembly map. Our second main result is Theorem 1.9, showing that these two assembly maps are equal on the level of homotopy groups.

In the following, we give an informal description of the construction of the two homology theories entering (1.1).

Starting from classical Paschke duality, we further explain the development of ideas leading to the construction of the map in (1.1). We then state the precise version of our Paschke duality result as Theorem 1.5, and finally discuss the comparison of assembly maps.

We emphasize that this paper is not the first to treat the topic of equivariant Paschke duality and comparisons of assembly maps, most current are the papers [2] and [26]. We explain more about this in Remarks 1.12 and 1.13.

Constructions with the coefficients. For facts about C^*-categories and their K-theory, we will generally refer to [3] and [9] which were written to provide the necessary background for the present paper, [7] and [12]. Both K-homology functors occuring in (1.1) depend on the choice of a G-C^*-category $\mathbf{C}$, i.e., an object of $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ (see [3, Sec. 3] or [9, Def. 2.6] for $C^*\mathbf{Cat}^{\mathrm{nu}}$). We use the symbol $\mathbf{MC}$ in order to denote the multiplier category of $\mathbf{C}$ [9, Def. 3.1].

In Definition 2.15, we describe an exact sequence

$$0 \to \mathbf{C}^{(G)}_{\mathrm{std}} \to \mathbf{MC}^{(G)}_{\mathrm{std}} \to \mathbf{Q}^{(G)}_{\mathrm{std}} \to 0$$

of G-C^*-categories (see [3, Def. 8.5] or [9, Def. 13.2] for the notion of an exact sequence) defining the Calkin G-C^*-category $\mathbf{Q}^{(G)}_{\mathrm{std}}$. These constructions depend functorially on $\mathbf{C}$ for non-degenerate morphisms.

Example 1.1. In the case of trivial coefficients, $\mathbf{C}$ is the G-C^*-category $\mathbf{Hilb}_c(\mathbb{C})$ of Hilbert spaces and compact operators with trivial G-action. The multiplier category of $\mathbf{Hilb}_c(\mathbb{C})$ can be identified with the category $\mathbf{Hilb}(\mathbb{C})$ of Hilbert spaces and all bounded operators [9, Lem. 8.1]. By specializing Definition 2.15, the G-C^*-category $\mathbf{C}^{(G)}_{\mathrm{std}}$ turns out to be the category $\mathbf{Hilb}(\mathbb{C})^{(G)}_{\mathrm{std}}$ of all pairs (H,ρ) of a Hilbert space H with a unitary G-representation ρ that are isomorphic to $(L^2(G)\otimes H', \lambda\otimes \mathrm{id}_{H'})$, where λ is the left-regular representation and H' is some auxiliary Hilbert space. The morphisms $(H_0,\rho_0)\to(H_1,\rho_1)$ in $\mathbf{Hilb}_c(\mathbb{C})^{(G)}_{\mathrm{std}}$ are all compact operators $H_0\to H_1$.

The G-C^*-category $\mathbf{MC}^{(G)}_{\mathrm{std}}$ is the category $\mathbf{Hilb}(\mathbb{C})^{(G)}_{\mathrm{std}}$ which has the same objects, but its morphism spaces are the bigger spaces of all bounded linear operators. In both cases, the G-action fixes objects and acts by conjugation on the morphism spaces. The G-C^*-category $\mathbf{Q}^{(G)}_{\mathrm{std}}$ is the Calkin category

$$\mathbf{Hilb}(\mathbb{C})^{(G)}_{\mathrm{std}}/\mathbf{Hilb}_c(\mathbb{C})^{(G)}_{\mathrm{std}}.$$

Its objects are the objects (H,ρ) of $\mathbf{Hilb}(\mathbb{C})^{(G)}_{\mathrm{std}}$, and its morphism spaces are the quotient spaces of bounded operators by compact operators with the induced G-action. In particular, the endomorphism algebra of each object (H,ρ) is the usual Calkin algebra $Q(H)$ of H with the G-action by conjugation, hence the name. □

Example 1.2. More generally, for a G-C^*-algebra A we consider the G-C^*-category $\mathbf{C}=\mathbf{Hilb}_c(A)$ of Hilbert A-modules and compact operators. Its multiplier category is the category $\mathbf{Hilb}(A)$ of Hilbert A-modules and all adjointable operators [9, Lem. 8.1]. The G-action on both categories is described explicitly in [9, Ex. 2.10].

If A is unital, then the associated G-C^*-category $\mathbf{Hilb}_c(A)^{(G)}_{\mathrm{std}}$ consists of pairs (H,ρ) of a Hilbert A-module together with a unitary G-action ρ such that H is isomorphic to an orthogonal sum of a family of finitely generated projective A-modules indexed by a free G-set. Since G acts non-trivially on $\mathbf{Hilb}_c(A)$ the details are slightly more complicated to describe, see Definition 2.15. □

Analytic *K*-homology. The construction of the equivariant analytic K-homology functor $K^{G,\mathrm{An}}_{\mathbf{C}}$ with coefficients in $\mathbf{C}$ employs the ∞-categorical version

$$\mathrm{kk}^G : \mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}}) \to \mathrm{KK}^G$$

of the KK-functor from [12, Def. 1.8] and its extension to C^*-categories

$$\begin{tikzcd} \mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}}) \arrow[rr, "\mathrm{kk}^G"] \arrow[dr, "\mathrm{incl}"'] & & \mathrm{KK}^G \\ & \mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}) \arrow[ur, "\mathrm{kk}^G_{C^*\mathbf{Cat}}"'] & \end{tikzcd} \tag{1.2}$$

introduced in [12, Def. 1.29], where incl interprets a G-C^*-algebra as a G-C^*-category with a single object. The mapping spectrum functor of the stable ∞-category KK^G will be denoted by

$$\mathrm{KK}^G(-,-)\colon\ \mathrm{KK}^{G^{\mathrm{op}}} \times \mathrm{KK}^G \to \mathbf{Sp}\,.$$

In order to simplify the notation, we drop the symbols kk^G or $\mathrm{kk}^G_{C^*\mathbf{Cat}}$ when we express the value of a functor F defined on KK^G on a G-C^*-algebra A or a G-C^*-category $\mathbf{C}$. By [12, Prop. 3.5], if A is a separable G-C^*-algebra and B is σ-unital, then the homotopy groups $\pi_*\mathrm{KK}^G(A,B)$ are canonically isomorphic to the classical equivariant KK^G-groups of Kasparov [25] associated to A, B.

The equivariant analytic K-homology functor $K^{G,\mathrm{An}}_{\mathbf{C}}$ is defined by the formula

$$K^{G,\mathrm{An}}_{\mathbf{C}}\colon\ G\mathbf{LCH}^{\mathrm{prop}}_+ \to \mathbf{Sp}\,,\quad X \mapsto \mathrm{KK}^G(C_0(X), \mathbf{Q}^{(G)}_{\mathrm{std}})\,. \tag{1.3}$$

The domain of this functor is the category $G\mathbf{LCH}^{\mathrm{prop}}_+$ of locally compact Hausdorff G-spaces with partially defined proper maps. Equivalently, $G\mathbf{LCH}^{\mathrm{prop}}_+$ is the Gelfand dual of the category $GC^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{comm}}$ of non-unital commutative G-C^*-algebras. The connection with the notation from [12, Def. 1.15] is given by

$$K^{G,\mathrm{An}}_{\mathbf{C}} = K^{G,\mathrm{an}}_{\mathbf{Q}^{(G)}_{\mathrm{std}}}\,, \tag{1.4}$$

In particular, $K^{G,\mathrm{An}}_{\mathbf{C}}$ is different from $K^{G,\mathrm{an}}_{\mathbf{C}}$ — we apologize for this notational inconvenience.

In view of (1.4), the basic properties of $K^{G,\mathrm{an}}$ listed in [12, Thm. 1.15] imply corresponding properties of $K^{G,\mathrm{An}}_{\mathbf{C}}$. In particular, the functor $K^{G,\mathrm{An}}_{\mathbf{C}}$ is homotopy invariant, is excisive for closed decompositions of second countable spaces with proper action (this restriction is due to the usage of [12, Prop. 1.12.1]), and it annihilates spaces of the form $[0,\infty) \times X$.

Example 1.3. Let us consider the coefficients $\mathbf{C} = \mathbf{Hilb}_c(A)$ for a unital G-C^*-algebra A. For a G-space X which is homotopy equivalent to a G-finite CW-complex with finite stabilizers, Proposition 10.15 provides a natural isomorphism

$$\pi_* K^{G,\mathrm{An}}_{\mathbf{C}}(X) \cong KK^G_{*-1}(C_0(X), A)\,. \tag{1.5}$$

This isomorphism identifies our definition of equivariant analytic K-homology with the classical definition given by the right hand side of (1.5), up to a shift of degrees. □

In order to deal correctly with non-G-compact spaces in $G\mathbf{LCH}_+^{\mathrm{prop}}$, we will consider the locally finite version $K_{\mathbf{C}}^{G,\mathrm{An},\mathrm{lf}}$ of $K_{\mathbf{C}}^{G,\mathrm{An}}$ which is defined as follows. If X is in $G\mathbf{LCH}_+^{\mathrm{prop}}$ and U is an open G-invariant subset of X with G-compact closure, then we have a morphism $X \to U$ in $G\mathbf{LCH}_+^{\mathrm{prop}}$ given by the partially defined map $X \supset U \xrightarrow{\mathrm{id}_U} U$ which corresponds to the extension-by-zero homomorphism $C_0(U) \to C_0(X)$ on the level of commutative G-C^*-algebras. We define

$$K_{\mathbf{C}}^{G,\mathrm{An},\mathrm{lf}}(X) := \lim_{U \subseteq X} K_{\mathbf{C}}^{G,\mathrm{An}}(U)\,, \tag{1.6}$$

where the limit runs over all open subsets U of X with G-compact closure. Using right Kan extensions, one can turn this prescription into the definition of a functor

$$K_{\mathbf{C}}^{G,\mathrm{An},\mathrm{lf}} : G\mathbf{LCH}_+^{\mathrm{prop}} \to \mathbf{Sp}\,, \tag{1.7}$$

see [6, Sec. 7.1.2] for a similar construction. We have a natural transformation

$$c : K_{\mathbf{C}}^{G,\mathrm{An}} \to K_{\mathbf{C}}^{G,\mathrm{An},\mathrm{lf}} \tag{1.8}$$

of functors from $G\mathbf{LCH}_+^{\mathrm{prop}}$ to $\mathbf{Sp}$. The functor $K_{\mathbf{C}}^{G,\mathrm{An},\mathrm{lf}}$ is homotopy invariant. Its restriction to second countable spaces with proper G-action is excisive for closed decompositions. Finally, it sends countable disjoint unions to products. If X is G-compact, then the canonical map $c_X : K_{\mathbf{C}}^{G,\mathrm{An}}(X) \to K_{\mathbf{C}}^{G,\mathrm{An},\mathrm{lf}}(X)$ is an equivalence. We refer to Proposition 10.16 for a calculation of the values of $K_{\mathbf{C}}^{G,\mathrm{An},\mathrm{lf}}$ on more general spaces.

The functors $K_{\mathbf{C}}^{G,\mathrm{An}}$ and $K_{\mathbf{C}}^{G,\mathrm{An},\mathrm{lf}}$ depend functorially on the coefficient G-C^*-category $\mathbf{C}$ for non-degenerate morphisms.

Remark 1.4. Using the equivariant E-theory functor [4, Def. 3.22], one could define a version of analytic K-homology

$$E_{\mathbf{C}}^{G,\mathrm{An}} : G\mathbf{LCH}_+^{\mathrm{prop}} \to \mathbf{Sp}\,, \quad X \mapsto \mathrm{EE}^G(C_0(X), \mathbf{Q}_{\mathrm{std}}^{(G)})$$

with better formal properties. Since the E-theory functor sends all exact sequences of C^*-categories to fibre sequences, in the case of $\mathbf{C} = \mathbf{Hilb}_c(A)$ for a unital G-C^*-algebra A we have the analogue of (1.5)

$$E_{\mathbf{C}}^{G,\mathrm{An}}(X) \simeq \Sigma\mathrm{EE}^G(C_0(X), A)$$

without any restriction on X. Furthermore, the functor $E_{\mathbf{C}}^{G,\mathrm{An}}$ is excisive for arbitrary invariant closed decompositions, i.e., we can drop the assumptions of properness of the G-action and second countability needed for $K_{\mathbf{C}}^{G,\mathrm{An}}$. Finally, since the E-theory functor preserves filtered colimits of G-C^*-algebras, the functor $E_{\mathbf{C}}^{G,\mathrm{An}}$ is already locally finite, i.e., the analogue $E_{\mathbf{C}}^{G,\mathrm{An}} \to E_{\mathbf{C}}^{G,\mathrm{An},\mathrm{lf}}$ of the comparison map (1.8) is an equivalence (see [13, Prop. 3.30] for an analogous statement).

The comparison functor $\mathrm{KK}^G \to \mathrm{EE}^G$ induces a transformation $K_{\mathbf{C}}^{G,\mathrm{An}} \to E_{\mathbf{C}}^{G,\mathrm{An}}$ which is an equivalence on spaces which are homotopy equivalent to

G-finite G-simplicial complexes with finite stabilizers. Composing the Paschke morphism (1.17) below with this comparison map we get a Paschke morphism with target $E^{G,\mathrm{An}}_{\mathbf{C}}$. Furthermore, our main Theorem 1.5 on the Paschke equivalence implies a similar result involving $E^{G,\mathrm{An}}_{\mathbf{C}}$.

Here are our three reasons to prefer $K^{G,\mathrm{An}}_{\mathbf{C}}$. First of all, this is the analytic K-homology functor considered in the classical literature. Secondly, working with $K^{G,\mathrm{An}}_{\mathbf{C}}$ provides a finer result. Finally, and this is our main reason, in the application to assembly maps we need reduced crossed products with G which descend to equivariant KK-theory, but not to equivariant E-theory by the lack of exactness of $- \rtimes_r G$. □

Coarse K-homology. We now turn to a brief description of the equivariant local K-homology functor $K^{G,\mathcal{X}}_{\mathbf{C}}$. For our purposes, the functor $K^{G,\mathcal{X}}_{\mathbf{C}}$ is most naturally defined on the category $G\mathbf{UBC}$ of G-uniform bornological coarse spaces [10, Def. 9.9]. This category comes with a cone-at-∞ functor $\mathcal{O}^\infty : G\mathbf{UBC} \to G\mathbf{BC}$ (see Definition 4.5), where $G\mathbf{BC}$ denotes the category of G-bornological coarse spaces [10, Def. 2.1]. We define our equivariant local K-homology as the composition of $\mathcal{O}^\infty$ with the equivariant coarse homology theory $K\mathbf{C}\mathcal{X}^G_{G_{can,max}} : G\mathbf{BC} \to \mathbf{Sp}$. This functor is the twist (see Definition 4.7) of the equivariant coarse K-homology $K\mathbf{C}\mathcal{X}^G : G\mathbf{BC} \to \mathbf{Sp}$ constructed in [7] (see also Definition 3.4) by the object $G_{can,max}$ in $G\mathbf{BC}$.

In order to construct $K\mathcal{X}\mathbf{C}^G$, we must assume that the coefficient G-C^*-category $\mathbf{C}$ satisfies further axioms, namely that it is effectively additive and admits countable AV-sums (see Definitions 2.3 and 2.2). The coefficient category $\mathbf{Hilb}_c(A)$ for a G-C^*-algebra A satisfies these axioms by [9, Lem. 7.9] since it admits all small AV-sums.

We define the equivariant local K-homology functor by

$$K^{G,\mathcal{X}}_{\mathbf{C}} := K\mathbf{C}\mathcal{X}^G_{G_{can,max}} \circ \mathcal{O}^\infty : G\mathbf{UBC} \to \mathbf{Sp}\,. \tag{1.9}$$

This composition is an equivariant local homology theory, i.e. it is homotopy invariant, excisive for closed decompositions, u-continuous, and vanishes on spaces of the form $[0,\infty) \otimes X$, see Proposition 4.6.

The functor $K\mathbf{C}\mathcal{X}^G$ and therefore also $K^{G,\mathcal{X}}_{\mathbf{C}}$ depend also functorially on the coefficient category $\mathbf{C}$ for non-degenerate morphisms.

A common domain for $K^{G,\mathrm{An}}_{\mathbf{C}}$ and $K^{G,\mathcal{X}}_{\mathbf{C}}$. By now, the functors $K^{G,\mathrm{An}}_{\mathbf{C}}$ and $K^{G,\mathcal{X}}_{\mathbf{C}}$ can not be compared. They are invariants of different objects: locally compact Hausdorff G-spaces on the one hand, and G-uniform bornological coarse spaces on the other hand. In order to compare their domains, we consider the functor

$$\iota^{\mathrm{top}} : G\mathbf{UBC} \to G\mathbf{LCH}^{\mathrm{prop}}_{+}$$

from (6.1). It is uniquely characterized by the equalities

$$C_0(X) = C_0(\iota^{\mathrm{top}}(X)) \tag{1.10}$$

for all X in $G\mathbf{UBC}$, where the C^*-algebra $C_0(X)$ on the left-hand side consists of the bounded uniformly continuous functions which become arbitrary small outside of sufficiently large bounded subsets. The symbol $C_0(\iota^{\mathrm{top}}(X))$ has the usual meaning.

We let $G\mathbf{Simpl}^{\mathrm{prop}}_{\mathrm{fin}}$ denote the category of G-finite G-simplicial complexes with finite stabilizers and equivariant proper simplicial maps. Equipping G-simplicial complexes with the spherical path metric provides a functor

$$G\mathbf{Simpl}^{\mathrm{prop}}_{\mathrm{fin}} \to G\mathbf{UBC}\,.$$

We can summarize our first main result, slightly informally, by the following diagram.

$$\begin{tikzcd}
& G\mathbf{Simpl}^{\mathrm{prop}}_{\mathrm{fin}} \arrow[dl] \arrow[dr] & \\
G\mathbf{UBC} \arrow[rr, "\iota^{\mathrm{top}}"] \arrow[rr, phantom, "\overset{p}{\Longrightarrow}"', shift right=4ex] \arrow[dr, bend right, "K^{G,\mathcal{X}}_{\mathbf{C}}"'] & & G\mathbf{LCH}^{\mathrm{prop}}_{+} \arrow[dl, bend left, "K^{G,\mathrm{An}}_{\mathbf{C}}"] \\
& \mathbf{Sp} &
\end{tikzcd}$$

The Paschke transformation p will be constructed as a natural transformation filling the lower triangle. Equivalently, naturality of p can be stated by saying that it makes the lower square lax-commutative. We then show that the Paschke transformation renders the large square commutative. In other words, the Paschke transformation becomes a natural equivalence when restricted to G-finite G-simplicial complexes with finite stabilizers. In addition, the Paschke transformation is natural in the coefficient category $\mathbf{C}$ for non-degenerate morphisms. We will state our main theorem more formally as Theorem 1.5 below.

A review of classical Paschke duality. In order to motivate the definitions involved in the above diagram, we now review some aspects of classical Paschke duality. Based on the seminal work of Paschke [32], the general theme of Paschke duality is to express the analytic K-homology

$$K^{\mathrm{an}}_*(X) := \mathrm{KK}_*(C_0(X), \mathbb{C})$$

in terms of the K-theory of a C^*-algebra naturally associated to X, which is then often referred to as the Paschke dual algebra of X.

Classically, this is implemented as follows. Let X be a proper metric space and $\phi\colon C_0(X) \to B(H)$ be a homomorphism of C^*-algebras, where H is a separable Hilbert space. To this data one associates an exact sequence of C^*-algebras

$$0 \to C(H,\phi) \to D(H,\phi) \to Q(H,\phi) \to 0 \tag{1.11}$$

where $D(H,\phi)$ is the C^*-subalgebra of $B(H)$ generated by the controlled and pseudolocal operators and $C(H,\phi)$, called the Roe algebra, is its ideal generated by the operators which are in addition locally compact.

If (H,ϕ) is sufficiently large (very ample in classical terminology or absorbing in the sense of Definition 11.1) and non-degenerate (meaning that $\overline{\phi(C_0(X))H} = H$), then the K-theory of $Q(H,\phi)$ is a well-behaved invariant of X. More precisely, for a proper map $f: X \to X'$ and absorbing non-degenerate representations (H,ϕ) and (H',ϕ') for X and X' respectively, there exists a unitary, controlled and pseudolocal isometry $(H',\phi') \cong (H,\phi\circ f^*)$ called a covering, which is unique up to conjugation by unitaries in $D(H',\phi')$. This covering induces a homomorphism $D(H,\phi) \to D(H',\phi')$ preserving the respective Roe algebras and therefore a homomorphism $Q(H,\phi) \to Q(H',\phi')$ between the quotients. For $f = \mathrm{id}_X$, this shows that the K-theory of $Q(H,\phi)$ is independent of the choice of an absorbing representation (H,ϕ). We recall here that Voiculescu's Theorem grants the existence of such absorbing representations. Furthermore, setting

$$K_*^{\mathcal{X}}(X) := K_*^{C^*\mathrm{Alg}}(Q(H,\phi))$$

for any choice of an absorbing non-degenerate representation (H,ϕ), one obtains a functor

$$K_*^{\mathcal{X}}(-)\colon \mathbf{Met}^{\mathrm{prop}} \to \mathbf{Ab}^{\mathbb{Z}}$$

defined on the category of proper metric spaces and proper maps and taking values in graded abelian groups. The superscript $\mathcal{X}$ indicates the coarse geometric origin of the construction, whose implementation was initiated by Roe [34]. The functor $K_*^{\mathcal{X}}(-)$ is homotopy invariant and admits Mayer–Vietoris sequences. In addition, there is a natural Paschke duality isomorphism

$$K_*^{\mathcal{X}}(X) \cong K_{*-1}^{\mathrm{an}}(X) \tag{1.12}$$

given by a concrete cycle level construction, see [23] for details. So up to suspension $Q(H,\phi)$ is the Paschke dual of $C_0(X)$.

The Paschke transformation following Quiao–Roe. The paper [33] discusses a systematic approach to the isomorphism (1.12), whose basic idea we now adapt to the equivariant situation. We continue to assume that the G-space X is equipped with an absorbing non-degenerate representation $\phi\colon C_0(X) \to B(H,\rho)$ where H is a separable Hilbert space equipped with a unitary G-action ρ. The idea is to derive the isomorphism in (1.12) from a multiplication map

$$\mu_X\colon C_0(X) \otimes Q^G(X) \to Q(H)\,, \tag{1.13}$$

$$Q^G(X) := Q^G(H,\rho,\phi) := D^G(H,\rho,\phi)/C^G(H,\rho,\phi)\,,$$

where $D^G(H,\rho,\phi)$ and $C^G(H,\rho,\phi)$ are defined as in the non-equivariant case by just adding the condition that the controlled generators are G-invariant. Furthermore $Q(H) = Q(H,\rho)$ is the Calkin algebra of (H,ρ) with the induced G-action. Using the multiplication map (1.13), one may define a Paschke morphism as the composition

$$p_X^{(H,\rho,\phi)} : \mathrm{KK}(\mathbb{C}, Q^G(X)) \xrightarrow{\delta_X} \mathrm{KK}^G(C_0(X), C_0(X) \otimes Q^G(X)) \qquad (1.14)$$
$$\xrightarrow{\mu_X} \mathrm{KK}^G(C_0(X), Q(H)) .$$

The map $\delta_X := C_0(X) \otimes -$ is the exterior product in equivariant KK-theory and is called the diagonal morphism. We note that the algebras $Q^G(X)$ and $Q(H)$ are not separable, which is the reason why E-theory instead of KK-theory is used in [33]. However, the equivariant KK-theory of [12] is well-defined for all G-C^*-algebras, so we can safely work with this version rather than with E-theory.

With this more abstract definition, how can one show that the Paschke morphism induces an isomorphism on K-groups, at least for suitable spaces X? Our strategy to answer this question is as follows. Suppose one could show that the maps $p_X^{(H,\rho,\phi)}$ in (1.14) were the components of a natural transformation of functors with values in the ∞-category of spectra, and that both the domain and target of the Paschke transformation are homotopy invariant and excisive[1] as functors in X. Then for G-finite G-CW-complexes X, by induction over the number of G-cells, one can reduce the verification that $p_X^{(H,\rho,\phi)}$ is an equivalence to the cases of G-orbits, i.e., of spaces of the form G/H, where H runs over the subgroups of G appearing as stabilizer of the G-action on X. While in the non-equivariant case only the trivial case $X =*$ is to be treated, the verification that the Paschke maps are equivalences on general G-orbits is a non-trivial matter.

The above strategy will indeed be the essential idea of the proof of our main Theorem 1.5 below. The first difficulty to overcome is to show that the Paschke maps $p_X^{(H,\rho,\phi)}$ are indeed the components of a natural transformation, in particular, to show that the spectrum $\mathrm{KK}(\mathbb{C}, Q^G(X))$ appearing in the domain of the Paschke map, is a homotopy invariant and excisive functor in X (at the moment is not even a functor in any obvious manner). The origin of the problem is that in order to define $Q^G(X) = Q^G(H, \rho, \phi)$, one has to *choose* an absorbing non-degenerate representation (H, ρ, ϕ), and for a morphism $X \to X'$ one has to *choose* a covering in order to define the map $\mathrm{KK}(\mathbb{C}, Q^G(X)) \to \mathrm{KK}(\mathbb{C}, Q^G(X'))$. Defined in this way, the resulting map of spectra depends on these choices and is, at best, unique up to an unspecified homotopy, which is not sufficient for our purposes.

The Paschke transformation in our setup. Our key idea to overcome these functoriality issues is to work with the category of all representations. In fact, the categories of such representations themselves depend on the space in a strictly functorial manner. Their use hence circumvents the need to find absorbing representations. The idea to work with the whole category of representations is

[1]This is the spectrum analogue of the property of admitting Mayer–Vietoris sequences for group-valued functors

not new; it has first been exploited in [6] in order to define a spectrum-valued coarse K-homology functor $K\mathcal{X}$.

In the present paper, as indicated earlier, we work with its equivariant generalization, the equivariant coarse K-homology functor

$$K\mathbf{C}\mathcal{X}^G : G\mathbf{BC} \to \mathbf{Sp}$$

introduced in [7]. Again, the symbol $\mathbf{C}$ refers to its dependence on a coefficient G-C^*-category $\mathbf{C}$. In the case of trivial coefficients, it is shown in [8, Thm. 6.1] that this functor is equivalent to the classical definition of equivariant coarse K-homology in terms of Roe algebras. More precisely, if the G-space X is nice, and $C^G(X) := C^G(H,\rho,\phi)$ with (H,ρ,ϕ) ample, we have a natural equivalence

$$K\mathbf{C}\mathcal{X}^G(X) \simeq K^{C^*\mathrm{Alg}}(C^G(X)) .$$

By construction, see Definition 3.4, for X in $G\mathbf{BC}$ we have

$$K\mathbf{C}\mathcal{X}^G(X) = \mathrm{KK}(\mathbb{C}, \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) ,$$

where $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)$ is a C^*-category of equivariant locally finite X-controlled objects in $\mathbf{C}$, see Definition 3.2 for the details. The endomorphism algebras of the objects of $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)$ are natural analogues of the Roe algebras $C(H,\rho,\phi)$.

We now indicate the relation between the functor $X \mapsto K^{G,\mathcal{X}}_{\mathbf{C}}(X)$ and the association $X \mapsto \mathrm{KK}(\mathbb{C}, Q^G(X))$ appearing in the source of the Paschke morphism (1.14). Recall from (1.9) that $K^{G,\mathcal{X}}_{\mathbf{C}}$ is defined as a composition of $K\mathbf{C}\mathcal{X}^G$ with the functor $\mathcal{O}^\infty(-) \otimes G_{can,max}$ on G-uniform bornological coarse spaces.

If X is in $G\mathbf{UBC}$, then the cone-at-∞ $\mathcal{O}^\infty(X)$ is the G-set $\mathbb{R} \times X$ with a certain G-bornological coarse structure described in Definition 4.4. It contains the underlying G-bornological coarse space of X as the subspace $\{0\} \times X$. We further consider the cone $\mathcal{O}(X)$ in $G\mathbf{BC}$ defined as the subset $[0,\infty) \times X$ with the induced structures. The inclusion $X \to \mathcal{O}(X)$ induces an inclusion of categories

$$\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X \otimes G_{can,max}) \to \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(X) \otimes G_{can,max}) \tag{1.15}$$

to be thought of as the analog of the inclusion $C^G(X) \to D^G(X)$ in the classical situation, see Section 10 for more details. The resulting quotient C^*-category $\mathbf{Q}(X)$ is then our version of the algebra $Q^G(X)$, and we have natural equivalence

$$K^{G,\mathcal{X}}_{\mathbf{C}}(X) \simeq \mathrm{KK}(\mathbb{C}, \mathbf{Q}(X)) . \tag{1.16}$$

We refer to Lemma 6.1 for more details and necessary additions. We construct a multiplication map (see (6.12))

$$\mu_X : C_0(X) \otimes \mathbf{Q}(X) \to \mathbf{Q}^{(G)}_{\mathrm{std}} .$$

In complete analogy to the earlier described Paschke morphism (1.14), we define our version of the Paschke morphism as the composition:

$$p_X : \mathrm{KK}(\mathbb{C}, \mathbf{Q}(X)) \xrightarrow{\delta_X} \mathrm{KK}^G(C_0(X), C_0(X) \otimes \mathbf{Q}(X)) \xrightarrow{\mu_X} \mathrm{KK}^G(C_0(X), \mathbf{Q}^{(G)}_{\mathrm{std}}) . \tag{1.17}$$

The main result of this paper is then the following theorem.

Theorem 1.5. *We assume that* $\mathbf{C}$ *is effectively additive and admits countable AV-sums.*

(1) *The morphisms in* (1.17) *assemble into a natural transformation of spectrum-valued functors on* $G\mathbf{UBC}$
$$p : K_{\mathbf{C}}^{G,\mathcal{X}} \to K_{\mathbf{C}}^{G,\mathrm{An}} \circ \iota^{\mathrm{top}} \tag{1.18}$$
that is natural in the coefficient category $\mathbf{C}$ *for non-degenerate morphisms.*

(2) *If* X *is in* $G\mathbf{UBC}$ *and homotopy equivalent to a* G*-finite* G*-simplicial complex with finite stabilizers, then*
$$p_X : K_{\mathbf{C}}^{G,\mathcal{X}}(X) \to K_{\mathbf{C}}^{G,\mathrm{An}}(\iota^{\mathrm{top}}(X))$$
is an equivalence.

(3) *If* $\mathbf{C}$ *admits all very small AV-sums,* G *is finite,* X *is in* $G\mathbf{UBC}$ *and homotopy equivalent to a countable finite-dimensional* G*-simplicial complex, then*
$$p_X^{\mathrm{lf}} : K_{\mathbf{C}}^{G,\mathcal{X}}(X) \to K_{\mathbf{C}}^{G,\mathrm{An,lf}}(\iota^{\mathrm{top}}(X))$$
is an equivalence.

We refer again to Definitions 2.2 and 2.3 for the conditions on $\mathbf{C}$ appearing in the statement above, and recall that the coefficient category $\mathbf{Hilb}_c(A)$, for A a G-C^*-algebra, satisfies these conditions. In Assertion 1.5.3, we use the transformation $c : K_{\mathbf{C}}^{G,\mathrm{An}} \to K_{\mathbf{C}}^{G,\mathrm{An,lf}}$ from (1.8) and set $p^{\mathrm{lf}} := c \circ p$.

Definition 1.6. *The transformation* p *in* (1.18) *is called the Paschke transformation.*

The proof of Assertion 1.5.1 will be finished in Section 7, and the proof of Assertions 1.5.2 and 1.5.3 will be completed in Section 9. Once p is constructed, which is not at all trivial, the verification that it is an equivalence under additional conditions follows the route described above, i.e. by reducing it to the case of orbits. The verification that p is indeed an equivalence on G-orbits with finite stabilizers also turns out to be quite involved and uses a lot of the properties of the K-theory functor for C^*-categories obtained in [9].

In the case of trivial coefficients and under the assumption of the existence of an absorbing representation (H, ρ, ϕ), we can compare the version of the Paschke morphism $p_X^{(H,\rho,\phi)}$ from (1.14) with the newly defined Paschke morphism p_X from (1.17) (in particular their domains): Indeed, in Proposition 11.2 we show that there is a commutative diagram
$$\begin{tikzcd} K_{\mathbf{C}}^{G,\mathcal{X}}(X) \arrow[r, "\gamma"] \arrow[d, "p_X"] & \mathrm{KK}(\mathbb{C}, Q^G(X)) \arrow[d, "p_X^{(H,\rho,\phi)}"] \\ K_{\mathbf{C}}^{G,\mathrm{An}}(X) & \mathrm{KK}^G(C_0(X), Q(H)) \arrow[l, "\simeq"'] \end{tikzcd}$$
so that, under the assumption that p_X is an equivalence, γ is an equivalence if and only if $p_X^{(H,\rho,\phi)}$ is.

Assembly maps. Our original motivation to show the Paschke duality theorem above was the wish to write out a complete proof for the fact the homotopy theoretic assembly map of Davis–Lück [16] and the analytic assembly map appearing in the Baum–Connes conjecture are equivalent. Such an equivalence was asserted in [21], but the details of the proof given in this reference remained sparse. While we were preparing this paper, a comparison of the two assembly maps was recently also carried out by Kranz [26] with methods different from ours, see Remark 1.13.

Homotopy theoretic assembly maps are generally defined for any equivariant homology theory $G\mathbf{Orb} \to \mathbf{M}$ with cocomplete target $\mathbf{M}$ and a family $\mathcal{F}$ of subgroups, see Definition 12.1. Our comparison concerns the functor

$$K\mathbf{C}^G : G\mathbf{Orb} \to \mathbf{Sp}\,, \quad S \mapsto K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(S_{min,max})\,, \tag{1.19}$$

see Definition 12.2. Note that the twist is different from the one used in the Definition (1.9) of $K^{G,\mathcal{X}}_{\mathbf{C}}$, namely it is $G_{can,min}$ rather than $G_{can,max}$. For appropriate choice of coefficients $\mathbf{C}$, the functor $K\mathbf{C}^G$ is equivalent to the functor introduced by Davis–Lück, see Remark 10.12.

The equivariant homology theory $K\mathbf{C}^G$ canonically extends to a functor

$$K\mathbf{C}^G : G\mathbf{Top} \to \mathbf{Sp}$$

denoted by the same symbol, see Definition 10.3. For any family of subgroups $\mathcal{F}$ of G the homotopy theoretic assembly map can be described as the map

$$\mathrm{Asmbl}^h_{\mathbf{C},\mathcal{F}} : K\mathbf{C}^G(E_{\mathcal{F}}G^{\mathrm{CW}}) \to K\mathbf{C}^G(*)$$

induced by the projection $E_{\mathcal{F}}G^{\mathrm{CW}} \to *$, where $E_{\mathcal{F}}G^{\mathrm{CW}}$ is a G-CW-complex representing the homotopy type of the classifying space of G with respect to the family $\mathcal{F}$.

For the following we assume that $\mathcal{F} \subseteq \mathbf{Fin}$. We define

$$RK^{G,\mathrm{An}}_{\mathbf{C}}(E_{\mathcal{F}}G^{\mathrm{CW}}) := \mathrm{colim}_{W\subseteq E_{\mathcal{F}}G^{\mathrm{CW}}} K^{G,\mathrm{An}}_{\mathbf{C}}(W),$$

where the colimit runs over the G-finite subcomplexes of $E_{\mathcal{F}}G^{\mathrm{CW}}$. In Definition 12.12 we construct an analytic assembly map

$$\mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C},\mathcal{F}} : RK^{G,\mathrm{An}}_{\mathbf{C}}(E_{\mathcal{F}}G^{\mathrm{CW}}) \to \Sigma\mathrm{KK}(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)\,, \tag{1.20}$$

where the C^*-category $\mathbf{C}^{(G)}_{\mathrm{std}}$ is defined in Definition 2.15 and the reduced crossed product for C^*-categories is as introduced in [9, Thm. 12.1].

The assembly maps $\mathrm{Asmbl}^h_{\mathbf{C},\mathcal{F}}$ and $\mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C},\mathcal{F}}$ depend naturally on the coefficient category $\mathbf{C}$ for non-degenerate morphisms.

In Definition 12.8, we construct a spectrum-valued version of the classical Kasparov assembly map

$$\mu^{\mathrm{Kasp}}_{A,\mathcal{F}} : RK^{G,\mathrm{an}}_{A}(E_{\mathcal{F}}G^{\mathrm{CW}}) \to \mathrm{KK}(\mathbb{C}, A \rtimes_r G) \tag{1.21}$$

which functorially depends on A in KK^G. We consider the spectrum-valued refinement (1.21) of Kasparov's assembly map as an interesting result in its own right. In view of the definition of the domain, one has to construct a family of such assembly maps indexed by the G-finite subcomplexes W of $E_{\mathcal{F}}G^{\mathrm{CW}}$ which is compatible with inclusions. While it is easy to lift Kasparov's construction to a map of spectra for each such W individually, and it is also easy to obtain the required compatibility on the level of homotopy groups, it is a non-trivial matter to enhance the compatibility to the spectrum level. We obtain this enhancement in the form of the natural transformation (12.17).

For a G-C^*-category $\mathbf{C}$ let $\mathbf{C}^u$ denote the full unital G-C^*-subcategory of unital objects. In Proposition 16.3, we show the following comparison result.

Proposition 1.7. *We have an equivalence between the assembly maps* $\mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C},\mathcal{F}}$ *from* (1.20) *and* $\Sigma\mu^{\mathrm{Kasp}}_{(\mathbf{C}^u)^{(G)},\mathcal{F}}$ *from* (1.21).

Example 1.8. In the case of a unital G-C^*-algebra A and for $\mathbf{C} := \mathbf{Hilb}_c(A)$, it follows from (12.18) and Proposition 1.7 that the assembly map $\mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C},\mathcal{F}}$ is equivalent to $\Sigma\mu^{\mathrm{Kasp}}_{A,\mathcal{F}}$. □

The following theorem (whose proof will be completed at the end of Section 14) now provides a comparison of the Davis–Lück and Baum–Connes assembly maps on the level of homotopy groups. As indicated earlier, a version of this result has recently been shown also by [26] with completely different methods.

Theorem 1.9. *We assume that* $\mathbf{C}$ *is effectively additive and admits countable AV-sums. We have a commutative square*

$$\begin{tikzcd} K\mathbf{C}^G_*(E_{\mathcal{F}}G^{\mathrm{CW}}) \arrow[r, "\pi_*\mathrm{Asmbl}^h_{\mathbf{C},\mathcal{F}}"] \arrow[d, dash, "\cong"'] & K\mathbf{C}^G_*(*) \arrow[d, "\cong"] \\ RK^{G,\mathrm{An}}_{\mathbf{C},*+1}(E_{\mathcal{F}}G^{\mathrm{CW}}) \arrow[r, "\pi_{*+1}\mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C},\mathcal{F}}"] & \mathrm{KK}_*(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G) \end{tikzcd} \tag{1.22}$$

in which all terms are natural in $\mathbf{C}$ *for non-degenerate morphisms.*

The left vertical equivalence in (1.22) is, in a non-obvious manner, a consequence of our Paschke Duality Theorem 1.5. If A is a G-C^*-algebra, then $\mathbf{C} := \mathbf{Hilb}_c(A)$ admits all small AV-sums (this follows from [9, Thm. 8.4]) and hence satisfies the assumption of Theorem 1.9.

We believe that our method can be upgraded to provide a commutative diagram on the spectrum level, but carrying this out would involve to control further large coherence diagrams. We refrain from doing this additional step at this point, but emphasize that the passage to a statement about homotopy groups is really only in the very final step where one filters $E_{\mathcal{F}}G^{\mathrm{CW}}$ through G-finite subcomplexes. For any G-finite X in place of $E_{\mathcal{F}}G^{\mathrm{CW}}$, the diagram in Theorem 1.9 commutes already before applying homotopy groups. In particular, the square in (1.22) commutes before applying homotopy groups when there is a G-finite model of $E_{\mathcal{F}}G^{\mathrm{CW}}$. It is just that we have not worked out that the homotopies for varying X can be obtained in a compatible way. This problem is not visible to homotopy groups, and hence one obtains Theorem 1.9 irrespective of this issue.

We note that it is important to consider the reduced crossed product in the target for the approach presented here. While the construction of the analytic assembly map easily lifts to the maximal crossed product our method unfortunately does not generalize to produce the corresponding comparison of assembly maps also for the maximal crossed product.

Further remarks. Finally, we explain some relations to previous works on (equivariant) Paschke duality and the analytical assembly map. We begin with Paschke duality.

Remark 1.10. Valette established a non-commutative generalization of the classical Paschke duality [36] whose statement we briefly recall here. We consider a C^*-algebra B with a strictly positive element. Then we have an exact sequence

$$0 \to B \otimes K(\ell^2) \to \mathcal{M}^s(B) \xrightarrow{\pi} \mathcal{Q}^s(B) \to 0\,,$$

where $\mathcal{M}^s(B)$ is the stable multiplier algebra and the stable Calkin algebra $\mathcal{Q}^s(B)$ is defined as the quotient. In place of $\phi : C_0(X) \to B(H)$ above, we now consider a unital separable nuclear C^*-algebra A with a representation $\tau : A \to B(\ell^2)$ such that $\tau(A) \cap K(\ell^2) = \{0\}$ and set $\phi : A \xrightarrow{1\otimes\tau} \mathcal{M}^s(B) \xrightarrow{\pi} \mathcal{Q}^s(B)$. We further replace $Q(H,\phi)$ from above by the commutant $Q(A,\phi,B) := \phi(A)'$ of the image of ϕ. The proof of the following result employs Kasparov's generalization of Voiculescu's Theorem.

Proposition 1.11 ([36, Prop. 3])**.** *We have an isomorphism*

$$\mathrm{KK}_*(\mathbb{C}, Q(A,\phi,B)) \cong \mathrm{KK}_{*-1}(A,B)$$

which is natural in A and B.

In this statement, KK_* denote Kasparov's KK-groups. Note that the right-hand side in the original statement of Valette is expressed in terms of Ext-groups which are isomorphic to the KK_*-groups under the given assumptions on A and B. If B is in addition σ-unital, then by [12, Prop. 1.20] the KK-group on the right-hand side coincides with the KK-group obtained from the spectrum-valued KK-theory constructed in [12].

See also [35, Thm. 3.2] for a related result. □

Remark 1.12. Our Theorem 1.5 is similar in spirit to [2, Thm. 1.5]. But while Theorem 1.5 produces a natural transformation between spectrum-valued functors which becomes an equivalence when evaluated on spaces satisfying suitable finiteness conditions, [2, Thm. 1.5] states an isomorphism between K-theory groups for a fixed space. While the class of spaces to which [2, Thm. 1.5] applies is larger than the class of spaces for which Theorem 1.5 provides an equivalence, our theorem allows to treat more general coefficients.

But even in the case where both theorems are applicable the technical details of their statements are quite different so that at the moment it is difficult to compare them in a precise way. In the following, we explain this problem in greater detail.

The space X in [2, Thm. 1.5] (denoted by Z in the reference) is a metric space with an isometric proper cocompact action of G. In order to fit into our theorem, we must require that it is homotopy equivalent to a G-finite G-simplicial complex. The domain of the Paschke map in [2, Thm. 1.5] is the K-theory of a certain C^*-algebra $Q^G(H, \rho, \phi)$, where H is a sufficiently large Hilbert C^*-module over a commutative unital C^*-algebra A. In order to compare with our theorem, we would restrict the coefficients to the special case $\mathbf{C} = \mathbf{Hilb}_c(A)$. We then could ask whether we have

$$K_*^{C^*\mathrm{Cat}}(\mathbf{Q}(X)) \cong K_*^{C^*\mathrm{Alg}}(Q(H, \rho, \phi)),$$

see (1.16). The construction of a comparison map could proceed similarly as the construction of the map γ in Proposition 11.2 once we know that (H, ρ, ϕ) is absorbing in the sense of the natural generalization of Definition 11.1 to controlled Hilbert A-modules.

On the positive side, in the case $\mathbf{C} = \mathbf{Hilb}_c(A)$, the targets of the two Paschke duality maps in [2, Thm. 1.5] and Theorem 1.5 are equivalent in view of

$$K_{\mathbf{C}}^{G,\mathrm{An}}(X) \simeq \mathrm{KK}^G(C_0(X), \mathbf{Q}_{\mathrm{std}}^{(G)}) \overset{\text{Prop. 10.15}}{\simeq} \Sigma\mathrm{KK}^G(C_0(X), A)$$

provided X is homotopy equivalent to a G-finite G-CW-complex. □

Remark 1.13. As mentioned earlier, in [26] Kranz also provides an identification of the Davis–Lück assembly map and the Kasparov assembly map. In fact, the contribution of Kranz is a comparison of the Davis–Lück assembly map with the version of the assembly map introduced by Meyer–Nest [30]. The latter is compared in [30] with Kasparov's assembly map employing work of Chabert–Echterhoff [15]. In Section 15, we will give a detailed account of the argument of Kranz using the ∞-categorical language of equivariant KK-theory developed in [12]. As an application, in Theorem 16.1 we give an argument (which is independent of Chabert–Echterhoff [15]) that the Kasparov assembly map is an equivalence for compactly induced coefficient categories or algebras. □

2. Constructions with C^*-categories

In order to fix size issues, we choose a sequence of four Grothendieck universes whose sets will be called very small, small, large, and very large, respectively. The group G, bornological coarse spaces or G-topological spaces belong to the very small universe. The categories of these objects, the coefficient C^*-categories, the categories of controlled objects, and the values of the K-theory functor $K^{C^*\mathrm{Cat}}$ will belong to the small universe. The categories of spectra $\mathbf{Sp}$ and KK^G are large, but locally small. They are objects of a category of stable ∞-categories $\mathbf{CAT}_\infty^{ex}$ which is itself very large.

We let $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ denote the category of small not necessarily unital C^*-categories with G-action and equivariant functors, and $\mathrm{Fun}(BG, C^*\mathbf{Cat})$ be the subcategory of unital C^*-categories and functors preserving units. Both versions of K-homology considered in the present paper depend on the choice of a coefficient C^*-category $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$.

Example 2.1. We let $\mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}})$ denote the full subcategory of $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ of C^*-algebras with G-action considered as single object categories. We furthermore set

$$\mathrm{Fun}(BG, C^*\mathbf{Alg}) := \mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}}) \cap \mathrm{Fun}(BG, C^*\mathbf{Cat})\,.$$

Our basic example of a coefficient category is the category $\mathbf{C} = \mathbf{Hilb}_c(A)$ of Hilbert A-modules and compact operators for A in $\mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}})$, see Example 1.3. □

Below we will consider conditions on $\mathbf{C}$ in $C^*\mathbf{Cat}^{\mathrm{nu}}$ which involve orthogonal sums of possibly infinite families $(C_i)_{i\in I}$ of objects of $\mathbf{C}$. Let $(C, (e_i)_{i\in I})$ be a pair of an object of $\mathbf{C}$ and a family of mutually orthogonal isometries $e_i : C_i \to C$ in the multiplier category $\mathbf{MC}$ of $\mathbf{C}$.

Definition 2.2 ([9, Def. 3.1])**.** *We say that $(C, (e_i)_{i\in I})$ represents an AV-sum of the family $(C_i)_{i\in I}$ if $\sum_{i\in I} e_i e_i^*$ converges strictly to id_C in $\mathbf{MC}$.*

Let p be an orthogonal projection on an object C in a C^*-category, i.e., an endomorphism of C satisfying $p^* = p$ and $p^2 = p$. A morphism $u : C' \to C$ represents the image of p if u is an isometry, i.e., $u^*u = \mathrm{id}_{C'}$, and $uu^* = p$. We say that p is effective if it admits an image. In the present paper, we will only consider orthogonal projections, and therefore we will omit the word orthogonal from now on. We refer to [9, 2.16-2.19] for more details.

Definition 2.3 ([7, Def. 3.12])**.** *We say that $\mathbf{C}$ is effectively additive if for every object C of $\mathbf{C}$ and mutually orthogonal family of effective projections $(p_i)_{i\in I}$ on C in $\mathbf{MC}$ such that $\sum_{i\in I} p_i$ converges strictly to a projection p in $\mathbf{MC}$, the latter is also effective in $\mathbf{MC}$.*

If $\mathbf{C}$ admits all small AV-sums or is idempotent complete, then it is effectively additive. If $\mathbf{C}$ is in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$, then we will apply the notions introduced above to the underlying C^*-category obtained by forgetting the G-action.

In general, the category $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ may contain objects which admit an identity morphism. These objects are called unital. We note that automorphisms of $\mathbf{C}$ preserve unital objects.

Definition 2.4. *For $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$, we let $\mathbf{C}^u$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat})$ denote the full subcategory of unital objects in $\mathbf{C}$.*

Example 2.5. Let A be in $\mathrm{Fun}(BG, C^*\mathbf{Alg})$ and $\mathbf{C} = \mathbf{Hilb}_c(A)$ as in Example 2.1. Then $\mathbf{C}^u = \mathbf{Hilb}(A)^{\mathrm{proj,fg}}$ is the full subcategory of $\mathbf{Hilb}(A)$ of finitely generated projective Hilbert A-modules. □

For the moment, let $\mathbf{D}$ be in $\mathrm{Fun}(BG, C^*\mathbf{Cat})$. Our main example will be the multiplier category $\mathbf{MC}$ of $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$. We fix the following notation convention concerning the G-action on $\mathbf{D}$. If D is an object of $\mathbf{D}$ and g is in G, then we let gD denote the object obtained by applying g to D. Similarly, if A is a morphism in $\mathbf{D}$, then we write gA for the morphism obtained by applying g to A.

Definition 2.6. *A G-object in $\mathbf{D}$ is a pair (D,ρ) of an object in $\mathbf{D}$ and a family $\rho=(\rho_g)_{g\in G}$ of unitaries $\rho_g\colon D\to gD$ such that $g\rho_h\,\rho_g=\rho_{gh}$ for all h,g in G.*

Example 2.7. If G acts trivially on $\mathbf{D}$, then the datum of a G-object (D,ρ) in $\mathbf{D}$ is the same as an object D of $\mathbf{D}$ together with a homomorphism $\rho\colon G\to \mathrm{Aut}_{\mathbf{D}}(D)$, $g\mapsto \rho_g^{-1}$, such that $\rho_{g^{-1}}=\rho_g^*$. □

Definition 2.8. *The category of G-objects in $\mathbf{D}$ is the C^*-category with G-action $\mathbf{D}^{(G)}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat})$ defined as follows:*

(1) *objects: The objects of $\mathbf{D}^{(G)}$ are the G-objects in $\mathbf{D}$.*
(2) *morphisms: The morphisms in $\mathbf{D}^{(G)}$ are given by*

$$\mathrm{Hom}_{\mathbf{D}^{(G)}}((D,\rho),(D',\rho')) := \mathrm{Hom}_{\mathbf{D}}(D,D')\,. \tag{2.1}$$

(3) *composition and involution: The composition and involution are inherited from $\mathbf{D}$.*
(4) *G-action:*
    (a) *objects: G fixes the objects of $\mathbf{D}^{(G)}$.*
    (b) *morphisms: g in G acts on a morphism $A\colon (D,\rho)\to(D',\rho')$ by*

$$g\cdot A := \rho_g^{\prime,-1}\, gA\,\rho_g\,. \tag{2.2}$$

Note that we use the notation $g\cdot -$ in order to denote the G-action on morphisms between G-objects which should not be confused with the original action denoted by $g-$.

Associated to $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ we have two derived objects $\mathbf{C}^u$ and $(\mathbf{C}^u)^{(G)}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat})$. In the following, we will show that they are related by a canonical zig-zag of fully faithful functors. To this end we construct a third object $\hat{\mathbf{C}}^{u,(G)}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat})$.

(1) objects: The G-set of objects of $\hat{\mathbf{C}}^{u,(G)}$ is the union of the G-sets of objects of $\mathbf{C}^u$ and $(\mathbf{C}^u)^{(G)}$.
(2) morphisms: The morphism spaces of $\hat{\mathbf{C}}^{u,(G)}$ are defined such that $\mathbf{C}^u$ and $(\mathbf{C}^u)^{(G)}$ are fully faithfully embedded. If C is in $\mathbf{C}^u$ and (C',ρ') is in $(\mathbf{C}^u)^{(G)}$, then we define $\mathrm{Hom}_{\hat{\mathbf{C}}^{u,(G)}}(C,(C',\rho')) := \mathrm{Hom}_{\mathbf{C}}(C,C')$ and $\mathrm{Hom}_{\hat{\mathbf{C}}^{u,(G)}}((C',\rho'),C) := \mathrm{Hom}_{\mathbf{C}}(C',C)$.
(3) The composition and the involution are inherited from $\mathbf{C}$.
(4) G-action: The G-action is defined such that both the inclusions $\mathbf{C}^u\to\hat{\mathbf{C}}^{u,(G)}$ and $(\mathbf{C}^u)^{(G)}\to\hat{\mathbf{C}}^{u,(G)}$ are G-equivariant.

    If $\hat f\colon C\to(C',\rho')$ is a morphism in $\mathrm{Hom}_{\hat{\mathbf{C}}^{u,(G)}}(C,(C',\rho'))$ given by $f\colon C\to C'$ in $\mathbf{C}$, then $g\hat f\colon gC\to(C',\rho')$ is given by $\rho_g^{\prime,-1}\circ gf\colon gC\to C'$. Similarly, if $\hat h\colon (C',\rho')\to C$ is a morphism in $\mathrm{Hom}_{\hat{\mathbf{C}}^{u,(G)}}((C',\rho'),C)$ given by $h\colon C'\to C$, then $g\hat h\colon C'\to gC$ is given by $gh\circ\rho'_g\colon C'\to gC$.

Definition 2.9. *We say that G weakly fixes the objects of $\mathbf{C}^u$ if for every object C of $\mathbf{C}^u$ there exists a refinement (C,ρ) to an object of $(\mathbf{C}^u)^{(G)}$.*

In other words, G weakly fixes the objects of $\mathbf{C}^u$ if and only if the canonical functor

$$\lim_{BG}^{C^*\mathbf{Cat}_{2,1}} \mathbf{C}^u \to \mathrm{Res}^G(\mathbf{C}^u)$$

from the 2-categorical G-fixed points of $\mathbf{C}^u$ to $\mathbf{C}^u$ with G-action forgotten is essentially surjective.

Lemma 2.10.

(1) *The inclusion $\mathbf{C}^u \to \hat{\mathbf{C}}^{u,(G)}$ is a unitary equivalence.*
(2) *If G weakly fixes the objects of $\mathbf{C}^u$, then the inclusion $(\mathbf{C}^u)^{(G)} \to \hat{\mathbf{C}}^{u,(G)}$ is a unitary equivalence.*

Proof. By construction both inclusion functors are fully faithful. We now argue that they are essentially surjective. We start with the inclusion of $\mathbf{C}^u$. We consider an object (C,ρ) in $(\mathbf{C}^u)^{(G)}$. Then C is in $\mathbf{C}^u$ and id_C gives a unitary isomorphism $C \to (C,\rho)$ in $\hat{\mathbf{C}}^{u,(G)}$.

We now consider the inclusion of $(\mathbf{C}^u)^{(G)}$. Let C be an object of $\mathbf{C}^u$. By assumption there exists an object (C,ρ) in $(\mathbf{C}^u)^{(G)}$ and again id_C gives a unitary isomorphism $C \to (C,\rho)$ in $\hat{\mathbf{C}}^{u,(G)}$. □

For a G-C^*-category $\mathbf{C}$ and a G-bornological space X we will introduce the notion of X-controlled G-objects in $\mathbf{C}$. To this end, we recall that a G-bornology on a G-set X is a G-invariant subset of the power set $\mathcal{P}_X$ of X which is closed under forming finite unions, subsets, and which contains all one-point subsets. A G-bornological space is a pair $(X,\mathcal{B})$ of a G-set X with a G-bornology $\mathcal{B}$ whose elements will be called the bounded subsets of X. If $(X,\mathcal{B})$ and $(X',\mathcal{B}')$ are G-bornological spaces and $f : X \to X'$ is an equivariant map of underlying G-sets, then f is called proper if $f^{-1}(\mathcal{B}') \subseteq \mathcal{B}$. By $G\mathbf{Born}$ denote the category of very small G-bornological spaces and proper maps. We refer to [10] for more details. We will usually use the shorter notation X for G-bornological spaces. To any G-set S we can associate the following objects in $G\mathbf{Born}$.

(1) S_{min} is S equipped with the minimal bornology consisting of the finite subsets. The map $S \mapsto S_{min}$ is functorial for morphisms of G-sets with finite fibres.
(2) S_{max} is S equipped with the maximal bornology consisting of all subsets of S. We have a functor $G\mathbf{Set} \to G\mathbf{Born}$ given on objects by $S \mapsto S_{max}$.

Let X be in $G\mathbf{Born}$.

Definition 2.11. *A subset L of X is called locally finite if $B \cap L$ is finite for every bounded subset in X.*

The following definition is an expanded version of [7, Def. 4.6]. Let X be in $G\mathbf{Born}$.

Definition 2.12. *A locally finite X-controlled G-object in $\mathbf{C}$ is a triple (C,ρ,μ), where:*

(1) *(C,ρ) is an object in $\mathbf{MC}^{(G)}$.*
(2) *μ is an invariant, finitely additive measure on X with values in projections in $\mathrm{End}_{\mathbf{MC}}(C)$ such that the following properties hold:*
 (a) *$\mu(X) = \mathrm{id}_C$.*
 (b) *$\mu(\{x\})$ is effective and belongs to $\mathbf{C}$ for all x in X.*
 (c) *C is the orthogonal AV-sum of the images of the family of projections $(\mu(\{x\}))_{x\in X}$.*
 (d) *The subset $\mathrm{supp}(\mu)$ of X is locally finite.*

Remark 2.13. In this remark, we explain Condition 2 in more detail. It first of all says that μ is a function from the power set $\mathcal{P}_X$ of X to the set of projections in $\mathrm{End}_{\mathbf{MC}}(C)$ such that for all Y, Z in $\mathcal{P}_X$ with $Y \subseteq Z$ we have $\mu(Z) = \mu(Y) + \mu(Z \setminus Y)$. The invariance condition of μ means that

$$g \cdot \mu(Y) = \mu(gY) \tag{2.3}$$

for all g in G and subsets Y of X.

Condition 2c says that $\sum_{x\in X} \mu(\{x\})$ converges strictly to id_C.

The support of μ is the subset

$$\mathrm{supp}(\mu) := \{x \in X \mid \mu(\{x\}) \neq 0\}$$

of X.

The Conditions 2b and 2d together imply that $\mu(B)$ belongs to the ideal $\mathbf{C}$ of $\mathbf{MC}$ for every bounded subset B of X. □

Let $\mathbf{C}$ be in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ and X be in $G\mathbf{Born}$.

Definition 2.14. *We define $\mathbf{C}^{(G)}_{\mathrm{lf}}(X)$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat})$ as follows:*

(1) *objects: The objects of $\mathbf{C}^{(G)}_{\mathrm{lf}}(X)$ are the locally finite X-controlled G-objects in $\mathbf{C}$.*
(2) *morphisms: The morphisms in $\mathbf{C}^{(G)}_{\mathrm{lf}}(X)$ are given by*

$$\mathrm{Hom}_{\mathbf{C}^{(G)}_{\mathrm{lf}}(X)}((C,\rho,\mu),(C',\rho',\mu')) := \mathrm{Hom}_{\mathbf{MC}^{(G)}}((C,\rho),(C',\rho'))\,.$$

(3) *composition, involution and G-action: The composition, involution and the G-action are induced from $\mathbf{MC}^{(G)}$.*

We have a fully faithful forgetful functor

$$\mathcal{F} : \mathbf{C}^{(G)}_{\mathrm{lf}}(X) \to \mathbf{MC}^{(G)}\,, \quad (C,\rho,\mu) \mapsto (C,\rho)\,. \tag{2.4}$$

Definition 2.15.

(1) *We define $\mathbf{MC}^{(G)}_{\mathrm{std}}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat})$ as the full subcategory of $\mathbf{MC}^{(G)}$ of objects which are isomorphic to objects of the form $\mathcal{F}((C,\rho,\mu))$ for some object (C,ρ,μ) in $\mathbf{C}^{(G)}_{\mathrm{lf}}(Y_{min})$ for some free G-set Y.*
(2) *We let $\mathbf{C}^{(G)}_{\mathrm{std}}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ denote the G-invariant ideal of $\mathbf{MC}^{(G)}_{\mathrm{std}}$ of morphisms belonging to $\mathbf{C}$.*

(3) *We define the quotient*

$$\mathbf{Q}^{(G)}_{\mathrm{std}} := \frac{\mathbf{MC}^{(G)}_{\mathrm{std}}}{\mathbf{C}^{(G)}_{\mathrm{std}}} \tag{2.5}$$

in $\mathrm{Fun}(BG, C^*\mathbf{Cat})$.

Remark 2.16. Let us assume for simplicity that $\mathbf{C}$ is effectively additive. Applying Definition 2.3 to the empty family of projections on an object C shows that $\mathbf{C}$ admits zero objects since the zero projection on C must be effective. It can happen that $\mathbf{C}^u$ only consists of zero objects. In this case, $\mathbf{C}^{(G)}_{\mathrm{lf}}(X)$ consists of zero objects for any X in $G\mathbf{Born}$. Furthermore, the categories $\mathbf{C}^{(G)}_{\mathrm{std}}$, $\mathbf{MC}^{(G)}_{\mathrm{std}}$, and $\mathbf{Q}^{(G)}_{\mathrm{std}}$ consist of zero objects. □

Lemma 2.17. *The inclusion* $\mathbf{C}^{(G)}_{\mathrm{std}} \to \mathbf{MC}^{(G)}_{\mathrm{std}}$ *presents* $\mathbf{MC}^{(G)}_{\mathrm{std}}$ *as the multiplier category of* $\mathbf{C}^{(G)}_{\mathrm{std}}$.

Proof. We have a fully faithful forgetful functor $\mathbf{C}^{(G)}_{\mathrm{std}} \to \mathbf{C}$ which sends (C, ρ) to C. It induces a fully faithful functor $\mathbf{M}(\mathbf{C}^{(G)}_{\mathrm{std}}) \to \mathbf{MC}$. This functor has an obvious factorization $\mathbf{M}(\mathbf{C}^{(G)}_{\mathrm{std}}) \to \mathbf{MC}^{(G)}_{\mathrm{std}} \to \mathbf{MC}$, where the first functor is the identity on objects. Since the composition and the second functor are fully faithful, so is the first which is therefore an isomorphism. □

For a G-C^*-algebra A in $\mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}})$ we consider the G-C^*-category $\mathbf{C} := \mathbf{Hilb}_c(A)$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$. The following constructions will be used later to compare K-theoretic constructions involving, e.g., $\mathbf{Q}^{(G)}_{\mathrm{std}}$ with constructions involving A directly.

For $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ we let $\mathbf{MC}^{(G)}_{\mathrm{std},+}$ denote the full subcategory of $\mathbf{MC}^{(G)}$ of objects (C, ρ) which belong to $\mathbf{MC}^{(G)}_{\mathrm{std}}$ or $(\mathbf{C}^u)^{(G)}$. We furthermore let $\mathbf{C}^{(G)}_{\mathrm{std},+}$ denote the ideal in $\mathbf{MC}^{(G)}_{\mathrm{std},+}$ of morphisms which belong to $\mathbf{C}$.

Example 2.18. For a G-C^*-algebra A in $\mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}})$ and $\mathbf{C} := \mathbf{Hilb}_c(A)$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ we let $\hat{A}$ be the object of $\mathbf{C}$ given by A with the right-multiplication and the scalar product $\langle a, b\rangle_{\hat{A}} = a^*b$. Left multiplication identifies A with $\mathrm{End}_{\mathbf{C}}(\hat{A})$. For g in G we have a $\mathbb{C}$-linear map $\kappa_g : \hat{A} \to \hat{A}$ given by the action of g^{-1} on A, i.e., $\kappa_g(a) := {}^{g^{-1}}a$. This map is a unitary multiplier isomorphism $\hat{A} \to g\hat{A}$ in $\mathbf{C}$. The family $\kappa := (\kappa_g)_{g\in G}$ refines $\hat{A}$ to an object $(\hat{A}, \kappa)$ of $\mathbf{C}^{(G)}$. Moreover, the identification $A \cong \mathrm{End}_{\mathbf{C}^{(G)}}((\hat{A}, \kappa))$ is equivariant.

If A is unital, then the object $(\hat{A}, \kappa)$ belongs to $(\mathbf{C}^u)^{(G)}$ and hence to $\mathbf{MC}^{(G)}_{\mathrm{std},+}$. In this case, we have a zig-zag of equivariant inclusions

$$A \to \mathbf{MC}^{(G)}_{\mathrm{std},+} \leftarrow \mathbf{MC}^{(G)}_{\mathrm{std}}, \quad A \to \mathbf{C}^{(G)}_{\mathrm{std},+} \leftarrow \mathbf{C}^{(G)}_{\mathrm{std}}.$$

The left functors sends the unique object of A to the object $(\hat{A}, \kappa)$ and identify A with $\mathrm{End}_{\mathbf{MC}^{(G)}}((\hat{A}, \kappa))$ or $\mathrm{End}_{\mathbf{C}^{(G)}}((\hat{A}, \kappa))$, respectively. □

Recall the definitions of a Morita equivalence [9, 16.7], of a relative Morita equivalence [9, Def. 17.1], and of a weak Morita equivalence [9, Def. 18.3] between C^*-categories. In the equivariant case, an equivariant functor is a Morita equivalence or weak Morita equivalence if it has the respective property after forgetting the G-action. In addition, we will need in the following a stronger version of the notion of a relative Morita equivalence which we call a split relative Morita equivalence. Let $\phi : \mathbf{D} \to \mathbf{E}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$.

Definition 2.19. *We say that ϕ is a split relative Morita equivalence if there exists a diagram*

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{D} \arrow[r] \arrow[d, "\phi"] & \mathbf{D}' \arrow[r, "p"] \arrow[d] & \mathbf{D}'/\mathbf{D} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbf{E} \arrow[r] & \mathbf{E}' \arrow[r, "q"] & \mathbf{E}'/\mathbf{E} \arrow[r] & 0 \end{tikzcd} \tag{2.6}$$

in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ *with horizontal exact sequences such that the two right vertical functors are Morita equivalences between unital C^*-categories and the functors p and q admit right-inverses.*

Let $\mathbf{C}$ be in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$.

Lemma 2.20.

(1) $\mathbf{MC}^{(G)}_{\mathrm{std}} \to \mathbf{MC}^{(G)}_{\mathrm{std},+}$ *is a Morita equivalence.*

(2) $\mathbf{C}^{(G)}_{\mathrm{std}} \to \mathbf{C}^{(G)}_{\mathrm{std},+}$ *is a split relative Morita equivalence.*

(3) *If A is in* $\mathrm{Fun}(BG, C^*\mathbf{Alg})$ *and $\mathbf{C} = \mathbf{Hilb}_c(A)$, then $A \to \mathbf{C}^{(G)}_{\mathrm{std},+}$ has a factorization into the Morita equivalence $A \to (\mathbf{C}^u)^{(G)}$ followed by the weak Morita equivalence $(\mathbf{C}^u)^{(G)} \to \mathbf{C}^{(G)}_{\mathrm{std},+}$.*

Proof. We start with the Assertion 1. The inclusion $\mathbf{MC}^{(G)}_{\mathrm{std}} \to \mathbf{MC}^{(G)}_{\mathrm{std},+}$ is fully faithful. We will show that every object of $\mathbf{MC}^{(G)}_{\mathrm{std},+}$ is a summand of an object of $\mathbf{MC}^{(G)}_{\mathrm{std}}$. It suffices to show this for objects of $(\mathbf{C}^u)^{(G)}$. Thus let (C', ρ') be an object of $(\mathbf{C}^u)^{(G)}$. Then using the fact that $\mathbf{C}$ admits countable AV-sums one can construct an object (C, ρ, μ) in $\mathbf{C}^{(G)}_{\mathrm{lf}}(G_{min})$ such that there exists an isometry $u : C' \to C$ in $\mathbf{MC}$ representing an image of $\mu(\{e\})$. For C we must take an AV-sum of the family $(gC')_{g \in G}$. We consider u as an isometry $u : (C', \rho') \to (C, \rho)$ in $\mathbf{MC}^{(G)}_{\mathrm{std},+}$ with $(C, \rho) \in \mathrm{Ob}(\mathbf{MC}^{(G)}_{\mathrm{std}})$. It realizes (C', ρ') as a summand of the object (C, ρ) of $\mathbf{MC}^{(G)}_{\mathrm{std}}$. This finishes the proof of Assertion 1.

Let $\mathbf{C}^{(G),\sharp}_{\mathrm{std}}$ and $\mathbf{C}^{(G),\sharp}_{\mathrm{std},+}$ be the C^*-categories obtained from $\mathbf{C}^{(G)}_{\mathrm{std}}$ and $\mathbf{C}^{(G)}_{\mathrm{std},+}$ by adjoining units to all non-unital objects. We then have a diagram of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{C}^{(G)}_{\mathrm{std}} \arrow[r] \arrow[d] & \mathbf{C}^{(G),\sharp}_{\mathrm{std}} \arrow[r, "p"] \arrow[d] & \mathbf{C}^{(G),\sharp}_{\mathrm{std}}/\mathbf{C}^{(G)}_{\mathrm{std}} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbf{C}^{(G)}_{\mathrm{std},+} \arrow[r] & \mathbf{C}^{(G),\sharp}_{\mathrm{std},+} \arrow[r, "p_+"] & \mathbf{C}^{(G),\sharp}_{\mathrm{std},+}/\mathbf{C}^{(G)}_{\mathrm{std},+} \arrow[r] & 0. \end{tikzcd}$$

Since the objects of $(\mathbf{C}^u)^{(G)}$ are unital they represent zero objects in $\mathbf{C}^{(G),\sharp}_{\mathrm{std},+}/\mathbf{C}^{(G)}_{\mathrm{std},+}$. We conclude that the right vertical morphism is a Morita equivalence. Since the morphisms $u : (C', \rho') \to (C, \rho)$ from the argument for Assertion 1 actually belong to $\mathbf{C}^{(G),\sharp}_{\mathrm{std},+}$ we conclude that the middle arrow is a Morita equivalence, too. The projections p and p_+ have obvious splits.

In order to show Assertion 3, first note that if (C, ρ) is an object of $(\mathbf{C}^u)^{(G)}$, then C is a finitely generated projective A-module and hence a summand of a finite sum of copies of A. This implies that $A \to (\mathbf{C}^u)^{(G)}$ is a Morita equivalence. In order to show that the second morphism $(\mathbf{C}^u)^{(G)} \to \mathbf{C}^{(G)}_{\mathrm{std},+}$ is a weak Morita equivalence, we first observe that it is fully faithful. We then use that the morphisms in $\mathbf{C}^{(G)}_{\mathrm{std},+}$ are compact operators between Hilbert C^*-modules. A compact operator can be approximated arbitrary well by an operator which factorizes over a finitely generated projective A-module, i.e., an object of $\mathbf{C}^u$. This implies that the set of objects of $(\mathbf{C}^u)^{(G)}$ is weakly generating in $\mathbf{C}^{(G)}_{\mathrm{std},+}$. □

Recall the definition of flasque G-C^*-categories [9, Def. 11.3].

Lemma 2.21. *If* $\mathbf{C}$ *admits countable AV-sums, then* $\mathbf{MC}^{(G)}_{\mathrm{std}}$ *is flasque.*

Proof. We claim that $\mathbf{C}^{(G)}_{\mathrm{std}}$ also admits countable AV-sums. Then $\mathbf{M}(\mathbf{C}^{(G)}_{\mathrm{std}})$ is flasque by [9, Ex. 11.5]. We finally use Lemma 2.17 in order to conclude that $\mathbf{MC}^{(G)}_{\mathrm{std}}$ is flasque.

We show the claim. We consider a countable family $(C_i, \rho_i)_{i\in I}$ of objects in $\mathbf{C}^{(G)}_{\mathrm{std}}$. For every g in G we can choose an AV-sum $(C_g, (e_i^{gC_i})_{i\in I})$ of the family $(gC_i)_{i\in I}$ in $\mathbf{C}$. We set $C := C_e$ and let $u_g : C_g \to gC$ be the canonical multiplier unitary such that $g(e_i^{C_i,*})u_g e_i^{gC_i} = \mathrm{id}_{gC_i}$ for all i in I. Then $\rho := (u_g \circ \bigoplus_{i\in I} \rho_i)_{g\in G}$ defines a multiplier cocycle on C such that we have $(C, \rho) \in \mathbf{C}^{(G)}$. We now show that $(C, \rho) \in \mathbf{C}^{(G)}_{\mathrm{std}}$. By assumption, for every i in I we can refine the pair (C_i, ρ_i) to an object (C_i, ρ_i, μ_i) in $\mathbf{C}^{(G)}_{\mathrm{lf}}(X_i)$ for some free G-set X_i. Then (C, ρ, μ) belongs to $\mathbf{C}^{(G)}_{\mathrm{lf}}(X)$, where $X = \bigsqcup_{i\in I} X_i$ and the measure μ is given by $\mu(Y) := \bigoplus_{i\in I} \mu_i(Y \cap X_i)$ for all subsets Y of X. Since X is again a free G-set we conclude that (C, ρ) belongs to $\mathbf{C}^{(G)}_{\mathrm{std}}$.

By construction, the sum $\sum_{i\in I} e_i^{C_i} e_i^{C_i,*}$ strictly converges to $\mathrm{id}_{(C,\rho)}$ in $\mathbf{MC}^{(G)}_{\mathrm{std}}$. By Lemma 2.17 it also strictly converges in $\mathbf{M}(\mathbf{C}^{(G)}_{\mathrm{std}})$. Therefore, the pair (C,ρ) represents the AV-sum of the family $(C_i,\rho_i)_{i\in I}$ in $\mathbf{C}^{(G)}_{\mathrm{std}}$. $\square$

If $\mathbf{K}$ is in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$, then we can form the reduced crossed product $\mathbf{K} \rtimes_r G$ introduced in [9, Thm. 12.1]. We use the explicit description of the algebraic crossed product $\mathbf{K} \rtimes^{\mathrm{alg}} G$ and the notation introduced in [3, Def. 5.1]. Recall that the maximal crossed product is defined in [3, Def. 5.9] as the completion of the pre-C^*-category $\mathbf{K} \rtimes^{\mathrm{alg}} G$. In contrast, the reduced crossed product $\mathbf{K} \rtimes_r G$ is defined in [9, Def. 12.9] as the completion of $\mathbf{K} \rtimes^{\mathrm{alg}} G$ in the norm induced by a specific representation on a W^*-category $\mathbf{L}^2(G, \mathbf{WMK})$ [9, Def. 12.2], where $\mathbf{WMK}$ is the universal W^*-envelope of the multiplier category $\mathbf{MK}$ defined in [9, Def. 2.33]. In order to define $\mathbf{L}^2(G, \mathbf{WMK})$, we must assume that $\mathbf{K}$ admits countable AV-sums. The W^*-category $\mathbf{L}^2(G, \mathbf{WMK})$ has the same objects as $\mathbf{K}$, and the morphisms are given by

$$\mathrm{Hom}_{\mathbf{L}^2(G,\mathbf{WMK})}(K,K') \cong \mathrm{Hom}_{\mathbf{WMK}}\Big(\bigoplus_{g\in G} gK, \bigoplus_{g\in G} gK'\Big)\,. \tag{2.7}$$

Let $(e_h^K)_{h\in G}$ be the family of isometries $e_h^K : hK \to \bigoplus_{g\in G} gK$ witnessing the sum $\bigoplus_{g\in G} gK$. On generators the representation $\mathbf{K} \rtimes^{\mathrm{alg}} G \to \mathbf{L}^2(G, \mathbf{WMK})$ is then defined according to [9, (12.8)] by

$$(f,g) \mapsto \sum_{h\in G} e^{K'}_{hg^{-1}}\, hf\, e_h^{K,*} \tag{2.8}$$

(note that $f : K \to g^{-1}K'$), where the sum converges strictly.

In the present paper, we in particular need the reduced crossed product $\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G$ for $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$. In the following, by specializing the general description above, we describe this crossed product and a part of its multipliers explicitly, thereby introducing notation which will be employed later in the paper. We assume that $\mathbf{C}$ is effectively additive and admits countable AV-sums. In the proof of Lemma 2.21, we saw that $\mathbf{C}^{(G)}_{\mathrm{std}}$ also admits countable AV-sums. The objects of $\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G$ are the objects of $\mathbf{C}^{(G)}_{\mathrm{std}}$. The C^*-category $\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G$ is the completion of the image the functor $\sigma : \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes^{\mathrm{alg}} G \to \mathbf{L}^2(G, \mathbf{WMC}^{(G)}_{\mathrm{std}})$. The W^*-category $\mathbf{L}^2(G, \mathbf{WMC}^{(G)}_{\mathrm{std}})$ has the same objects as $\mathbf{C}^{(G)}_{\mathrm{std}}$. Since the functor $\mathbf{WMC}^{(G)}_{\mathrm{std}} \to \mathbf{WMC}$ induced by $\mathbf{C}^{(G)}_{\mathrm{std}} \to \mathbf{C}$ is fully faithful and that G fixes the objects of $\mathbf{C}^{(G)}_{\mathrm{std}}$, by specializing (2.7) can identify the morphism spaces of $\mathbf{L}^2(G, \mathbf{WMC}^{(G)}_{\mathrm{std}})$ with

$$\mathrm{Hom}_{\mathbf{L}^2(G,\mathbf{WMC}^{(G)}_{\mathrm{std}})}((C,\rho),(C',\rho')) \cong \mathrm{Hom}_{\mathbf{WMC}}\Big(\bigoplus_{g\in G} C, \bigoplus_{g\in G} C'\Big)\,,$$

where $(\bigoplus_{g\in G} C, (e_l)_{l\in G})$ and $(\bigoplus_{g\in G} C', (e'_l)_{l\in G})$ represent AV-sums of the constant families $(C)_{g\in G}$ and $(C')_{g\in G}$, respectively.

We can now describe the functor σ explicitly specializing (2.8) where we use that the G-action in morphisms in $\mathbf{C}^{(G)}_{\mathrm{std}}$ is given by $(h, f) \mapsto h \cdot f$. On objects σ acts as the identity. Furthermore, σ sends the morphism $(f, g) \colon (C, \rho) \to (C', \rho')$ in $\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes^{\mathrm{alg}} G$ to

$$\sigma(f, h) := \sum_{l \in G} e'_{lh^{-1}} \, l \cdot f \, e^*_l \colon \bigoplus_{g \in G} C \to \bigoplus_{g \in G} C' \,. \tag{2.9}$$

If $\mathbf{L}$ is a closed wide subcategory of a C^*-category $\mathbf{H}$, then the idealizer of $\mathbf{L}$ in $\mathbf{H}$ is the maximal wide subcategory of $\mathbf{H}$ containing $\mathbf{L}$ as an ideal. It consists of all morphisms of $\mathbf{H}$ which preserve $\mathbf{L}$ by left- and right composition.

Definition 2.22. *We define the unital C^*-category $\mathbf{U}$ to be the idealizer of $\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G$ in $\mathbf{L}^2(G, \mathbf{WMC}^{(G)}_{\mathrm{std}})$.*

We will understand $\mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)$ as the idempotent completion relative to $\mathbf{U}$, see [9, Def. 17.5]. Therefore, objects in $\mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)$ are triples (C, ρ, p), where p is a projection on (C, ρ) in $\mathbf{U}$.

Using formula (2.9), we see that σ extends canonically to a functor

$$\sigma \colon \mathbf{MC}^{(G)}_{\mathrm{std}} \rtimes^{\mathrm{alg}} G \to \mathbf{U}$$

given by the same formula. By the universal property of the maximal crossed product it further extends to a morphism

$$\sigma \colon \mathbf{MC}^{(G)}_{\mathrm{std}} \rtimes G \to \mathbf{U} \,. \tag{2.10}$$

Let $\phi \colon \mathbf{C} \to \mathbf{C}'$ be a morphism in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$.

Definition 2.23 ([9, Def. 3.11])**.** *The morphism ϕ is called non-degenerate if for every two objects C_0, C_1 of $\mathbf{C}$ the linear subspaces*

$$\phi(\mathrm{End}_{\mathbf{C}}(C_1))\mathrm{Hom}_{\mathbf{C}'}(\phi(C_0), \phi(C_1)) \quad \textit{and} \quad \mathrm{Hom}_{\mathbf{C}'}(\phi(C_0), \phi(C_1))\phi(\mathrm{End}_{\mathbf{C}}(C_0))$$

are dense in $\mathrm{Hom}_{\mathbf{C}'}(\phi(C_0), \phi(C_1))$.

We will consider the chain of subcategories

$$C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg,add}} \subseteq C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg},\omega\mathrm{add,eadd}} \subseteq C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg}} \subseteq C^*\mathbf{Cat}^{\mathrm{nu}} \,, \tag{2.11}$$

where

(1) $C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg}}$ is the wide subcategory of $C^*\mathbf{Cat}^{\mathrm{nu}}$ of non-degenerate morphisms,
(2) $C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg},\omega\mathrm{add,eadd}}$ is full subcategory of $C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg}}$ of effectively additive objects which admit countable AV-sums,
(3) $C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg,add}}$ is full subcategory of $C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg}}$ of objects which admit all small AV-sums.

By [9, Prop. 3.16] a non-degenerate morphism $\phi : \mathbf{C} \to \mathbf{C}'$ naturally induces a morphism $\mathbf{M}\phi : \mathbf{MC} \to \mathbf{MC}'$ of the associated multiplier categories and, again by non-degeneracy, it restricts to a unital morphism $\phi^u : \mathbf{C}^u \to \mathbf{C}'^{,u}$ of full subcategories of unital objects. This implies that the constructions of $\mathbf{C}^{(G)}_{\mathrm{std}}$, $\mathbf{MC}^{(G)}_{\mathrm{std}}$, $\mathbf{Q}^{(G)}_{\mathrm{std}}$, $\mathbf{C}^u$, $(\mathbf{C}^u)^{(G)}$ and $\mathbf{C}^{(G)}_{\mathrm{lf}}$ extend to functors on $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg,eadd,}\omega\mathrm{add}})$. Further, ϕ induces a morphism $\mathbf{L}^2(G, \mathbf{WMC}^{(G)}_{\mathrm{std}}) \to \mathbf{L}^2(G, \mathbf{WMC}'^{,(G)}_{\mathrm{std}})$ (see the proof of [9, Lem. 12.10]) and hence $\mathbf{U}$ and $\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G$ also extend to such functors.

3. G-bornological coarse spaces and $K\mathbf{C}\mathcal{X}^G$

We fix $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$. In the present section, we recall the construction of the equivariant coarse homology theory

$$K\mathbf{C}\mathcal{X}^G : G\mathbf{BC} \to \mathbf{Sp}$$

introduced in [7] (see Definition 3.4) which will give rise to the equivariant local K-homology $K^{G,\mathcal{X}}_{\mathbf{C}}$ described in Definition 4.9.

In order to define the functor $K\mathbf{C}\mathcal{X}^G$ the coefficient category $\mathbf{C}$ must be effectively additive (Definition 2.3). If $\mathbf{C}$ also admits countable AV-sums (Definition 2.2), then $K\mathbf{C}\mathcal{X}^G$ is an equivariant coarse homology theory. Finally, in order to ensure strong additivity of $K\mathbf{C}\mathcal{X}^G$ by [7, Thm.11.1] we must assume the existence of all very small AV-sums.

Example 3.1. For every A in $\mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}})$ the category $\mathbf{Hilb}_c(A)$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ admits all small AV-sums and is idempotent complete, hence is in particular effectively additive. It therefore satisfies all the conditions listed above. □

Let X be a set. Subsets of $X \times X$ will be called entourages on X. The set $\mathcal{P}_{X\times X}$ of all entourages is a monoid with involution, where the composition of the entourages U and V is the entourage

$$U \circ V := \mathrm{pr}_{14}[(U \times V) \cap (X \times \mathrm{diag}(X) \times X)]\,,$$

the unit is the entourage $\mathrm{diag}(X)$, and the involution is given by the formula

$$U^* := \{(y,x) \mid (x,y) \in U\}\,.$$

The monoid $\mathcal{P}_{X\times X}$ acts on $\mathcal{P}_X$ by

$$(U, Y) \mapsto U[Y] := \mathrm{pr}_1[U \cap (X \times Y)]\,. \tag{3.1}$$

A G-coarse structure $\mathcal{C}$ on a G-set X is by definition a G-invariant submonoid of $\mathcal{P}_{X\times X}$ which is closed under taking subsets, applying the involution, and forming finite unions, and in which the subset of G-invariant entourages $\mathcal{C}^G$ is cofinal with respect to the inclusion relation. A G-coarse space is a pair $(X, \mathcal{C})$ of a G-set and a G-coarse structure. If $(X, \mathcal{C})$ and $(X', \mathcal{C}')$ are two G-coarse spaces and $f : X \to X'$ is an equivariant map of the underlying G-sets, then f is controlled if $(f \times f)(\mathcal{C}) \subseteq \mathcal{C}'$. Finally, a coarse structure $\mathcal{C}$ is compatible with a bornology $\mathcal{B}$ if $\mathcal{C}[\mathcal{B}] \subseteq \mathcal{B}$.

The category $G\mathbf{BC}$ of G-bornological coarse spaces was introduced in [10, Def. 2.1]. Its objects are triples $(X,\mathcal{C},\mathcal{B})$ of a very small G-set X with a G-coarse structure $\mathcal{C}$ and a G-bornology $\mathcal{B}$ which is compatible with $\mathcal{C}$. Morphisms are maps of G-sets which are controlled and proper. We usually use the shorter notation X for G-bornological coarse spaces.

Let X be in $G\mathbf{BC}$. Then we can consider the category

$$\mathbf{C}^G_{\mathrm{lf}}(X) := \lim_{BG} \mathbf{C}^{(G)}_{\mathrm{lf}}(X) \tag{3.2}$$

in $C^*\mathbf{Cat}$, where $\mathbf{C}^{(G)}_{\mathrm{lf}}(X)$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat})$ is as introduced in Definition 2.14. Explicity, $\mathbf{C}^G_{\mathrm{lf}}(X)$ is the wide subcategory of $\mathbf{C}^{(G)}_{\mathrm{lf}}(X)$ consisting of the G-invariant morphisms, i.e., morphisms A satisfying $g \cdot A = A$ for all g in G, where the G-action is given by formula (2.2). Note that this construction does not use the coarse structure yet, but this will be the case in the following.

If Y, Y' are two subsets of X and U is an entourage of X, then we say that Y' is U-separated from Y if $Y' \cap U[Y] = \emptyset$, see (3.1) for the definition of the U-thickening $U[Y]$ of Y. We say that a morphism $A : (C,\rho,\mu) \to (C',\rho',\mu')$ in $\mathbf{C}^G_{\mathrm{lf}}(X)$ is U-controlled if $\mu'(Y')A\mu(Y) = 0$ for all pairs of subsets Y', Y of X such that Y' is U-separated from Y.

Definition 3.2. *We define $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)$ in $C^*\mathbf{Cat}$ as follows:*

1. *objects: The objects of $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)$ are the objects of $\mathbf{C}^G_{\mathrm{lf}}(X)$.*
2. *morphisms: The space of morphisms $\mathrm{Hom}_{\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)}((C,\rho,\mu),(C',\rho',\mu'))$ is the closed subspace of $\mathrm{Hom}_{\mathbf{C}^G_{\mathrm{lf}}(X)}((C,\rho,\mu),(C',\rho',\mu'))$ generated by those morphisms which are U-controlled for some coarse entourage U of X.*
3. *composition and involution: The composition and the involution of $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)$ are inherited from $\mathbf{C}^G_{\mathrm{lf}}(X)$.*

One must check that the composition defined in Point 3 preserves the morphism spaces defined in Point 2. We refer to [7, Sec. 4] for the argument.

Let $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ be effectively additive.

Definition 3.3. *We define a functor*

$$\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}} : G\mathbf{BC} \to C^*\mathbf{Cat}$$

as follows:

1. *objects: The functor $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}$ sends X in $G\mathbf{BC}$ to $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)$ in $C^*\mathbf{Cat}$.*
2. *morphisms: The functor $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}$ sends a morphism $f : X \to X'$ in $G\mathbf{BC}$ to the functor $f_* : \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X) \to \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X')$ defined as follows:*
   (a) *objects: $f_*(C,\rho,\mu) := (C,\rho,f_*\mu)$.*
   (b) *morphisms: $f_*(A) := A$.*

For the verification that f_* is well-defined we again refer to [7, Sec. 4]. It is at this point where we need the assumption that $\mathbf{C}$ is effectively additive.

Using the functors from (1.2) for the trivial group we define the topological K-theory functor for C^*-categories as the composition

$$K^{C^*\mathrm{Cat}}: C^*\mathbf{Cat}^{\mathrm{nu}} \overset{\mathrm{kk}_{C^*\mathbf{Cat}}}{\to} \mathrm{KK} \overset{\mathrm{KK}(\mathbb{C},-)}{\to} \mathbf{Sp}. \tag{3.3}$$

The functor (3.3) is equivalent to the functors considered in [24], [6, Sec. 8.5], [9, Sec. 14]. Note that here we consider C^*-algebras like $\mathbb{C}$ as C^*-categories with a single object.

Let $\mathbf{C}$ be in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ be effectively additive.

Definition 3.4. *We define the functor $K\mathbf{C}\mathcal{X}^G$ as the composition*

$$K\mathbf{C}\mathcal{X}^G: G\mathbf{BC} \xrightarrow{\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}} C^*\mathbf{Cat} \xrightarrow{K^{C^*\mathrm{Cat}}} \mathbf{Sp}.$$

For the definition of the notion of an equivariant coarse homology theory we refer to [10, Def. 3.10]. References for additional properties are:

(1) strongly additive: [10, Def. 3.12]
(2) strongness: [10, Def. 4.19]
(3) continuity: [10, Def. 5.15] .

The following theorem is shown in [7, Sec. 6] (and [7, Sec. 11] for strong additivity).

Theorem 3.5. *If $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ is effectively additive and admits countable AV-sums, then $K\mathbf{C}\mathcal{X}^G$ is an equivariant coarse homology theory which is in addition strong and continuous. If $\mathbf{C}$ admits all very small AV-sums, then $K\mathbf{C}\mathcal{X}^G$ is strongly additive.*

By construction the functors $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}$ and $K\mathbf{C}\mathcal{X}^G$ depend functorially on the coefficient category $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg,eadd},\omega\mathrm{add}})$.

4. G-uniform bornological coarse spaces, cones and $K^{G,\mathcal{X}}_{\mathbf{C}}$

A G-uniform structure on X is a G-invariant subset $\mathcal{U}$ of $\mathcal{P}_{X\times X}$ consisting of entourages containing the diagonal, which is closed under taking supersets, finite intersections, compositions, and the involution, and which has the property that every U in $\mathcal{U}$ contains a G-invariant element of $\mathcal{U}$ and admits V in $\mathcal{U}$ with $V\circ V \subseteq U$. A G-uniform space is a pair $(X,\mathcal{U})$ of a G-set and a G-uniform structure. Let $(X,\mathcal{U})$ and $(X',\mathcal{U}')$ be G-uniform spaces and $f: X \to X'$ be a G-invariant map of the underlying sets. Then f is uniform if $(f\times f)^{-1}(\mathcal{U}') \subseteq \mathcal{U}$. A uniform structure $\mathcal{U}$ is compatible with a coarse structure if $\mathcal{U}\cap\mathcal{C} \neq \emptyset$.

Let $G\mathbf{UBC}$ denote the category of G-uniform bornological coarse spaces introduced in [10, Def. 9.9]. Objects are tuples $(X,\mathcal{C},\mathcal{B},\mathcal{U})$ such that $(X,\mathcal{C},\mathcal{B})$ is a G-bornological coarse space and $\mathcal{U}$ is a G-uniform structure compatible with $\mathcal{C}$. Morphisms are morphisms of G-bornological coarse spaces which are in addition uniform. We will usually use the shorter notation X for G-uniform bornological coarse spaces. We have canonical forgetful functors

$$G\mathbf{UBC} \to G\mathbf{BC}, \quad G\mathbf{UBC} \to G\mathbf{Top} \tag{4.1}$$

which forget the uniform structure or take the underlying G-topological space, respectively.

If not said differently, we will consider all subsets of $\mathbb{R}^n$ as objects of $G\mathbf{UBC}$ with the trivial G-action and the structures induced by the standard metric.

The categories $G\mathbf{BC}$ and $G\mathbf{UBC}$ have monoidal structures $\otimes$ which are the cartesian structure on the underlying G-uniform and G-coarse spaces (see [10, Ex. 2.17] for the case of $G\mathbf{BC}$) such that the forgetful functor $G\mathbf{UBC} \to G\mathbf{BC}$ is symmetric monoidal in the canonical way. The bornology on $X \otimes X'$ is generated by the subsets $B \times B'$ for all bounded subsets B of X and B' of X', respectively.

Let X be in $G\mathbf{UBC}$.

Definition 4.1. *X is flasque if it is a retract of $[0, \infty) \otimes X$.*

Note that this definition is a little more restrictive than the definition given in [5, Text before Def. 3.10]. The same argument as for [6, Lem. 3.28] in the non-equivariant case shows that the underlying G-bornological coarse space of X is flasque in the generalized sense.

The notion of homotopy in the category $G\mathbf{UBC}$ is defined in the usual manner using the interval functor $X \mapsto [0, 1] \otimes X$.

Recall the definitions of uniformly or coarsely excisive pairs from [5, Def. 3.3] and [5, Def. 3.5].

Let $E : G\mathbf{UBC} \to \mathbf{M}$ be a functor whose target is a stable ∞-category.

Definition 4.2.

(1) *E is homotopy invariant if it sends the projection $[0, 1] \otimes X \to X$ to an equivalence for every X in $G\mathbf{UBC}$.*

(2) *E satisfies closed excision if $E(\emptyset) \simeq 0$ and for every uniformly and coarsely excisive pair (Y, Z) of invariant closed subsets of some X in $G\mathbf{UBC}$ such that $X = Y \cup Z$ the square*

$$\begin{tikzcd} E(Y \cap Z) \arrow[r] \arrow[d] & E(Y) \arrow[d] \\ E(Z) \arrow[r] & E(X) \end{tikzcd}$$

is a push-out square.

(3) *E vanishes on flasques if $E(X) \simeq 0$ for any flasque X in $G\mathbf{UBC}$.*

(4) *E is u-continuous if for every X in $G\mathbf{UBC}$ we have $\mathrm{colim}_V E(X_V) \simeq E(X)$, where V runs over $\mathcal{C}^G \cap \mathcal{U}$, and X_V is obtained from X by replacing its coarse structure $\mathcal{C}$ on X by the coarse structure generated by V.*

Let X be in $G\mathbf{UBC}$ with uniform structure $\mathcal{U}$. Note that $\mathcal{U}$ and $\mathcal{P}_{X \times X}$ are posets with respect to the inclusion relation.

Definition 4.3. *A scale for X is a non-increasing function $\psi : \mathbb{R} \to \mathcal{P}(X \times X)^G$ with the following properties:*

(1) *If t is in $(-\infty, 0]$, then $\psi(t) = X \times X$.*

(2) *For every V in $\mathcal{U}$, there exists t_0 in $\mathbb{R}$ such that $\psi(t) \subseteq V$ for all t in $[t_0, \infty)$.*

Definition 4.4. *We define the geometric cone-at-∞ of X to be the object $\mathcal{O}^\infty(X)$ in G**BC** given as follows:*

1. *The underlying G-set of $\mathcal{O}^\infty(X)$ is $\mathbb{R}\times X$.*
2. *The bornology of $\mathcal{O}^\infty(X)$ is generated by the subsets $[-r,r]\times B$ for all r in $(0,\infty)$ and bounded subsets B of X.*
3. *The coarse structure is generated by the entourages $U\cap U_\psi$ for all scales ψ, where U is a coarse entourage of $\mathbb{R}\otimes X$ and*

$$U_\psi := \{((s,x),(t,y))\in(\mathbb{R}\times X)\times(\mathbb{R}\times X)\mid (x,y)\in\psi(\max\{s,t\})\}\,. \tag{4.2}$$

We furthermore define the cone $\mathcal{O}(X)$ of X to be the subset $[0,\infty)\times X$ of $\mathcal{O}^\infty(X)$ with the induced structures.

Definition 4.5. *We define functors*

$$\mathcal{O}^\infty,\mathcal{O}:\ G\mathbf{UBC}\to G\mathbf{BC}$$

as follows:

1. *objects: The functors send X in G**UBC** to $\mathcal{O}^\infty(X)$ or $\mathcal{O}(X)$, respectively.*
2. *morphisms: The functors send a morphism $f:X\to X'$ in G**UBC** to the morphism $\mathcal{O}^\infty(X)\to\mathcal{O}^\infty(X')$ or $\mathcal{O}(X)\to\mathcal{O}(X')$ given by $\mathrm{id}_{\mathbb{R}}\times f$ or $\mathrm{id}_{[0,\infty)}\times f$, respectively.*

The definition of the functors for morphisms in Point 2 needs a justification which is given e.g. by a specialization of the argument for [6, Lem. 5.15].

For X in G**UBC**, we have a natural sequence of maps in G**UBC**

$$X\to\mathcal{O}(X)\to\mathcal{O}^\infty(X)\to\mathbb{R}\otimes X \tag{4.3}$$

called the cone sequence. Here the first map is given by $x\mapsto(0,x)$, the second map is the inclusion, and the third map is the identity on the underlying sets.

Let $E:\ G\mathbf{BC}\to\mathbf{M}$ be a functor with target a stable ∞-category. Then we consider the functors

$$\begin{aligned} E\mathcal{O}^\infty &:= E\circ\mathcal{O}^\infty:\ G\mathbf{UBC}\to\mathbf{M} \\ E\mathcal{O} &:= E\circ\mathcal{O}:\ G\mathbf{UBC}\to\mathbf{M}\,. \end{aligned} \tag{4.4}$$

Proposition 4.6. *We assume that E is a coarse homology theory which is in addition strong. Then the functors $E\mathcal{O}^\infty$ and $E\mathcal{O}$ have the following properties:*

1. *homotopy invariance,*
2. *closed excision,*
3. *vanishing on flasques,*
4. *u-continuous.*

Moreover, the cone sequence (4.3) *induces a fibre sequence of functors*

$$E\to E\mathcal{O}\to E\mathcal{O}^\infty\xrightarrow{\partial^{\mathrm{Cone}}}\Sigma E\,. \tag{4.5}$$

This proposition follows from the results stated in [5, Sec. 9] (which are stated there in the non-equivariant case, but the same proof applies here). In particular, the list of properties of the functors is given by [5, Lem. 9.6] and the cone sequence follows from [5, (9.1)]. Note that we consider E in (4.5) as a functor on $G\mathbf{UBC}$ by using the first forgetful functor in (4.1).

Let Y be in $G\mathbf{BC}$ and $E : G\mathbf{BC} \to \mathbf{M}$ be some functor.

Definition 4.7 ([10, (10.17)]). *We define the twist E_Y of E by Y as the functor*

$$E_Y : G\mathbf{BC} \to \mathbf{M}, \quad E_Y(X) := E(X \otimes Y).$$

The following has been shown in [10, Lem. 4.17 & 11.25]:

Lemma 4.8. *If E is a coarse homology theory, then so is its twist E_Y. If E is strong, then so is E_Y.*

We apply this construction to the equivariant coarse homology theory $K\mathbf{C}\mathcal{X}^G$ from Definition 3.4. The group G gives rise to the G-bornological coarse spaces $G_{can,min}$ [10, Ex. 2.4] and also $G_{can,max}$. Here *min* and *max* refer to the minimal (finite subsets) and maximal (all subsets) bornologies, and the canonical coarse structure *can* is the minimal G-coarse structure such that G_{can} is a connected G-coarse space. It is generated by the entourages $\{(g,h)\}$ for all (g,h) in $G \times G$. Later we will in particular consider the coarse homology theories $K\mathbf{C}\mathcal{X}^G_{G_{can,max}}$ and $K\mathbf{C}\mathcal{X}^G_{G_{can,min}}$ obtained from $K\mathbf{C}\mathcal{X}^G$ by twisting with $G_{can,max}$ and $G_{can,min}$, respectively.

Let $\mathbf{C}$ be in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ be effectively additive.

Definition 4.9. *We define the equivariant local K-homology functor*

$$K^{G,\mathcal{X}}_{\mathbf{C}} : G\mathbf{UBC} \to \mathbf{Sp}$$

as the composition

$$K^{G,\mathcal{X}}_{\mathbf{C}} : G\mathbf{UBC} \xrightarrow{\mathcal{O}^\infty} G\mathbf{BC} \xrightarrow{K\mathbf{C}\mathcal{X}^G_{G_{can,max}}} \mathbf{Sp}.$$

The following proposition lists the properties of the functor $K^{G,\mathcal{X}}_{\mathbf{C}}$. It is a consequence of Theorem 3.5 and Proposition 4.6.

Proposition 4.10. *If $\mathbf{C}$ is effectively additive and admits all countable AV-sums, then the functor $K^{G,\mathcal{X}}_{\mathbf{C}}$ has the following properties:*

(1) *closed excision,*
(2) *homotopy invariant,*
(3) *u-continuous,*
(4) *vanishing on flasques.*

Note that the functor $K^{G,\mathcal{X}}_{\mathbf{C}}$ depends functorially on coefficient category $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg,eadd},\omega\mathrm{add}})$.

5. Locality and pseudolocality

For a set X, we let $\ell^\infty(X)$ denote the C^*-algebra of all bounded functions $X \to \mathbb{C}$ with the supremum norm $\|f\| := \sup_{x\in X} |f(x)|$.

For an entourage U on X and a subset W, we define the U-variation on W of a function $f : X \to \mathbb{C}$ by

$$\mathrm{Var}_U(f, W) := \sup_{(x,y)\in U\cap(W\times W)} |f(x) - f(y)| \,.$$

Let $\mathcal{Y}$ be a filtered family of subsets in X, ordered by inclusion.

Definition 5.1.

(1) *The C^*-algebra $\ell^\infty(\mathcal{Y})$ of functions vanishing away from $\mathcal{Y}$ is defined as the sub-C^*-algebra of $\ell^\infty(X)$ of functions f satisfying*

$$\lim_{Y\in\mathcal{Y}} \|f_{X\setminus Y}\| = 0 \,.$$

(2) *For a coarse space X with coarse structure $\mathcal{C}$, we define the algebra of bounded functions with vanishing variation away from $\mathcal{Y}$ as*

$$\ell^\infty_{\mathcal{Y}}(X) := \{f \in \ell^\infty(X) \mid \forall U \in \mathcal{C} : \lim_{Y\in\mathcal{Y}} \mathrm{Var}_U(f, X \setminus Y) = 0\} \,.$$

If X is a coarse space, then $\mathcal{Y}$ is a big family if for every Y in $\mathcal{Y}$ and coarse entourage U of X the thickening $U[Y]$ is again contained in a member of $\mathcal{Y}$ [6, Def. 3.2]. If $\mathcal{Y}$ is a big family, then we have $\ell^\infty(\mathcal{Y}) \subseteq \ell^\infty_{\mathcal{Y}}(X)$.

For $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ and X in $G\mathbf{Born}$, we consider the G-C^*-category $\mathbf{C}^{(G)}_{\mathrm{lf}}(X)$ introduced in Definition 2.14. Let (C, ρ, μ) be an object in $\mathbf{C}^{(G)}_{\mathrm{lf}}(X)$. We then extend the projection-valued measure μ to a homomorphism of C^*-algebras

$$\mu : \ell^\infty(X) \to \mathrm{End}_{\mathbf{MC}}(C)$$

which sends f in $\ell^\infty(X)$ to

$$\mu(f) := \int_X f d\mu \,. \tag{5.1}$$

Remark 5.2. This integral can be interpreted as follows. For every x in X, we can choose a representative $u_x : C_x \to C$ of the image in $\mathbf{MC}$ of the projection $\mu(\{x\})$ on C. By Definition 2.12.2c

$$(C, (u_x)_{x\in X}) \tag{5.2}$$

represents the AV-sum of the family $(C_x)_{x\in X}$. Using that f is bounded and that the family $(u_x)_{x\in X}$ is mutually orthogonal we conclude using [9, Lem. 7.8] that the sum

$$\mu(f) := \sum_{x\in X} u_x f(x) u_x^*$$

strictly converges in $\mathbf{MC}$. □

One checks that μ is a homomorphism of C^*-algebras and that $\mu(\chi_Y) = \mu(Y)$ for the characteristic function χ_Y of a subset Y of X. Furthermore, using the equivariance (2.3) of μ, one checks that ϕ is equivariant in the sense that

$$g^{-1} \cdot \mu(f) = \mu(g^* f) \tag{5.3}$$

for all g in G, see (2.2) for notation.

Let X be in $G\mathbf{BC}$ and $\mathcal{Y}$ be a big family on X. Let $(C,\rho,\mu),(C',\rho',\mu')$ be objects of $\mathbf{C}^{(G)}_{\mathrm{lf}}(X)$ and $A : (C,\rho,\mu) \to (C',\rho',\mu')$ be a morphism in this C^*-category. The argument for the following commutator estimate is taken from [33], see also [13, Lemma 3.9].

Lemma 5.3. *If f is in $\ell^\infty_{\mathcal{Y}}(X)$ and A is U-controlled for some coarse entourage U, then*

$$\lim_{Y\in\mathcal{Y}} \|\mu'(X\setminus Y)\,(\mu'(f)A - A\mu(f))\,\mu(X\setminus Y)\| = 0\,.$$

Proof. Let ϵ in $(0,\infty)$ be given and set $\eta := \epsilon/4\|A\|$. We then choose Y in $\mathcal{Y}$ such that $\mathrm{Var}_U(f, X\setminus Y') \le \eta$ for each Y' in $\mathcal{Y}$ with $Y \subseteq Y'$. We define the partition $(S_k)_{k\in\mathbb{Z}}$ of $X\setminus Y$ by

$$S_k := \{x \in X\setminus Y \mid (k-1)\eta \le f(x) < k\eta\}\,.$$

Since f is bounded, only finitely many of these sets are non-empty. If k,l are in $\mathbb{Z}$, then $x\in S_k$ and $y \in S_l$ implies $|f(x)-f(y)| \ge (|k-l|-1)\eta$. Since the U-variation of f on $X\backslash Y = \bigcup_{k\in\mathbb{Z}} S_k$ is bounded by η, the condition $|k-l|\ge 2$ implies that $S_k \cap U[S_l] = U[S_k]\cap S_l = \emptyset$. Since A is U-controlled we can conclude that $\mu'(S_k)A\mu(S_l) = 0$.

We set

$$\tilde f := \chi_Y \cdot f + \eta \sum_{k\in\mathbb{Z}} k\cdot \chi_{S_k}\,.$$

Then by construction $\|\tilde f - f\| \le \eta$ and hence

$$\|(\mu'(f)A - A\mu(f)) - (\mu'(\tilde f)A - A\mu(\tilde f))\| \le 2\eta\|A\| = \frac{\epsilon}{2}\,. \tag{5.4}$$

Since A is U-controlled, we have

$$\begin{aligned}\mu'(X\setminus U[Y])&(\mu'(\tilde f)A - A\mu(\tilde f))\mu(X\setminus U[Y]) \\ &= \eta \sum_{k\in\mathbb{Z}} k\cdot \mu'(X\setminus U[Y])(\mu'(S_k)A - A\mu(S_k))\mu(X\setminus U[Y])\,.\end{aligned} \tag{5.5}$$

Inserting the identities $\mu(X\setminus Y) = \sum_{k\in\mathbb{Z}} \mu(S_k)$ and $\mu'(X\setminus Y) = \sum_{k\in\mathbb{Z}}\mu'(S_k)$ and using that $\mu'(S_k)A\mu(S_l) = 0$ whenever $|k-l|\ge 2$, we get

$$\begin{aligned}&\sum_{k\in\mathbb{Z}} k(\mu'(S_k)A - A\mu(S_k)) \\ &= \mu'(X\setminus Y)\sum_{k\in\mathbb{Z}}(\mu'(S_k)A\mu(S_{k-1}) - \mu'(S_k)A\mu(S_{k+1}))\mu(X\setminus Y)\,.\end{aligned}$$

The right-hand side is an operator with norm bounded by $2\|A\|$. Using

$$\mu(X\setminus U[Y]) = \mu(X\setminus U[Y])\mu(X\setminus Y)$$

and plugging the above equality into (5.5), we get

$$\|\mu'(X\setminus U[Y])(\mu'(\tilde f)A - A\mu(\tilde f))\mu(X\setminus U[Y])\| \le 2\eta\|A\| = \frac{\epsilon}{2}.$$

Combining this with (5.4), we see that

$$\|\mu'(X \setminus Y')(\mu'(f)A - A\mu(f))\mu(X \setminus Y')\| \leq \varepsilon$$

for all Y' in $\mathcal{Y}$ with $U[Y] \subseteq Y'$. □

Recall Definition 3.2 of the C^*-category $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)$.

Corollary 5.4. *For a morphism $A : (C, \rho, \mu) \to (C', \rho', \mu')$ in $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)$ and f in $\ell^\infty_{\mathcal{Y}}(X)$, we have*

$$\lim_{Y \in \mathcal{Y}} \|\mu'(X \setminus Y)(\mu'(f)A - A\mu(f))\mu(X \setminus Y)\| = 0\,.$$

Proof. We use that A can be approximated in norm by U-controlled equivariant morphisms A' and apply Lemma 5.3 to the approximants A'. □

If Y is an invariant subset of X, then we define the wide subcategory $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(Y \subseteq X)$ of $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)$ (see [7, Def. 5.5]) such that for objects (C, ρ, μ) and (C', ρ', μ') in $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(Y \subseteq X)$

$$\begin{aligned}&\mathrm{Hom}_{\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(Y \subseteq X)}((C, \rho, \mu), (C', \rho', \mu')) \\ &\qquad := \mu'(Y)\mathrm{Hom}_{\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)}((C, \rho, \mu),(C', \rho', \mu'))\mu(Y)\,.\end{aligned}$$

Similarly, for an invariant big family $\mathcal{Y} = (Y_i)_{I \in I}$ on X (see [10, Def. 3.5]) we have the wide subcategory

$$\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{Y} \subseteq X) := \overline{\bigcup_{i \in I} \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(Y_i \subseteq X)} \tag{5.6}$$

of $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)$ (the union and closure are both taken on the level of morphisms). By [7, Lem. 5.9] we know that $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{Y} \subseteq X)$ is an ideal in $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)$.

Corollary 5.5. *For a morphism $A : (C, \rho, \mu) \to (C', \rho', \mu')$ in $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)$ and f in $\ell^\infty_{\mathcal{Y}}(X)$ for an invariant big family $\mathcal{Y}$ on X, we have $\mu'(f)A - A\mu(f) \in \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{Y} \subseteq X)$.*

Let X be in $G\mathbf{UBC}$ and $\mathcal{B}$ denote the bornology of X.

Definition 5.6.

(1) *We let $C_u(X) \subseteq \ell^\infty(X)$ be the sub-algebra of uniformly continuous functions on X.*
(2) *We set $C_0(X) := C_u(X) \cap \ell^\infty(\mathcal{B})$.*

Note the discussion [13, 3.13] about the difference of $C_0(X)$ and the possibly smaller C^*-algebra $C_u(\mathcal{B})$ generated by uniformly continuous functions supported on bounded subsets.

Recall the cone construction $\mathcal{O} : G\mathbf{UBC} \to G\mathbf{BC}$ introduced in Definition 4.5. For X in $G\mathbf{UBC}$, we consider $\mathcal{O}(X) \otimes G_{can,max}$ in $G\mathbf{BC}$. The underlying G-set of this G-bornological coarse space is $[0,\infty)\times X\times G$. We let $\pi : [0,\infty)\times X\times G \to X$ be the projection. It induces a homomorphism

$$\pi^* : \ell^\infty(X) \to \ell^\infty(\mathcal{O}(X)\otimes G_{can,max}).$$

In the following, $\mathcal{B}$ denotes the bornology of $\mathcal{O}(X)\otimes G_{can,max}$.

Lemma 5.7. *The homomorphism π^* restricts to a homomorphism*

$$\pi^* : C_0(X) \to \ell^\infty_{\mathcal{B}}(\mathcal{O}(X)\otimes G_{can,max})\,.$$

Proof. Let f be in $C_0(X)$ and V be a coarse entourage of $\mathcal{O}(X)\otimes G_{can,max}$. For every ϵ in $(0,\infty)$, we must find a bounded subset A of $\mathcal{O}(X)\otimes G_{can,max}$ such that $\mathrm{Var}_V(\pi^* f, X\setminus A)\le\epsilon$.

We can find a bounded subset B of X such that $\|\chi_{X\setminus B} f\|\le \frac{\epsilon}{2}$. By uniform continuity we can further find a uniform entourage U of X such that $\mathrm{Var}_U(f,X)\le\epsilon$. There exists t in $(0,\infty)$ such that $((s,x,g),(s',x',g'))\in V$ and $s\ge t$ or $s'\ge t$ implies $(x,x')\in U$. It follows that $\mathrm{Var}_V(\pi^* f, Y_t)\le\epsilon$, where $Y_t := [t,\infty)\times X\times G$.

We also have $\|\chi_{\pi^{-1}(X\setminus B)}\pi^* f\|\le\frac{\epsilon}{2}$ so that actually

$$\mathrm{Var}_V(\pi^* f, Y_t\cup\pi^{-1}(X\setminus B)\|\le\epsilon.$$

Finally note that $A := (\mathcal{O}(X)\otimes G_{can,max})\setminus(Y_t\cup\pi^{-1}(X\setminus B))$ is a bounded subset of $\mathcal{O}(X)\otimes G_{can,max}$. □

Let (C,ρ,μ) be an object of $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(X)\otimes G_{can,max})$. Using (5.1) we define the homomorphism

$$\phi : \ell^\infty(X)\to \mathrm{End}_{\mathbf{MC}}(C)\,,\quad f\mapsto\phi(f) := \mu(\pi^* f)\,. \tag{5.7}$$

Let $A : (C,\rho,\mu)\to(C',\rho',\mu')$ be a morphism in $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(X)\otimes G_{can,max})$. Recall that A is in particular a multiplier morphism from C to C'. Our next result states that A is pseudolocal (in the sense of [23, Def. 12.3.1] if one replaces the ideal of compact operators in all bounded operators by the ideal $\mathbf{C}$ in the multiplier category $\mathbf{MC}$ and we consider the objects of $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(X)\otimes G_{can,max})$ as X-controlled via (5.7)). Let ϕ' be defined as in (5.7), but for the object (C',ρ',μ').

Lemma 5.8. *For f in $C_0(X)$, the difference $A\phi(f)-\phi'(f)A$ belongs to* $\mathbf{C}$.

Proof. Recall that $\mathcal{B}$ denotes the bornology of $\mathcal{O}(X)\otimes G_{can,max}$. By Lemma 5.7 we have

$$\pi^* f\in\ell_{\mathcal{B}}(\mathcal{O}(X)\otimes G_{can,max})\,.$$

By Corollary 5.5 we have

$$A\phi(f)-\phi'(f)A\in\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{B}\subseteq\mathcal{O}(X)\otimes G_{can,max})\,.$$

By local finiteness of the objects of $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(X)\otimes G_{can,max})$ we conclude that

$$A\phi(f)-\phi'(f)A : C\to C'$$

is a morphism in $\mathbf{C}$. □

We consider the big family

$$\mathcal{Z} := (Z_n)_{n\in\mathbb{N}}\,, \quad Z_n := [0,n]\times X\times G\,. \tag{5.8}$$

on $\mathcal{O}(X)\otimes G_{can,max}$.

Let $A:\ (C,\rho,\mu)\to(C',\rho',\mu')$ be a morphism in $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{Z}\subseteq\mathcal{O}_\tau(X)\otimes G_{can,max})$, see (5.6). Our next result shows that it locally belongs to $\mathbf{C}$. Let $\mathcal{B}_X$ denote the bornology of X.

Lemma 5.9. *For f in $\ell^\infty(\mathcal{B}_X)$, we have $\phi'(f)A\in\mathbf{C}$ and $A\phi(f)\in\mathbf{C}$.*

Proof. It suffices to show that $\phi'(f)A\in\mathbf{C}$. In order to deduce $A\phi(f)\in\mathbf{C}$, we then use the involution.

We fix ϵ in $(0,\infty)$. Then we can find A' in $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{Z}\subseteq\mathcal{O}(X)\otimes G_{can,max})$ and n in $\mathbb{N}$ such that $\|A-A'\|\le\frac{\epsilon}{2\|f\|}$ and $\mu(Z_n)A'\mu(Z_n)=A'$. We can furthermore find a bounded subset B of X such that $\|\chi_{X\setminus B}f\|\le\frac{\epsilon}{2\|A\|}$. We set $f' := \chi_B f$. Then $\|\phi'(f)A-\phi'(f')A'\|\le\epsilon$. Since ϵ can be taken arbitrary small and $\mathbf{C}$ is closed in $\mathbf{MC}$ it suffices to show that $\phi'(f')A'\in\mathbf{C}$. But $\phi'(f')A'$ is supported on the bounded set $[0,n]\times B\times G$ of $\mathcal{O}(X)\otimes G_{can,max}$. Hence, $\phi'(f')A'\in\mathbf{C}$ by local finiteness of (C',ρ',μ'). $\square$

6. Construction of the Paschke morphism

To X in $G\mathbf{UBC}$ we can associate the commutative G-C^*-algebra $C_0(X)$ introduced in Definition 5.6. Since a morphism $f\,:\,X\to X'$ in $G\mathbf{UBC}$ is uniform and proper it induces a homomorphism $f^*\,:\,C_0(X')\to C_0(X)$ given by precomposition. We therefore get a functor

$$C_0\,:\,G\mathbf{UBC}\to(GC^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{comm}})^{\mathrm{op}}\,,\qquad X\mapsto C_0(X)\,.$$

Using Gelfand duality $(GC^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{comm}})^{\mathrm{op}}\simeq G\mathbf{LCH}^{\mathrm{prop}}_{+}$ we thus get a functor

$$\iota^{\mathrm{top}}\,:\,G\mathbf{UBC}\to G\mathbf{LCH}^{\mathrm{prop}}_{+} \tag{6.1}$$

uniquely characterized by the equality (1.10).

The main result of the present section is the description of the Paschke morphism for a given space X in $G\mathbf{UBC}$. The general idea for its construction via a multiplication map like μ_X as below, but with completely different technical details otherwise, has been used at various places, see e.g. [37, Sec. 6.5] or [38, Sec. 6.4]. In the Section 7, we will provide a refinement of this construction to a natural transformation of functors defined on $G\mathbf{UBC}^{\mathrm{prop}}$.

We start with a description of the following intermediate constructions which go into the construction of the Paschke morphism:

(1) The functor $X\mapsto\mathbf{Q}(X)$ from $G\mathbf{UBC}$ to $C^*\mathbf{Cat}^{\mathrm{nu}}$,
(2) the tensor product $C_0(X)\otimes\mathbf{Q}(X)$,
(3) the multiplication morphism $\mu_X:\ C_0(X)\otimes\mathbf{Q}(X)\to\mathbf{Q}^{(G)}_{\mathrm{std}}$,
(4) the diagonal morphism $\delta_X:\ \mathrm{KK}(\mathbb{C},\mathbf{Q}(X))\to\mathrm{KK}^G(C_0(X),C_0(X)\otimes\mathbf{Q}(X))$.

Using the cone functor $\mathcal{O}$ introduced in Definition 4.5 we define the functor

$$G\mathbf{UBC} \to G\mathbf{BC}\,, \quad X \mapsto \mathcal{O}(X) \otimes G_{can,max}\,. \tag{6.2}$$

For an effectively additive $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$, composing (6.2) with $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}$ from Definition 3.3 we get a functor

$$G\mathbf{UBC} \to C^*\mathbf{Cat}\,, \quad X \mapsto \mathbf{D}(X) := \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(X) \otimes G_{can,max})\,. \tag{6.3}$$

We furthermore have the subfunctor

$$G\mathbf{UBC} \to C^*\mathbf{Cat}^{\mathrm{nu}}\,, \quad X \mapsto \mathbf{C}(X) := \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{Z} \subseteq \mathcal{O}(X) \otimes G_{can,max}) \tag{6.4}$$

(see (5.6) and (5.8) for notation) such that $\mathbf{C}(X)$ is a closed ideal in $\mathbf{D}(X)$. Note that $\mathbf{C}(X)$ is our replacement for $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X \otimes G_{can,max})$ which can be considered as a subcategory of $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(X) \otimes G_{can,max})$ of objects which are supported on $\{0\} \times X \times G$, but which is not an ideal (these two C^*-categories actually have the same K-theory as will be used and also explained further below in Diagram (6.8)). Our choice of notation $\mathbf{C}(X)$ and $\mathbf{D}(X)$ should indicate that these C^*-categories are our versions of the Roe algebra and the algebra of pseudolocal operators. We refer to Section 10 for more details. By forming quotients of C^*-categories we finally define the functor

$$G\mathbf{UBC} \to C^*\mathbf{Cat}^{\mathrm{nu}}\,, \quad X \mapsto \mathbf{Q}(X) := \frac{\mathbf{D}(X)}{\mathbf{C}(X)}\,. \tag{6.5}$$

The functors $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{Q}$ depend functorially on the G-C^*-category $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg,eadd},\omega\mathrm{add}})$ since $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}$ has this property.

Recall the functor $K^{G,\mathcal{X}}_{\mathbf{C}}$ from Definition 4.9, and the K-theory functor $K^{C^*\mathrm{Cat}}$ for C^*-categories from (3.3).

We assume that $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ is effectively additive and admits countable AV-sums.

Lemma 6.1. *We have a canonical equivalence of functors*

$$K^{G,\mathcal{X}}_{\mathbf{C}} \simeq K^{C^*\mathrm{Cat}} \circ \mathbf{Q} \colon G\mathbf{UBC} \to \mathbf{Sp}\,. \tag{6.6}$$

Proof. We have a natural (naturality here and below refers to X in $G\mathbf{UBC}$) commutative diagram of C^*-categories

$$\begin{tikzcd}
& \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X \otimes G_{can,max}) \arrow[r] \arrow[d] & \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(X) \otimes G_{can,max}) \arrow[d, equal] & & \\
0 \arrow[r] & \mathbf{C}(X) \arrow[r] & \mathbf{D}(X) \arrow[r] & \mathbf{Q}(X) \arrow[r] & 0
\end{tikzcd} \tag{6.7}$$

where the top horizontal and left vertical morphisms are induced from canonical inclusions of bornological coarse spaces. We apply $K^{C^*\mathrm{Cat}}$ to Diagram (6.7). Since $K^{C^*\mathrm{Cat}}$ sends exact sequences in $C^*\mathbf{Cat}^{\mathrm{nu}}$ to fibre sequences in $\mathbf{Sp}$ ([12, Thm. 1.32.5] or [9, Prop. 14.7]) we get a natural morphism of fibre sequences

$$\begin{tikzcd} K^{C^*\mathrm{Cat}}(\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X \otimes G_{can,max})) \arrow[r] \arrow[d, "\simeq"] & K^{C^*\mathrm{Cat}}(\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(X) \otimes G_{can,max})) \arrow[r] \arrow[d, equal] & P \arrow[d, "\simeq"] \\ K^{C^*\mathrm{Cat}}(\mathbf{C}(X)) \arrow[r] & K^{C^*\mathrm{Cat}}(\mathbf{D}(X)) \arrow[r] & K^{C^*\mathrm{Cat}}(\mathbf{Q}(X)) \end{tikzcd} \tag{6.8}$$

in **Sp**, where P is defined as the cofibre of the left upper horizontal morphism. In order to see that the left vertical morphism is an equivalence, we argue as in the proof of [7, Thm. 7.2]. For every n in $\mathbb{N}$, the inclusion $X \otimes G_{can,max} \to Z_n$ (see (5.8)) is a coarse equivalence and hence induces an equivalence

$$K^{C^*\mathrm{Cat}}(\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X \otimes G_{can,max})) \stackrel{def}{=} K\mathbf{C}\mathcal{X}^G(X \otimes G_{can,max}) \xrightarrow{\simeq} K\mathbf{C}\mathcal{X}^G(Z_n)\,.$$

The inclusion

$$\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(Z_n) \to \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(Z_n \subseteq \mathcal{O}(X) \otimes G_{can,max})$$

is a unitary equivalence by [7, Lem. 6.10(2)] and therefore induces an equivalence

$$K\mathbf{C}\mathcal{X}^G(Z_n) \stackrel{def}{=} K^{C^*\mathrm{Cat}}(\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(Z_n)) \xrightarrow{\simeq} K^{C^*\mathrm{Cat}}(\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(Z_n \subseteq \mathcal{O}(X) \otimes G_{can,max}))\,.$$

We therefore get an equivalence

$$K^{C^*\mathrm{Cat}}(\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X \otimes G_{can,max})) \xrightarrow{\simeq} \mathrm{colim}_{n\in\mathbb{N}} K^{C^*\mathrm{Cat}}(\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(Z_n \subseteq \mathcal{O}(X) \otimes G_{can,max}))\,.$$

Finally using (5.6), (6.4) and the fact that $K^{C^*\mathrm{Cat}}$ preserves filtered colimits (see [9, Thm. 14.4]) we get the equivalence

$$K^{C^*\mathrm{Cat}}(\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X \otimes G_{can,max})) \xrightarrow{\simeq} K^{C^*\mathrm{Cat}}(\mathbf{C}(X))$$

appearing as the left vertical arrow in (6.6).

It follows that the right vertical morphism is an equivalence, too.

Using the Definition 3.4 of $K\mathbf{C}\mathcal{X}^G$ we get a natural morphism of fibre sequences

$$\begin{tikzcd} K^{C^*\mathrm{Cat}}(\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X \otimes G_{can,max})) \arrow[r] \arrow[d, equal] & K^{C^*\mathrm{Cat}}(\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(X) \otimes G_{can,max})) \arrow[r] \arrow[d, equal] & P \arrow[d, "\simeq"] \\ K\mathbf{C}\mathcal{X}^G_{G_{can,max}}(X) \arrow[r] & K\mathbf{C}\mathcal{X}^G_{G_{can,max}}(\mathcal{O}(X)) \arrow[r] & K\mathbf{C}\mathcal{X}^G_{G_{can,max}}(\mathcal{O}^\infty(X)) \end{tikzcd} \tag{6.9}$$

where, by inserting definitions, we have rewritten the lower sequence as an instance of the cone sequence (4.5) applied to $E := K\mathbf{C}\mathcal{X}^G_{G_{can,max}}$.

Composing the inverse of the right vertical equivalence in (6.9) with the right vertical equivalence in (6.8) and invoking Definition 4.9 yields the natural equivalence

$$K^{G,\mathcal{X}}_{\mathbf{C}}(X) \simeq K^{C^*\mathrm{Cat}}(\mathbf{Q}(X))\,. \tag{6.10}$$

as desired. □

In the present paper, $\otimes$ denotes the maximal tensor product of C^*-categories [12, Def. 7.2]. By [12, Prop. 1.21] the stable ∞-category category KK^G has a presentably symmetric monoidal structure induced by the maximal tensor product of C^*-algebras, and by [12, Thm. 1.35] the functor $\mathrm{kk}^G_{C^*\mathbf{Cat}}$ has a symmetric monoidal refinement. We define the functor

$$-\hat{\otimes}-\colon \mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}})\times \mathrm{KK} \xrightarrow{\mathrm{kk}^G\times \mathrm{Res}^{\{1\}}_G} \mathrm{KK}^G\times \mathrm{KK}^G \xrightarrow{\otimes} \mathrm{KK}^G\,, \qquad (6.11)$$

where $\otimes$ is structure map of the symmetric monoidal structure of KK^G and $\mathrm{Res}^{\{1\}}_G$ is the restriction induced by the projection $G\to\{1\}$ from [12, Thm. 1.22] (on C^*-algebras $\mathrm{Res}^{\{1\}}_G$ is given by equipping a C^*-algebra with the trivial G-action). Using that KK^G is presentably symmetric monoidal category and $\mathrm{Res}^{\{1\}}_G$ preserves small colimits we see that $\hat{\otimes}$ preserves small colimits in its second variable.

Let A be in $\mathrm{Fun}(BG, C^*\mathbf{Alg})$ and $\mathbf{Q}$ be in $C^*\mathbf{Cat}^{\mathrm{nu}}$.

Lemma 6.2. *We have an equivalence*

$$A\hat{\otimes}\,\mathrm{kk}_{C^*\mathbf{Cat}}(\mathbf{Q})\simeq \mathrm{kk}^G_{C^*\mathbf{Cat}}(A\otimes \mathrm{Res}^{\{1\}}_G(\mathbf{Q}))$$

which is natural in A and $\mathbf{Q}$.

Proof. The chain of natural equivalences

$$\begin{aligned} A\hat{\otimes}\,\mathrm{kk}_{C^*\mathbf{Cat}}(\mathbf{Q}) &\stackrel{def}{\simeq} \mathrm{kk}^G(A)\otimes \mathrm{Res}^{\{1\}}_G(\mathrm{kk}_{C^*\mathbf{Cat}}(\mathbf{Q}))\\ &\stackrel{(1)}{\simeq} \mathrm{kk}^G_{C^*\mathbf{Cat}}(A)\otimes \mathrm{kk}^G_{C^*\mathbf{Cat}}(\mathrm{Res}^{\{1\}}_G(\mathbf{Q}))\\ &\stackrel{(2)}{\simeq} \mathrm{kk}^G_{C^*\mathbf{Cat}}(A\otimes \mathrm{Res}^{\{1\}}_G(\mathbf{Q}))\end{aligned}$$

(see [12, Thm. 1.22] for (1) and [12, Thm. 1.35] for (2)) gives the desired equivalence, where in the last two lines we implicitly consider A as a G-C^*-category with a single object. $\square$

From now on, in order to simplify the notation, we will write $\mathbf{Q}$ instead of $\mathrm{Res}^{\{1\}}_G(\mathbf{Q})$.

For X in $G\mathbf{UBC}$, we have the objects $C_0(X)$ in $\mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}})$ and $\mathbf{Q}(X)$ in $C^*\mathbf{Cat}^{\mathrm{nu}}$ and can thus define $C_0(X)\otimes\mathbf{Q}(X)$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$, where consider the left tensor factor as a C^*-category. The objects of this category are the objects of $\mathbf{Q}(X)$, and the morphism spaces are certain completions of the algebraic tensor products of the morphism spaces of $\mathbf{Q}(X)$ with $C_0(X)$. For concreteness, we will work with the maximal tensor product [12, Def. 7.2].

Recall the Definition 2.15.3 of $\mathbf{Q}^{(G)}_{\mathrm{std}}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$. We define the multiplication morphism

$$\mu_X\colon C_0(X)\otimes\mathbf{Q}(X)\to \mathbf{Q}^{(G)}_{\mathrm{std}} \qquad (6.12)$$

in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ as follows.

(1) objects: The morphism μ_X sends the object (C,ρ,μ) to the object (C,ρ) of $\mathbf{Q}^{(G)}_{\mathrm{std}}$. Note that (C,ρ) belongs to $\mathbf{Q}^{(G)}_{\mathrm{std}}$ since the underlying G-set of $\mathcal{O}(X)\otimes G_{can,max}$ is a free G-set (see (6.3), (6.4) and (6.5)).
(2) morphisms: The morphism μ_X is defined on morphisms uniquely by the universal property of the maximal tensor product of C^*-categories such that it sends the morphism $f\otimes[A]$ in $C_0(X)\otimes\mathbf{Q}(X)$ with $A:(C',\rho',\mu')\to(C,\rho,\mu)$ to the morphism $[\phi(f)A]$ in $\mathbf{Q}^{(G)}_{\mathrm{std}}$. Here the brackets $[-]$ indicate classes in the respective quotients (6.5) and (2.5), and $\phi(f)$ is defined in (5.7).

    To see that this map is well-defined note that if A is in $\mathbf{C}(X)$, then $\phi(f)A\in\mathbf{C}^{(G)}_{\mathrm{std}}$ by Lemma 5.9. Further, by Lemma 5.8 we have $[\phi(f)A]=[A\phi'(f)]$ which implies that this prescription is compatible with the composition and the involution.

Finally, we define the diagonal morphism δ_X as the composition

$$\begin{aligned}\delta_X:\ &\mathrm{KK}(\mathbb{C},\mathbf{Q}(X))\\ &\simeq\ \mathrm{KK}(\mathrm{kk}_{C^*\mathbf{Cat}}(\mathbb{C}),\mathrm{kk}_{C^*\mathbf{Cat}}(\mathbf{Q}(X)))\\ &\xrightarrow{C_0(X)\hat{\otimes}-}\ \mathrm{KK}^G(C_0(X)\hat{\otimes}\,\mathrm{kk}_{C^*\mathbf{Cat}}(\mathbb{C}),C_0(X)\hat{\otimes}\,\mathrm{kk}_{C^*\mathbf{Cat}}(\mathbf{Q}(X)))\\ &\overset{!}{\simeq}\ \mathrm{KK}^G(\mathrm{kk}^G_{C^*\mathbf{Cat}}(C_0(X)\otimes\mathbb{C}),\mathrm{kk}^G_{C^*\mathbf{Cat}}(C_0(X)\otimes\mathbf{Q}(X)))\\ &\simeq\ \mathrm{KK}^G(C_0(X),C_0(X)\otimes\mathbf{Q}(X))\,.\end{aligned}\tag{6.13}$$

The last equivalence is given by the identification $C_0(X)\otimes\mathbb{C}\cong C_0(X)$, and the equivalence marked by ! uses Lemma 6.2

We now define the Paschke morphism whose existence was claimed in Theorem 1.5.1. We assume that $\mathbf{C}$ in $\mathrm{Fun}(BG,C^*\mathbf{Cat}^{\mathrm{nu}})$ is effectively additive and admits countable AV-sums.

Definition 6.3. *The Paschke morphism for X in $G\mathbf{UBC}$ is defined as the composition*

$$\begin{aligned}p_X:K^{G,\mathcal{X}}_{\mathbf{C}}(X)\ &\overset{(6.6),(3.3)}{\simeq}\ \mathrm{KK}(\mathbb{C},\mathbf{Q}(X))\\ &\xrightarrow{\delta_X}\ \mathrm{KK}^G(C_0(X),C_0(X)\otimes\mathbf{Q}(X))\\ &\xrightarrow{\mu_X}\ \mathrm{KK}^G(C_0(X),\mathbf{Q}^{(G)}_{\mathrm{std}})\\ &\overset{(1.3)}{\simeq}\ K^{G,\mathrm{An}}_{\mathbf{C}}(\iota^{\mathrm{top}}(X))\,.\end{aligned}\tag{6.14}$$

Note that from this definition is not clear that the Paschke morphism is natural in X. The naturality will be discussed in the next Section 7.

7. Naturality of the Paschke morphism

In this subsection, we discuss the naturality of the Paschke morphism from Definition 6.3. More precisely, we will construct a natural transformation whose component on X in $G\mathbf{UBC}$ is the Paschke morphism of Definition 6.3. Note that naturality in the ∞-categorical sense is more than the existence of a filler for the square

$$\begin{tikzcd} K_{\mathbf{C}}^{G,\mathcal{X}}(X) \arrow[r, "f_*"] \arrow[d, "p_X"] & K_{\mathbf{C}}^{G,\mathcal{X}}(X') \arrow[d, "p_{X'}"] \\ K_{\mathbf{C}}^{G,\mathrm{An}}(\iota^{\mathrm{top}}(X)) \arrow[r, "f_*"] & K_{\mathbf{C}}^{G,\mathrm{An}}(\iota^{\mathrm{top}}(X')) \end{tikzcd} \tag{7.1}$$

for all morphisms $f : X \to X'$, in $G\mathbf{UBC}$. The existence of such a filler can indeed be easily seen by considering the big diagram (7.2) below. In order to produce the data of a natural transformation, we must reformulate the construction of the Paschke morphisms appropriately. The main problem is that $\mathrm{KK}^G(C_0(X), C_0(X) \otimes \mathbf{Q}(X))$ is not a functor on X so that δ_X and μ_X can not be interpreted as natural transformations separately.

We assume that $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ is effectively additive and admits countable AV-sums. In order to get an idea what we have to do to get the existence of a filler of (7.1), we first consider the diagram

$$\begin{tikzcd} \mathrm{KK}(\mathbb{C}, \mathbf{Q}(X)) \arrow[r, "\delta_X"] \arrow[ddd, "\mathrm{KK}^G(-,\mathbf{Q}(f))"'] \arrow[ddr, bend right, "\delta_{X'}"] & \mathrm{KK}^G(C_0(X), C_0(X) \otimes \mathbf{Q}(X)) \arrow[r, "\mu_X"] \arrow[d, "\mathrm{KK}^G(f^*,-)"] & \mathrm{KK}^G(C_0(X), \mathbf{Q}^{(G)}_{\mathrm{std}}) \arrow[ddd, "\mathrm{KK}^G(f^*,-)"] \\ & \mathrm{KK}^G(C_0(X'), C_0(X) \otimes \mathbf{Q}(X)) \arrow[ddr, bend left, "\mu_X"] & \\ & \mathrm{KK}^G(C_0(X'), C_0(X') \otimes \mathbf{Q}(X)) \arrow[u, "\mathrm{KK}^G(-,f^*)"'] \arrow[d, "\mathrm{KK}^G(-,\mathbf{Q}(f)"] & \\ \mathrm{KK}(\mathbb{C}, \mathbf{Q}(X')) \arrow[r, "\delta_X"] & \mathrm{KK}^G(C_0(X'), C_0(X') \otimes \mathbf{Q}(X')) \arrow[r, "\mu_X"] & \mathrm{KK}^G(C_0(X'), \mathbf{Q}^{(G)}_{\mathrm{std}}) \end{tikzcd} \tag{7.2}$$

all of whose cells have essentially obvious fillers. This already implies that the Paschke morphism is natural on the level of homotopy categories.

Remark 7.1. Our idea for showing that the Paschke morphism is an equivalence is to reduce this by homotopy invariance to G-simplicial complexes, and then by excision to G-orbits where it can be verified by an explicit calculation. The excision step requires a natural transformation on the spectrum level. If one is only interested in homotopy groups, then it would be sufficient to know the compatibility of the Paschke map with the Mayer–Vietoris boundary maps which is an immediate consequence of the spectrum-valued naturality. So even if we were finally only interested in the Paschke isomorphism on the level of homotopy groups we would still need the spectrum level natural transformation for the proof that it is an isomorphism.

For similar reasons, the spectrum-valued version is also crucial in the proof of our second Theorem 1.9 comparing the two assembly maps, though the latter is indeed a statement on the level of homotopy groups. □

In the following remarks about general ∞-categorical constructions, we prepare the actual construction of the natural Paschke transformation.

Remark 7.2. For a category $\mathcal{C}$ let $\mathbf{Tw}(\mathcal{C})$, denote the twisted arrow category. Objects are morphisms $C \to C'$ in $\mathcal{C}$, and morphisms $(C_0 \to C'_0) \to (C_1 \to C'_1)$ are commutative diagrams

$$\begin{tikzcd} C_0 \arrow[r] & C'_0 \arrow[d] \\ C_1 \arrow[u] \arrow[r] & C'_1 \end{tikzcd} \tag{7.3}$$

We have a canonical functor

$$(\mathrm{ev}, \mathrm{ev}') : \mathbf{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \quad (C \to C') \mapsto (C, C').$$

If $F, G : \mathcal{C} \to \mathcal{D}$ are two functors to a stable ∞-category, then we can express the spectrum of natural transformations between F and G as

$$\mathrm{nat}(F, G) \simeq \lim_{\mathbf{Tw}(\mathcal{C})} \mathrm{map}_{\mathcal{D}}(F \circ \mathrm{ev}, G \circ \mathrm{ev}'). \tag{7.4}$$

We refer to [17, 18] where this is discussed even in the more general case of $\mathcal{C}$ being an ∞-category. □

Remark 7.3. Recall that our universe in which we do homotopy theory is the one of small sets. The corresponding categories then belong to the large universe. A locally small, large presentable stable ∞-category $\mathcal{C}$ is enriched and tensored over $\mathbf{Sp}$. We thus a functor

$$\mathcal{C} \times \mathbf{Sp} \to \mathcal{C}, \quad (C, E) \mapsto C \wedge E \tag{7.5}$$

preserving small colimits in both variables and such that

$$- \wedge S \simeq \mathrm{id}_{\mathcal{C}}. \tag{7.6}$$

Furthermore, for every object C_0 in $\mathcal{C}$ we have an adjunction

$$C_0 \wedge - : \mathbf{Sp} \leftrightarrows \mathcal{C} : \mathrm{map}_{\mathcal{C}}(C_0, -). \tag{7.7}$$

The counit of the adjunction in (7.7) is a natural transformation

$$C_0 \wedge \mathrm{map}_{\mathcal{C}}(C_0, -) \to \mathrm{id}_{\mathcal{C}}(-) \tag{7.8}$$

of endofunctors of $\mathcal{C}$. □

Remark 7.4. Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be ∞-categories and $- \hat{\otimes} - : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ be a functor. We consider ∞-categories $\mathcal{I}$, $\mathcal{J}$ and natural transformations of functors $(F \xrightarrow{\alpha} F') : \mathcal{I} \to \mathcal{C}$ and $(G \xrightarrow{\beta} G') : \mathcal{J} \to \mathcal{D}$. Then we get a natural transformation of functors

$$(F \times G \xrightarrow{\alpha \times \beta} F' \times G') : \mathcal{I} \times \mathcal{J} \to \mathcal{C} \times \mathcal{D},$$

and by composition with $- \hat{\otimes} -$ a natural transformation

$$(F \hat{\otimes} G \xrightarrow{\alpha \hat{\otimes} \beta} F' \hat{\otimes} G') : \mathcal{I} \times \mathcal{J} \to \mathcal{E}, \tag{7.9}$$

where we write $F \hat{\otimes} G$ for $(- \hat{\otimes} -)\circ(F \times G)$. $\square$

Applying (7.5) to $\mathcal{C} = \mathrm{KK}$ we get a functor

$$(B,E) \mapsto B \wedge E : \mathrm{KK} \times \mathbf{Sp} \to \mathrm{KK}\,.$$

In the following, we specialize B to $\mathrm{kk}(\mathbb{C})$. We then have a functor $(A,E) \mapsto A \wedge E$ given as the composition

$$\mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}}) \times \mathbf{Sp} \xrightarrow{\mathrm{id}\times(\mathrm{kk}(\mathbb{C})\wedge -)} \mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}}) \times \mathrm{KK} \xrightarrow{-\hat{\otimes}-} \mathrm{KK}^G\,, \tag{7.10}$$

where $\hat{\otimes}$ is as in (6.11). Note that

$$A \wedge S \overset{(7.10)}{\simeq} A \hat{\otimes} (\mathrm{kk}(\mathbb{C}) \wedge S) \overset{(7.6)}{\simeq} A \hat{\otimes} \mathrm{kk}_{C^*\mathbf{Cat}}(\mathbb{C})$$
$$\overset{Lem.6.2}{\simeq} \mathrm{kk}^G(A \otimes \mathrm{Res}^G_{\{1\}}(\mathbb{C})) \simeq \mathrm{kk}^G(A)\,.$$

Since the functor $-\hat{\otimes}-$ in (6.11) preserves small colimits in its second variable, the functor in (7.10) is essentially uniquely determined by the equivalence $A \wedge S \simeq \mathrm{kk}^G(A)$ and the fact that it preserves small colimits in the second variable. Furthermore, by the adjunction (7.7) we have a natural equivalence

$$\mathrm{map}_{\mathbf{Sp}}(E, \mathrm{KK}^G(A,B)) \simeq \mathrm{KK}^G(A \wedge E, B) \tag{7.11}$$

for E in $\mathbf{Sp}$, A in $\mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}})$, and B in KK^G.

We consider the functor

$$F : G\mathbf{UBC}^{\mathrm{op}} \times \mathrm{KK} \xrightarrow{C_0(-)\times \mathrm{KK}(\mathbb{C},-)} \mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}}) \times \mathbf{Sp} \xrightarrow{-\wedge-,(7.10)} \mathrm{KK}^G \tag{7.12}$$

written as

$$(X,B) \mapsto C_0(X) \wedge \mathrm{KK}(\mathbb{C}, B)\,.$$

We further consider the functor

$$H : G\mathbf{UBC}^{\mathrm{op}} \times \mathrm{KK} \xrightarrow{C_0(-)\times \mathrm{id}(-)} \mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}}) \times \mathrm{KK} \xrightarrow{-\hat{\otimes}-} \mathrm{KK}^G$$

written as

$$(X,B) \mapsto C_0(X) \hat{\otimes} B\,.$$

We now construct the diagonal transformation

$$(F \xrightarrow{\tilde{\delta}} H) : G\mathbf{UBC}^{\mathrm{op}} \times \mathrm{KK} \to \mathrm{KK}^G\,. \tag{7.13}$$

Its specialization at X in $G\mathbf{UBC}$ and B in KK is a morphism

$$\tilde{\delta}_{X,B} : C_0(X) \wedge \mathrm{KK}(\mathbb{C}, B) \to C_0(X) \hat{\otimes} B \tag{7.14}$$

in KK^G. Inserting (7.10) into the definition (7.12) of F, we get

$$F = (- \hat{\otimes} -)\circ(C_0(-) \times \mathrm{kk}(\mathbb{C}) \wedge \mathrm{KK}(\mathbb{C}, -))\,.$$

We now obtain $\tilde{\delta}$ in (7.13) by specializing (7.9) to the transformations

$$(C_0(-) \xrightarrow{\mathrm{id}} C_0(-)) : G\mathbf{UBC}^{\mathrm{op}} \to \mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}})$$

and

$$(\mathrm{kk}(\mathbb{C}) \wedge \mathrm{KK}(\mathbb{C}, -) \to \mathrm{id}(-)) : \mathrm{KK} \to \mathrm{KK}$$

given by (7.8).

We define the functor

$$Q : G\mathbf{UBC} \to \mathrm{KK}, \quad X \mapsto Q(X) := \mathrm{kk}_{C^*\mathbf{Cat}}(\mathbf{Q}(X)), \tag{7.15}$$

see (6.5) for $\mathbf{Q}(X)$. Then we consider the functor

$$\mathbf{Tw}(G\mathbf{UBC})^{\mathrm{op}} \to G\mathbf{UBC}^{\mathrm{op}} \times \mathrm{KK}, \quad (X \to X') \mapsto (X', Q(X)). \tag{7.16}$$

The pull-back of $\tilde{\delta}$ in (7.13) along (7.16) yields a natural transformation

$$(\delta : C_0(-') \wedge \mathrm{KK}(\mathbb{C}, Q(-)) \to C_0(-') \hat{\otimes} Q(-)) : \mathbf{Tw}(G\mathbf{UBC})^{\mathrm{op}} \to \mathrm{KK}^G \tag{7.17}$$

whose evaluation at an object $f : X \to X'$ in $\mathbf{Tw}(G\mathbf{UBC})^{\mathrm{op}}$ is a morphism

$$\delta_f : C_0(X') \wedge \mathrm{KK}(\mathbb{C}, \mathbf{Q}(X)) \to C_0(X') \hat{\otimes} Q(X) \tag{7.18}$$

in KK^G. This is our version of the diagonal (6.13) as a natural transformation. In fact, under the canonical equivalence

$$\begin{aligned} &\mathrm{KK}^G(C_0(X) \wedge \mathrm{KK}(\mathbb{C}, \mathbf{Q}(X)), C_0(X) \hat{\otimes} Q(X)) \\ &\overset{(7.11)}{\simeq} \mathrm{map}(\mathrm{KK}(\mathbb{C}, \mathbf{Q}(X)), \mathrm{KK}^G(C_0(X), C_0(X) \otimes \mathbf{Q}(X)) \end{aligned} \tag{7.19}$$

the map δ_{id_X} in (7.18) corresponds to δ_X from (6.13).

We now construct the refinement (7.24) of the family of multiplication maps μ_X from (6.12) for all X in $G\mathbf{UBC}$. We start with the functor

$$\begin{aligned} \mathbf{Tw}(G\mathbf{UBC})^{\mathrm{op}} &\xrightarrow{C_0(-')\otimes\mathbf{Q}(-)} \mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}) \\ (X \to X') &\longmapsto C_0(X')\otimes\mathbf{Q}(X). \end{aligned}$$

We also consider $\mathbf{Q}^{(G)}_{\mathrm{std}}$ as a constant functor from $\mathbf{Tw}(G\mathbf{UBC}^{\mathrm{prop}})^{\mathrm{op}}$ to $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$. We first construct a natural transformation

$$(\tilde{\mu} : C_0(-') \otimes \mathbf{Q}(-) \to \mathbf{Q}^{(G)}_{\mathrm{std}}) : \mathbf{Tw}(G\mathbf{UBC})^{\mathrm{op}} \to \mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}). \tag{7.20}$$

For every object $f : X \to X'$ in $\mathbf{Tw}(G\mathbf{UBC})^{\mathrm{op}}$, we must define a functor

$$\tilde{\mu}_f : C_0(X') \otimes \mathbf{Q}(X) \to \mathbf{Q}^{(G)}_{\mathrm{std}}. \tag{7.21}$$

This construction extends the construction of μ_X in (6.12) which will be recovered as $\mu_X = \tilde{\mu}_{\mathrm{id}_X}$.

(1) objects: The functor $\tilde{\mu}_f$ sends the object (C, ρ, μ) in $C_0(X') \otimes \mathbf{Q}(X)$ (hence an object of $\mathbf{Q}(X)$) to the object (C, ρ) in $\mathbf{Q}^{(G)}_{\mathrm{std}}$.
(2) morphisms: If $[A] : (C', \rho', \mu') \to (C, \rho, \mu)$ is a morphism in $\mathbf{Q}(X)$ and h is in $C_0(X')$, then $\tilde{\mu}_f(h \otimes [A]) := [\phi(f^*h)A]$, see (5.7) for the definition of ϕ.

The argument that the functor $\tilde\mu_f$ is well-defined is the same as for μ_X. We now check that $\tilde\mu := (\tilde\mu_f)_{f\in\mathbf{Tw}(G\mathbf{UBC})^{\mathrm{op}}}$ is a natural transformation. We consider a morphism $f \to g$ in $\mathbf{Tw}(G\mathbf{UBC})^{\mathrm{op}}$, see (7.3). Since we work with the opposite of the twisted arrow category, it is given by a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"] & X' \\ Y \arrow[r, "g"] & Y' \arrow[u, "\beta"'] \end{tikzcd} \tag{7.22}$$

We must show that

$$\begin{tikzcd} C_0(X') \otimes \mathbf{Q}(X) \arrow[rr, "\beta^*\otimes\mathbf{Q}(\alpha)"] \arrow[dr, "\tilde\mu_f"'] & & C_0(Y') \otimes \mathbf{Q}(Y) \arrow[dl, "\tilde\mu_g"] \\ & \mathbf{Q}^{(G)}_{\mathrm{std}} & \end{tikzcd}$$

commutes.

(1) objects: Let (C,ρ,μ) be an object in $C_0(X') \otimes \mathbf{Q}(X)$. Then we have the equality

$$\tilde\mu_g((\beta^* \otimes \mathbf{Q}(\alpha))(C,\rho,\mu)) = \tilde\mu_g(C,\rho,\alpha_*\mu) = (C,\rho) = \tilde\mu_f(C,\rho,\mu)\,.$$

(2) morphisms: Let $[A] \colon (C',\rho',\mu') \to (C,\rho,\mu)$ be a morphism in $\mathbf{Q}(X)$ and h be in $C_0(X')$. Then we have the equality

$$\begin{aligned}\tilde\mu_g((\beta^* \otimes \mathbf{Q}(\alpha))(h \otimes [A])) = \tilde\mu_g(\beta^*h \otimes [\alpha_*A]) &= [\phi(g^*(\beta^*(h)))\alpha_*A] \\ &= [(\alpha_*\phi)(g^*\beta^*h)A]\,.\end{aligned}$$

On the other hand,

$$\tilde\mu_f(h \otimes [A]) = [\phi(f^*h)A]\,.$$

The desired equality

$$[\phi(f^*h)A] = [(\alpha_*\phi)(g^*\beta^*h)A]$$

now follows from the identity

$$(\alpha_*\phi)(g^*\beta^*h) = \phi(\alpha^*g^*\beta^*h) = \phi(f^*h)$$

since $\alpha^*g^*\beta^*h = f^*h$ by the commutativity of (7.22).

We post-compose the transformation in (7.20) with the functor $\mathrm{kk}^G_{C^*\mathbf{Cat}}$ and get a natural transformation

$$(\mathrm{kk}^G(\tilde\mu) \colon \mathrm{kk}^G_{C^*\mathbf{Cat}}(C_0(-') \otimes \mathbf{Q}(-)) \to Q^{(G)}_{\mathrm{std}}) \colon \mathbf{Tw}(G\mathbf{UBC})^{\mathrm{op}} \to \mathrm{KK}^G\,, \tag{7.23}$$

where we use the abbreviation

$$Q^{(G)}_{\mathrm{std}} := \mathrm{kk}^G_{C^*\mathbf{Cat}}(\mathbf{Q}^{(G)}_{\mathrm{std}})\,.$$

Composing the transformation (7.23) with the equivalence

$$C_0(-') \mathbin{\hat\otimes} Q(-) \simeq \mathrm{kk}^G_{C^*\mathbf{Cat}}(C_0(-') \otimes \mathbf{Q}(-))$$

given by Lemma 6.2 (see (7.15) for the notation $Q(-)$) we get a natural transformation

$$(\mu : C_0(-') \hat{\otimes} Q(-) \to \tilde{Q}^{(G)}_{\mathrm{std}}) : \mathbf{Tw}(G\mathbf{UBC})^{\mathrm{op}} \to \mathrm{KK}^G . \tag{7.24}$$

The composition of (7.17) and (7.24) then gives a natural transformation

$$(\mu\circ\delta : C_0(-') \wedge \mathrm{KK}(\mathbb{C}, Q(-)) \to C_0(-') \otimes Q(-) \to Q^{(G)}_{\mathrm{std}}) : \mathbf{Tw}(G\mathbf{UBC})^{\mathrm{op}} \to \mathrm{KK}^G$$

whose value at the object $f : X \to X'$ is the morphism

$$\mu_f \circ \delta_f : C_0(X') \wedge \mathrm{KK}(\mathbb{C}, Q(X)) \to C_0(X') \otimes Q(X) \to Q^{(G)}_{\mathrm{std}} .$$

Equivalently, by (7.4) and since the target functor is constant we can interpret this as a map of spectra

$$S \to \mathrm{KK}^G(\mathrm{colim}_{\mathbf{Tw}(G\mathbf{UBC})^{\mathrm{op}}} C_0(-') \wedge \mathrm{KK}(\mathbb{C}, Q(-)), Q^{(G)}_{\mathrm{std}}) . \tag{7.25}$$

Note that $\mathbf{Tw}(G\mathbf{UBC})^{\mathrm{op}}$ is small and the presentable category KK^G admits all small colimits. We now use the chain of canonical equivalences

$$\begin{aligned}
&\mathrm{KK}^G(\mathrm{colim}_{\mathbf{Tw}(G\mathbf{UBC})^{\mathrm{op}}} C_0(-') \wedge \mathrm{KK}(\mathbb{C}, Q(-)), Q^{(G)}_{\mathrm{std}}) \\
\simeq\ & \lim_{\mathbf{Tw}(G\mathbf{UBC})} \mathrm{KK}^G(C_0(-') \wedge \mathrm{KK}(\mathbb{C}, Q(-)), Q^{(G)}_{\mathrm{std}}) \\
\overset{(7.11)}{\simeq}\ & \lim_{\mathbf{Tw}(G\mathbf{UBC})} \mathrm{map}(\mathrm{KK}(\mathbb{C}, Q(-)), \mathrm{KK}^G(C_0(-'), Q^{(G)}_{\mathrm{std}})) \\
\overset{(7.4)}{\simeq}\ & \mathrm{nat}(\mathrm{KK}(\mathbb{C}, Q(-)), \mathrm{KK}^G(C_0(-), Q^{(G)}_{\mathrm{std}})) ,
\end{aligned}$$

where nat denotes the spectrum of natural transformations between functors from $G\mathbf{UBC}$ to $\mathbf{Sp}$. Therefore, (7.25) provides a map

$$S \to \mathrm{nat}(\mathrm{KK}(\mathbb{C}, Q(-)), \mathrm{KK}^G(C_0(-), Q^{(G)}_{\mathrm{std}})) .$$

This is the desired natural transformation

$$p : \mathrm{KK}(\mathbb{C}, Q(-)) \to \mathrm{KK}^G(C_0(-), Q^{(G)}_{\mathrm{std}}) \tag{7.26}$$

of functors from $G\mathbf{UBC}$ to $\mathbf{Sp}$. It follows from the identifications of δ_{id_X} with δ_X by (7.19) and of $\tilde{\mu}_{\mathrm{id}_X}$ with μ_X stated after (7.21) that the evaluation of p at X in $G\mathbf{UBC}$ is equivalent to the morphism p_X from (6.14).

Recall that we use the notation

$$\mathrm{KK}(\mathbb{C}, Q(X)) \simeq \mathrm{KK}(\mathbb{C}, \mathbf{Q}(X)) \simeq K^{G,\mathcal{X}}_{\mathbf{C}}(X) ,$$

and

$$\mathrm{KK}^G(C_0(X), Q^{(G)}_{\mathrm{std}}) \simeq \mathrm{KK}^G(C_0(X), \mathbf{Q}^{(G)}_{\mathrm{std}}) \simeq K^{G,\mathrm{An}}_{\mathbf{C}}(\iota^{\mathrm{top}}(X)) .$$

Therefore, (7.26) is the desired Paschke transformation

$$p : K^{G,\mathcal{X}}_{\mathbf{C}} \to K^{G,\mathrm{An}}_{\mathbf{C}} \circ \iota^{\mathrm{top}} .$$

By construction, we see that the Paschke transformation is natural in the coefficient category $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg,eadd},\omega\mathrm{add}})$. This finishes the proof of Theorem 1.5.1.

8. Reduction to G-orbits

In this section, we reduce the verification of the Assertions 1.5.2 and 1.5.3 to the case of G-orbits. A discrete G-uniform bornological coarse space is a G-set with the minimal coarse and bornological structures and the discrete uniform structure. An object Y of $G\mathbf{Set}$ can canonically be considered as a discrete object in $G\mathbf{UBC}$ which we will also denote by Y. Alternatively we may use the more informative, but lengthier notation $Y_{min,min,disc}$, where the first *min* indicates the minimal coarse structure, the second *min* the minimal bornology, and finally *disc* the discrete uniform structure. Note that the construction $Y \mapsto Y_{min,min,disc}$ is functorial only for maps between G-sets with finite fibres.

Let $\mathcal{F}$ denote a family of subgroups of G. We will be mainly interested in the family **Fin** of finite subgroups, but the following proposition is valid for any family $\mathcal{F}$. We let $G_{\mathcal{F}}\mathbf{Set}$ be the category of very small G-sets with stabilizers in $\mathcal{F}$.

Let X be in $G\mathbf{UBC}$. We assume that $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ is effectively additive and admits countable AV-sums and recall the Definition 6.3 of the Paschke morphism.

Proposition 8.1. *Assume:*

(1) *The Paschke morphism for S is an equivalence for every S in $G_{\mathcal{F}}\mathbf{Orb}$.*
(2) *X is homotopy equivalent to a G-finite G-simplicial complex with stabilizers in $\mathcal{F}$ and with structures induced by its spherical path metrics.*

Then the Paschke morphism for X is an equivalence.

Proof. We argue by induction on the dimension n of the G-simplicial complex in Assumption 8.1.2. In order to simplify the notation, we drop the functor ι^{top} from the notation if we apply $K^{G,\mathrm{An}}_{\mathbf{C}}$ to an object of $G\mathbf{UBC}$.

Assume that $n = 0$ and that K is in $G\mathbf{UBC}$ such that K is a 0-dimensional G-finite G-simplicial complex with stabilizers in $\mathcal{F}$. For every orbit S in $G\backslash K$, we consider the closed invariant partition $(S, K \setminus S)$ of K. Applying excision for the functors $K^{G,\mathcal{X}}_{\mathbf{C}}$ and $K^{G,\mathrm{An}}_{\mathbf{C}}$ we get the respective projections $q^{\mathcal{X}}_S : K^{G,\mathcal{X}}_{\mathbf{C}}(K) \to K^{G,\mathcal{X}}_{\mathbf{C}}(S)$ and $q^{\mathrm{An}}_S : K^{G,\mathrm{An}}_{\mathbf{C}}(\iota^{\mathrm{top}}(K)) \to K^{G,\mathrm{An}}_{\mathbf{C}}(\iota^{\mathrm{top}}(S))$ for all S in $G\backslash K$. We have a commutative square

$$\begin{tikzcd}
K^{G,\mathcal{X}}_{\mathbf{C}}(K) \arrow[r, "\oplus_S q^{\mathcal{X}}_S", "\simeq"'] \arrow[d, "p_K"] & \bigoplus_{S\in G\backslash K} K^{G,\mathcal{X}}_{\mathbf{C}}(S) \arrow[d, "\oplus_S p_S", "\simeq"'] \\
K^{G,\mathrm{An}}_{\mathbf{C}}(\iota^{\mathrm{top}}(K)) \arrow[r, "\oplus_S q^{\mathrm{An}}_S", "\simeq"'] & \bigoplus_{S\in G\backslash K} K^{G,\mathrm{An}}_{\mathbf{C}}(\iota^{\mathrm{top}}(S))
\end{tikzcd}$$

Since we assume that $G\backslash K$ is finite the horizontal morphisms are equivalences by excision. Furthermore, the right vertical morphism is an equivalence by Assumption 8.1.1. Consequently, the left vertical morphism is an equivalence.

Let n be in $\mathbb{N}$ and assume that we have shown that p_K is an equivalence provided K is G-finite G-simplicial complex of dimension n with stabilizers in

$\mathcal{F}$ and with structures induced by its spherical path metrics. Let then X be in $G\mathbf{UBC}$ and assume that there exists a homotopy equivalence $X \to K$. By the naturality of the Paschke transformation we can consider the commutative square

$$\begin{tikzcd} K^{G,\mathcal{X}}_{\mathbf{C}}(X) \arrow[r, "\simeq"] \arrow[d, "p_X"] & K^{G,\mathcal{X}}_{\mathbf{C}}(K) \arrow[d, "p_K", "\simeq"'] \\ K^{G,\mathrm{An}}_{\mathbf{C}}(\iota^{\mathrm{top}}(X)) \arrow[r, "\simeq"] & K^{G,\mathrm{An}}_{\mathbf{C}}(\iota^{\mathrm{top}}(K)) \end{tikzcd}$$

Since the functors and $K^{G,\mathrm{An}}_{\mathbf{C}}$ and $K^{G,\mathcal{X}}_{\mathbf{C}}$ are homotopy invariant by [12, Thm. 1.15] and Proposition 4.10, respectively, the horizontal morphisms are equivalences. By assumption the right vertical morphism in an equivalence, too. Consequently, the left vertical morphism is also an equivalence.

We now show the induction step. Assume that K in $G\mathbf{UBC}$ is such that K is a G-finite G-simplicial complex of dimension n with stabilizers in $\mathcal{F}$ with structures induced by its spherical path metrics. Let Y be the closed $1/2$-neighbourhood of the $(n-1)$-skeleton K_{n-1} of K and set $Z := K \setminus \mathrm{int}(Y)$. Then (Y, Z) is a closed decomposition of K.

We can consider Y, Z and $Y \cap Z$ as objects in $G\mathbf{UBC}$ with the induced structures. We then have the following commutative diagram

$$\begin{tikzcd} K^{G,\mathcal{X}}_{\mathbf{C}}(Y \cap Z) \arrow[rrr] \arrow[ddd] \arrow[dr, "p_{Y\cap Z}", "\simeq"'] & & & K^{G,\mathcal{X}}_{\mathbf{C}}(Z) \arrow[dl, "p_Z"', "\simeq"] \arrow[ddd] \\ & K^{G,\mathrm{An}}_{\mathbf{C}}(\iota^{\mathrm{top}}(Y \cap Z)) \arrow[r] \arrow[d] & K^{G,\mathrm{An}}_{\mathbf{C}}(\iota^{\mathrm{top}}(Z)) \arrow[d] & \\ & K^{G,\mathrm{An}}_{\mathbf{C}}(\iota^{\mathrm{top}}(Y)) \arrow[r] & K^{G,\mathrm{An}}_{\mathbf{C}}(\iota^{\mathrm{top}}(K)) & \\ K^{G,\mathcal{X}}_{\mathbf{C}}(Y) \arrow[rrr] \arrow[ur, "\simeq", "p_Y"'] & & & K^{G,\mathcal{X}}_{\mathbf{C}}(K) \arrow[ul, "p_{(K,\tau_K)}"] \end{tikzcd} \quad . \tag{8.1}$$

Since Y, Z and $Y \cap Z$ are homotopy equivalent in $G\mathbf{UBC}$ to G-finite G-simplicial complexes of dimension $< n$ with stabilizers in $\mathcal{F}$ their Paschke morphisms are equivalences by the induction hypothesis. Since the functors $K^{G,\mathrm{An}}_{\mathbf{C}} \circ \iota^{\mathrm{top}}$ and $K^{G,\mathcal{X}}_{\mathbf{C}}$ are excisive for this closed decomposition (for $K^{G,\mathrm{An}}_{\mathbf{C}}$ we use [12, Prop. 5.1.2]) the inner and the outer square are push-out squares. Alltogether we can then conclude that the Paschke morphism p_K is an equivalence, too. □

In order to prepare the proof of Theorem 1.5.3, we replace the Paschke morphism p in Proposition 8.1 by the locally finite version p^{lf} with target $K^{G,\mathrm{An,lf}}_{\mathbf{C}}$. In Assumption 8.1.1, we further replace $G_{\mathcal{F}}\mathbf{Orb}$ by $G_{\mathcal{F}}\mathbf{Set}$. Note that this is a stronger assumption. Let X be in $G\mathbf{UBC}$. The argument for Proposition 8.1 then also shows the following statement.

Proposition 8.2. *Assume:*

(1) *The Paschke morphism* $p_S^{\mathrm{lf}} : K_{\mathbf{C}}^{G,\mathcal{X}}(S) \to K_{\mathbf{C}}^{G,\mathrm{An,lf}}(\iota^{\mathrm{top}}(S))$ *is an equivalence for every countable* S *in* $G_{\mathcal{F}}\mathbf{Set}$.
(2) X *is homotopy equivalent to a countable, finite-dimensional* G*-simplicial complex with stabilizers in* $\mathcal{F}$ *and with structures induced by its spherical path metrics.*

Then the Paschke morphism $p_X^{\mathrm{lf}} : K_{\mathbf{C}}^{G,\mathcal{X}}(X) \to K_{\mathbf{C}}^{G,\mathrm{An,lf}}(\iota^{\mathrm{top}}(X))$ *is an equivalence.*

Proof. Using the stronger Assumption 8.2.1 instead of Assumption 8.1.1 one can redo the proof of Proposition 8.1 for p^{lf} avoiding the step where we decompose the zero-dimensional complex K into a finite union of G-orbits. □

In the following lemma, we show that Assumption 8.1.1 implies Assumption 8.2.1 provided G is finite and $\mathbf{C}$ admits all very small AV-sums.

Lemma 8.3. *We assume that* G *is finite and that* $\mathbf{C}$ *admits all very small orthogonal AV-sums. If the Paschke morphism* p_T *is an equivalence for every* T *in* $G\mathbf{Orb}$*, then the Paschke morphism* p_S^{lf} *is an equivalence for every countable* S *in* $G\mathbf{Set}$.

Proof. The functor $K_{\mathbf{C}}^{G,\mathrm{An,lf}} \circ \iota^{\mathrm{top}}$ sends countable disjoint unions into products. Hence, we have an equivalence

$$K_{\mathbf{C}}^{G,\mathrm{An,lf}}(S_{disc}) \simeq \prod_{T \in G\backslash S} K_{\mathbf{C}}^{G,\mathrm{An,lf}}(\iota^{\mathrm{top}}(T_{disc})) . \tag{8.2}$$

If G is finite, then we have an equality $G_{can,max} = G_{max,max}$. Recall the notion of the free union from [10, Ex. 2.16]. As in the proof of [10, Lem. 3.13], by exploiting the equality $G_{can,max} = G_{max,max}$, we have an isomorphism

$$\begin{aligned} S_{min,min} \otimes G_{can,max} &\cong \Big(\bigsqcup_{T \in G\backslash S}^{\mathrm{free}} T_{min,min} \Big) \otimes G_{can,max} \\ &\cong \bigsqcup_{T \in G\backslash S}^{\mathrm{free}} (T_{min,min} \otimes G_{can,max}) \end{aligned} \tag{8.3}$$

in $G\mathbf{BC}$. The additional assumption on $\mathbf{C}$ implies that $K\mathbf{C}\mathcal{X}^G$ is strongly additive by [7, Thm. 11.1], see also Theorem 3.5. It therefore sends free unions to products. Applying now $K\mathbf{C}\mathcal{X}^G$ to (8.3) and using Definition 4.9 we consequently have an equivalence

$$K_{\mathbf{C}}^{G,\mathcal{X}}(S_{min,min,disc}) \simeq \prod_{T \in G\backslash S} K_{\mathbf{C}}^{G,\mathrm{An,lf}}(\iota^{\mathrm{top}}(T_{min,min,disc})) \tag{8.4}$$

arising in the following way:

$$\begin{aligned}
K\mathbf{C}^{G,\mathcal{X}}(S_{min,min,disc}) &= K\mathbf{C}\mathcal{X}^G_{G_{can,max}}(\mathcal{O}^\infty(S_{min,min,disc})) \\
&\overset{!}{\simeq} \Sigma K\mathbf{C}\mathcal{X}^G_{G_{can,max}}(S_{min,min,disc}) \\
&\overset{(8.3)}{\simeq} \Sigma K\mathbf{C}\mathcal{X}^G\Big(\coprod_{T\in G\backslash S}^{\mathrm{free}}(T_{min,min}\otimes G_{can,max})\Big) \\
&\simeq \Sigma \prod_{T\in G\backslash S} K\mathbf{C}\mathcal{X}^G(T_{min,min}\otimes G_{can,max}) \\
&\overset{!}{\simeq} \prod_{T\in G\backslash S} K\mathbf{C}^{G,\mathcal{X}}(T_{min,min,disc}) \\
&\overset{\prod_T p_T}{\simeq} \prod_{T\in G\backslash S} K\mathbf{C}^{G,\mathrm{An}}(\iota^{\mathrm{top}}(T_{min,min,disc})) \\
&\simeq \prod_{T\in G\backslash S} K\mathbf{C}^{G,\mathrm{An,lf}}(\iota^{\mathrm{top}}(T_{min,min,disc}))\,.
\end{aligned}$$

Here we use [10, Prop. 9.35] for the equivalences marked by !. By naturality of the Paschke transformation, under the equivalences (8.2) and (8.4) the Paschke morphism p_S^{lf} corresponds to the product of the Paschke morphisms p_T for the G-orbits T in S. If the latter are equivalences, then p_S is an equivalence. □

At the moment we do not know whether this lemma generalizes to infinite groups, possibly with restrictions on allowed stabilizers.

Combining Proposition 8.2 with Lemma 8.3 we get the following result.

Corollary 8.4. *Assume:*

(1) *G is finite.*
(2) **C** *admits all very small AV-sums.*
(3) *The Paschke morphism p_T is an equivalence for every T in G***Orb**.
(4) *X is homotopy equivalent to a countable, finite-dimensional G-simplicial complex with structures induced by its spherical path metric.*

Then the Paschke morphism $p_X^{\mathrm{lf}}\colon K\mathbf{C}^{G,\mathcal{X}}(X)\to K\mathbf{C}^{G,\mathrm{An,lf}}(\iota^{\mathrm{top}}(X))$ is an equivalence.

Remark 8.5. We cannot expect that the Paschke morphism is an equivalence for spaces which are not proper G-spaces. More precisely, we do not expect that Assumption 8.1.1 is satisfied if $\mathcal{F}$ contains infinite subgroups.

Indeed, assume that $S = G/H$ with H infinite. Then we have

$$\begin{aligned} K^{G,\mathcal{X}}_{\mathbf{C}}((G/H)_{min,min,disc}) &\stackrel{\text{def.}}{\simeq} K\mathbf{C}\mathcal{X}^G_{G_{can,max}}(\mathcal{O}^\infty((G/H)_{min,min,disc}))) \\ &\stackrel{(1)}{\simeq} \Sigma K\mathbf{C}\mathcal{X}^G_{G_{can,max}}((G/H)_{min,min}) \\ &\stackrel{\text{def.}}{\simeq} \Sigma K\mathbf{C}\mathcal{X}^G((G/H)_{min,min} \otimes G_{can,max}) \\ &\stackrel{(2)}{\simeq} 0\,, \end{aligned} \tag{8.5}$$

where the equivalence (1) is an instance of [10, Prop. 9.35] since $(G/H)_{min,min,disc}$ is discrete. In order to see the equivalence (2), we use that the functor $K\mathbf{C}\mathcal{X}^G$ is continuous: We refer to [10, Def. 5.15] for the definition of this notion and to [7, Thm. 6.3] for the fact. Continuity implies that the value of $K\mathbf{C}\mathcal{X}^G(X)$ for any X in $G\mathbf{BC}$ is given as a colimit of the values $K\mathbf{C}\mathcal{X}^G(L)$ over the locally finite invariant subsets L of X. We now observe that if H is infinite, then $(G/H)_{min,min}\otimes G_{can,max}$ does not admit any non-empty invariant locally finite subset. Indeed, if L would be such a subset, then on the one hand $(eH \times G) \cap L$ is finite, but the infinite group H acts freely on this set on the other hand.

In contrast, the spectrum

$$K^{G,\mathrm{An}}_{\mathbf{C}}((G/H)_{disc}) \simeq \mathrm{KK}^G(C_0((G/H)_{disc}), \mathbf{Q}^{(G)}_{\mathrm{std}})$$

does not vanish in general. As an example we consider the case $G = H$, and we further specialize to $\mathbf{C} = \mathbf{Hilb}^G_c(A)$ for a unital G-C^*-algebra A. By Proposition 10.15 we have an equivalence

$$K^{G,\mathrm{An}}_{\mathbf{C}}((G/H)_{disc}) \simeq \Sigma\mathrm{KK}^G(\mathbb{C}, A)\,.$$

We claim that this spectrum is non-trivial if we take $A = \mathbb{C}$ with the trivial G-action. Indeed, in this case we have the class $\mathrm{id}_{\mathrm{kk}^G(\mathbb{C})}$ in $\mathrm{KK}^G_0(\mathbb{C},\mathbb{C})$ and $\mathrm{id}_{\mathrm{kk}^G(\mathbb{C})} \simeq 0$ if and only if $\mathrm{KK}^G(\mathbb{C},\mathbb{C}) \simeq 0$. Since $\mathrm{kk}^G(\mathbb{C})$ is the tensor unit of KK^G we have $\mathrm{KK}^G(\mathbb{C},\mathbb{C}) \simeq 0$ if and only if $\mathrm{KK}^G \simeq 0$. But since

$$\mathrm{KK}^G(C_0(G), \mathbb{C}) \simeq K^{C^*\mathrm{Alg}}(\mathbb{C}) \simeq KU$$

by [12, Thm. 1.23] this never happens. □

Consider Y in $G\mathbf{LCH}^{\mathrm{prop}}_+$. At various places we will use the following properties of this functor.

Lemma 8.6. *If Y is homotopy equivalent to a G-finite G-CW-complex with finite stabilizers, then:*

1. $\mathrm{KK}^G(C_0(Y), -)$ *sends exact sequences in* $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ *to fibre sequences.*
2. $\mathrm{KK}^G(C_0(Y), -)$ *annihilates flasque objects in* $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$.
3. $\mathrm{KK}^G(C_0(Y), -)$ *sends relative Morita equivalences to equivalences.*

Proof. By [12, Prop. 1.26] the object $\mathrm{kk}^G(C_0(Y))$ is G-proper and therefore ind-G-proper in the sense of [12, Def. 1.25]. The assertions now follow from [12, Thm. 1.32]. □

Let X be in $G\mathbf{UBC}$. Then we have the multiplication map (6.12)

$$\mu_X^{\mathbf{Q}} : C_0(X) \otimes \mathbf{Q}(X) \to \mathbf{Q}^{(G)}_{\mathrm{std}} .$$

We add a superscript $\mathbf{Q}$ since we are going to consider other versions of this map which will be distinguished by other choices for this superscript. The main ingredient in the verification that $\mu_X^{\mathbf{Q}}$ is well-defined was Lemma 5.8 saying that for a morphism $A : (C, \rho, \mu) \to (C', \rho', \mu')$ in $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(X) \otimes G_{can,max})$ we have $\phi'(f)A - A\phi(f) \in \mathbf{C}$ for all f in $C_0(X)$. If X is discrete, then we actually have $\phi'(f)A - A\phi(f) = 0$ for all such f. This has the effect that in the construction of μ_X in (6.12) on morphisms (see Item 2 in the list below (6.12)) we do not have to go to the quotients in order to ensure compatibility with the composition.

From now one we assume that X is discrete. Using the observation just made we can lift $\mu_X^{\mathbf{Q}}$ to a multiplication map

$$\mu_X^{\mathbf{D}} : C_0(X) \otimes \mathbf{D}(X) \to \mathbf{MC}^{(G)}_{\mathrm{std}} , \quad f \otimes A \mapsto fA ,$$

where $\mathbf{D}(X)$ is defined in (6.3). Using in addition Lemma 5.9 and the definition (6.4) of $\mathbf{C}(X)$ the map $\mu_X^{\mathbf{D}}$ restricts to a map $\mu_X^{\mathbf{C}}$ so that we get a morphism of exact sequences in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$

$$\begin{tikzcd} 0 \arrow[r] & C_0(X) \otimes \mathbf{C}(X) \arrow[r] \arrow[d, "\mu_X^{\mathbf{C}}"] & C_0(X) \otimes \mathbf{D}(X) \arrow[r] \arrow[d, "\mu_X^{\mathbf{D}}"] & C_0(X) \otimes \mathbf{Q}(X) \arrow[r] \arrow[d, "\mu_X^{\mathbf{Q}}"] & 0 \\ 0 \arrow[r] & \mathbf{C}^{(G)}_{\mathrm{std}} \arrow[r] & \mathbf{MC}^{(G)}_{\mathrm{std}} \arrow[r] & \mathbf{Q}^{(G)}_{\mathrm{std}} \arrow[r] & 0 \end{tikzcd} \tag{8.6}$$

Here in the upper line we used (6.5) and that $C_0(X) \otimes -$ (involving the maximal tensor product) preserves exact sequences of C^*-categories by [12, Prop. 7.23.1].

In the definition (6.13) of the diagonal morphism δ_X, we could replace $\mathbf{Q}(X)$ by $\mathbf{C}(X)$ or $\mathbf{D}(X)$. Using the obvious naturality of the construction of δ_X in this variable we get a commutative diagram

$$\begin{tikzcd} \mathrm{KK}(\mathbb{C}, \mathbf{C}(X)) \arrow[r] \arrow[d, "\delta_X^{\mathbf{C}}"] & \mathrm{KK}(\mathbb{C}, \mathbf{D}(X)) \arrow[r] \arrow[d, "\delta_X^{\mathbf{D}}"] & \mathrm{KK}(\mathbb{C}, \mathbf{Q}(X)) \arrow[d, "\delta_X^{\mathbf{Q}}"] \\ \mathrm{KK}^G(C_0(X), C_0(X) \otimes \mathbf{C}(X)) \arrow[r] & \mathrm{KK}^G(C_0(X), C_0(X) \otimes \mathbf{D}(X)) \arrow[r] & \mathrm{KK}^G(C_0(X), C_0(X) \otimes \mathbf{Q}(X)) \end{tikzcd} . \tag{8.7}$$

Recall that we assume that X is discrete. We now in addition assume that X is G-finite and has finite stabilizers. Using the exactness of the upper horizontal sequence in (8.6) and (6.5) we can conclude with Lemma 8.6.1 that the horizontal sequences are segments of fibre sequences. Applying $\mathrm{KK}^G(C_0(X), -)$ to (8.6) and composing the resulting morphism of fibre sequences with the morphism

(8.7) we get the morphism of fibre sequences

$$\begin{tikzcd} \mathrm{KK}(\mathbb{C},\mathbf{C}(X)) \arrow[r] \arrow[d, "p_X^{\mathbf{C}}"] & \mathrm{KK}(\mathbb{C},\mathbf{D}(X)) \arrow[r] \arrow[d, "p_X^{\mathbf{D}}"] & \mathrm{KK}(\mathbb{C},\mathbf{Q}(X)) \arrow[d, "p_X^{\mathbf{Q}}"] \\ \mathrm{KK}^G(C_0(X),\mathbf{C}^{(G)}_{\mathrm{std}}) \arrow[r] & \mathrm{KK}^G(C_0(X),\mathbf{MC}^{(G)}_{\mathrm{std}}) \arrow[r] & \mathrm{KK}^G(C_0(X),\mathbf{Q}^{(G)}_{\mathrm{std}}) \end{tikzcd} \tag{8.8}$$

where $p_X^{\mathbf{Q}}$ is the Paschke morphism (6.14).

For a family of subgroups $\mathcal{F}$, we denote by $G_{\mathcal{F}}\mathbf{Set}$ the full subcategory of $G\mathbf{Set}$ of G-sets with stabilizers in $\mathcal{F}$. Let Y be a discrete object of $G\mathbf{UBC}$.

Proposition 8.7.

(1) *We have* $\mathrm{KK}(\mathbb{C},\mathbf{D}(Y))\simeq 0$.

(2) *If Y is in $G_{\mathbf{Fin}}\mathbf{Set}$ and $G\backslash Y$ is finite, then* $\mathrm{KK}^G(C_0(Y),\mathbf{MC}^{(G)}_{\mathrm{std}})\simeq 0$.

Proof. We have the chain of equivalences:

$$\begin{aligned} \mathrm{KK}(\mathbb{C},\mathbf{D}(Y)) &\overset{(6.3)\ \&\ \text{Def. } 3.4}{\simeq} K\mathbf{C}\mathcal{X}^G_{G_{can,max}}(\mathcal{O}(Y)) \\ &\simeq 0 \end{aligned}$$

since the cone $\mathcal{O}(Y)$ of a discrete object in $G\mathbf{UBC}$ is a flasque object in $G\mathbf{BC}$ by [10, Ex. 9.25] and the coarse homology theory $K\mathbf{C}\mathcal{X}^G_{G_{can,max}}$ vanishes on flasques.

Since $\mathbf{MC}^{(G)}_{\mathrm{std}}$ is flasque by Lemma 2.21 we conclude Assertion 2 with Lemma 8.6.2. □

Using Proposition 8.7 and the morphism of fibre sequences (8.8) we get the following corollary.

Corollary 8.8. *If X is in $G_{\mathbf{Fin}}\mathbf{Orb}$, then we have a commutative square*

$$\begin{tikzcd} \Omega\mathrm{KK}(\mathbb{C},\mathbf{Q}(X)) \arrow[r, "\simeq"] \arrow[d, "\Omega p_X^{\mathbf{Q}}"] & \mathrm{KK}(\mathbb{C},\mathbf{C}(X)) \arrow[d, "p_X^{\mathbf{C}}"] \\ \Omega\mathrm{KK}^G(C_0(X),\mathbf{Q}^{(G)}_{\mathrm{std}}) \arrow[r, "\simeq"] & \mathrm{KK}^G(C_0(X),\mathbf{C}^{(G)}_{\mathrm{std}}) \end{tikzcd}$$

In particular, the Paschke morphism for X in $G_{\mathbf{Fin}}\mathbf{Orb}$ is an equivalence if and only if the morphism $p_X^{\mathbf{C}} := \mu_X^{\mathbf{C}}\circ\delta_X^{\mathbf{C}}$ is an equivalence.

In view of Corollary 8.8 and Proposition 8.1 and Corollary 8.4[2], the following proposition finishes the proof of the Theorems 1.5.2 and 1.5.3. We assume that $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ is effectively additive and admits countable AV-sums.

Proposition 8.9. *If X is in $G_{\mathbf{Fin}}\mathbf{Orb}$, then*

$$p_X^{\mathbf{C}}:\ \mathrm{KK}(\mathbb{C},\mathbf{C}(X))\to\mathrm{KK}^G(C_0(X),\mathbf{C}^{(G)}_{\mathrm{std}}) \tag{8.9}$$

is an equivalence.

The whole of Section 9 is devoted to the proof of this proposition.

[2]This corollary is needed only for Theorem 1.5.3.

9. Verification of the Paschke equivalence on G-orbits

We assume that $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ is effectively additive and admits countable AV-sums. We fix a finite subgroup H of G and consider the G-set G/H in $G_{\mathbf{Fin}}\mathbf{Orb}$. As a first step we construct an explicit functor Θ in $C^*\mathbf{Cat}^{\mathrm{nu}}$ and show in Proposition 9.3 that $p^{\mathbf{C}}_{G/H}$ is an equivalence if and only if $K^{C^*\mathrm{Cat}}(\Theta)$ is an equivalence. In the second step, we then verify in Proposition 9.6 that $K^{C^*\mathrm{Cat}}(\Theta)$ is an equivalence.

We form the G-bornological coarse space $(G/H)_{min,min} \otimes G_{can,max}$. It contains the locally finite subset

$$X := G(H,e)\,, \tag{9.1}$$

the G-orbit of the point (H,e) in $G/H \times G$. Note that in contrast to the example in Remark 8.5 the group H is finite. We equip X with the bornological coarse structures induced from $(G/H)_{min,min} \otimes G_{can,max}$. The map $g \mapsto g(H,e)$ is a G-equivariant bijection of sets between G and X which will be used below to name points and subsets of X. The induced bornology on X is the minimal one. The induced G-coarse structure reflects the information about the finite subgroup H and is in general smaller than the canonical coarse structure on G. For instance, the subset H is a coarse component of X.

The following lemma states that the inclusion $X \to (G/H)_{min,min} \otimes G_{can,max}$ is a continuous equivalence in the sense of [11, Sec. 7].

Lemma 9.1. *The inclusion $X \to (G/H)_{min,min} \otimes G_{can,max}$ induces an equivalence $E(X) \to E((G/H)_{min,min} \otimes G_{can,max})$ for any continuous equivariant coarse homology theory E.*

Proof. For Y in $G\mathbf{BC}$, we let $\mathrm{LF}(Y)$ denote the poset of G-invariant locally finite subsets. Let L be in $\mathrm{LF}((G/H)_{min,min} \otimes G_{can,max})$. Then $L_0 := L \cap (\{H\} \times G)$ is a finite set which we will sometimes consider as a subset of G. Since every G-orbit in L meets L_0 we have $L = GL_0$.

We claim that for every L in $\mathrm{LF}((G/H)_{min,min} \otimes G_{can,max})$ the inclusion $i : X \to L \cup X$ is a coarse equivalence. Indeed, we can construct an inverse equivalence $p : L \cup X \to X$. The map p is the identity on X, and it sends a point $g(H,h)$ (with h in $L_0 \setminus \{e\}$) in $L \setminus X$ to $g(H,e)$ in X. Then $p \circ i = \mathrm{id}_X$ and $i \circ p$ is close to the identity. In order to see the second assertion, note that L_0 is finite and therefore $\mathrm{diag}(G/H) \times \{(gh,g) \mid h \in L_0, g \in G\}$ is a coarse entourage of $(G/H)_{min,min} \otimes G_{can,max}$. We then use that

$$(\mathrm{id}_X, i \circ p)(\mathrm{diag}(L \cup X)) \subseteq \mathrm{diag}(G/H) \times \{(gh,g) \mid h \in L_0, g \in G\}\,.$$

If E is any equivariant coarse homology theory, then the canonical morphism

$$E(X) \to \mathrm{colim}_{L \in \mathrm{LF}((G/H)_{min,min} \otimes G_{can,max})} E(L)$$

is an equivalence since the elements of $\mathrm{LF}((G/H)_{min,min} \otimes G_{can,max})$ containing X are cofinal and for those elements the inclusions $X \to L$ are coarse equivalences. Since we assume in addition that E is continuous, the canonical morphism

$$\operatorname{colim}_{L\in \mathrm{LF}((G/H)_{min,min}\otimes G_{can,max})} E(L) \to E((G/H)_{min,min}\otimes G_{can,max})$$

is an equivalence. Hence, the composition of these equivalences is an equivalence

$$E(X) \to E((G/H)_{min,min}\otimes G_{can,max})\,. \qquad \square$$

Using the inclusion

$$i\colon X \to G/H \times G \to Z_0 \tag{9.2}$$

(see (5.8) for the notation Z_0 as a subspace of $\mathcal{O}((G/H)_{min,min})\otimes G_{can,max})$

$$i_*\colon \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X) \to \mathbf{C}(G/H) \tag{9.3}$$

(where we use (6.4) for $\mathbf{C}(G/H) := \mathbf{C}((G/H)_{min,min,disc}))$ we get an inclusion which identifies $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)$ with the full subcategory of objects of $\mathbf{C}(G/H)$ supported on $i(X)$.

In the following, $\mathrm{Idem}(\mathrm{Res}^G_H(\mathbf{C}^{(G)}_{\mathrm{std}})\rtimes H)$ is the relative idempotent completion using the embedding of $\mathrm{Res}^G_H(\mathbf{C}^{(G)}_{\mathrm{std}})\rtimes H$ as an ideal into $\mathrm{Res}^G_H(\mathbf{MC}^{(G)}_{\mathrm{std}})\rtimes H$, [9, Def. 17.5]. In order to keep the notation readable[3], in contrast to the reference we will not indicate the bigger unital category by a superscript. Recall the notation for morphisms in crossed products from [3, Def. 5.1]. In the formulas below, e.g., in order to interpret the term $\mu(H)$ in (9.5), we use the bijection between G and X mentioned above.

Definition 9.2. *We define the functor*

$$\Theta\colon \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X) \to \mathrm{Idem}(\mathrm{Res}^G_H(\mathbf{C}^{(G)}_{\mathrm{std}})\rtimes H) \tag{9.4}$$

as follows:

(1) *objects: Θ sends the object (C,ρ,μ) in $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)$ to the object (C,ρ,π) in the category $\mathrm{Idem}(\mathrm{Res}^G_H(\mathbf{C}^{(G)}_{\mathrm{std}})\rtimes H)$, where the orthogonal projection π on (C,ρ) is given by*

$$\pi := \frac{1}{|H|}\sum_{h\in H}(\mu(H),h)\,. \tag{9.5}$$

(2) *morphisms: Θ sends $A\colon (C,\rho,\mu)\to(C',\rho',\mu')$ in $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)$ to the morphism*

$$\pi'(A,e)\pi\colon (C,\rho,\pi)\to(C',\rho',\pi')$$

in $\mathrm{Idem}(\mathrm{Res}^G_H(\mathbf{C}^{(G)}_{\mathrm{std}})\rtimes H)$.

Note that $A\colon C\to C'$ belongs to $\mathbf{MC}$, but since H is a finite and hence bounded subset of X, the projection $\mu(H)$ belongs to $\mathbf{C}$ by the local finiteness of (C,ρ,μ). Therefore, $\pi'(A,e)\pi$ belongs to the ideal $\mathrm{Idem}(\mathrm{Res}^G_H(\mathbf{C}^{(G)}_{\mathrm{std}})\rtimes H)$ as stated. In order to see that Θ is compatible with the composition, note that the

[3] i.e., to avoid symbols like $\mathrm{Idem}^{\mathrm{Res}^G_H(\mathbf{MC}^{(G)}_{\mathrm{std}})\rtimes H)}(\mathrm{Res}^G_H(\mathbf{C}^{(G)}_{\mathrm{std}})\rtimes H)$

relations $A\mu(H) = \mu'(H)A$ (since H is a coarse component of X) and $h \cdot A = A$ for all h in H imply that $(A, e)\pi = \pi'(A, e)$.

Proposition 9.3. *The morphism* $p^{\mathbf{C}}_{G/H}$ *in* (8.9) *is an equivalence if and only if the morphism* $K^{C^*\mathrm{Cat}}(\Theta)$ *is an equivalence, where* Θ *is as in Definition* 9.2.

Proof. Recall that we consider G/H as the object $G/H_{min,min,disc}$ of $G\mathbf{UBC}$ so that $C_0(G/H)$ is given by Definition 5.6.2. In analogy to the diagonal morphism (6.13), we define

$$\delta' : \mathrm{KK}(\mathbb{C}, \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) \xrightarrow{C_0(G/H)\otimes -} \mathrm{KK}^G(C_0(G/H), C_0(G/H) \otimes \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) .$$

We then have a commutative diagram

$$\begin{tikzcd}
\mathrm{KK}(\mathbb{C}, \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) \arrow[r, "\mathrm{KK}(\mathbb{C}, i_*)"] \arrow[d, "\delta'"] \arrow[dd, bend right=60, "p'"'] & KK(\mathbb{C}, \mathbf{C}(G/H)) \arrow[d, "\delta^{\mathbf{C}}_{G/H}"] \arrow[dd, bend left=60, "p^{\mathbf{C}}_{G/H}"] \\
\mathrm{KK}^G(C_0(G/H), C_0(G/H) \otimes \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) \arrow[r, "C_0(G/H)\otimes i_*"] \arrow[d, "\mu'"] & \mathrm{KK}^G(C_0(G/H), C_0(G/H) \otimes \mathbf{C}(G/H)) \arrow[d, "\mu^{\mathbf{C}}_{G/H}"] \\
\mathrm{KK}^G(C_0(G/H), \mathbf{C}^{(G)}_{\mathrm{std}}) \arrow[r, equal] & \mathrm{KK}^G(C_0(G/K), \mathbf{C}^{(G)}_{\mathrm{std}})
\end{tikzcd} , \tag{9.6}$$

where $\mu' := \mu^{\mathbf{C}}_{G/H} \circ (C_0(G/H) \otimes i_*)$ and i_* is as in (9.3). The filler of the upper square is induced from the fact that (6.11) is a bifunctor. Implicitly we also used the Lemma 6.2 in order to relate $\hat{\otimes}$ and $\otimes$.

Lemma 9.4. *The morphism* $\mathrm{KK}(\mathbb{C}, i_*) : \mathrm{KK}(\mathbb{C}, \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) \to \mathrm{KK}(\mathbb{C}, \mathbf{C}(G/H))$ *is an equivalence.*

Proof. Using the definitions

$$K\mathbf{C}\mathcal{X}^G(-) := K^{C^*\mathrm{Cat}}(\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(-)) \quad \text{and} \quad K^{C^*\mathrm{Cat}}(-) := \mathrm{KK}(\mathbb{C}, -)$$

and (9.2) we can rewrite the morphism in question as

$$K\mathbf{C}\mathcal{X}^G(X) \to K\mathbf{C}\mathcal{X}^G((G/H)_{min,min} \otimes G_{can,max}) \to K^{C^*\mathrm{Cat}}(\mathbf{C}(G/H)) , \tag{9.7}$$

where the morphisms are induced by the canonical inclusions of C^*-categories. We have seen in the proof of Proposition 6.1 that the second morphism in (9.7) (it is an instance of the left vertical morphism in (6.8) applied to $(G/H)_{min,min,disc}$ in place of X) is an equivalence. The first morphism in (9.7) is induced by the inclusion $X \to (G/H)_{min,min} \otimes G_{can,max}$. Since $K\mathbf{C}\mathcal{X}^G$ is a continuous equivariant coarse homology theory it is an equivalence by Lemma 9.1. □

We continue with the proof of Proposition 9.3. We define $p' := \mu' \circ \delta'$. In view of (9.6) and Lemma 9.4, we conclude that

$$p' \simeq p^{\mathbf{C}}_{G/H} . \tag{9.8}$$

We consider the morphism $\epsilon : \mathbb{C} \to \mathbb{C} \rtimes H$ which sends 1 to the projection $\frac{1}{|H|} \sum_{h \in H} (1, h)$. Let furthermore $\iota : \mathbb{C} \to \mathrm{Res}^G_H(C_0(G/H))$ be the homomorphism sending z in $\mathbb{C}$ to $z\chi_H$, where χ_H is the characteristic function of the orbit

H in G/H. We then have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{KK}^G(C_0(G/H), C_0(G/H) \otimes \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) \arrow[r, "\mu'"] \arrow[d] \arrow[dd, bend right=60, "r^G_H"', "\simeq"] & \mathrm{KK}^G(C_0(G/H), \mathbf{C}^{(G)}_{\mathrm{std}}) \arrow[d] \arrow[dd, bend left=60, "r^G_H", "\simeq"'] \\
\mathrm{KK}^H(\mathrm{Res}^G_H(C_0(G/H)), \mathrm{Res}^G_H(C_0(G/H) \otimes \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X))) \arrow[r, "\mathrm{Res}^G_H(\mu')"] \arrow[d, "\iota^*"] & \mathrm{KK}^H(\mathrm{Res}^G_H(C_0(G/H)), \mathrm{Res}^G_H(\mathbf{C}^{(G)}_{\mathrm{std}})) \arrow[d, "\iota^*"] \\
\mathrm{KK}^H(\mathbb{C}, \mathrm{Res}^G_H(C_0(G/H) \otimes \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X))) \arrow[r, "\mathrm{Res}^G_H(\mu')"] \arrow[d, "- \rtimes H"] \arrow[dd, bend right=60, "j^H"', "\simeq"] & \mathrm{KK}^H(\mathbb{C}, \mathrm{Res}^G_H(\mathbf{C}^{(G)}_{\mathrm{std}})) \arrow[d, "- \rtimes H"] \arrow[dd, bend left=60, "j^H", "\simeq"'] \\
\mathrm{KK}(\mathbb{C} \rtimes H, (\mathrm{Res}^G_H(C_0(G/H) \otimes \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X))) \rtimes H) \arrow[r, "\mathrm{Res}^G_H(\mu') \rtimes H"] \arrow[d, "\epsilon^*"] & \mathrm{KK}(\mathbb{C} \rtimes H, \mathrm{Res}^G_H(\mathbf{C}^{(G)}_{\mathrm{std}}) \rtimes H) \arrow[d, "\epsilon^*"] \\
\mathrm{KK}(\mathbb{C}, (\mathrm{Res}^G_H(C_0(G/H) \otimes \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X))) \rtimes H) \arrow[r, "\mathrm{Res}^G_H(\mu') \rtimes H"] & \mathrm{KK}(\mathbb{C}, \mathrm{Res}^G_H(\mathbf{C}^{(G)}_{\mathrm{std}}) \rtimes H)
\end{tikzcd} . \tag{9.9}$$

The second and the last middle square commute by the associativity of the composition in KK^H and KK, respectively. The first and the third square commute since Res^G_H and $- \rtimes H$ are functors. In order to see that r^G_H and j^H are equivalences, we observe that ι and ϵ are instances of the units of the adjunctions in [12, Thm. 1.23.1 & 2] (induction and restriction ($\mathrm{Ind}^G_H \dashv \mathrm{Res}^G_H$) and the Green-Julg adjunction ($\mathrm{Res}^H \dashv - \rtimes H$)) and that r^G_H and j^H are precisely the corresponding equivalences of mapping spectra.

We furthermore have the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}})}(C_0(G/H), C_0(G/H)) \times \mathrm{KK}(\mathbb{C}, \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) \arrow[r, "\hat{\otimes}"] \arrow[d] & \mathrm{KK}^G(C_0(G/H), C_0(G/H) \otimes \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) \arrow[d] \arrow[dd, bend left=60, "r^G_H", "\simeq"'] \\
\mathrm{Hom}_{\mathrm{Fun}(BH, C^*\mathbf{Alg}^{\mathrm{nu}})}(\mathrm{Res}^G_H C_0(G/H), \mathrm{Res}^G_H C_0(G/H)) \times \mathrm{KK}(\mathbb{C}, \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) \arrow[r, "\hat{\otimes}"] \arrow[d, "\iota^* \times \mathrm{id}"] & \mathrm{KK}^H(\mathrm{Res}^G_H(C_0(G/H)), \mathrm{Res}^G_H(C_0(G/H)) \otimes \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) \arrow[d, "\iota^*"] \\
\mathrm{Hom}_{\mathrm{Fun}(BH, C^*\mathbf{Alg}^{\mathrm{nu}})}(\mathbb{C}, \mathrm{Res}^G_H C_0(G/H)) \times \mathrm{KK}(\mathbb{C}, \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) \arrow[r, "\hat{\otimes}"] \arrow[d, "(- \rtimes H) \times \mathrm{id}"] & \mathrm{KK}^H(\mathbb{C}, \mathrm{Res}^G_H(C_0(G/H)) \otimes \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) \arrow[d, "- \rtimes H"] \arrow[dd, bend left=60, "j^H", "\simeq"'] \\
\mathrm{Hom}_{C^*\mathbf{Alg}^{\mathrm{nu}}}(\mathbb{C} \rtimes H, \mathrm{Res}^G_H C_0(G/H) \rtimes H) \times \mathrm{KK}(\mathbb{C}, \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) \arrow[r, "\hat{\otimes}"] \arrow[d, "\epsilon^* \times \mathrm{id}"] & \mathrm{KK}(\mathbb{C} \rtimes H, (\mathrm{Res}^G_H(C_0(G/H)) \otimes \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) \rtimes H) \arrow[d, "\epsilon^*"] \\
\mathrm{Hom}_{C^*\mathbf{Alg}^{\mathrm{nu}}}(\mathbb{C}, \mathrm{Res}^G_H C_0(G/H) \rtimes H) \times \mathrm{KK}(\mathbb{C}, \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) \arrow[r, "\hat{\otimes}"] & \mathrm{KK}(\mathbb{C}, (\mathrm{Res}^G_H(C_0(G/H)) \otimes \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) \rtimes H)
\end{tikzcd} \tag{9.10}$$

In the targets of the two lower maps, we implicitly used the identification

$$(A \rtimes H) \otimes \mathbf{B} \cong (A \otimes \mathbf{B}) \rtimes H \tag{9.11}$$

for A in $\mathrm{Fun}(BH, C^*\mathbf{Alg}^{\mathrm{nu}})$ and $\mathbf{B}$ in $C^*\mathbf{Cat}^{\mathrm{nu}}$. The second and the last square commute since $\hat{\otimes}$ in (6.11) is a bifunctor. We now provide the fillers for the first and the third square. We consider the diagram

$$\begin{tikzcd}
\mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}}) \times C^*\mathbf{Alg}^{\mathrm{nu}} \arrow[r, "\otimes"] \arrow[d, "\mathrm{Res}^G_H \times \mathrm{id}"] & \mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}}) \arrow[r, "\mathrm{kk}^G"] \arrow[d, "\mathrm{Res}^G_H"] & \mathrm{KK}^G \arrow[d, "\mathrm{Res}^G_H"] \\
\mathrm{Fun}(BH, C^*\mathbf{Alg}^{\mathrm{nu}}) \times C^*\mathbf{Alg}^{\mathrm{nu}} \arrow[r, "\otimes"] & \mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}}) \arrow[r, "\mathrm{kk}^H"] & \mathrm{KK}^H
\end{tikzcd}$$

The left cell obviously commutes, and the right cell commutes by [12, Thm. 1.22]. We now extend using the universal property of $\mathrm{kk} : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathrm{KK}$ [12, Thm. 1.19] in order to get a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}}) \times \mathrm{KK} \arrow[r, "\hat{\otimes}"] \arrow[d, "\mathrm{Res}^G_H \times \mathrm{id}"] & \mathrm{KK}^G \arrow[d, "\mathrm{Res}^G_H"] \\
\mathrm{Fun}(BH, C^*\mathbf{Alg}^{\mathrm{nu}}) \times \mathrm{KK} \arrow[r, "\hat{\otimes}"] & \mathrm{KK}^H
\end{tikzcd}$$

This applied to morphism spaces yields the filler of the first middle square in (9.10). In order to justify the third middle square, we argue similarly. We consider the diagram

$$\begin{tikzcd}
\mathrm{Fun}(BH, C^*\mathbf{Alg}^{\mathrm{nu}}) \times C^*\mathbf{Alg}^{\mathrm{nu}} \arrow[r, "\otimes"] \arrow[d, "- \rtimes H \times \mathrm{id}"] & \mathrm{Fun}(BH, C^*\mathbf{Alg}^{\mathrm{nu}}) \arrow[r, "\mathrm{kk}^G"] \arrow[d, "- \rtimes H"] & \mathrm{KK}^H \arrow[d, "- \rtimes H"] \\
\mathrm{Fun}(BH, C^*\mathbf{Alg}^{\mathrm{nu}}) \times C^*\mathbf{Alg}^{\mathrm{nu}} \arrow[r, "\otimes"] & \mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}}) \arrow[r, "\mathrm{kk}"] & \mathrm{KK}
\end{tikzcd}$$

The left square commutes because of (9.11), and the right cell commutes by [12, Thm. 1.22]. We now extend using the universal property of $\mathrm{kk} : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathrm{KK}$ in [12, Thm. 1.19] in order to get a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}(BH, C^*\mathbf{Alg}^{\mathrm{nu}}) \times \mathrm{KK} \arrow[r, "\hat{\otimes}"] \arrow[d, "- \rtimes H \times \mathrm{id}"] & \mathrm{KK} \arrow[d, "- \rtimes H"] \\
\mathrm{Fun}(BH, C^*\mathbf{Alg}^{\mathrm{nu}}) \times \mathrm{KK} \arrow[r, "\hat{\otimes}"] & \mathrm{KK}
\end{tikzcd}$$

This square yields the of the third middle square in (9.10).

We specialize the diagram (9.10) at $\mathrm{id}_{C_0(G/H)}$ in

$$\mathrm{Hom}_{\mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}})}(C_0(G/H), C_0(G/H)) .$$

Then we get

$$
\begin{tikzcd}
\mathrm{KK}(\mathbb{C}, \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) \arrow[r, "\delta'"] \arrow[d, equal] & \mathrm{KK}^G(C_0(G/H), C_0(G/H) \otimes \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) \arrow[d] \arrow[dd, bend left=60, "r^G_H"', "\simeq"] \\
\mathrm{KK}(\mathbb{C}, \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) \arrow[r, "\mathrm{id}_{\mathrm{Res}^G_H C_0(G/H)}\hat{\otimes}"] \arrow[d, equal] & \mathrm{KK}^H(\mathrm{Res}^G_H(C_0(G/H)), \mathrm{Res}^G_H(C_0(G/H)) \otimes \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) \arrow[d, "\iota^*"] \\
\mathrm{KK}(\mathbb{C}, \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) \arrow[r, "\iota\hat{\otimes}"] \arrow[d, equal] & \mathrm{KK}^H(\mathbb{C}, \mathrm{Res}^G_H(C_0(G/H)) \otimes \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) \arrow[d, "-\rtimes H"] \arrow[dd, bend left=60, "j^H"', "\simeq"] \\
\mathrm{KK}(\mathbb{C}, \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) \arrow[r, "(\iota\rtimes H)\hat{\otimes}"] \arrow[d, equal] & \mathrm{KK}(C^*(H), (\mathrm{Res}^G_H(C_0(G/H)) \otimes \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) \rtimes H) \arrow[d, "\epsilon^*"] \\
\mathrm{KK}(\mathbb{C}, \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) \arrow[r, "\delta''"] & \mathrm{KK}(\mathbb{C}, (\mathrm{Res}^G_H(C_0(G/H) \otimes \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) \rtimes H)
\end{tikzcd}
,
\tag{9.12}
$$

where

$$
\begin{aligned}
\delta'' := \epsilon^*(\iota \rtimes H)\hat{\otimes}- :\ & \mathrm{KK}(\mathbb{C}, \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) \\
& \to \mathrm{KK}(\mathbb{C}, (\mathrm{Res}^G_H(C_0(G/H)) \rtimes H) \otimes \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X))) \\
& \simeq \mathrm{KK}(\mathbb{C}, (\mathrm{Res}^G_H(C_0(G/H)) \otimes \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) \rtimes H)\,.
\end{aligned}
$$

Composing (9.12) with (9.9) we get a commutative square

$$
\begin{tikzcd}
\mathrm{KK}(\mathbb{C}, \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) \arrow[r, "p'=\mu'\circ\delta'"] \arrow[d, equal] & \mathrm{KK}^G(C_0(G/H), \mathbf{C}^{(G)}_{\mathrm{std}}) \arrow[d, "j^H\circ r^G_H", "\simeq"'] \\
\mathrm{KK}(\mathbb{C}, \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) \arrow[r, "p'':=\mathrm{Res}^G_H(\mu')\rtimes H\circ\delta''"] & \mathrm{KK}(\mathbb{C}, \mathrm{Res}^G_H(\mathbf{C}^{(G)}_{\mathrm{std}}) \rtimes H)
\end{tikzcd}
$$

We therefore have an equivalence

$$
p'' \simeq p' \overset{(9.8)}{\simeq} p^{\mathbf{C}}_{G/H}\,. \tag{9.13}
$$

By construction the morphism p'' is induced by an explicit functor

$$
\Theta' :\ \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X) \to \mathrm{Res}^G_H(\mathbf{C}^{(G)}_{\mathrm{std}}) \rtimes H\,. \tag{9.14}
$$

Inserting all definitions we see that Θ' is given by follows:

(1) objects: Θ' sends the object (C,ρ,μ) in $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)$ to (C,ρ) in $\mathrm{Res}^G_H(\mathbf{C}^{(G)}_{\mathrm{std}})\rtimes H$.

(2) morphisms: The functor Θ' sends a morphism $A :\ (C,\rho,\mu) \to (C',\rho',\mu')$ in $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)$ to the morphism

$$
\pi' A \pi :\ (C,\rho) \to (C',\rho')
$$

in $\mathrm{Res}^G_H(\mathbf{C}^{(G)}_{\mathrm{std}}) \rtimes H$, where π is as in (9.5).

The observations made after the Definition 9.2 of Θ also show that Θ' is well-defined. Note, however, that Θ' is not full.

Let

$$c: \mathrm{Res}^G_H(\mathbf{C}^{(G)}_{\mathrm{std}}) \rtimes H \to \mathrm{Idem}(\mathrm{Res}^G_H(\mathbf{C}^{(G)}_{\mathrm{std}}) \rtimes H)$$

be the inclusion into the relative idempotent completion. We consider the two functors

$$\Theta, c\circ\Theta': \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X) \to \mathrm{Idem}(\mathrm{Res}^G_H(\mathbf{C}^{(G)}_{\mathrm{std}}) \rtimes H)$$

in $C^*\mathbf{Cat}^{\mathrm{nu}}$.

Recall the notion of a Murray von Neumann (MvN) equivalence [9, Def. 17.12].

Lemma 9.5. *There is a MvN equivalence $\Theta \to c\circ\Theta'$. In particular,*

$$K^{C^*\mathrm{Cat}}(\Theta) \simeq K^{C^*\mathrm{Cat}}(c\circ\Theta'): K^{C^*\mathrm{Cat}}(\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)) \to K^{C^*\mathrm{Cat}}(\mathrm{Idem}(\mathrm{Res}^G_H(\mathbf{C}^{(G)}_{\mathrm{std}}) \rtimes H)). \tag{9.15}$$

Proof. Applying [9, Rem. 17.13] to the inclusion of $\mathrm{Idem}(\mathrm{Res}^G_H(\mathbf{C}^{(G)}_{\mathrm{std}}) \rtimes H)$ as an ideal into $\mathrm{Idem}(\mathrm{Res}^G_H(\mathbf{MC}^{(G)}_{\mathrm{std}}) \rtimes H)$ it suffices to construct a natural transformation $v: \Theta \to c\circ\Theta'$ implemented by a family $(v_{(C,\rho,\mu)})_{(C,\rho,\mu)\in\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)}$ of partial isometries in $\mathrm{Idem}(\mathrm{Res}^G_H(\mathbf{MC}^{(G)}_{\mathrm{std}}) \rtimes H)$.

We define $v_{(C,\rho,\mu)}: (C,\rho,p) \to (C,\rho)$ to be the canonical inclusion. Since the formulas for the actions of Θ and Θ' on morphisms are equal, this family is indeed a natural transformation. □

We continue with the proof of Proposition 9.3. Since the homological functor $K^{C^*\mathrm{Cat}}$ is Morita invariant by [9, Thm. 16.18] the morphism

$$K^{C^*\mathrm{Cat}}(c): K^{C^*\mathrm{Cat}}(\mathrm{Res}^G_H(\mathbf{C}^{(G)}_{\mathrm{std}}) \rtimes H) \to K^{C^*\mathrm{Cat}}(\mathrm{Idem}(\mathrm{Res}^G_H(\mathbf{C}^{(G)}_{\mathrm{std}}) \rtimes H))$$

is an equivalence by [9, Prop. 17.8]. Therefore, $K^{C^*\mathrm{Cat}}(\Theta)$ is an equivalence if and only if $K^{C^*\mathrm{Cat}}(\Theta')$ is an equivalence. The Proposition 9.3 now follows from the combination of (9.13) and the fact that p'' is induced by the functor Θ'. □

Recall the Definition 9.2 of the functor Θ and that H denotes a finite subgroup of G. The next proposition finishes the proof of Proposition 8.9 and hence of Theorem 1.5.

Proposition 9.6. *$K^{C^*\mathrm{Cat}}(\Theta)$ is an equivalence.*

Proof. The proof of Proposition 9.6 is based on the factorization of Θ as described by the commutative diagram (9.17). The functors in this diagram will all induce equivalences in K-theory, but for different reasons. The rest of this section is devoted to the proof of Proposition 9.6 which is split in several lemmas.

Lemma 9.7. *The functor Θ is fully faithful.*

Proof. Recall that $X = G(H, e)$ is a subspace of $(G/H)_{min,min} \otimes G_{can,max}$, see (9.1). Let (C, ρ, μ) and (C', ρ', μ') be objects of $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)$. Then $\Theta(C, \rho, \mu) = (C, \rho, \pi)$ with π given by (9.5), and similarly $\Theta(C', \rho', \mu') = (C', \rho', \pi')$. Let

$$B : (C, \rho, \pi) \to (C', \rho', \pi')$$

by any morphism. We can write $B = \sum_{h \in H}(B_h, h)$, where $B_h : C \to C'$. The condition $\pi' B \pi = B$ implies that $B_h = \mu'(H) B_e \mu(H)$ and $h \cdot B_e = B_e$ for every h in H. Using [9, Lem. 7.8] we can define the morphism

$$A := \frac{1}{|H|} \sum_{g \in G} g \cdot B_e : (C, \rho, \mu) \to (C', \rho', \mu') \tag{9.16}$$

in $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)$. Then $\Theta(A) = B$. The formula (9.16) defines an inverse of Θ on the level of morphisms. □

In $\mathrm{Idem}(\mathrm{Res}^G_H(\mathbf{C}^{(G)}_{\mathrm{std}}) \rtimes H)$, we consider the full subcategory $\mathbf{D}$ of objects of the form $(C, \rho, (\mu(H), e))$, where (C, ρ, μ) is in $\mathbf{C}^{(G)}_{\mathrm{lf}}(X)$. We let furthermore $\mathbf{D}'$ be the full subcategory of $\mathrm{Idem}(\mathrm{Res}^G_H(\mathbf{C}^{(G)}_{\mathrm{std}}) \rtimes H)$ on objects of the form $(C, \rho, (\mu(Z), e))$, where (C, ρ, μ) is in $\mathbf{C}^{(G)}_{\mathrm{lf}}(Y)$ for some free G-set Y and Z is a H-invariant subset of Y. By Λ we denote the canonical inclusion of $\mathbf{D}$ into $\mathbf{D}'$. Below, the idempotent completions of $\mathbf{D}$ and $\mathbf{D}'$ are formed relative to the full subcategories of $\mathrm{Idem}(\mathrm{Res}^G_H(\mathbf{MC}^{(G)}_{\mathrm{std}}) \rtimes H)$ on objects from $\mathbf{D}$ or $\mathbf{D}'$, respectively. Then we have the following diagram

$$\begin{tikzcd}
\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X) \arrow[r, "\Phi"] \arrow[rrr, bend left, dotted, "\Theta"] & \mathrm{Idem}(\mathbf{D}) \arrow[r, dotted, "\mathrm{Idem}(\Lambda)"] & \mathrm{Idem}(\mathbf{D}') \arrow[r, "\Delta"] & \mathrm{Idem}(\mathrm{Res}^G_H(\mathbf{C}^{(G)}_{\mathrm{std}}) \rtimes H)\,, \\
& \mathbf{D} \arrow[u, "\Xi"] \arrow[r, "\Lambda"] & \mathbf{D}' \arrow[u, "\Psi"'] &
\end{tikzcd} \tag{9.17}$$

where Δ is again the canonical inclusion. The upper line is then a factorization of Θ as indicated.

In the following, we will show that all solid morphisms in (9.17) induce equivalences after applying $K^{C^*\mathrm{Cat}}$. It is clear that this implies that $K^{C^*\mathrm{Cat}}(\Theta)$ is an equivalence. To this end we use that $K^{C^*\mathrm{Cat}}$ sends unitary equivalences, Morita equivalences, relative idempotent completions, and weak Morita equivalences (see [9, Sec. 16–18]) to equivalences. In the following lemmas, we argue case by case that all solid arrows in the above diagram have one of these properties.

Recall the notion of a relative idempotent completion [9, Def. 17.5].

Lemma 9.8. *Ξ and Ψ are relative idempotent completions.*

Proof. This is true by construction. □

Therefore, $K^{C^*\mathrm{Cat}}(\Xi)$ and $K^{C^*\mathrm{Cat}}(\Psi)$ are equivalences by [9, Prop. 17.4].

Lemma 9.9. *Δ is a unitary equivalence in the sense of* [9, Def. 3.19].

Proof. It suffices to show the claim that every object of $\mathrm{Idem}(\mathbf{D}')$ admits a unitary isomorphism to an object of $\mathrm{Idem}(\mathrm{Res}^G_H(\mathbf{C}^{(G)}_{\mathrm{std}}) \rtimes H)$ in $\mathrm{Idem}(\mathrm{Res}^G_H(\mathbf{MC}^{(G)}_{\mathrm{std}}) \rtimes H)$. Since $\mathbf{D}'$ in particular contains all objects of the form $(C, \rho, (\mu(Y), e))$ for all free G-sets Y and all (C, ρ, μ) in $\mathbf{C}^{(G)}_{\mathrm{lf}}(Y)$, every object of $\mathrm{Res}^G_H(\mathbf{C}^{(G)}_{\mathrm{std}}) \rtimes H$ is unitarily isomorphic in $\mathrm{Res}^G_H(\mathbf{MC}^{(G)}_{\mathrm{std}}) \rtimes H$ to an object of $\mathbf{D}'$. This implies the claim by going over to the relative idempotent completions. □

Since $K^{C^*\mathrm{Cat}}$ is a homological functor by [9, Thm. 14.4] the morphism $K^{C^*\mathrm{Cat}}(\Delta)$ is an equivalence.

Lemma 9.10. *Φ is a Morita equivalence.*

Proof. The functor $\mathrm{Idem}(\Lambda)$ is fully faithful by construction. Since Θ is fully faithful by Lemma 9.7 and Δ is also fully faithful, we can conclude that Φ is fully faithful, too.

Let (C, ρ, μ) be an object of $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)$. Then we define

$$U := \frac{1}{\sqrt{|H|}} \sum_{h \in H} (\mu(\{h\}), h)$$

in $\mathrm{End}_{\mathbf{D}}((C, \rho, (\mu(H), e)))$. We calculate that

$$UU^* = (\mu(\{e\}), e), \quad U^*U = \pi,$$

where π is as in (9.5). This calculation shows that the projection π is MvN-equivalent to $(\mu(\{e\}), e)$. For h in H, we consider the unitary $V_h := (\mu(H), h^{-1})$ in $\mathrm{End}_{\mathbf{D}}((C, \rho, (\mu(H), e)))$. Then

$$V_h(\mu(\{e\}), e)V^*_h = (\mu(\{h\}), e).$$

So the projection $(\mu(\{h\}), e)$ is also MvN-equivalent to π for every h in H. Since the projections $((\mu(\{h\}), e))_{h \in H}$ are mutually orthogonal and $\sum_{h \in H}(\mu(\{h\}), e) = (\mu(H), e)$ we see that any object of $\mathbf{D}$ is an orthogonal summand of a finite orthogonal sum of objects in the essential image of Φ. This implies that also every object of $\mathrm{Idem}(\mathbf{D})$ is an orthogonal summand of a finite orthogonal sum of objects in the essential image of Φ. □

Since $K^{C^*\mathrm{Cat}}$ is Morita invariant by [9, Thm. 16.18] the morphism $K^{C^*\mathrm{Cat}}(\Phi)$ is an equivalence.

Lemma 9.11. *Λ is a weak Morita equivalence.*

Proof. The functor Λ is fully faithful by definition. Furthermore, $\mathbf{D}$ is unital since the identity on an object $(C, \rho, (\mu(H), e))$ of $\mathbf{D}$ is given by $(\mu(H), e)$ and $\mu(H)$ is in $\mathbf{C}$. It remains to show that the set of objects of $\mathbf{D}$ is weakly generating in $\mathbf{D}'$, see [9, Def. 18.1].

Let $(C, \rho, (\mu(Z), e))$ be any object of $\mathbf{D}'$, where (C, ρ, μ) is in $\mathbf{C}^{(G)}_{\mathrm{lf}}(Y)$ for some free G-set Y and Z is a H-invariant subset of Y. Let y be a point in Y. Then we can form the object $(C, \rho, (\mu(Hy), e))$ in $\mathbf{D}'$. We claim that this object is isomorphic to an object in $\mathbf{D}$. We consider the G-equivariant injection $i\colon X \to Y$ which sends (H, e) to y. We choose an image $u\colon C' \to C$ in $\mathbf{MC}$ of the projection $\mu(Gy)$. Then we define (C', ρ', μ') in $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X)$ by setting $\rho'_g = gu^*\rho_g u$ for every g in G and $\mu'(W) = u^*\mu(i(W))u$ for every subset W of X. Then we have an isomorphism

$$(u, e)\colon (C', \rho', (\mu'(H), e)) \to (C, \rho, (\mu(Hy), e))$$

in $\mathbf{D}'$. More generally, if Z is any finite H-invariant subset of Y (note that H is finite), then $(C, \rho, (\mu(Z), e))$ is isomorphic to a finite sum of objects in $\mathbf{D}$.

Let now $(A_j)_{j\in J}$ with $A_j\colon (C_j, \rho_j, p_j) \to (C, \rho, p)$ be a finite family of morphisms in $\mathbf{D}'$.

Let ϵ in $(0, \infty)$ be given. Since C is isomorphic to the AV-sum in $\mathbf{C}$ of the family of projections $(\mu(S))_{S\in H\backslash Y}$ the sum $\sum_{S\in H\backslash Y}\mu(S)$ converges strictly in $\mathbf{MC}$ to id_C. Since the morphisms A_j belong to $\mathbf{C}$ there exists a finite H-invariant subset Z of Y such that

$$\|A_j - (\mu(Z), e)A_j\| < \epsilon$$

for all j in J. □

By [9, Thm. 18.6] the morphism $K^{C^*\mathrm{Cat}}(\Lambda)$ is an equivalence.

Applying $K^{C^*\mathrm{Cat}}$ to the diagram in (9.17) and combining the results above we conclude the proof of Proposition 9.6. □

Therefore, the proofs of the Theorems 1.5.2 and 1.5.3 are also complete.

10. Calculation of the domain and target of the Paschke transformation

The domain of the Paschke transformation is the functor

$$K^{G,\mathcal{X}}_{\mathbf{C}}\colon G\mathbf{UBC} \to \mathbf{Sp}\,.$$

The first goal of this section is to describe its values on sufficiently nice spaces in terms of the equivariant homology theory

$$K\mathbf{C}^G\colon G\mathbf{Orb} \to \mathbf{Sp}$$

introduced in (1.19), see Definition 12.2 below for the technical description. Our final result is stated in Proposition 10.10.

In order to understand why the construction of the comparison map in Proposition 10.10 is difficult, note that on the one hand for X in $G\mathbf{UBC}$ the spectrum $K^{G,\mathcal{X}}_{\mathbf{C}}(X)$ is defined as the K-theory of an explicitly constructed C^*-category associated to X and the coefficient category $\mathbf{C}$. On the other hand the spectrum $K\mathbf{C}^G(X)$ is the value on the underlying G-topological space of X of the equivariant homology theory given by a spectrum-valued functor $K\mathbf{C}^G$ on the orbit category $G\mathbf{Orb}$ of G determined by $\mathbf{C}$. The construction of a natural map between $K^{G,\mathcal{X}}_{\mathbf{C}}(X)$ and $K\mathbf{C}^G(X)$ will involve a classification of functors with certain homological properties on subcategories of $G\mathbf{Top}$. This classification is related to Elmendorf's theorem and the techniques behind it.

The second theme of the present section is the calculation of the domain and target of the Paschke transformation. Our main example of a coefficient category is $\mathbf{C} = \mathbf{Hilb}_c(A)$ for a C^*-algebra A with an action of G. If A is unital, then one can express the values of the functors $K^{G,\mathcal{X}}_{\mathbf{C}}$ on G-orbits and of $K^{G,\mathrm{An}}_{\mathbf{C}}$ on sufficiently nice spaces directly in terms of constructions with the algebra A. The results are stated as Corollary 10.13 and Propositions 10.15 and 10.16.

We start with the statement of Elmendorf's theorem. Let $\mathbf{M}$ be a cocomplete stable ∞-category. In the present paper, we adopt the following simple definition which in some sense reverses the history of this notion.

Definition 10.1. *An equivariant* $\mathbf{M}$*-valued homology theory is a functor*

$$E: G\mathbf{Orb} \to \mathbf{M}.$$

Recall that a weak equivalence between topological spaces is a continuous map which induces a bijection between the sets of connected components and isomorphisms between the higher homotopy groups at all base points. We have a functor

$$\ell: \mathbf{Top} \to \mathbf{Spc} \tag{10.1}$$

which presents $\mathbf{Spc}$ as the localization of $\mathbf{Top}$ at the weak equivalences. We now consider the functor

$$Y^G: G\mathbf{Top} \to \mathbf{PSh}(G\mathbf{Orb}) \tag{10.2}$$

which sends X in $G\mathbf{Top}$ to the presheaf

$$S \mapsto \ell(\mathrm{Map}_{G\mathbf{Top}}(S_{disc}, X)),$$

where $\mathrm{Map}_{G\mathbf{Top}}(S_{disc}, X)$ in $\mathbf{Top}$ is the topological mapping space of equivariant maps. By definition, a map $f: X \to Y$ between G-topological spaces is an equivariant weak equivalence if it induces weak equivalences $\mathrm{Map}_{G\mathbf{Top}}(S_{disc}, X) \to \mathrm{Map}_{G\mathbf{Top}}(S_{disc}, Y)$ for all S in $G\mathbf{Orb}$.

Theorem 10.2 (Elmendorf's theorem)**.** *The functor* Y^G *presents* $\mathbf{PSh}(G\mathbf{Orb})$ *as the Dwyer-Kan localization of* $G\mathbf{Top}$ *at the equivariant weak equivalences.*

By the universal property of presheaves, the pull-back along the Yoneda embedding $\mathrm{yo}: G\mathbf{Orb} \to \mathbf{PSh}(G\mathbf{Orb})$ induces an equivalence

$$\mathrm{yo}^*: \mathrm{Fun}^{\mathrm{colim}}(\mathbf{PSh}(G\mathbf{Orb}), \mathbf{M}) \xrightarrow{\simeq} \mathrm{Fun}(G\mathbf{Orb}, \mathbf{M}).$$

Let $E: G\mathbf{Orb} \to \mathbf{M}$ be an equivariant homology theory. Its colimit preserving extension to presheaves is the left Kan-extension $\mathrm{yo}_!E: \mathbf{PSh}(G\mathbf{Orb}) \to \mathbf{M}$ of E along yo.

Definition 10.3. *The evaluation of E on G-topological spaces is defined as composition (which we will again denote by E)*

$$E: G\mathbf{Top} \xrightarrow{Y^G} \mathbf{PSh}(G\mathbf{Orb}) \xrightarrow{\mathrm{yo}_!E} \mathbf{M}. \tag{10.3}$$

If S is in $G\mathbf{Orb}$, then the value of the original functor E on S and the evaluation of E on the discrete G-space S_{disc} coincide so that there is no conflict of notation. The value of the equivariant homology theory on a general space X in $G\mathbf{Top}$ is given by the coend

$$E(X) := \int^{G\mathbf{Orb}} E \wedge \Sigma^\infty_+ Y^G(X)\,, \tag{10.4}$$

where $\wedge\colon\ \mathbf{M}\times\mathbf{Sp}\to\mathbf{M}$ is the tensor structure of $\mathbf{M}$ (the same as (7.5)) which exists by the cocompleteness and stability assumptions on $\mathbf{M}$.

We let $G\mathbf{UBC}^{\mathrm{pcc}}$ be the full subcategory of $G\mathbf{UBC}$ of G-uniform bornological coarse spaces which have the following properties:

(1) the underlying topological space is Hausdorff,
(2) the bornology is generated by relatively compact subsets,
(3) the coarse structure is generated by all entourages of the form $G(K\times K)$, where K is a relatively compact connected subset,
(4) G acts properly and cocompactly.

The category $G\mathbf{UBC}^{\mathrm{pcc}}$ contains all G-finite G-simplicial complexes with finite stabilizers with the structures induced by the spherical path metric. We consider the functor $\iota\colon\ G\mathbf{UBC}\to G\mathbf{Top}$ which takes the underlying G-topological space.

Lemma 10.4. *The restriction $\iota_{|G\mathbf{UBC}^{\mathrm{pcc}}}\colon\ G\mathbf{UBC}^{\mathrm{pcc}}\to G\mathbf{Top}$ is fully faithful.*

Proof. It is clear that $\iota_{|G\mathbf{UBC}^{\mathrm{pcc}}}$ is faithful. We must show that it is full. Let X, Y be in $G\mathbf{UBC}^{\mathrm{pcc}}$ and $f\colon X\to Y$ be an equivariant continuous map. We must show that it is controlled, uniformly continuous and proper.

We first show that f is proper. Let K be a relatively compact subset of Y and let $(x_\alpha)_\alpha$ be a net in $f^{-1}(K)$. Since K is relatively compact, and $G\backslash X$ is compact, we can assume by taking a subnet that $(f(x_\alpha))_\alpha$ and $([x_\alpha])_\alpha$ converge in Y and $G\backslash X$, respectively. By the latter there exists a family $(g_\alpha)_\alpha$ in G such that $(g_\alpha x_\alpha)_\alpha$ converges. Since then $(g_\alpha f(x_\alpha))_\alpha$ also converges and G acts properly on Y we can assume after taking a subnet that $(g_\alpha)_\alpha$ is constant. But this means that $(x_\alpha)_\alpha$ has a subnet converging in X, which shows that $f^{-1}(K)$ is relatively compact.

We claim that any invariant open entourage of the diagonal of X is uniform. The claim implies that f is uniformy continuous: Indeed, if V is any uniform entourage of Y, then by the axioms for a G-uniform structure there exists an invariant uniform entourage V' of Y such that $V'\subseteq V$. But then $(f\times f)^{-1}(V')$ is invariant and open, hence a uniform entourage of X by the claim. The relation $(f\times f)^{-1}(V')\subseteq (f\times f)^{-1}(V)$ implies that $(f\times f)^{-1}(V)$ is uniform.

We now show the claim. Assume by contradiction that U is not uniform. Then for every invariant uniform entourage V of X there exists (x_V, y_V) in $V\setminus U$. By compactness of the quotient we can assume, after taking a cofinal subnet $(V_\alpha)_\alpha$ of uniform entourages, that $[x_{V_\alpha}]\to[x]$ and $[y_{V_\alpha}]\to[y]$. We can find a net $(g_\alpha)_\alpha$ in G such that $g_\alpha x_{V_\alpha}\to x$ in X. But then also $g_\alpha y_{V_\alpha}\to x$ since X is Hausdorff and the net $(V_\alpha)_\alpha$ of uniform entourages is cofinal. Since U is G-invariant we have $(g_\alpha x_{V_\alpha}, g_\alpha y_{V_\alpha})\notin U$ for all α, and since U is open we conclude that also $(x,x) \notin U$. But this is impossible since U was an open neighbourhood of the diagonal.

We check on generators that f is controlled. Let K be a relatively compact connected subset of X and consider the generator $G(K\times K)$ of the coarse structure of X. Then $f(K)$ is relatively compact and connected, too. Therefore, $(f\times f)G(K\times K)=G(f(K)\times f(K))$ is a coarse entourage of Y. □

Recall that $K^{G,\mathcal{X}}_{\mathbf{C}}$ is defined on $G\mathbf{UBC}$. By the Lemma 10.4 we can restrict $K^{G,\mathcal{X}}_{\mathbf{C}}$ to a functor defined on the full subcategory $G\mathbf{UBC}^{\mathrm{pcc}}$ of $G\mathbf{Top}$. In contrast, the equivariant homology theory $K\mathbf{C}^G$ gives rise to a functor defined on all of $G\mathbf{Top}$ by Definition 10.3. Therefore, as a preparation we present a general result which helps to compare a functor with homological properties defined on some full subcategory of $G\mathbf{Top}$ with an associated equivariant homology theory.

Let $\mathbf{V}$ be a simplicial model category with weak equivalences W, homotopy equivalences W_h, and with functorial factorizations. The associated ∞-category of $\mathbf{V}$ is defined by $\mathbf{V}_\infty := \mathbf{V}[W^{-1}]$. We let $\ell : \mathbf{V}\to\mathbf{V}_\infty$ denote the canonical functor. We furthermore let $\mathbf{V}^{\mathrm{cf}}$ denote the full subcategory of cofibrant/fibrant objects in $\mathbf{V}$. The following lemma is of course well-known, but for lack of reference, we include a proof here.

Lemma 10.5. *The inclusion $\mathbf{V}^{\mathrm{cf}}\to\mathbf{V}$ induces an equivalence of Dwyer–Kan localizations $\mathbf{V}^{\mathrm{cf}}[W_h^{-1}]\simeq\mathbf{V}[W^{-1}]$.*

Proof. We consider the following square

$$\begin{tikzcd} \mathbf{V}^{\mathrm{cf}} \arrow[r, "\ell_h"] \arrow[d] & \mathbf{V}^{\mathrm{cf}}[W_h^{-1}] \arrow[d, dotted] \\ \mathbf{V} \arrow[r, "\ell"] & \mathbf{V}_\infty \end{tikzcd}$$

where the dotted arrow is obtained from the universal property of the localization ℓ_h. We claim that it is an equivalence as desired. In order to produce an inverse, we consider the square

$$\begin{tikzcd} \mathbf{V} \arrow[r, "\ell"] \arrow[d, "RL"] & \mathbf{V}_\infty \arrow[d, dotted] \\ \mathbf{V}^{\mathrm{cf}} \arrow[r, "\ell_h"] & \mathbf{V}^{\mathrm{cf}}[W_h^{-1}] \end{tikzcd}$$

where RL is the fibrant-cofibrant replacement functor. The dotted arrow is obtained from the universal property of ℓ since RL sends weak equivalences to homotopy equivalences. We have a diagram

$$\begin{tikzcd} & L \arrow[dl] \arrow[dr] & \\ \mathrm{id} & & RL \end{tikzcd}$$

of endofunctors of $\mathbf{V}$, where L and R are the fibrant and cofibrant replacement functors. It is sent by ℓ to a diagram of equivalences. Similarly, ℓ_h sends the restriction of this diagram to $\mathbf{V}^{\mathrm{cf}}$ to a diagram of equivalences. From this we can conclude that the two dotted arrows are inverse to each other. □

Let $E : \mathbf{V} \to \mathbf{M}$ be a homotopy invariant functor.

Lemma 10.6. *There exists a functor $E^\infty : \mathbf{V}_\infty \to \mathbf{M}$ such that the following square commutes:*

$$\begin{tikzcd} \mathbf{V}^{\mathrm{cf}} \arrow[r, "E_{|\mathbf{V}^{\mathrm{cf}}}"] \arrow[d] & \mathbf{M} \\ \mathbf{V} \arrow[r] & \mathbf{V}_\infty \arrow[u, "E^\infty"'] \end{tikzcd}$$

Proof. We obtain the desired square from

$$\begin{tikzcd} \mathbf{V}^{\mathrm{cf}} \arrow[r, "\ell_h"] \arrow[d] \arrow[rr, bend left, "E_{|\mathbf{V}^{\mathrm{cf}}}"] & \mathbf{V}^{\mathrm{cf}}[W_h^{-1}] \arrow[r, dotted] \arrow[d, "\simeq"] & \mathbf{M} \arrow[d, equal] \\ \mathbf{V} \arrow[r, "\ell"] & \mathbf{V}_\infty \arrow[r, "E^\infty"] & \mathbf{M} \end{tikzcd}$$

where the dotted arrow exists since E sends homotopy equivalences to equivalences. □

We consider some full subcategory $\mathbf{W}$ of $G\mathbf{Top}$ and let $E : \mathbf{W} \to \mathbf{M}$ be some functor. We assume that $\mathcal{F}$ is a family of subgroups of G and that $G_{\mathcal{F}}\mathbf{Orb} \subseteq \mathbf{W}$. We then define the equivariant homology theory $E^{\%,\mathcal{F}} : G\mathbf{Orb} \to \mathbf{M}$ as the left Kan extension of the functor $E_{|G_{\mathcal{F}}\mathbf{Orb}}$ along $i_{\mathcal{F}}$:

$$\begin{tikzcd} G_{\mathcal{F}}\mathbf{Orb} \arrow[r, "E_{|G_{\mathcal{F}}\mathbf{Orb}}"] \arrow[d, "i_{\mathcal{F}}"'] & \mathbf{M}\,. \\ G\mathbf{Orb} \arrow[ur, "E^{\%,\mathcal{F}}"'] & \end{tikzcd}$$

Following Definition 10.3 we will consider $E^{\%,\mathcal{F}}$ also as a functor $E^{\%,\mathcal{F}} : G\mathbf{Top} \to \mathbf{M}$.

We now use that $G\mathbf{Top}$ admits a simplicial model category structure with the weak equivalences as described after (10.2) and such that the notion of homotopy is the usual one. By Theorem 10.2 the functor $Y^G : G\mathbf{Top} \to \mathbf{PSh}(G\mathbf{Orb})$ is equivalent to the functor $G\mathbf{Top} \to G\mathbf{Top}_\infty$ in the notation introduced before Lemma 10.6. Let $j : \mathbf{W} \to G\mathbf{Top}$ denote the inclusion.

Lemma 10.7. *Assume:*

(1) $G_{\mathcal{F}}\mathbf{Orb} \subseteq \mathbf{W} \subseteq G\mathbf{Top}^{\mathrm{cf}}$

(2) $\mathbf{W}$ *is closed under taking the product with* $[0,1]$.
(3) E *is homotopy invariant.*

Then we have a canonical natural transformation of functors

$$j^*E^{\%,\mathcal{F}} \to E : \mathbf{W} \to \mathbf{M}\,.$$

Proof. Since j is fully faithful, we have an equivalence $E \stackrel{\simeq}{\to} j^* j_! E$. We claim that $j_!E$ is homotopy invariant. Let X be in $G\mathbf{Top}$. Then we must show that $(j_!E)([0,1]\times X) \to (j_!E)(X)$ is an equivalence. We use the point-wise formula for the left Kan extension in order to rewrite this map as

$$\operatorname{colim}_{(Y\to[0,1]\times X)\in \mathbf{W}_{/[0,1]\times X}} E(Y) \to \operatorname{colim}_{(Z\to X)\in\mathbf{W}_{/X}} E(Z)\,. \tag{10.5}$$

We now observe that the maps of the form $[0,1]\times Z \to [0,1]\times X$ for maps $Z \to X$ are cofinal in the index category of the left colimit. At this point we use that $\mathbf{W}$ is closed under taking products with an interval. Indeed, let $(a,b) : Y \to [0,1]\times X$ be a map. Then we consider the factorization

$$Y \stackrel{(a,\mathrm{id}_Y)}{\to} [0,1]\times Y \stackrel{(\mathrm{id}_{[0,1]},b)}{\to} [0,1]\times X\,.$$

Consequently, the morphism in (10.5) is equivalent to

$$\operatorname{colim}_{(Z\to X)\in\mathbf{W}_{/X}} E([0,1]\times Z) \to \operatorname{colim}_{(Z\to X)\in\mathbf{W}_{/X}} E(Z)\,.$$

This map is an equivalence since E is homotopy invariant. This finishes the proof of the claim.

By Lemma 10.6 we get a functor $(j_!E)^\infty : \mathbf{PSh}(G\mathbf{Orb}) \to \mathbf{M}$ fitting into the commutative square in

$$\begin{tikzcd}
 & & \mathbf{W} \arrow[dl] \arrow[ddl, "j"] \arrow[dr, "E"] & \\
G_{\mathcal{F}}\mathbf{Orb} \arrow[urr] \arrow[d, "i_{\mathcal{F}}"] & G\mathbf{Top}^{\mathrm{cf}} \arrow[rr, "(j_!E)_{|G\mathbf{Top}^{\mathrm{cf}}}"] \arrow[d] & & \mathbf{M} \\
G\mathbf{Orb} \arrow[r] \arrow[rrr, bend right, "\mathrm{yo}"'] & G\mathbf{Top} \arrow[rr, "Y^G"] & & \mathbf{PSh}(G\mathbf{Orb}) \arrow[u, "(j_!E)^\infty"']
\end{tikzcd} \tag{10.6}$$

Here the triangle involving $(j_!E)_{|G\mathbf{Top}^{\mathrm{cf}}}$ commutes since $j^*j_!E \simeq E$ as observed already above. The commutative diagram provides an equivalence $E_{|G_{\mathcal{F}}\mathbf{Orb}} \simeq i^*_{\mathcal{F}}\mathrm{yo}^*(j_!E)^\infty$. Applying the left Kan extension $\mathrm{yo}_! i_{\mathcal{F},!}$ we get an equivalence

$$\mathrm{yo}_! E^{\%,\mathcal{F}} \simeq \mathrm{yo}_! i_{\mathcal{F},!} E_{|G_{\mathcal{F}}\mathbf{Orb}} \simeq \mathrm{yo}_! i_{\mathcal{F},!} i^*_{\mathcal{F}} \mathrm{yo}^*(j_!E)^\infty\,.$$

The counit $\mathrm{yo}_! i_{\mathcal{F},!} i^*_{\mathcal{F}}\mathrm{yo}^* \to \mathrm{id}$ then yields the transformation $\mathrm{yo}_! E^{\%,\mathcal{F}} \to (j_!E)^\infty$. We finally apply $j^*(Y^G)^*$ and get the desired transformation

$$j^*E^{\%,\mathcal{F}} \to j^*(Y^G)^*(j_!E)^\infty \simeq E : \mathbf{W} \to \mathbf{M}\,,$$

where the second equivalence follows form the commutativity of a part of the diagram (10.6) above. □

Recall that $\mathbf{W}$ is a full subcategory of $G\mathbf{Top}$ and that $E\colon \mathbf{W} \to \mathbf{M}$ is some functor. We call E reduced if $E(\emptyset) \simeq 0$. We let $\mathbf{W}^{\mathrm{hfin}}_{\mathcal{F}}$ denote the full subcategory of $\mathbf{W}$ of spaces which are homotopy equivalent to a G-finite G-CW complex with stabilizers in $\mathcal{F}$.

Proposition 10.8. *Assume:*

(1) $\mathbf{W} \subseteq G\mathbf{Top}^{\mathrm{cf}}$ *and* $\mathbf{W}$ *contains all G-finite CW-complexes with stabilizers in $\mathcal{F}$.*
(2) $\mathbf{W}$ *is closed under taking the product with* $[0,1]$.
(3) *E is reduced, homotopy invariant, and excisive for cell attachments.*

Then the natural transformation from Lemma 10.7 *induces an equivalence*

$$(j^*E^{\%,\mathcal{F}})_{|\mathbf{W}^{\mathrm{hfin}}_{\mathcal{F}}} \to E_{|\mathbf{W}^{\mathrm{hfin}}_{\mathcal{F}}} .$$

Proof. We note that $j^*E^{\%,\mathcal{F}}\colon \mathbf{W} \to \mathbf{M}$ is reduced, homotopy invariant, and excisive for cell attachments.

We must show that $E^{\%,\mathcal{F}}(X) \to E(X)$ is an equivalence for all X in $\mathbf{W}^{\mathrm{hfin}}_{\mathcal{F}}$. Since $j^*E^{\%,\mathcal{F}}$ and E are homotopy invariant we can assume that X is a G-finite CW-complex with stabilizers in $\mathcal{F}$.

We then argue by induction by the number of cells. The assertion is clear for the empty G-CW-complex since both functors are reduced. Assume now that the assertion is true for the G-CW-complex Y, and that X is obtained from Y by a cell-attachement. Then we have a push-out diagram

$$\begin{tikzcd} G/K\times S^n \arrow[r]\arrow[d] & Y \arrow[d] \\ G/K\times D^{n+1} \arrow[r] & X \end{tikzcd}$$

where n is in $\mathbb{N}$ and K is a subgroup of G belonging to $\mathcal{F}$. The natural transformation induces a map of push-out diagrams

$$\begin{tikzcd} E^{\%,\mathcal{F}}(G/K\times S^n) \arrow[r]\arrow[d] & E^{\%,\mathcal{F}}(Y) \arrow[d] \\ E^{\%,\mathcal{F}}(G/K\times D^{n+1}) \arrow[r] & E^{\%,\mathcal{F}}(X) \end{tikzcd} \to \begin{tikzcd} E(G/K\times S^n) \arrow[r]\arrow[d] & E(Y) \arrow[d] \\ E(G/K\times D^{n+1}) \arrow[r] & E(X) \end{tikzcd}$$

which is implemented by equivalences at the two upper and the lower left corners by the induction hypothesis. We conclude that $E^{\%,\mathcal{F}}(X) \xrightarrow{\simeq} E(X)$. □

We now consider two functors $E, F\colon \mathbf{W} \to \mathbf{M}$ and assume that we are given an equivalence

$$\phi\colon E_{|G_{\mathcal{F}}\mathbf{Orb}} \to F_{|G_{\mathcal{F}}\mathbf{Orb}} .$$

Corollary 10.9. *Assume:*

(1) $\mathbf{W} \subseteq G\mathbf{Top}^{\mathrm{cf}}$ *and* $\mathbf{W}$ *contains all G-finite CW-complexes with stabilizers in $\mathcal{F}$.*
(2) $\mathbf{W}$ *is closed under taking the product with* $[0,1]$.
(3) *E and F are reduced, homotopy invariant, and excisive for cell attachments.*

Then ϕ extends to an equivalence

$$\tilde{\phi} \colon E_{|\mathbf{W}^{\mathrm{hfin}}_{\mathcal{F}}} \to F_{|\mathbf{W}^{\mathrm{hfin}}_{\mathcal{F}}} .$$

Proof. The equivalence ϕ induces an equivalence $\tilde{\phi} \colon E^{\%,\mathcal{F}} \xrightarrow{\simeq} F^{\%,\mathcal{F}}$. The desired equivalence is now given by the composition

$$E_{|\mathbf{W}^{\mathrm{hfin}}_{\mathcal{F}}} \xleftarrow{\simeq} (j^*E^{\%,\mathcal{F}})_{|\mathbf{W}^{\mathrm{hfin}}_{\mathcal{F}}} \xrightarrow{\tilde{\phi},\simeq} (j^*F^{\%,\mathcal{F}})_{|\mathbf{W}^{\mathrm{hfin}}_{\mathcal{F}}} \xrightarrow{\simeq} F_{|\mathbf{W}^{\mathrm{hfin}}_{\mathcal{F}}}$$

where the outer equivalences are supplied by Proposition 10.8 □

We let $G\mathbf{UBC}^{\mathrm{pcc},hfin}$ be the full subcategory of $G\mathbf{UBC}^{\mathrm{pcc}} \cap G\mathbf{Top}^{\mathrm{cf}}$ of G-spaces which are homotopy equivalent to G-finite G-CW complexes with stabilizers in **Fin**. We consider $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ which is effectively additive and admits countable AV-sums.

Proposition 10.10. *We have an equivalence*

$$K^{G,\mathcal{X}}_{\mathbf{C}}(-)_{|G\mathbf{UBC}^{\mathrm{pcc,hfin}}} \simeq \Sigma K\mathbf{C}^G(-)_{|G\mathbf{UBC}^{\mathrm{pcc,hfin}}} .$$

Proof. We start with the equivalence

$$\begin{aligned} K^{G,\mathcal{X}}_{\mathbf{C}}(S_{min,min,disc}) &\stackrel{\mathrm{def}}{=} K\mathbf{C}\mathcal{X}^G_{G_{can,max}}(\mathcal{O}^\infty(S_{min,min,disc})) \\ &\simeq \Sigma K\mathbf{C}\mathcal{X}^G_{G_{can,max}}(S_{min,min}) , \end{aligned}$$

where the second equivalence is given by the cone boundary [10, Prop. 9.35]. For every S in $G_{\mathbf{Fin}}\mathbf{Orb}$, the sets of invariant locally finite subsets $\mathrm{LF}(S_{min,max} \otimes G_{can,min})$ and $\mathrm{LF}(S_{min,min} \otimes G_{can,max})$ are equal. Using that $K\mathbf{C}\mathcal{X}^G$ is a continuous equivariant coarse homology theory we get the middle equivalence in

$$\begin{aligned} K\mathbf{C}\mathcal{X}^G_{G_{can,max}}((-)_{min,min}) &\stackrel{\mathrm{def}}{=} K\mathbf{C}\mathcal{X}^G((-)_{min,min} \otimes G_{can,max}) \\ &\simeq K\mathbf{C}\mathcal{X}^G((-)_{min,max} \otimes G_{can,min})) \stackrel{\mathrm{def}}{=} K\mathbf{C}^G(-) \end{aligned}$$

of functors on $G_{\mathbf{Fin}}\mathbf{Orb}$. We now apply Corollary 10.9 with $\mathbf{W} = G\mathbf{UBC}^{\mathrm{pcc}} \cap G\mathbf{Top}^{\mathrm{cf}}$, $\mathcal{F} = \mathbf{Fin}$, $E = K^{G,\mathcal{X}}_{\mathbf{C}}(-)$ and $F = \Sigma K\mathbf{C}^G(-)$ in order to get the desired equivalence. □

Using Proposition 10.10 we can express the domain of the Paschke transformation in terms of the equivariant homology theory $K\mathbf{C}^G$. In the following, we describe the values of this functor on G-orbits in some detail. We use remark environments in order to be able to refer to this discussion later.

Remark 10.11. We assume that $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ is effectively additive. By [7, Prop. 8.2.3] we have an explicit description of the values of the functor $K\mathbf{C}^G$ on G-orbits S:

$$K\mathbf{C}^G(S) \simeq K^{C^*\mathrm{Cat}}(\tilde{\bar{\mathbf{C}}}^{\mathrm{ctr}}_{\mathrm{lf}}(S_{min,max}) \rtimes_r G)\,. \tag{10.7}$$

Here $\tilde{\bar{\mathbf{C}}}^{\mathrm{ctr}}_{\mathrm{lf}}(S_{min,max})$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ is the C^*-category $\bar{\mathbf{C}}^{\mathrm{ctr}}_{\mathrm{lf}}(S_{min,max})$ with the G-action induced by functoriality by the actions of G on S and $\mathbf{C}$, and $- \rtimes_r G$ is the reduced crossed product for G-C^*-categories introduced in [9, Thm. 12.1]. Note that the objects of $\bar{\mathbf{C}}^{\mathrm{ctr}}_{\mathrm{lf}}(S_{min,max})$ are objects of $\mathbf{C}$ which are decomposed as AV-sums of S-indexed families of objects of $\mathbf{C}^u$ with finitely many non-zero terms, and morphisms are morphisms in $\mathbf{MC}$ which are diagonal with respect to this decomposition. We note that (10.7) implies that the functor $K\mathbf{C}^G$ is the functor defined in [9, Def. 19.12] for $\mathrm{Hg} = K^{C^*\mathrm{Cat}}$ and denoted there by $(K^{C^*\mathrm{Cat}})^G_{\mathbf{C}^u,r}$.

The right-hand side of the equivalence in (10.7) reflects the functorial dependence on S in an obvious manner. If one is not interested in functoriality, then one can give even simpler formulas. For a subgroup H of G, we have the equivalence

$$K\mathbf{C}^G(G/H) \overset{(10.7)}{\simeq} K^{C^*\mathrm{Cat}}(\bar{\mathbf{C}}^{\mathrm{ctr}}_{\mathrm{lf}}((G/H)_{min,max}) \rtimes_r G) \simeq K^{C^*\mathrm{Cat}}(\mathbf{C}^u \rtimes_r H)$$

by using [9, Cor. 19.13] and the Morita invariance of $K^{C^*\mathrm{Cat}}$. □

Remark 10.12. We continue the calculations from Remark 10.11 but now specialize further to the case $\mathbf{C} = \mathbf{Hilb}_c(A)$ for an A in $\mathrm{Fun}(BG, C^*\mathbf{Alg})$. Since A is unital, the inclusion $A \to \mathbf{Hilb}_c(A)^u$ is a Morita equivalence (combine [9, Ex. 16.9 & 18.15]) and therefore induces by [9, Prop. 16.11] (stating that $- \rtimes_r H$ preserves Morita equivalences) and [9, Thm. 16.18] (stating that $K^{C^*\mathrm{Cat}}$ is Morita invariant) an equivalence

$$K^{C^*\mathrm{Alg}}(A \rtimes_r H) \xrightarrow{\simeq} K^{C^*\mathrm{Cat}}(\mathbf{Hilb}_c(A)^u \rtimes_r H)\,.$$

So in this case

$$K\mathbf{C}^G(G/H) \simeq K^{C^*\mathrm{Alg}}(A \rtimes_r H)\,.$$

We see that the functor $K\mathbf{C}^G$ has the same values as the functor introduced in [16] (with additions by [24] or alternatively by [28])[4]. If A is unital and is equipped with the trivial G-action, then by [9, Prop. 19.18] the functor $K\mathbf{C}^G$ and the Davis–Lück functor are actually equivalent as functors. □

Using (8.5) and Proposition 10.10 combined with Remark 10.12 we can describe the values on the orbit category for the functor $K^{G,\mathcal{X}}_{\mathbf{Hilb}_c(A)}$ appearing in the domain of the Paschke morphism. Let A be in $\mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}})$.

[4]To be precise, in [16] only the case $A = \mathbb{C}$ is considered, but the generalization to unital C^*-algebras with trivial G-action is straightforward. The additions concern a correction in the construction of a K-theory functor for C^*-categories.

Corollary 10.13. *If A is unital, then for every subgroup H of G we have an equivalence*

$$K^{G,\mathcal{X}}_{\mathbf{Hilb}_c(A)}((G/H)_{min,min,disc}) \simeq \begin{cases} 0 & |H| = \infty\,, \\ \Sigma K^{C^*\mathrm{Alg}}(A \rtimes_r H) & |H| < \infty\,. \end{cases}$$

We now turn our attention to the target of the Paschke morphism. We show that in the case of $\mathbf{C} = \mathbf{Hilb}_c(A)$ for unital A, we can express the functor

$$K^{G,\mathrm{An}}_{\mathbf{C}}(-) \stackrel{(1.3)}{=} \mathrm{KK}^G(C_0(-), \mathbf{Q}^{(G)}_{\mathrm{std}})$$

in terms of the more familiar functor

$$K^{G,\mathrm{an}}_{A}(-) := \mathrm{KK}^G(C_0(-), A)$$

from $G\mathbf{LCH}^{\mathrm{prop}}_{+}$ to $\mathbf{Sp}$, see [12, Def. 1.14]. In order to state the results properly, we introduce the following notation.

Definition 10.14.

(1) *We let $G\mathbf{LCH}^{\mathrm{prop,hfin}}_{+}$ denote the full subcategory of $G\mathbf{LCH}^{\mathrm{prop}}_{+}$ on spaces which are homotopy equivalent in $G\mathbf{LCH}^{\mathrm{prop}}_{+}$ to G-finite G-CW complexes with finite stabilizers.*

(2) *We let $G\mathbf{LCH}^{\mathrm{prop},\sigma\mathrm{hfin}}_{\mathrm{2nd},+}$ denote the full subcategory of $G\mathbf{LCH}^{\mathrm{prop}}_{+}$ of second countable spaces with proper G-action which are homotopy equivalent in $G\mathbf{LCH}^{\mathrm{prop}}_{+}$ to countable G-CW complexes with proper G-action.*

Let A be in $\mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}})$.

Proposition 10.15. *If A is unital, then we have an equivalence of functors*

$$(\Sigma K^{G,\mathrm{an}}_{A})_{|G\mathbf{LCH}^{\mathrm{prop,hfin}}_{+}} \simeq (K^{G,\mathrm{An}}_{\mathbf{Hilb}_c(A)})_{|G\mathbf{LCH}^{\mathrm{prop,hfin}}_{+}}\,.$$

Proof. We abbreviate $\mathbf{C} := \mathbf{Hilb}_c(A)$. Using the notation of [12, Def. 1.14] we have the equality

$$K^{G,\mathrm{an}}_{\mathbf{Q}^{(G)}_{\mathrm{std}}}(-) = K^{G,\mathrm{An}}_{\mathbf{C}}(-)\,.$$

If X is in $G\mathbf{LCH}^{\mathrm{prop,hfin}}_{+}$, then by Lemma 8.6 the functor $\mathbf{B} \mapsto K^{G,\mathrm{an}}_{\mathbf{B}}(X)$ sends exact sequences in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ to fibre sequences of functors on $G\mathbf{LCH}^{\mathrm{prop,hfin}}_{+}$, annihilates flasques, and sends relative Morita equivalences to equivalences. By [12, Thm. 1.32.3] it also sends weak Morita equivalences to equivalences.

We apply the exactness property to the exact sequence

$$0 \to \mathbf{C}^{(G)}_{\mathrm{std}} \to \mathbf{MC}^{(G)}_{\mathrm{std}} \xrightarrow{\pi} \mathbf{Q}^{(G)}_{\mathrm{std}} \to 0\,. \tag{10.8}$$

Since $\mathbf{C}^{(G)}_{\mathrm{std}}$ admits countable AV-sums, we know by Lemma 2.21 that $\mathbf{MC}^{(G)}_{\mathrm{std}}$ is flasque. Therefore, $K^{G,\mathrm{an}}_{\mathbf{MC}^{(G)}_{\mathrm{std}}}(-) \simeq 0$ and the boundary map of the fibre sequence obtained by applying $K^{G,\mathrm{an}}_{-}$ to (10.8) is an equivalence

$$K^{G,\mathrm{An}}_{\mathbf{C}}(-) = K^{G,\mathrm{an}}_{\mathbf{Q}^{(G)}_{\mathrm{std}}}(-) \xrightarrow{\simeq} \Sigma K^{G,\mathrm{an}}_{\mathbf{C}^{(G)}_{\mathrm{std}}}(-) \tag{10.9}$$

of functors on $G\mathbf{LCH}^{\mathrm{prop,hfin}}_{+}$. We consider the zig-zag

$$A \to (\mathbf{C}^{u})^{(G)} \to \mathbf{C}^{(G)}_{\mathrm{std},+} \leftarrow \mathbf{C}^{(G)}_{\mathrm{std}} \tag{10.10}$$

in $\mathrm{Fun}(BG, C^{*}\mathbf{Cat}^{\mathrm{nu}})$, where by Lemma 2.20.3 the first map is a Morita equivalence, the second is a weak Morita equivalence, and the third one is a split relative Morita equivalence by Lemma 2.20.2. We therefore get an associated zig-zag of equivalences

$$K^{G,\mathrm{an}}_{A}(-) \xrightarrow{\simeq} K^{G,\mathrm{an}}_{(\mathbf{C}^{u})^{(G)}}(-) \xrightarrow{\simeq} K^{G,\mathrm{an}}_{\mathbf{C}^{(G)}_{\mathrm{std},+}}(-) \xleftarrow{\simeq} K^{G,\mathrm{an}}_{\mathbf{C}^{(G)}_{\mathrm{std}}}(-) \tag{10.11}$$

of functors on $G\mathbf{LCH}^{\mathrm{prop,hfin}}_{+}$.

Composing the equivalences in (10.9) and (10.11) we get the asserted equivalence. □

In the next proposition, we calculate the values of the functor $K^{G,\mathrm{An,lf}}_{\mathbf{C}}$ from (1.7). We will use the notation introduced in Definition 10.14.2. Let A be in $\mathrm{Fun}(BG, C^{*}\mathbf{Alg}^{\mathrm{nu}})$.

Proposition 10.16. *If A is unital and separable, then we have an equivalence*

$$(\Sigma K^{G,\mathrm{an}}_{A})_{|G\mathbf{LCH}^{\mathrm{prop},\sigma\mathrm{hfin}}_{\mathrm{2nd},+}} \simeq (K^{G,\mathrm{An,lf}}_{\mathbf{Hilb}_{c}(A)})_{|G\mathbf{LCH}^{\mathrm{prop},\sigma\mathrm{hfin}}_{\mathrm{2nd},+}} .$$

Proof. The argument is similar as for Proposition 10.15. However, if X is in $G\mathbf{LCH}^{\mathrm{prop},\sigma\mathrm{hfin}}_{\mathrm{2nd},+}$, then $\mathrm{kk}^{G}(C_0(X))$ is not ind-G-proper in general so that $\mathbf{B} \to K^{G,\mathrm{an}}_{\mathbf{B}}(X)$ does not send all exact sequences to fibre sequences, i.e., Lemma 8.6 is not directly applicable.

In analogy with (1.6), we can define the locally finite evaluation F^{lf} of any functor F on $G\mathbf{LCH}^{\mathrm{prop}}_{+}$ (with complete target) by

$$F^{\mathrm{lf}}(X) := \lim_{U \subseteq X} F(U) ,$$

where U runs over the open subsets of X with G-compact closure. We have a natural transformation $c_F : F \to F^{\mathrm{lf}}$, and the transformation $c_{F^{\mathrm{lf}}} : F^{\mathrm{lf}} \to (F^{\mathrm{lf}})^{\mathrm{lf}}$ is an equivalence by a cofinality argument.

We again abbreviate $\mathbf{C} := \mathbf{Hilb}_{c}(A)$. We will construct an equivalence

$$(\Sigma K^{G,\mathrm{an,lf}}_{A})_{|G\mathbf{LCH}^{\mathrm{prop},\sigma\mathrm{hfin}}_{\mathrm{2nd},+}} \simeq (K^{G,\mathrm{An,lf}}_{\mathbf{C}})_{|G\mathbf{LCH}^{\mathrm{prop},\sigma\mathrm{hfin}}_{\mathrm{2nd},+}} \tag{10.12}$$

and furthermore show that the canonical morphism $c_{K^{G,\mathrm{an}}_{A}}$ induces an equivalence

$$(K^{G,\mathrm{an}}_{A})_{|G\mathbf{LCH}^{\mathrm{prop},\sigma\mathrm{hfin}}_{\mathrm{2nd},+}} \xrightarrow{\simeq} (K^{G,\mathrm{an,lf}}_{A})_{|G\mathbf{LCH}^{\mathrm{prop},\sigma\mathrm{hfin}}_{\mathrm{2nd},+}} . \tag{10.13}$$

The asserted equivalence is then defined as the composition of the equivalences in (10.12) and (10.13).

We start with the construction of (10.12). We consider the following diagram in KK^G

$$\begin{tikzcd}
 & \mathrm{kk}^G_{C^*\mathbf{Cat}}(\mathbf{C}^{(G)}_{\mathrm{std}}) \arrow[r] \arrow[d, dotted, "i"] & \mathrm{kk}^G_{C^*\mathbf{Cat}}(\mathbf{MC}^{(G)}_{\mathrm{std}}) \arrow[r, "\mathrm{kk}^G_{C^*\mathbf{Cat}}(\pi)"] \arrow[d, equal] & \mathrm{kk}^G(\mathbf{Q}^{(G)}_{\mathrm{std}}) \arrow[d, equal] \\
\Sigma^{-1}\mathrm{kk}^G_{C^*\mathbf{Cat}}(\mathbf{Q}^{(G)}_{\mathrm{std}}) \arrow[r, "j"] & F(\pi) \arrow[r] & \mathrm{kk}^G_{C^*\mathbf{Cat}}(\mathbf{MC}^{(G)}_{\mathrm{std}}) \arrow[r, "\mathrm{kk}^G_{C^*\mathbf{Cat}}(\pi)"] & \mathrm{kk}^G_{C^*\mathbf{Cat}}(\mathbf{Q}^{(G)}_{\mathrm{std}})\,.
\end{tikzcd} \tag{10.14}$$

The lower part is a segment of a fibre sequence with $F(\pi)$ defined as the fibre of $\mathrm{kk}^G(\pi)$, where π is the quotient morphism $\mathbf{MC}^{(G)}_{\mathrm{std}} \to \mathbf{Q}^{(G)}_{\mathrm{std}}$. The upper composition vanishes since (10.8) is exact, but it is not necessarily part of a fibre sequence since $\mathrm{kk}^G_{C^*\mathbf{Cat}}$ is only conditionally exact. The dotted arrow and the corresponding square is then given by the universal property of the fibre.

We consider an ind-G-proper object P and apply the exact functor

$$\mathrm{KK}^G(P,-)\colon\ \mathrm{KK}^G \to \mathbf{Sp}$$

to (10.14). We then get the following diagram in $\mathbf{Sp}$ (as usual we drop the symbol $\mathrm{kk}^G_{C^*\mathbf{Cat}}$ if we insert objects in $\mathrm{KK}^G(-,-)$)

$$\begin{tikzcd}
 & \mathrm{KK}^G(P,\mathbf{C}^{(G)}_{\mathrm{std}}) \arrow[r] \arrow[d, dotted, "i_*", "\simeq"'] & \mathrm{KK}^G(P,\mathbf{MC}^{(G)}_{\mathrm{std}}) \arrow[r, "\pi_*"] \arrow[d, equal] & \mathrm{KK}^G(P,\mathbf{Q}^{(G)}_{\mathrm{std}}) \arrow[d, equal] \\
\Sigma^{-1}\mathrm{KK}^G(P,\mathbf{Q}^{(G)}_{\mathrm{std}}) \arrow[r, "j_*", "\simeq"'] & \mathrm{KK}^G(P,F(\pi)) \arrow[r] & \mathrm{KK}^G(P,\mathbf{MC}^{(G)}_{\mathrm{std}}) \arrow[r, "\pi_*"] & \mathrm{KK}^G(P,\mathbf{Q}^{(G)}_{\mathrm{std}})\,.
\end{tikzcd} \tag{10.15}$$

By [12, Thm. 1.32.5] the upper sequence becomes a fibre sequence, too. Therefore, the dotted arrow becomes an equivalence. Furthermore, $\mathbf{MC}^{(G)}_{\mathrm{std}}$ is flasque by Lemma 2.21 so that $\mathrm{KK}^G(P,\mathbf{MC}^{(G)}_{\mathrm{std}}) \simeq 0$ by [12, Thm. 1.32.7], and j_* becomes an equivalence.

We consider the following two natural transformations

$$\Sigma^{-1}K^{G,\mathrm{An}}_{\mathbf{C}} \stackrel{\mathrm{def}}{=} \Sigma^{-1}K^{G,\mathrm{an}}_{\mathbf{Q}^{(G)}_{\mathrm{std}}} \xrightarrow{j_*} K^{G,\mathrm{an}}_{F(\pi)} \tag{10.16}$$

and

$$K^{G,\mathrm{an}}_{A} \stackrel{(10.11)}{\simeq} K^{G,\mathrm{an}}_{\mathbf{C}^{(G)}_{\mathrm{std}}} \xrightarrow{i_*} K^{G,\mathrm{an}}_{F(\pi)} \tag{10.17}$$

of $\mathbf{Sp}$-valued functors on $G\mathbf{LCH}^{\mathrm{prop}}_{+}$, where i_* and j_* are induced by the morphisms i and j in (10.14). Since by [12, Prop. 1.26] the restriction of $\mathrm{kk}^G \circ C_0(-)$ to $G\mathbf{LCH}^{\mathrm{prop,hfin}}_{+}$ takes values in ind-G-proper objects, the restrictions of j_* in (10.16) and i_* in (10.17) to $G\mathbf{LCH}^{\mathrm{prop,hfin}}_{+}$ are equivalences.

We apply the $(-)^{\mathrm{lf}}$-construction to the transformations in (10.16) and (10.17) and get transformations

$$\Sigma^{-1}K_{\mathbf{C}}^{G,\mathrm{An},\mathrm{lf}} \to K_{F(\pi)}^{G,\mathrm{an},\mathrm{lf}} \tag{10.18}$$

and

$$K_{A}^{G,\mathrm{an},\mathrm{lf}} \to K_{F(\pi)}^{G,\mathrm{an},\mathrm{lf}} . \tag{10.19}$$

We now show that the evaluations of (10.18) and (10.19) at X in $G\mathbf{LCH}^{\mathrm{prop},\sigma\mathrm{hfin}}_{\mathrm{2nd},+}$ are equivalences. By homotopy invariance of the domains and targets we can assume that X is a countable G-CW-complex with proper G-action. By local compactness, it admits a cofinal family of open subsets U with G-compact closure belonging to $G\mathbf{LCH}^{\mathrm{prop},\mathrm{hfin}}_{+}$. This implies that j_* in (10.16) and i_* in (10.17) become equivalences after evaluation at such U. We get the equivalences (10.18) and (10.19) as limits of equivalences. The desired equivalence (10.12) is now defined as the suspension of the composition

$$\begin{aligned}(K_{A}^{G,\mathrm{an},\mathrm{lf}})_{|G\mathbf{LCH}^{\mathrm{prop},\sigma\mathrm{hfin}}_{\mathrm{2nd},+}} &\xrightarrow{(10.19),\simeq} (K_{F(\pi)}^{G,\mathrm{an},\mathrm{lf}})_{|G\mathbf{LCH}^{\mathrm{prop},\sigma\mathrm{hfin}}_{\mathrm{2nd},+}} \\ &\xleftarrow{\simeq,(10.18)} (\Sigma^{-1}K_{\mathbf{C}}^{G,\mathrm{An},\mathrm{lf}})_{|G\mathbf{LCH}^{\mathrm{prop},\sigma\mathrm{hfin}}_{\mathrm{2nd},+}} .\end{aligned} \tag{10.20}$$

It now remains to show that the canonical transformation (10.13) is an equivalence. We can again assume that X is a countable G-CW-complex with proper G-action. We let $(U_n)_{n\in\mathbb{N}}$ be an exhaustion of X by an increasing family of invariant open subsets with G-compact closure. Then setting $Y_n := X \setminus U_n$ the family $(Y_n)_{n\in\mathbb{N}}$ is a decreasing family of closed invariant subsets of X with $\bigcap_{n\in\mathbb{N}} Y_n = \emptyset$. We get a diagram of maps

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
K_A^{G,\mathrm{an}}(Y_{n+1}) \arrow[r] \arrow[d] & K_A^{G,\mathrm{an}}(X) \arrow[r] \arrow[d] & K_A^{G,\mathrm{an}}(U_{n+1}) \arrow[d] \\
K_A^{G,\mathrm{an}}(Y_{n}) \arrow[r] \arrow[d] & K_A^{G,\mathrm{an}}(X) \arrow[r] \arrow[d] & K_A^{G,\mathrm{an}}(U_{n}) \arrow[d] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

whose horizontal pieces are fibre sequences by [12, Thm. 1.15.3]. Here we use that the inclusions $Y_n \to X$ are split-closed by [12, Prop. 5.1.1] and our topological assumptions on X. We now consider the fibre sequence obtained as the limit of this diagram in the vertical direction. Using that A is separable and [12, Thm. 1.15.6] the limit of the left column vanishes. Hence, we get an equivalence

$$K_A^{G,\mathrm{an}}(X) \xrightarrow{\simeq} \lim_{n\in\mathbb{N}} K_A^{G,\mathrm{an}}(U_{n+1}) \simeq K_A^{G,\mathrm{an,lf}}(X)$$

as desired. □

11. Comparison with classical constructions

As explained already in the introduction the classical definition of the domain of the Paschke morphism does not involve a C^*-category of controlled Hilbert spaces but it involves the choice of a single sufficiently large continuously controlled Hilbert space. So in order to compare the approach of the present paper with the classical one we specialize to the case of trivial coefficients characterized by $\mathbf{C} = \mathbf{Hilb}_c(\mathbb{C})$ and $\mathbf{MC} = \mathbf{Hilb}(\mathbb{C})$. According to Definition 2.8 the objects of $\mathbf{Hilb}(\mathbb{C})^{(G)}$ are pairs (H,ρ) of a Hilbert space H and a unitary representation $\rho : G^{\mathrm{op}} \to U(H)$, $g \mapsto \rho_g$. The morphisms are given by $\mathrm{Hom}_{\mathbf{Hilb}(\mathbb{C})^{(G)}}((H,\rho),(H',\rho')) = B(H,H')$, the bounded linear operators from H to H'. The group G fixes the objects of $\mathbf{Hilb}(\mathbb{C})^{(G)}$ and acts on the morphisms by $g\cdot A := \rho_g^{\prime,-1} A \rho_g$.

We consider a second countable proper metric space X with an isometric action of the group G. In the following, we construct an exact sequence of C^*-categories

$$0 \to \mathbf{C}^G(X) \to \mathbf{D}^G(X) \to \mathbf{Q}^G(X) \to 0\,. \tag{11.1}$$

We start with the definition of a C^*-category $\mathbf{B}(X)$ with G-action. Its objects are triples (H,ρ,ϕ), where (H,ρ) is in $\mathbf{Hilb}(\mathbb{C})^{(G)}$ such that H is separable and $\phi : C_0(X) \to B(H)$ is homomorphism of C^*-algebras satisfying the following properties:

(1) The representation ϕ is equivariant, i.e., we have $g^{-1}\cdot\phi(f) = \phi(g^*f)$ for all f in $C_0(X)$ and g in G, see (5.3).

(2) The representation ϕ is non-degenerate in the sense that $\overline{\phi(C_0(X))H} = H$.

(3) There exists an equivariant unitary isomorphism $(H,\rho) \cong (L^2(G)\otimes H', \lambda\otimes \mathrm{id}_H)$, where λ is the left-regular representation of G on $L^2(G)$ and H' is some auxiliary separable Hilbert space.

The morphisms of $\mathbf{B}(X)$ are inherited from $\mathbf{Hilb}(\mathbb{C})^{(G)}$. The group G fixes the objects of $\mathbf{B}(X)$ and acts on morphisms as in $\mathbf{Hilb}(\mathbb{C})^{(G)}$.

Let (H,ρ,ϕ) and (H',ρ',ϕ') be objects of $\mathbf{B}(X)$. An operator A in $B(H,H')$ is called locally compact if $\phi'(f)A$ and $A\phi(f)$ belong to $K(H,H')$ for all f in $C_0(X)$, where $K(H,H')$ denotes the set of compact linear operators from H to H'. Further, A is called pseudolocal if $\phi'(f)A - A\phi(f) \in K(H,H')$ for all f in $C_0(X)$. Finally, it is called controlled if there exists R in $(0,\infty)$ such that $d(\mathrm{supp}(f'),\mathrm{supp}(f)) \geq R$ implies that $\phi'(f)A\phi(f) = 0$. The C^*-category $\mathbf{C}^G(X)$ is the wide C^*-subcategory of $\mathbf{B}(X)$ generated by the invariant, locally compact and controlled operators. Similarly the C^*-category $\mathbf{D}^G(X)$ is generated by the invariant, pseudolocal and controlled operators. Finally $\mathbf{Q}^G(X)$ is defined as the quotient, see (11.1). If (H, ρ, ϕ) is an object of $\mathbf{B}(X)$, then the corresponding endomorphism algebras form an exact sequence

$$0 \to C^G(H, \rho, \phi) \to D^G(H, \rho, \phi) \to Q^G(H, \rho, \phi) \to 0$$

which is the equivariant generalization of (1.11) from the introduction.

Definition 11.1. *An object (H, ρ, ϕ) of $\mathbf{D}^G(X)$ is called absorbing if for every other (H', ρ', ϕ') in $\mathbf{D}^G(X)$ there exists an isometry $u : (H', \rho', \phi') \to (H, \rho, \phi)$ in $\mathbf{D}^G(X)$.*

The existence of absorbing objects in the case of trivial G follows from [22, Lem. 7.7].[5] For the following discussion, we assume that we can choose an absorbing object (H, ρ, ϕ). We set $Q^G(X) := Q^G(H, \rho, \phi)$ and let $Q(H)$ be the Calkin algebra of H with the induced G-action. With these choices we can define the Paschke morphism

$$p_X^{(H,\rho,\phi)} := \mu_X \circ \delta_X : \mathrm{KK}(\mathbb{C}, Q^G(X)) \to \mathrm{KK}^G(C_0(X), Q(H))$$

as in (1.14). We can consider X as an object of $G\mathbf{UBC}$ with the structures induced by the metric. We furthermore assume that X is homotopy equivalent to a G-compact G-CW-complex with finite stabilizers. The following proposition asserts that the Paschke morphism p_X from (1.17) is compatible with $p_X^{(H,\rho,\phi)}$.

Proposition 11.2. *There exists a commutative square*

$$\begin{tikzcd} K\mathbf{C}^{G,\mathcal{X}}(X) \arrow[r, "\gamma"] \arrow[d, "p_X"] & \mathrm{KK}(\mathbb{C}, Q^G(X)) \arrow[d, "p_X^{(H,\rho,\phi)}"] \\ K\mathbf{C}^{G,\mathrm{An}}(\iota^{\mathrm{top}}(X)) & \mathrm{KK}^G(C_0(X), Q(H)) \arrow[l, "\simeq"'] \end{tikzcd} \tag{11.2}$$

Proof. We use the identifications

$$K\mathbf{C}^{G,\mathcal{X}}(X) \overset{\text{Lem. 6.1}}{\simeq} \mathrm{KK}(\mathbb{C}, \mathbf{Q}(X))$$

and

$$K\mathbf{C}^{G,\mathrm{An}}(\iota^{\mathrm{top}}(X)) \overset{(1.3)}{=} \mathrm{KK}^G(C_0(X), \mathbf{Q}^{(G)}_{\mathrm{std}})\,.$$

The objects of $\mathbf{Q}(X)$ (and also of $\mathbf{D}(X)$ and $\mathbf{C}(X)$, see (6.3) and (6.4)) are the objects of $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(X) \otimes G_{can,max})$. If (H', ρ', μ') is such an object, we get the object (H', ρ', ϕ') of $\mathbf{D}^G(X)$ with ϕ' as in (5.7). Note that since X is second countable and has the bornology of relatively compact subsets, the Hilbert space H' is separable by the local finiteness conditions (see Definition 2.12) on (H', ρ', μ'). Furthermore, using that $X \times G$ is a free G-set we see that (H', ρ') is a multiple of the regular representation of G on $L^2(G)$. Since we assume that (H, ρ, ϕ) is absorbing there exists an isometry $u' : (H', \rho', \phi') \to (H, \rho, \phi)$ in $\mathbf{D}^G(X)$.

[5] We neither know a reference nor have a proof for the existence of absorbing objects in the equivariant case in full generality, see Remark 11.4.

We consider the category $\mathbf{D}^u(X)$ consisting of pairs $((H',\rho',\mu'),u')$ of an object (H',ρ',μ') in $\mathbf{D}(X)$ and an isometry u as above. A morphism

$$A\colon ((H',\rho',\mu'),u') \to ((H'',\rho'',\mu''),u'')$$

is a morphism $A\colon (H',\rho',\mu') \to (H'',\rho'',\mu'')$ in $\mathbf{D}(X)$. We define $\mathbf{C}^u(X)$ and $\mathbf{Q}^u(X)$ similarly. Then we have a diagram of maps of exact sequences of C^*-categories

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{C}(X) \arrow[r] & \mathbf{D}(X) \arrow[r] & \mathbf{Q}(X) \arrow[r] & 0 \\
0 \arrow[r] & \mathbf{C}^u(X) \arrow[r] \arrow[u] \arrow[d] & \mathbf{D}^u(X) \arrow[r] \arrow[u] \arrow[d] & \mathbf{Q}^u(X) \arrow[r] \arrow[u] \arrow[d] & 0 \\
0 \arrow[r] & C^G(X) \arrow[r] & D^G(X) \arrow[r] & Q^G(X) \arrow[r] & 0
\end{tikzcd}$$

where in the lower sequence we consider the C^*-algebras as C^*-categories with a single object. The upper vertical functors just forget the embedding u' and are unitary equivalences. The definition of the lower vertical functors on the objects is clear. The functor $\mathbf{D}^u(X) \to D^G(X)$ sends a morphism $A\colon ((H',\rho',\mu'),u') \to ((H'',\rho'',\mu''),u'')$ to $u''Au'^{,*}$. The other functors are defined similarly. Since $K^{C^*\mathrm{Cat}}$ sends unitary equivalences to equivalences, we get the following morphism

$$\begin{tikzcd}
K^{C^*\mathrm{Cat}}(\mathbf{C}(X)) \arrow[r] \arrow[d, "\simeq"', "\alpha"] & K^{C^*\mathrm{Cat}}(\mathbf{D}(X)) \arrow[r] \arrow[d] & K^{C^*\mathrm{Cat}}(\mathbf{Q}(X)) \arrow[d, "\gamma"] \\
K^{C^*\mathrm{Alg}}(C^G(X)) \arrow[r] & K^{C^*\mathrm{Alg}}(D^G(X)) \arrow[r] & K^{C^*\mathrm{Alg}}(Q^G(X))
\end{tikzcd}$$

of fibre sequences. The right vertical map is the map γ in the square (11.2). The map α is an equivalence by [8, Thm. 6.1], but this will not be used here.

If X is homotopy equivalent to a G-finite G-CW complex with finite stabilizers, then the functor $\mathrm{KK}^G(C_0(X),-)$ sends exact sequences in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ to fibre sequences by a combination of [12, Prop. 1.26] and [12, Thm. 1.32.5]. The lower horizontal map in (11.2) is induced by the functor $Q(H) \to \mathbf{Q}^{(G)}_{\mathrm{std}}$ which just views (H,ρ) as an object of $\mathbf{Q}^{(G)}_{\mathrm{std}}$. In order to show that it is an equivalence, we consider the map of fibre sequences obtained by applying $\mathrm{KK}^G(C_0(X),-)$ to the map of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & K(H) \arrow[r] \arrow[d] & B(H) \arrow[r] \arrow[d] & Q(H) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbf{Hilb}_c(\mathbb{C})^{(G)}_{\mathrm{std}} \arrow[r] & \mathbf{Hilb}(\mathbb{C})^{(G)}_{\mathrm{std}} \arrow[r] & \mathbf{Q}^{(G)}_{\mathrm{std}} \arrow[r] & 0
\end{tikzcd} \tag{11.3}$$

The vertical maps send the unique object of the domain to the object (H,ρ). We have $\mathrm{KK}^G(C_0(X), B(H)) \simeq 0$ by [12, Cor. 6.22], and we also have

$$\mathrm{KK}^G(\mathbb{C}, \mathbf{Hilb}(\mathbb{C})^{(G)}_{\mathrm{std}}) \simeq 0$$

by [12, Thm. 1.32.7] since $\mathbf{Hilb}(\mathbb{C})^{(G)}_{\mathrm{std}}$ is flasque by Lemma 2.21.

We will show that the left vertical map in (11.3) induces an equivalence after applying $\mathrm{KK}^G(C_0(X), -)$. We let $\mathbf{Hilb}_c(\mathbb{C})^{(G),\mathrm{sep}}_{\mathrm{std}}$ and $\mathbf{Hilb}(\mathbb{C})^{(G),\mathrm{sep}}_{\mathrm{std}}$ denote the full subcategories of $\mathbf{Hilb}_c(\mathbb{C})^{(G)}_{\mathrm{std}}$ and $\mathbf{Hilb}(\mathbb{C})^{(G)}_{\mathrm{std}}$, respectively, of separable Hilbert spaces. Then we have a factorization of the left vertical morphism in (11.3) as

$$K(H) \to \mathbf{Hilb}_c(\mathbb{C})^{(G),\mathrm{sep}}_{\mathrm{std}} \to \mathbf{Hilb}_c(\mathbb{C})^{(G)}_{\mathrm{std}} . \tag{11.4}$$

We claim that first morphism is an idempotent completion relative to the ideal inclusion $K(H) \to B(H)$, and therefore a relative Morita equivalence by [9, Prop. 17.8]. In order to see the claim, note we have an equivariant unitary isomorphism $(H,\rho) \cong (L^2(G) \otimes H', \lambda \otimes \mathrm{id}_{H'})$. Since (H,ρ,ϕ) is absorbing we can in addition assume that $\dim(H') = \infty$. Since every separable Hilbert space is isomorphic to a subspace of H' we see that every object of $\mathbf{Hilb}(\mathbb{C})^{(G),\mathrm{sep}}_{\mathrm{std}}$ admits an isometry to (H,ρ). We now consider the square

$$\begin{tikzcd} K(H) \arrow[r] \arrow[d] & B(H) \arrow[d] \\ \mathbf{Hilb}_c(\mathbb{C})^{(G),\mathrm{sep}}_{\mathrm{std}} \arrow[r] & \mathbf{Hilb}(\mathbb{C})^{(G),\mathrm{sep}}_{\mathrm{std}} \end{tikzcd}$$

where the horizontal maps are ideal inclusions. By the observation above the right vertical map presents $\mathbf{Hilb}(\mathbb{C})^{(G),\mathrm{sep}}_{\mathrm{std}}$ as the idempotent completion of $B(H)$.

The second morphism in (11.4) is easily seen to be a weak Morita equivalence. Since $\mathrm{KK}^G(C_0(X), -)$ sends both relative Morita equivalences and weak Morita equivalences to equivalences by [12, Thm. 1.32.8] and [12, Thm. 1.32.3], respectively, the left vertical morphism in (11.3) induces an equivalence after applying $\mathrm{KK}^G(C_0(X), -)$. This together with the fact that this functor annihilates $B(H)$ and $\mathbf{Hilb}(\mathbb{C})^{(G)}_{\mathrm{std}}$ implies that

$$\mathrm{KK}^G(C_0(X), Q(H)) \to \mathrm{KK}^G(C_0(X), \mathbf{Q}^{(G)}_{\mathrm{std}})$$

is an equivalence. This explains the lower horizontal equivalence in (11.2).

It is obvious from the definitions of the Paschke morphisms in (1.14) and Definition 6.3 that the diagram commutes. □

In the following, we assume that X satisfies the assumptions of Theorem 1.5.2 such that p_X is an equivalence.

Corollary 11.3. *The morphism γ is an equivalence if and only if $p_X^{(H,\rho,\phi)}$ is an equivalence.*

This says that in all cases where the classical Paschke morphism $p_X^{(H,\rho,\phi)}$ is an equivalence it is equivalent to our morphism p_X as a spectrum map. An independent proof[6] that γ is an equivalence would then allow us to conclude from Theorem 1.5.2 that $p_X^{(H,\rho,\phi)}$ is an equivalence.

Remark 11.4. This is a remark about the existence of absorbing objects an in Definition 11.1. First of all the discussion above depends on the existence of an absorbing object in $\mathbf{D}^G(X)$ for which we neither have a reference nor a proof. Related results are [37, Lem. 4.5.5 & Prop. 4.5.14]. They are adapted for the approach based on localization algebras but do not imply directly what we need. A similar remark applies to [2, Thm. 1.3].

In the non-equivariant case, the existence of absorbing objects is settled in [22, Lem. 7.7] by an application of Voiculescu's Theorem.

We furthermore do not know a reference for the fact that $p_X^{(H,\rho,\phi)}$ is an equivalence. In fact, [2, Thm. 1.5] states a Paschke duality isomorphism in the equivariant case. But it is not obvious how to identify the targets and the maps in [2, Thm. 1.5] with $p_X^{(H,\rho,\phi)}$. □

12. Homotopy theoretic and analytic assembly maps

In this section, we describe the homotopy theoretic and the analytic assembly maps which we will eventually compare in Theorem 1.9. The homotopy theoretic assembly introduced in Definition 12.2 is a standard construction from equivariant homotopy theory [16]. For the historic development of the analytic assembly map, we refer to [19]. Our Definition 12.12 is a spectrum valued refinement of the assembly map of [25, 1] which is new in this form.

We begin with the homotopy theoretic assembly map. Let $G\mathbf{Orb}$ denote the orbit category of G and $\mathbf{M}$ be some cocomplete stable ∞-category. Recall that by Definition 10.1 an equivariant $\mathbf{M}$-valued homology theory is simply a functor

$$E : G\mathbf{Orb} \to \mathbf{M}\,.$$

Let $\mathcal{F}$ be a family of subgroups of G. By $G_{\mathcal{F}}\mathbf{Orb}$ we denote the full subcategory of the orbit category $G\mathbf{Orb}$ of transitive G-sets with stabilizers in the family $\mathcal{F}$. Since $*$ is a final object of $G\mathbf{Orb}$ we have a natural transformation $E \to \underline{E(*)}$ in $\mathrm{Fun}(G\mathbf{Orb}, \mathbf{M})$. This transformation induces the homotopy theoretic assembly map:

Definition 12.1. *The homotopy theoretic assembly map for E and $\mathcal{F}$ is the canonical morphism*

$$\mathrm{Asmbl}^h_{E,\mathcal{F}} : \mathrm{colim}_{G_{\mathcal{F}}\mathbf{Orb}} E \to E(*)$$

in $\mathbf{M}$.

[6] We do not know a reference for such a proof.

Recall that we can evaluate the equivariant homology theory E on G-topological spaces using (10.3). For every X in $G\mathbf{Top}$, we get a morphism

$$\mathrm{Asmbl}^h_{E,X} : E(X) \to E(*) \tag{12.1}$$

which is induced by the projection $X \to *$. We let $E_{\mathcal{F}}G^{\mathrm{CW}}$ be a G-CW complex representing the homotopy type of the classifying space for the family $\mathcal{F}$. It is characterized essentially uniquely by the condition that

$$Y^G(E_{\mathcal{F}}G^{\mathrm{CW}})(S) \simeq \begin{cases} \emptyset & S \notin G_{\mathcal{F}}\mathbf{Orb}\,, \\ * & S \in G_{\mathcal{F}}\mathbf{Orb}\,. \end{cases} \tag{12.2}$$

As a consequence of (10.4) we then get the equivalence $E(E_{\mathcal{F}}G^{\mathrm{CW}}) \simeq \mathrm{colim}_{G_{\mathcal{F}}\mathbf{Orb}}E$, and under this identification we have the equivalence

$$\mathrm{Asmbl}^h_{E,\mathcal{F}} \simeq \mathrm{Asmbl}^h_{E,E_{\mathcal{F}}G^{\mathrm{CW}}} \tag{12.3}$$

of assembly maps. Further below, in the special case of the functor $E = \hat{K}^G_A$ introduced in Definition 15.10 for A in KK^G we will use the notation

$$\mu^{DL}_{A,X} := \mathrm{Asmbl}^h_{\hat{K}^G_A,X}\,, \quad \mu^{DL}_{A,\mathcal{F}} := \mathrm{Asmbl}^h_{\hat{K}^G_A,\mathcal{F}} \tag{12.4}$$

indicating that $\mu^{DL}_{A,\mathcal{F}}$ is the assembly map introduced by Davis–Lück [16].

We have a functor

$$\iota : G\mathbf{Orb} \to G\mathbf{BC}\,, \quad S \mapsto S_{min,max}\,, \tag{12.5}$$

where $S_{min,max}$ is the G-set S equipped with the minimal coarse structure and the maximal bornology. For a coefficient category $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ which is effectively additive and admits countable AV-sums, we have an equivariant coarse K-homology functor

$$K\mathbf{C}\mathcal{X}^G_{G_{can,min}} : G\mathbf{BC} \to \mathbf{Sp}$$

(see Definition 3.4 for $K\mathbf{C}\mathcal{X}^G$ and Definition 4.7 for the twist of an equivariant coarse homology theory by an object of $G\mathbf{BC}$, in the present case by $G_{can,min}$). The following is the technical definition of the functor described in (1.19).

Definition 12.2. *We define the functor*

$$K\mathbf{C}^G : G\mathbf{Orb} \xrightarrow{\iota} G\mathbf{BC} \xrightarrow{K\mathbf{C}\mathcal{X}^G_{G_{can,min}}} \mathbf{Sp}\,.$$

We now apply the definitions of assembly maps explained above to the functor $K\mathbf{C}^G$ in place of E and introduce a shorter notation.

Definition 12.3. *The homotopy theoretic assembly map associated to G, $\mathcal{F}$ and $\mathbf{C}$ is defined to be the map*

$$\mathrm{Asmbl}^h_{\mathbf{C},\mathcal{F}} := \mathrm{Asmbl}^h_{K\mathbf{C}^G,\mathcal{F}} : \mathrm{colim}_{G_{\mathcal{F}}\mathbf{Orb}} K\mathbf{C}^G \to K\mathbf{C}^G(*)\,.$$

More generally, for every X in $G\mathbf{Top}$, specializing (12.1), we have the morphism

$$\mathrm{Asmbl}^h_{\mathbf{C},X} := \mathrm{Asmbl}^h_{K\mathbf{C}^G,X} : K\mathbf{C}^G(X) \to K\mathbf{C}^G(*) \tag{12.6}$$

induced by the projection $X \to *$. Since $K\mathbf{C}\mathcal{X}^G$ depends naturally on the coefficient category $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg,eadd},\omega\mathrm{add}})$, see (2.11) for the definition of this category, so do the assembly maps $\mathrm{Asmbl}^h_{\mathbf{C},X}$ and $\mathrm{Asmbl}^h_{\mathbf{C},\mathcal{F}}$.

We now turn to the analytic assembly map whose final definition will be stated in Definition 12.12. We start with introducing the notation for its domain. Recall that $G\mathbf{LCH}^{\mathrm{prop}}_+$ is the category of locally compact Hausdorff G-spaces with partially defined proper maps.

Definition 12.4. *We denote by $G\mathbf{LCH}^{\mathrm{prop}}_{+,\mathrm{pc}}$ the full category of $G\mathbf{LCH}^{\mathrm{prop}}_+$ of spaces on which G acts properly and cocompactly.*

We will describe the analytic assembly map $\mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C},\mathcal{F}}$ associated to $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ and a family $\mathcal{F}$ contained in $\mathbf{Fin}$. In analogy to (12.1), we will further describe a natural transformation

$$\mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C}} : K^{G,\mathrm{An}}_{\mathbf{C}}(-) \to \underline{\Sigma\mathrm{KK}(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)}$$

of functors from $G\mathbf{LCH}^{\mathrm{prop}}_{+,\mathrm{pc}}$ to $\mathbf{Sp}$. Note that for infinite G the morphism

$$\mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C},X} : K^{G,\mathrm{An}}_{\mathbf{C}}(X) \to \Sigma\mathrm{KK}(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G) \tag{12.7}$$

can not simply be induced by a map $X \to *$ since $*$ and therefore this map are not in the category $G\mathbf{LCH}^{\mathrm{prop}}_{+,\mathrm{pc}}$. If $E_{\mathcal{F}}G^{\mathrm{CW}}$ happens to be in $G\mathbf{LCH}^{\mathrm{prop}}_{+,\mathrm{pc}}$, then we will have an equivalence $\mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C},\mathcal{F}} \simeq \mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C},E_{\mathcal{F}}G^{\mathrm{CW}}}$ in analogy to (12.3).

The classical definition of the analytic assembly map is based on a construction of a family $(\mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C},X,*})_{X\in G\mathbf{LCH}^{\mathrm{prop}}_{+,\mathrm{pc}}}$ of homomorphisms in $\mathbf{Ab}^{\mathbb{Z}}$

$$\mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C},X,*} : K^{G,\mathrm{An}}_{\mathbf{C},*}(X) = \mathrm{KK}^G_*(C_0(X), \mathbf{Q}^{(G)}_{\mathrm{std}}) \to \mathrm{KK}_{*-1}(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)\,, \tag{12.8}$$

which implement a natural transformation

$$K^{G,\mathrm{An}}_{\mathbf{C},*}(-) \to \underline{\mathrm{KK}_{*-1}(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)} \tag{12.9}$$

of functors from $G\mathbf{LCH}^{\mathrm{prop}}_{+,\mathrm{pc}}$ to $\mathbf{Ab}^{\mathbb{Z}}$.

In the following, we describe the details of the construction of $\mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C},X,*}$ in (12.8) thereby lifting it to the spectrum level. The construction has three steps. The first is an application of functor $-\rtimes G$ from [12, Thm. 1.22.3], where $\rtimes$ without subscript refers to the maximal crossed product. The second is a pull-back along the Kasparov projection given by (12.16) below. The last step consists of changing target categories (12.20).

The following discussion will be used to get rid of the choice of cut-off functions involved in the Kasparov projection. Here we can take full advantage of the ∞-categorical set-up. We let

$$\mathcal{R}\colon\ G\mathbf{LCH}^{\mathrm{prop}^{\mathrm{op}}}_{+,\mathrm{pc}} \to \mathbf{Set}$$

be the following functor:

(1) objects: The functor $\mathcal{R}$ sends X to the set $\mathcal{R}(X)$ of all functions χ in $C_c(X)$ such that
$$\sum_{g\in G} g^*\chi^2 = 1\,. \tag{12.10}$$
(2) morphisms: The functor $\mathcal{R}$ sends a morphism $f\colon\ X \to X'$ in $G\mathbf{LCH}^{\mathrm{prop}}_{+,\mathrm{pc}}$ to the map $\mathcal{R}(f)\colon\ \mathcal{R}(X') \to \mathcal{R}(X)$ which sends χ' in $\mathcal{R}(X')$ to $f^*\chi'$ in $\mathcal{R}(X)$.

For χ in $\mathcal{R}(X)$, we define the Kasparov projection
$$p_\chi := \sum_{g\in G}(\chi\cdot g^*\chi, g) \tag{12.11}$$
in $C_0(X)\rtimes G$. Note that this sum has finitely many non-zero terms.

If $f\colon\ X\to X'$ is a morphism in $G\mathbf{LCH}^{\mathrm{prop}}_{+,\mathrm{pc}}$ and χ' is in $\mathcal{R}(X')$, then we have the relation
$$(f^*\rtimes G)(p_{\chi'}) = p_{f^*\chi'}\,.$$
Hence, we get a natural transformation of contravariant $\mathbf{Set}$-valued functors
$$\mathcal{R}(-)\to \mathrm{Hom}_{C^*\mathbf{Alg}^{\mathrm{nu}}}(\mathbb{C}, C_0(-)\rtimes G)$$
on $G\mathbf{LCH}^{\mathrm{prop}}_{+,\mathrm{pc}}$ which sends χ in $\mathcal{R}(X)$ to the homomorphism
$$\mathbb{C}\ni\lambda\mapsto\lambda p_\chi\in C_0(X)\rtimes G\,.$$
Composing with kk we get a natural transformation of $\mathbf{Spc}$-valued functors
$$\ell'\mathcal{R}(-)\to\Omega^\infty\mathrm{KK}(\mathbb{C}, C_0(X)\rtimes G)\,,$$
where $\ell'\colon\ \mathbf{Set}\to\mathbf{Spc}$ is the canonical inclusion. Using the $(\Sigma^\infty_+,\Omega^\infty)$-adjunction we can interpret the result as a transformation
$$\Sigma^\infty_+\ell'\mathcal{R}(-)\to\mathrm{KK}(\mathbb{C}, C_0(-)\rtimes G) \tag{12.12}$$
of $\mathbf{Sp}$-valued functors.

Let $E\colon\ G\mathbf{LCH}^{\mathrm{prop}^{\mathrm{op}}}_{+,\mathrm{pc}}\to\mathbf{M}$ be any functor to a cocomplete target. We have a functor
$$q\colon\ G\mathbf{LCH}^{\mathrm{prop}}_{+,\mathrm{pc}}\times\Delta\to G\mathbf{LCH}^{\mathrm{prop}}_{+,\mathrm{pc}}$$
which sends $(X,[n])$ to $X\times\Delta^n$ with the G-action only on the first factor. We define the homotopification of E by
$$\mathcal{H}(E) := q_*q^*E\colon\ (G\mathbf{LCH}^{\mathrm{prop}}_{+,\mathrm{pc}})^{\mathrm{op}}\to\mathbf{Sp}\,,$$
where q^* is the pull-back along q and q_* is the right-adjoint of q^*, the right Kan-extension functor. The unit of the adjunction (q^*,q_*) provides a natural transformation $E\to\mathcal{H}(E)$. We say that E is homotopy invariant if the projection

$X\times\Delta^1\to X$ induces an equivalence $E(X)\xrightarrow{\simeq}E(X\times\Delta^1)$. A proof of the following lemma is for instance implicitly given in the proof of [14, Lem. 7.5]

Lemma 12.5 (cf. [14, Lem. 7.5])**.**

(1) *$\mathcal{H}(E)$ is homotopy invariant.*
(2) *E is homotopy invariant if and only if the canonical morphism $E\to\mathcal{H}(E)$ is an equivalence.*

Let S denote the sphere spectrum and $\underline{S}:\ G\mathbf{LCH}^{\mathrm{prop}^{\mathrm{op}}}_{+,\mathrm{pc}}\to\mathbf{Sp}$ be the constant functor with value S.

Lemma 12.6. *The projection $\mathcal{R}\to\underline{*}$ induces an equivalence $\mathcal{H}(\Sigma^\infty_+\ell'\mathcal{R})\simeq\underline{S}$.*

Proof. By the pointwise formula for the left Kan extension $q_!$ we must show that the projection $\mathcal{R}\to\underline{*}$ induces for every X in $G\mathbf{LCH}^{\mathrm{prop}}_{+,\mathrm{pc}}$ an equivalence

$$\mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}}\Sigma^\infty_+\ell'\mathcal{R}(X\times\Delta^n)\xrightarrow{\simeq}S\,.$$

Since $\Sigma^\infty_+:\ \mathbf{Spc}\to\mathbf{Sp}$ preserves colimits it actually suffices to show that

$$\mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}}\ell'\mathcal{R}(X\times\Delta^n)\xrightarrow{\simeq}*$$

in $\mathbf{Spc}$. For a simplicial set W, the colimit $\mathrm{colim}_{\Delta^{\mathrm{op}}}\ell'W$ is given by $\ell(|W|)$, where ℓ is as in (10.1) and $|W|$ in $\mathbf{Top}$ is the geometric realization of W. Since the geometric realization of the total space of a trivial Kan fibration over a point is contractible it therefore suffices to show that the map of simplicial sets $\mathcal{R}(X\times\Delta^-)\to*$ is a trivial Kan fibration. So we must show that for every n in $\mathbb{N}$ a function χ in $\mathcal{R}(X\times\partial\Delta^n)$ can be extended to a function $\tilde\chi$ in $\mathcal{R}(X\times\Delta^n)$.

For the case $n=0$, we observe that for any X in $G\mathbf{LCH}^{\mathrm{prop}}_{+,\mathrm{pc}}$ we have $\mathcal{R}(X)\neq\emptyset$. For $n\geq1$, using barycentric coordinates, we can write a point in Δ^n in the form σt where σ is in $[0,1]$ and t is in $\partial\Delta^n$. Then an extension of χ is given by

$$\tilde\chi(x,\sigma t):=\sqrt{\sigma\chi(x,t)^2+(1-\sigma)\chi(x,t_0)^2}\,,$$

where t_0 is the zero'th vertex of the simplex. □

We now use that $\mathrm{KK}(\mathbb{C},C_0(-)\rtimes G)$ is a homotopy invariant $\mathbf{Sp}$-valued functor. Applying $\mathcal{H}$ to (12.12) we get a transformation

$$\begin{aligned}\epsilon:\ \underline{S}\overset{\text{Lem. 12.6}}{\simeq}\mathcal{H}(\Sigma^\infty_+\ell'\mathcal{R})&\xrightarrow{\mathcal{H}(12.12)}\mathcal{H}(\mathrm{KK}(\mathbb{C},C_0(-)\rtimes G))\\&\xleftarrow{\simeq,\text{Lem. 12.5.2}}\mathrm{KK}(\mathbb{C},C_0(-)\rtimes G)\,.\end{aligned}\tag{12.13}$$

Let A be an object of KK^G and consider the functor from [12, Def. 1.14]:

$$K^{G,\mathrm{an}}_A:=\mathrm{KK}(C_0(-),A):\ G\mathbf{LCH}^{\mathrm{prop}}_+\to\mathbf{Sp}\,.\tag{12.14}$$

We have the maximal[7] crossed product functor $- \rtimes G$ [12, Thm. 1.22.3] whose action on mapping spectra induces the following natural transformation

$$- \rtimes G \,:\, K_A^{G,\mathrm{an}}(-) = \mathrm{KK}^G(C_0(-), A) \to \mathrm{KK}(C_0(-) \rtimes G, A \rtimes G)\,. \qquad (12.15)$$

of functors from $G\mathbf{LCH}^{\mathrm{prop}}_{+,\mathrm{pc}}$ to $\mathbf{Sp}$. The composition of morphisms in KK provides a natural transformation

$$\mathrm{KK}(\mathbb{C}, C_0(-) \rtimes G) \to \mathrm{map}_{\mathbf{Sp}}(\mathrm{KK}(C_0(-) \rtimes G, A \rtimes G), \mathrm{KK}(\mathbb{C}, A \rtimes G))\,.$$

We interpret its pre-composition with (12.13) as a natural transformation

$$\epsilon^* :\, \mathrm{KK}(C_0(-) \rtimes G, A \rtimes G) \to \underline{\mathrm{KK}(\mathbb{C}, A \rtimes G)} \qquad (12.16)$$

of functors on $G\mathbf{LCH}^{\mathrm{prop}}_{+,\mathrm{pc}}$ with values in $\mathbf{Sp}$. The composition of (12.15) and (12.16) is a natural transformation

$$\mu^{\mathrm{Kasp}}_{A,-,\max} :\, K_A^{G,\mathrm{an}}(-) \to \underline{\mathrm{KK}(\mathbb{C}, A \rtimes G)} \qquad (12.17)$$

of functors from $G\mathbf{LCH}^{\mathrm{prop}}_{+,\mathrm{pc}}$ to $\mathbf{Sp}$. We now assume $\mathcal{F} \subseteq \mathbf{Fin}$. In general, $E_{\mathcal{F}}G^{\mathrm{CW}}$ does not belong to $G\mathbf{LCH}^{\mathrm{prop}}_{+,\mathrm{pc}}$ so that we can not apply $K_A^{G,\mathrm{an}}$ or $\mu^{\mathrm{Kasp}}_{-,A,\max}$ to $E_{\mathcal{F}}G^{\mathrm{CW}}$ directly. Therefore, we adopt the following definition.

Definition 12.7. *We let*

$$RK_A^{G,\mathrm{an}} :\, G\mathbf{Top} \to \mathbf{Sp}$$

be the left Kan extension of $(K_A^{G,\mathrm{an}})_{|G\mathbf{LCH}^{\mathrm{prop}}_{+,\mathrm{pc}}}$ *along the inclusion*

$$G\mathbf{LCH}^{\mathrm{prop}}_{+,\mathrm{pc}} \to G\mathbf{Top}\,.$$

In particular, we have the diagram

$$\begin{tikzcd} G\mathbf{LCH}^{\mathrm{prop}}_{+,\mathrm{pc}} \arrow[rr, "K_A^{G,\mathrm{an}}"] \arrow[dr] & \Rightarrow & \mathbf{Sp}\,. \\ & G\mathbf{Top} \arrow[ur, dashed, "RK_A^{G,\mathrm{an}}"'] & \end{tikzcd}$$

The following definition introduces the spectrum-valued refinement of the classical Kasparov assembly map as introduced in [25, 1].

Definition 12.8. *The Kasparov assembly map associated to G, $\mathcal{F}$ and A is defined as the map*

$$\mu^{\mathrm{Kasp}}_{A,\mathcal{F},\max} :\, RK_A^{G,\mathrm{an}}(E_{\mathcal{F}}G^{\mathrm{CW}}) \to \mathrm{KK}(\mathbb{C}, A \rtimes G)$$

induced by the natural transformation in (12.17)*. We further define*

$$\mu^{\mathrm{Kasp}}_{A,\mathcal{F}} :\, RK_A^{G,\mathrm{an}}(E_{\mathcal{F}}G^{\mathrm{CW}}) \to \mathrm{KK}(\mathbb{C}, A \rtimes_r G)$$

as the composition of $\mu^{\mathrm{Kasp}}_{A,\mathcal{F},\max}$ *with the canonical morphism* $A \rtimes G \to A \rtimes_r G$.

[7]In the present paper, we use the convention to denote the maximal crossed product by $\rtimes$ and the reduced by $\rtimes_r$.

Note that both versions of the Kasparov assembly map are, by construction, natural in the coefficient object A in KK^G.

Using the functor $\mathrm{kk}_{C^*\mathbf{Cat}} : \mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}) \to \mathrm{KK}^G$ we consider the Kasparov assembly map as depending on a coefficient C^*-category with G-action in place of A. Recall that we drop $\mathrm{kk}_{C^*\mathbf{Cat}}$ from the notation.

Consider a morphism $\mathbf{C} \to \mathbf{D}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat})$.

Lemma 12.9. *If $\mathbf{C} \to \mathbf{D}$ is a Morita equivalence, then the induced morphism $\mu^{\mathrm{Kasp}}_{\mathbf{C},\mathcal{F}} \to \mu^{\mathrm{Kasp}}_{\mathbf{D},\mathcal{F}}$ is an equivalence.*

Proof. By the functoriality of the Kasparov assembly map we have a commutative square

$$\begin{tikzcd} RK^{G,\mathrm{an}}_{\mathbf{C}}(E_{\mathcal{F}}G^{\mathrm{CW}}) \arrow[r, "\mu^{\mathrm{Kasp}}_{\mathbf{C},\mathcal{F}}"] \arrow[d] & \mathrm{KK}(\mathbb{C}, \mathbf{C} \rtimes_r G) \arrow[d] \\ RK^{G,\mathrm{an}}_{\mathbf{D}}(E_{\mathcal{F}}G^{\mathrm{CW}}) \arrow[r, "\mu^{\mathrm{Kasp}}_{\mathbf{D},\mathcal{F}}"] & \mathrm{KK}(\mathbb{C}, \mathbf{D} \rtimes_r G) \end{tikzcd}$$

It suffices to show that the vertical morphisms are equivalences. We start with the left vertical morphism. Note that

$$RK^{G,\mathrm{an}}_{\mathbf{C}}(E_{\mathcal{F}}G^{\mathrm{CW}}) \simeq \mathrm{colim}_{W \subseteq E_{\mathcal{F}}G^{\mathrm{CW}}} \mathrm{KK}^G(C_0(W), \mathbf{C}),$$

where W runs over the G-finite subcomplexes of $E_{\mathcal{F}}G^{\mathrm{CW}}$. By [12, Prop. 1.26] the objects $\mathrm{kk}^G(C_0(W))$ of KK^G are G-proper and hence ind-G-proper (recall that we assume that the family $\mathcal{F}$ is contained in **Fin**). By [12, Thm. 1.32.8] the functor $\mathrm{KK}^G(C_0(W), -)$ sends relative Morita equivalences to equivalences. Hence, the left vertical arrow in the square above is equivalent to the colimit of equivalences

$$\mathrm{colim}_{W \subseteq E_{\mathcal{F}}G^{\mathrm{CW}}} \mathrm{KK}^G(C_0(W), \mathbf{C}) \to \mathrm{colim}_{W \subseteq E_{\mathcal{F}}G^{\mathrm{CW}}} \mathrm{KK}^G(C_0(W), \mathbf{D})$$

and hence itself an equivalence.

The right vertical arrow in the square is an equivalence since $- \rtimes_r G$ preserves Morita equivalences by [9, Prop. 16.11], and $\mathrm{KK}(\mathbb{C}, -)$ sends Morita equivalences to equivalences by [9, Thm. 16.18]. □

Example 12.10. Assume that A is an object in $\mathrm{Fun}(BG, C^*\mathbf{Alg})$ and set $\mathbf{C} := \mathbf{Hilb}_c(A)$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$. Then by Lemma 2.20.3 we have a Morita equivalence $A \to (\mathbf{C}^u)^{(G)}$ induced by the canonical inclusion. We then have an equivalence

$$\mu^{\mathrm{Kasp}}_{A,\mathcal{F}} \xrightarrow{\simeq} \mu^{\mathrm{Kasp}}_{(\mathbf{C}^u)^{(G)},\mathcal{F}} \tag{12.18}$$

by Lemma 12.9. □

We will now derive the analytic assembly map (12.7) associated to $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$. The composition of the two transformations $- \rtimes G \to - \rtimes_r G$

and id → Idem yields a morphism of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes G \arrow[r] \arrow[d] & \mathbf{MC}^{(G)}_{\mathrm{std}} \rtimes G \arrow[r] \arrow[d] & \mathbf{Q}^{(G)}_{\mathrm{std}} \rtimes G \arrow[r] \arrow[d, "\mathrm{ctc}"] & 0 \\ 0 \arrow[r] & \mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G) \arrow[r] & \mathrm{Idem}(\mathbf{U}) \arrow[r] & \frac{\mathrm{Idem}(\mathbf{U})}{\mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)} \arrow[r] & 0 \end{tikzcd} \tag{12.19}$$

where the middle vertical arrow is the composition of (2.10) with the inclusion $\mathbf{U} \to \mathrm{Idem}(\mathbf{U})$. In the upper line, we also used that the functor $- \rtimes G$ is exact. The functor ctc will be called the change of target categories functor.

The change of target categories functor ctc in (12.19) yields the first morphism in the following composition. The second is the boundary map associated to the second exact sequence in (12.19). Finally, the left-pointing morphism is an equivalence by the Moria invariance of $\mathrm{KK}(\mathbb{C}, -) = K^{C^*\mathrm{Cat}}$ [9, Thm. 16.18]:

$$\mathrm{KK}(\mathbb{C}, \mathbf{Q}^{(G)}_{\mathrm{std}} \rtimes G) \xrightarrow{\mathrm{ctc}} \mathrm{KK}(\mathbb{C}, \frac{\mathrm{Idem}(\mathbf{U})}{\mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)}) \tag{12.20}$$
$$\longrightarrow \Sigma\mathrm{KK}(\mathbb{C}, \mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)) \xleftarrow{\simeq} \Sigma\mathrm{KK}(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)\,.$$

We now specialize the assembly maps introduced in Definition 12.8 to $A = \mathrm{kk}^G(\mathbf{Q}^{(G)}_{\mathrm{std}})$, but we will drop the symbol kk^G in order to shorten the formulas. We use that $K^{G,\mathrm{an}}_{\mathbf{Q}^{(G)}_{\mathrm{std}}} = K^{G,\mathrm{An}}_{\mathbf{C}}$, compare (12.14) and (1.3).

Definition 12.11. *We define the natural transformation*

$$\mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C},-} : K^{G,\mathrm{An}}_{\mathbf{C}}(-) \xrightarrow{\mu^{\mathrm{Kasp}}_{\mathbf{Q}^{(G)}_{\mathrm{std}},-,\max}} \underline{\mathrm{KK}(\mathbb{C}, \mathbf{Q}^{(G)}_{\mathrm{std}} \rtimes G)} \xrightarrow{(12.20)} \underline{\Sigma\mathrm{KK}(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)}$$

of functors from $G\mathbf{LCH}^{\mathrm{prop}}_{+,\mathrm{pc}}$ to $\mathbf{Sp}$. We then define

$$\mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C},X,*} := \pi_*(\mathrm{Asmbl}^{an}_{\mathbf{C},X})\,.$$

We now use Definition 12.7 for $RK^{G,\mathrm{An}}_{\mathbf{C}} = RK^{G,\mathrm{an}}_{\mathbf{Q}^{(G)}_{\mathrm{std}}}$.

Definition 12.12. *The analytic assembly map associated to G, $\mathcal{F}$ and $\mathbf{C}$ is defined as the map*

$$\mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C},\mathcal{F}} : RK^{G,\mathrm{An}}_{\mathbf{C}}(E_{\mathcal{F}}G^{\mathrm{CW}}) \to \Sigma\mathrm{KK}(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)$$

induced by the natural transformation $\mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C}}$ *in Definition* 12.11.

The assembly maps $\mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C}}$ and $\mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C},\mathcal{F}}$ depend naturally on the coefficient category $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg,eadd},\omega\mathrm{add}})$.

13. C^*-categorical model for the homotopy theoretic assembly map

The homotopy theoretic assembly map $\mathrm{Asmbl}^h_{\mathbf{C},\mathcal{F}}$ in Definition 12.3 is defined in terms of the equivariant homology theory $K\mathbf{C}^G$. On the other hand, the analytic assembly map $\mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C},\mathcal{F}}$ is constructed in Definition 12.12 in terms of KK-theory. Our goal is to compare these two assembly maps. As a first step, in this section we will construct an assembly map Asmbl^Θ_X induced by an explicit functor Θ_X between C^*-categories and show that it is equivalent to the homotopy theoretic assembly map $\mathrm{Asmbl}^h_{\mathbf{C},X}$ on G-finite G-simplicial complexes. Asmbl^Θ_X also depends on $\mathbf{C}$, but we drop this subscript from the notation in order to simplify the notation.

Let $\mathbf{C}$ be in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ and assume that it is effectively additive and admits countable AV-sums. In the following, we will use the C^*-category $\mathbf{U}$ defined in Definition 2.22 which contains $\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G$ as an ideal, and the morphism $\sigma : \mathbf{MC}^{(G)}_{\mathrm{std}} \rtimes G \to \mathbf{U}$ from (2.10). Recall the Definition 3.3 of the functor $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}} : G\mathbf{BC} \to C^*\mathbf{Cat}$. Let X be in $G\mathbf{BC}$. For an object (C, ρ, μ) in $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X \otimes G_{can,min})$, we use the abbreviation

$$\mu_g := \mu(X \otimes \{g\}) \tag{13.1}$$

denoting a projection in $\mathbf{MC}$ on C. We refer to Proposition 13.2 below for the verifications related with the following definition.

Definition 13.1. *We define a functor*

$$\Theta_X : \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X \otimes G_{can,min}) \to \mathrm{Idem}(\mathbf{U}).$$

as follows:

(1) *objects: The functor Θ_X sends the object (C, ρ, μ) in $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X \otimes G_{can,min})$ to the object (C, ρ, p) in* $\mathrm{Idem}(\mathbf{U})$*, where*

$$p := \sigma(\mu_e, e). \tag{13.2}$$

(2) *morphisms: The functor Θ_X sends a morphism $A : (C, \rho, \mu) \to (C', \rho', \mu')$ in $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X \otimes G_{can,min})$ to the morphism*

$$\Theta_X(A) := \sum_{g \in G} \sigma(\mu'_{g^{-1}} A \mu_e, g) : (C, \rho, p) \to (C', \rho', p') \tag{13.3}$$

in $\mathrm{Idem}(\mathbf{U})$.

For the interpretation of the infinite sum in (13.3), we refer to the proof of Lemma 13.3 below. Let $G\mathbf{BC}_{\mathrm{bd}}$ denote the full category of $G\mathbf{BC}$ of bounded G-bornological coarse spaces.

Proposition 13.2.

(1) *For every X in $G\mathbf{BC}$, the functor Θ_X is well-defined.*

(2) *The family* $(\Theta_X)_{X\in G\mathbf{BC}}$ *is a natural transformation*

$$\Theta:\ \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(-\otimes G_{can,min})\to \underline{\mathrm{Idem}(\mathbf{U})} \tag{13.4}$$

of functors from $G\mathbf{BC}$ *to* $C^*\mathbf{Cat}^{\mathrm{nu}}$.

(3) *The transformation* Θ *restricts to a transformation*

$$\Theta:\ \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(-\otimes G_{can,min})\to \underline{\mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}}\rtimes_r G)} \tag{13.5}$$

of functors from $G\mathbf{BC}_{\mathrm{bd}}$ *to* $C^*\mathbf{Cat}^{\mathrm{nu}}$.

Proof. We start with Assertion 13.2.1. Since $X\times G$ is a free G-set and (C,ρ,μ) is locally finite it follows that (C,ρ) belongs to $\mathbf{C}^{(G)}_{\mathrm{std}}$. Furthermore, p belongs to $\mathbf{U}$ since μ_e belongs to $\mathbf{MC}$. Consequently, (C,ρ,p) is a well-defined object in $\mathrm{Idem}(\mathbf{U})$.

The following lemma finishes the verification that Θ_X is a well-defined functor between C^*-categories and therefore proves Assertion 13.2.1.

Lemma 13.3. *The formula* (13.3) *determines an isometric map* $\Theta_X(-)$ *on morphism spaces which is compatible with the composition and the involution.*

Proof. We first observe that if $A:\ (C,\rho,\mu)\to(C',\rho',\mu')$ has controlled propagation then the sum in (13.3) has finitely many non-zero terms which all belong to $\mathbf{U}$ since A belongs to $\mathbf{MC}$.

It follows from Definition 2.12.2c and [9, Lem. 7.10] that C is isomorphic to the orthogonal AV-sum of the images of the family of projections $(\mu_g)_{g\in G}$. Using [9, Lem. 7.8] we therefore get a multiplier isometry

$$u:\ C\to\bigoplus_{g\in G} C\,,\quad u:=\sum_{g\in G} e_g\mu_g\,, \tag{13.6}$$

where the sum converges strictly. We have an analogous multiplier isometry $u':\ C'\to\bigoplus_{g\in G} C'$. Still assuming that A is controlled, we calculate by using (2.3) (saying that $g\cdot\mu_h=\mu_{gh}$ for all g,h in G) and the G-invariance of A (saying that $g\cdot A=A$ for all g in G) that

$$\Theta_X(A):=\sum_{g\in G}\sigma(\mu'_{g^{-1}}A\mu_e,g)=u'Au^*\,. \tag{13.7}$$

Since $\mathbf{U}$ is closed in $\mathbf{L}^2(G,\mathbf{C}^{(G)}_{\mathrm{std}})$, this formula shows that Θ_X extends by continuity to an isometric map defined on all morphisms in $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X\otimes G_{can,min})$ with values in $\mathbf{U}$. Using the equality

$$uu^*=p \tag{13.8}$$

and (13.3) we see that

$$p'\Theta_X(A)=\Theta_X(A)p=\Theta_X(A)\,.$$

Altogether we obtain an isometric map

$$\Theta_X(-):\ \mathrm{Hom}_{\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X\otimes G_{can,min})}((C,\rho,\mu),(C',\rho',\mu'))\to\mathrm{Hom}_{\mathrm{Idem}(\mathbf{U})}((C,\rho,p),(C',\rho',p'))\,.$$

We finally show that $\Theta_X(-)$ is compatible with the composition and the involution. Let $A : (C,\rho,\mu) \to (C',\rho',\mu')$ and $A' : (C',\rho',\mu') \to (C'',\rho'',\mu'')$ be morphisms in $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X \otimes G_{can,min})$. Since Θ_X is continuous, as shown above, we can assume for simplicity that the morphisms are controlled. We then calculate using that u and u' are isometries and (13.7) that

$$\Theta_X(A')\Theta_X(A) = \Theta_X(A'A)\,, \quad \Theta_X(A)^* = \Theta_X(A^*)\,. \qquad \square$$

In order to see the Assertion 13.2.2, we consider a map $f : X \to X'$ in $G\mathbf{BC}$. Then $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(f)(C,\rho,\mu) = (C,\rho,f_*\mu)$. We observe by inspection of the definitions that

$$\Theta_{X'}(\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(f)(C,\rho,\mu)) = \Theta_X(C,\rho,\mu)\,, \quad \Theta_X(\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(f)(A)) = \Theta_X(A)\,.$$

We finally show Assertion 13.2.3. If X is bounded, then $X \times \{g\}$ is a bounded subset of $X \otimes G_{can,min}$ for every g in G. Consequently, μ_g belongs to $\mathbf{C}$, see the explanations in Remark 2.13. Every summand of (13.3) is a morphism in $\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G$. Since $\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G$ is closed in $\mathbf{U}$ we conclude that Θ_X takes values in the wide subcategory $\mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)$ of $\mathrm{Idem}(\mathbf{U})$, provided X is bounded. $\square$

For X in $G\mathbf{UBC}_{\mathrm{bd}}$, we will also write

$$\Theta_X : \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{Z} \subseteq \mathcal{O}(X) \otimes G_{can,min}) \to \underline{\mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)} \qquad (13.9)$$

for the restriction of the functor $\Theta_{\mathcal{O}(X)}$ to the ideal $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{Z} \subseteq \mathcal{O}(X) \otimes G_{can,min})$, see (5.6) for the notation.

We now apply the functor $K^{C^*\mathrm{Cat}}(-) = \mathrm{KK}(\mathbb{C},-)$ to the transformations (13.4) and (13.5). Using the Definition 3.4 of $K\mathbf{C}\mathcal{X}^G$ in order to rewrite the domain and the Morita invariance of $K^{C^*\mathrm{Cat}}(-)$ together with [9, Prop. 17.4 & 17.8] in order to remove $\mathrm{Idem}(-)$ in the target we get the assertions of the following corollary.

Corollary 13.4.

(1) *We have a natural transformation*

$$\theta : K\mathbf{C}\mathcal{X}^G_{G_{can,min}} \to \underline{\mathrm{KK}(\mathbb{C},\mathbf{U})} \qquad (13.10)$$

of functors from $G\mathbf{BC}$ to $\mathbf{Sp}$.

(2) *We have a natural transformation*

$$\theta : K\mathbf{C}\mathcal{X}^G_{G_{can,min}} \to \underline{\mathrm{KK}(\mathbb{C},\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)} \qquad (13.11)$$

of functors from $G\mathbf{BC}_{\mathrm{bd}}$ to $\mathbf{Sp}$.

Proposition 13.5. *The morphism*

$$\theta_* : K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(*) \to \mathrm{KK}(\mathbb{C},\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G) \qquad (13.12)$$

is an equivalence.

Proof. The proof is very similar to the proof of Proposition 9.6. But the difference is that here G is infinite while in Proposition 9.6 H was finite. By definition the morphism in question is

$$\begin{aligned} K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(*) &\simeq K^{C^*\mathrm{Cat}}(\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(G_{can,min})) \\ &\xrightarrow{K^{C^*\mathrm{Cat}}(\Theta_*)} K^{C^*\mathrm{Cat}}(\mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)) . \end{aligned}$$

We will construct a factorization of Θ_* as

$$\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(G_{can,min}) \to \mathbf{D} \to \mathrm{Idem}(\mathbf{D}) \to \mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G) ,$$

where $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(G_{can,min}) \to \mathbf{D}$ is a weak Morita equivalence, $\mathbf{D} \to \mathrm{Idem}(\mathbf{D})$ is a relative idempotent completion, and $\mathrm{Idem}(\mathbf{D}) \to \mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)$ is a unitary equivalence. Since $K^{C^*\mathrm{Cat}}$ sends functors with any of these properties to equivalences [9, Sec. 14 -16] the assertion then follows.

Lemma 13.6. *Θ_* is fully faithful.*

Proof. By Lemma 13.3 the functor Θ_* is an isometric inclusion on morphisms. It remains to show that it is surjective.

Let (C,ρ,μ) and (C',ρ',μ') be two objects of $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(G_{can,min})$. Note that $\Theta_*(C,\rho,\mu) = (C,\rho,p)$ and $\Theta_*(C',\rho',\mu') = (C'\rho',p')$ in $\mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)$. Let $A\colon (C,\rho,p) \to (C'\rho',p')$ be a morphism in $\mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)$. We will construct a morphism $\hat{A}\colon (C,\rho,\mu) \to (C',\rho',\mu')$ in $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(G_{can,min})$ such that $\Theta_*(\hat{A}) = A$.

Note that A is a morphism $(C,\rho) \to (C',\rho')$ in $\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G$ which in addition satisfies $\tilde{p}'A\tilde{p} = A$. There is a unique family $(A_g)_{g\in G}$ of morphisms $A_g\colon C \to C$ in $\mathbf{C}$ such that

$$A = \sum_{g\in G} \sigma(A_g, g) ,$$

where the sum converges in norm in $\mathbf{U}$. From the equality

$$\sum_{g\in G} \sigma(A_g, g) = A = p'Ap = \sum_{g\in G} \sigma(\mu'_{g^{-1}} A_g \mu_e, g)$$

we conclude that

$$\mu'_{g^{-1}} A_g \mu_e = A_g \tag{13.13}$$

for all g in G. Using the notation from (13.7) we define

$$\hat{A} := u^{\prime,*} \sum_{g\in G} \sigma(A_g, g) u$$

in $\mathrm{Hom}_{\mathbf{MC}}(C,C')$. Inserting all definitions we get $\hat{A} = \sum_{k\in G}\sum_{g\in G} k\cdot A_g$ where the g-sum converges in norm and the k-sum converges strictly. This formula shows that $\hat{A}$ is G-invariant. Furthermore, by (13.13) for every g in G the support of $\sum_{k\in G} k\cdot A_g$ is the coarse entourage $G(\{(g^{-1},e)\})$ of $G_{can,min}$. It follows that

$\hat{A}$ can be approximated in norm by controlled and invariant operators, i.e., we have $\hat{A} \in \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(G_{can,min})$.

By construction we have $\Theta_*(\hat{A}) = A$. This finishes the verification that Θ_* is full faithful. □

For every free G-set Y, every subset F of Y, and every object (C, ρ, μ) in $\mathbf{C}^{(G)}_{\mathrm{lf}}(Y_{min})$ we can consider the projection $p_F := \sigma(\mu(F), e)$ on (C, ρ) considered as an object of $\mathbf{U}$. We let $\mathbf{D}$ be the full subcategory of $\mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)$ of objects of the form (C, ρ, p_F) for some choice of Y, F and (C, ρ, μ) as above. We can consider $Y = G$ and $F = \{e\}$. Then $p = p_{\{e\}}$ so that $\mathbf{D}$ contains the image of Θ_*.

Recall the notion of a weak Morita equivalence from [9, Def. 18.3].

Lemma 13.7. *The functor* $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(G_{can,min}) \to \mathbf{D}$ *is a weak Morita equivalence.*

Proof. It follows from Lemma 13.6 that the morphism in question is fully faithful. It remains to show that set of objects $\Theta_*(\mathrm{Ob}(\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(G_{can,min})))$ is weakly generating [9, Def. 16.1]. In order to simplify the notation, we write $p_g := p_{\{g\}}$ and note that $p = p_e$. We have

$$\sigma(\mathrm{id}_C, g) p_e \sigma(\mathrm{id}_C, g)^* = p_g\,.$$

This shows that for every g in G and (C, ρ, μ) in $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(G_{can,min})$ the object (C, ρ, p_g) in $\mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)$ is isomorphic to the object (C, ρ, p) which belongs to the image of Θ_*. Furthermore, for every finite subset F of some free G-set Y and (C, ρ, μ) in $\mathbf{C}^{(G)}_{\mathrm{lf}}(Y_{min})$ the object (C, ρ, p_F) is isomorphic to a finite sum of objects in the image of Θ_*.

Let now (C, ρ, p_F) be any object of $\mathbf{D}$ and $(A_i)_{i \in I}$ be a finite family of morphisms in $\mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes G)$ with target (C, ρ, p_F). Let ϵ be in $(0, \infty)$. We write $A_i = \sum_{h \in G} \sigma(A_{i,h}, h)$ where the $A_{i,h}$ belong to $\mathbf{C}$. Since these sums converge in norm and I is finite there exists a finite subset F' of G such that $\|A_i - \sum_{h \in F'} \sigma(A_{i,h}, h)\| \le \epsilon/2$ for all i in I. Since $\sum_{y \in Y} \mu(\{y\})$ converges strictly to id_C we can find a finite subset F'' of Y such that

$$\|A_{i,g} - \mu(g^{-1}F'')A_{i,g}\| \le \frac{\epsilon}{2|F'|}$$

for all i in I and g in F'. Then $\|A_i - p_{F''}A_i\| \le \epsilon$ for all i in I. □

Recall the definition [9, Def. 17.5] of a relative idempotent completion. In the following, we let $\mathbf{E}$ be the full subcategory of $\mathrm{Idem}(\mathbf{U})$ with the same objects as $\mathbf{D}$. Then $\mathbf{D}$ is an ideal in $\mathbf{E}$ and the idempotent completion $\mathbf{D} \to \mathrm{Idem}(\mathbf{D})$ is understood relative to $\mathbf{E}$. We summarize this in the following corollary:

Corollary 13.8. *The functor* $\mathbf{D} \to \mathrm{Idem}(\mathbf{D})$ *is a relative idempotent completion.*

Lemma 13.9. *The inclusion* $\mathrm{Idem}(\mathbf{D}) \to \mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)$ *is a unitary equivalence.*

Proof. We apply the characterization of unitary equivalence given in [9, Rem. 3.20.3]. We consider the square

$$\begin{tikzcd} \mathrm{Idem}(\mathbf{D}) \arrow[r] \arrow[d] & \mathrm{Idem}(\mathbf{E}) \arrow[d] \\ \mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G) \arrow[r] & \mathrm{Idem}(\mathbf{U}) \end{tikzcd}$$

Its horizontal morphisms are ideal inclusions by construction. It remains to show that the right vertical morphism is a unitary equivalence in $C^*\mathbf{Cat}$. In fact, it is fully faithful by definition. Since $\mathbf{E}$ contains the all objects of the form $(C, \rho, \tilde{p}_Y) = (C, \rho)$ for free G-sets Y and (C, ρ, μ) in $\mathbf{C}^{(G)}_{\mathrm{lf}}(Y_{min})$ conclude that it is also essentially surjective. □

This finishes the proof of Proposition 13.5. □

We now apply the cone sequence (4.5) to the functor $K\mathbf{C}\mathcal{X}^G_{G_{can,min}}$ and obtain a boundary map

$$\partial^{\mathrm{Cone}} : K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(\mathcal{O}^\infty(-)) \to \Sigma K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(-)$$

between functors from $G\mathbf{UBC}$ to $\mathbf{Sp}$.

Definition 13.10. *We denote by $G\mathbf{UBC}_{\mathrm{bd}}$ the full subcategory of $G\mathbf{UBC}$ of bounded G-uniform bornological coarse spaces.*

We have a forgetful functor $G\mathbf{UBC}_{\mathrm{bd}} \to G\mathbf{BC}_{\mathrm{bd}}$ which we always drop from the notation. We can also restrict the cone boundary transformation along the inclusion $G\mathbf{UBC}_{\mathrm{bd}} \to G\mathbf{UBC}$.

Let X be in $G\mathbf{UBC}_{\mathrm{bd}}$. We use the Corollary 13.4.2 in order to see the that the natural transformation defined below takes values in the correct target.

Definition 13.11. *We define the natural transformation*

$$\mathrm{Asmbl}^\Theta := \theta \circ \partial^{\mathrm{Cone}} : K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(\mathcal{O}^\infty(-)) \to \underline{\Sigma \mathrm{KK}(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)}$$

of functors from $G\mathbf{UBC}_{\mathrm{bd}}$ to $\mathbf{Sp}$.

We consider the functor

$$\tilde{\iota} : G\mathbf{Set} \to G\mathbf{UBC}_{\mathrm{bd}}\,, \quad S \mapsto S_{min,max,disc}\,, \tag{13.14}$$

where *disc* stands for the discrete uniform structure. A G-simplicial complex is a simplicial complex with a simplicial G-action. We assume that if g in G fixes a point in the interior of a simplex, then it fixes the whole simplex pointwise. This can always be ensured by going over to a barycentric subdivision. We let $G\mathbf{Simpl}$ denote the category of G-simplicial complexes and simplicial equivariant maps.

Let $G\mathbf{Simpl}^{\text{fin-dim}}$ be the full subcategory of $G\mathbf{Simpl}$ of finite-dimensional G-simplicial complexes. We have a natural functor

$$\tilde{s} : G\mathbf{Simpl} \to G\mathbf{UBC}_{\mathrm{bd}}$$

which sends a G-simplicial complex X to the G-uniform bornological coarse space $\tilde{s}(X)$ given by X with the coarse and the uniform structures induced by the spherical path metric, and with the maximal bornology. We have a commutative diagram of canonical functors

$$\begin{tikzcd}
G\mathbf{Set} \arrow[rr, "\tilde{\iota}"] \arrow[dr, "(1)"] & & G\mathbf{UBC}_{\mathrm{bd}} \arrow[dr, "r"] & \\
& G\mathbf{Simpl}^{\text{fin-dim}} \arrow[ur, "s"] \arrow[rr, "t"] \arrow[dr, "f"] & & G\mathbf{Top} \\
& & G\mathbf{Simpl} \arrow[uu, "\tilde{s}"] \arrow[ur, "\tilde{t}"] &
\end{tikzcd} \tag{13.15}$$

where arrow (1) interprets a G-set as a zero-dimensional G-simplicial set, and arrow r sends a uniform bornological coarse space to the underlying G-topological space.

Proposition 13.12.

(1) *The transformation* $\tilde{\iota}^*\partial^{\mathrm{Cone}} : \tilde{\iota}^* K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(\mathcal{O}^\infty(-)) \to \tilde{\iota}^* r^* \Sigma K\mathbf{C}^G(-)$ *of functors from* $G\mathbf{Set}$ *to* $\mathbf{Sp}$ *is an equivalence.*

(2) *We have an equivalence*

$$s^* K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(\mathcal{O}^\infty(-)) \simeq t^*\Sigma K\mathbf{C}^G(-) \tag{13.16}$$

of functors from $G\mathbf{Simpl}^{\text{fin-dim}}$ *to* $\mathbf{Sp}$.

(3) *We have a commutative square of natural transformations*

$$\begin{tikzcd}
t^*\Sigma K\mathbf{C}^G(-) \arrow[r, "t^*\Sigma\mathrm{Asmbl}^h_{\mathbf{C}}"] \arrow[d, dash, "\simeq"', "(13.16)"] & \underline{\Sigma K\mathbf{C}^G(*)} \arrow[d, dash, "(13.5)"', "\simeq"] \\
s^* K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(\mathcal{O}^\infty(-)) \arrow[r, "s^*\mathrm{Asmbl}^\Theta"] & \underline{\Sigma \mathrm{KK}(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)}
\end{tikzcd} \tag{13.17}$$

between functors from $G\mathbf{Simpl}^{\text{fin-dim}}$ *to* $\mathbf{Sp}$ *which depends naturally on the coefficient category* $\mathbf{C}$ *in* $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg,eadd},\omega\mathrm{add}})$.

Proof. By [10, Prop. 9.35] for every S in $G\mathbf{Set}$ we have an equivalence

$$\partial^{\mathrm{Cone}} : K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(\mathcal{O}^\infty(\tilde{\iota}(S)) \xrightarrow{\simeq} \Sigma K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(\tilde{\iota}(S))\,.$$

If L is a locally finite subset of $S_{min,max} \otimes G_{can,min}$, then $L \cap (S \times \{e\})$ is finite. It follows that L is a finite union of G-orbits. By continuity of $K\mathbf{C}\mathcal{X}^G$ we have

$$\Sigma K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(\tilde{\iota}(S)) \simeq \bigoplus_{T\in G\backslash S} \Sigma K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(\iota(T))\,,$$

where ι is as in (12.5). By Definition 12.2

$$\Sigma K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(\iota(T)) \simeq \Sigma K\mathbf{C}^G(r(\tilde{\iota}(T)))\,.$$

Since the homology theory $K\mathbf{C}^G$ sends disjoint unions of orbits to sums we conclude that

$$\bigoplus_{T\in G\backslash S} \Sigma K\mathbf{C}^G(r(\tilde{\iota}(T))) \simeq \Sigma K\mathbf{C}^G(r(\tilde{\iota}(S)))\,.$$

Combining these equivalences we get Assertion 13.12.1.

We now show Assertion 13.12.2. Note that $K\mathbf{C}^G(-)$ in the statement is the evaluation of an equivariant homology theory defined on all of $G\mathbf{Top}$ by (10.3). The other functor $K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(\mathcal{O}^\infty(-))$ is defined on $G\mathbf{UBC}$. By restricting $K\mathbf{C}^G(-)$ along the forgetful functor $G\mathbf{UBC} \to G\mathbf{Top}$ we can consider them on the same domain $G\mathbf{UBC}$. Assertion 13.12.1 then provides an equivalence between the further restrictions of both functors to zero-dimensional simplicial complexes. We then argue that this natural equivalence canonically extends to an equivalence between these functors at least on $G\mathbf{Simpl}^{\text{fin-dim}}$ since they are both homotopy invariant and excisive for cell-attachements.

We will construct the desired equivalence by induction with respect to the dimension. We let $G\mathbf{Simpl}_{\le n}$ be the full subcategory of G-simplicial complexes of dimension $\le n$. We let s_n and t_n denote the restrictions of s and t to $G\mathbf{Simpl}_{\le n}$.

The case of zero-dimensional simplicial complexes is done by Assertion 1.

We assume now that we have constructed an equivalence

$$s^*_{n-1}K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(\mathcal{O}^\infty(-)) \simeq t^*_{n-1}\Sigma K\mathbf{C}^G(-) \tag{13.18}$$

for $n \ge 1$. The induction step exploits the fact that $s_n(X)$ in $G\mathbf{Simpl}_{\le n}$ has a canonical decomposition (Y, Z) in $G\mathbf{UBC}_{\mathrm{bd}}$, where Z is the disjoint union of $2/3$-scaled n-simplices, and Y is the complement of the disjoint union of the interiors of the $1/3$-scaled n-simplices (see the pictures in [6, P. 80]). We equip the subspaces Y and Z with the uniform bornological coarse structures induced from $s_n(X)$.

Since both functors $K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(\mathcal{O}^\infty(-))$ and $\Sigma K\mathbf{C}^G(-)$ are excisive for such decompositions we get push-out squares

$$\begin{tikzcd} K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(\mathcal{O}^\infty(Y\cap Z)) \arrow[r] \arrow[d] & K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(\mathcal{O}^\infty(Z)) \arrow[d, dotted] \\ K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(\mathcal{O}^\infty(Y)) \arrow[r, dotted] & K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(\mathcal{O}^\infty(X)) \end{tikzcd} \tag{13.19}$$

and

$$\begin{tikzcd} \Sigma K\mathbf{C}^G(Y\cap Z) \arrow[r] \arrow[d] & \Sigma K\mathbf{C}^G(Z) \arrow[d, dotted] \\ \Sigma K\mathbf{C}^G(Y) \arrow[r, dotted] & \Sigma K\mathbf{C}^G(X) \end{tikzcd} \tag{13.20}$$

We now use that both functors are homotopy invariant. The projection of Z to the G-set Z_0 of barycenters is a homotopy equivalence in $G\mathbf{Top}$ and $G\mathbf{UBC}_{\mathrm{bd}}$. Similarly, there is a projection of Y to the $(n-1)$-skeleton X_{n-1} of X and a projection of $Y \cap Z$ to a disjoint union $(Y \cap Z)_{n-1}$ of boundaries of the n-simplices. These two maps are homotopy equivalences in $G\mathbf{Top}$ and $G\mathbf{UBC}_{\mathrm{bd}}$. These projections identify the bold parts of the push-out squares above canonically with the respective bold parts of the push-out squares below:

$$\begin{tikzcd} K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(\mathcal{O}^\infty((Y\cap Z)_{n-1})) \arrow[r] \arrow[d] & K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(\mathcal{O}^\infty(Z_0)) \arrow[d, dotted] \\ K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(\mathcal{O}^\infty(X_{n-1})) \arrow[r, dotted] & K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(\mathcal{O}^\infty(X)) \end{tikzcd} \tag{13.21}$$

and

$$\begin{tikzcd} \Sigma K\mathbf{C}^G((Y\cap Z)_{n-1}) \arrow[r] \arrow[d] & \Sigma K\mathbf{C}^G(Z_0) \arrow[d, dotted] \\ \Sigma K\mathbf{C}^G(Y_{n-1}) \arrow[r, dotted] & \Sigma K\mathbf{C}^G(X) \end{tikzcd} \tag{13.22}$$

The induction hypothesis now provides an equivalence between the bold parts of (13.21) and (13.22). This equivalence then provides the desired equivalence of push-outs

$$K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(\mathcal{O}^\infty(X)) \simeq \Sigma K\mathbf{C}^G(X)\,.$$

The whole construction is functorial in X. To see this interpret the symbols X, Y, Z as placeholders for entries of diagram valued functors.

Remark 13.13. In order to give a more formal argument for naturality, we could proceed as in the proof of Corollary 10.9. Let $q : G\mathbf{Set} \to G\mathbf{Simpl}$ be the canonical inclusion. Then we have a counit morphism

$$q_!q^*\tilde{s}^*K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(\mathcal{O}^\infty(-)) \to \tilde{s}^*K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(\mathcal{O}^\infty(-))\,.$$

Using excision and homotopy invariance one checks that

$$f^*q_!q^*\tilde{s}^*K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(\mathcal{O}^\infty(-)) \to s^*K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(\mathcal{O}^\infty(-))\,.$$

is an equivalence. Since $K\mathbf{C}^G$ is an equivariant homology theory the counit

$$q_!q^*\tilde{t}^*K\mathbf{C}^G \xrightarrow{\simeq} \tilde{t}^*K\mathbf{C}^G$$

is an equivalence. Finally, applying $q_!$ to the equivalence from Assertion 1. we get the equivalence

$$q_!q^*\tilde{s}^*K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(\mathcal{O}^\infty(-)) \xrightarrow{\simeq} q_!q^*\tilde{t}^*K\mathbf{C}^G\,.$$

The desired equivalence is now given by

$$\begin{aligned} s^*K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(\mathcal{O}^\infty(-)) &\xleftarrow{\simeq} f^*q_!q^*\tilde{s}^*K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(\mathcal{O}^\infty(-)) \\ &\xrightarrow{\simeq} f^*q_!q^*\tilde{t}^*K\mathbf{C}^G \xrightarrow{\simeq} t^*K\mathbf{C}^G\,. \end{aligned}$$

□

Assertion 13.12.3. becomes obvious if we expand the square (13.17) as follows

$$
\begin{tikzcd}
t^*\Sigma K\mathbf{C}^G(-) \arrow[r, "t^*\mathrm{Asmbl}^h_{\mathbf{C}}"] & \Sigma K\mathbf{C}^G(*) \arrow[dr, "{\theta_*,(13.12)}", "\simeq"'] & \\
s^*K\mathbf{C}\mathcal{X}^G_{c,G_{can,min}}(\mathcal{O}^\infty(-)) \arrow[u, "\simeq", "(13.16)"'] \arrow[r] \arrow[rr, bend right, "s^*\mathrm{Asmbl}^\Theta"'] & K\mathbf{C}\mathcal{X}^G_{c,G_{can,min}}(\mathcal{O}^\infty(*)) \arrow[u, "\simeq", "{\partial^{\mathrm{Cone}},(13.16)}"'] \arrow[r, "\mathrm{Asmbl}^\Theta_*"'] & \Sigma\mathrm{KK}(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)
\end{tikzcd}
\tag{13.23}
$$

The left horizontal maps in the square are induced by the natural transformation $(-) \to \underline{\mathtt{const}_*}$ (see (12.6) for $\mathrm{Asmbl}^h_{\mathbf{C}}$), and the upper-left square commutes by the naturality statement in Assertion 13.12.2. The upper right triangle commutes by the definition of Asmbl^Θ_*, and finally the lower triangle commutes by the naturality of Asmbl^Θ. □

14. C^*-categorical model for the analytic assembly map

At the end of this section we finish the proof of Theorem 1.9.

The analytic assembly map $\mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C}}$ in Definition 12.11 was obtained using a construction on the level of spectrum-valued KK-theory. If we precompose this assembly map with the Paschke transformation from Theorem 1.6, then we get a functor whose domain is also expressed through the coarse K-homology functor $K\mathbf{C}\mathcal{X}^G$ and therefore in terms of C^*-categories of controlled objects. In the present section, we construct an assembly map Asmbl^Λ in terms of a natural functor Λ between C^*-categories which models this composition. We then relate Asmbl^Λ with both Asmbl^Θ and $\mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C}}$. The intermediate objects also depend on $\mathbf{C}$, but we again drop this subscript in their notation in order to simplify the notation.

Definition 14.1. *We let $G\mathbf{UBC}_{\mathrm{pc}}$ denote the full subcategory of $G\mathbf{UBC}$ of G-uniform bornological coarse spaces which have the bornology of relative compact subsets and whose underlying G-topological space belongs to $G\mathbf{LCH}^{\mathrm{prop}}_{+,\mathrm{pc}}$ introduced in Definition* 12.4.

We consider $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ and assume that it is effectively additive and admits countable AV-sums. Let X be in $G\mathbf{UBC}_{\mathrm{pc}}$ and choose χ in $\mathcal{R}(X)$, where the functor $\mathcal{R}$ is as in (12). If (C, ρ, μ) is an object in $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(X) \otimes G_{can,max})$, then we can consider the homomorphism $\phi : C_0(X) \to \mathrm{End}_{\mathbf{MC}}(C)$ defined in (5.7). The sum

$$
p_\chi := \sum_{m \in G} \sigma(\phi(\chi)\phi(m^*\chi), m) \tag{14.1}
$$

has finitely many non-zero terms and defines a projection on (C, ρ) considered as an object in the C^*-category $\mathbf{U}$ described in the Definition 2.22, where σ is as in (2.10). We refer to Proposition 14.3 for the necessary verifications related with the following definition.

Definition 14.2. *We define a functor*

$$\Lambda_{(X,\chi)} : \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(X) \otimes G_{can,max}) \to \mathrm{Idem}(\mathbf{U})$$

in $C^*\mathbf{Cat}^{\mathrm{nu}}$ as follows:

(1) *objects: The functor $\Lambda_{(X,\chi)}$ sends the object (C,ρ,μ) in $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(X) \otimes G_{can,max})$ to the object (C,ρ,p_χ) in* $\mathrm{Idem}(\mathbf{U})$*, where p_χ is as in* (14.1).

(2) *morphisms: The functor $\Lambda_{(X,\chi)}$ sends a morphism $A : (C,\rho,\mu) \to (C',\rho',\mu')$ in $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(X) \otimes G_{can,max})$ to the morphism*

$$\Lambda_{(X,\chi)}(A) := \sum_{m\in G} \sigma(\phi'(m^*\chi)A\phi(\chi), m) \tag{14.2}$$

in $\mathrm{Idem}(\mathbf{U})$.

We refer to the proof of Lemma 14.4 below for the interpretation of the infinite sum in (14.2).

In order to state the naturality of $\Lambda_{(X,\chi)}$, we introduce the category $G\mathbf{UBC}^{\mathcal{R}}_{\mathrm{pc}}$ given by the Grothendieck construction of the functor $\mathcal{R}$. Its objects are pairs (X,χ) of an object X in $G\mathbf{UBC}_{\mathrm{pc}}$ and χ in $\mathcal{R}(X)$, and a morphism $f : (X,\chi) \to (X',\chi')$ in $G\mathbf{UBC}^{\mathcal{R}}_{\mathrm{pc}}$ is a morphism $f : X \to X'$ in $G\mathbf{UBC}_{\mathrm{pc}}$ such that $f^*\chi' = \chi$. We have a forgetful functor $G\mathbf{UBC}^{\mathcal{R}}_{\mathrm{pc}} \to G\mathbf{UBC}_{\mathrm{pc}}$ which we will not write explicitly in formulas.

Proposition 14.3.

(1) *For every (X,χ) in $G\mathbf{UBC}^{\mathcal{R}}_{\mathrm{pc}}$, the functor $\Lambda_{(X,\chi)}$ is well-defined.*

(2) *The family $(\Lambda_{(X,\chi)})_{(X,\chi)\in G\mathbf{UBC}^{\mathcal{R}}_{\mathrm{pc}}}$ is a natural transformation*

$$\Lambda : \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(-) \otimes G_{can,max}) \to \underline{\mathrm{Idem}(\mathbf{U})}$$

of functors from $G\mathbf{UBC}^{\mathcal{R}}_{\mathrm{pc}}$ to $\mathbf{Sp}$.

(3) *The transformation restricts to a natural transformation*

$$\Lambda : \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{Z} \subseteq \mathcal{O}(-) \otimes G_{can,max}) \to \underline{\mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)} \tag{14.3}$$

of functors from $G\mathbf{UBC}^{\mathcal{R}}_{\mathrm{pc}}$ to $\mathbf{Sp}$.

Proof. The structure of this proof is the same as for Proposition 13.2.

We first observe that (C,ρ,p_χ) is an object of $\mathrm{Idem}(\mathbf{U})$.

Lemma 14.4. *The formula* (14.2) *determines a continuous map of morphism spaces which is compatible with the composition and the involution.*

Proof. In analogy to (13.6), for every (C,ρ,μ) in $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(X) \otimes G_{can,max})$ we consider the isometry

$$v : C \to \bigoplus_{g\in G} C\,, \quad v := \sum_{g\in G} e_g \phi(g^{-1,*}\chi)\,. \tag{14.4}$$

Then similarly as (13.7) we have

$$\Lambda_{(X,\chi)}(A) = v' A v^* \tag{14.5}$$

and

$$p_\chi = v v^* \tag{14.6}$$

in analogy to (13.8). □

This finishes the verification of Assertion 14.3.1. We continue with Assertion 14.3.2. Let $f : (X, \chi) \to (X', \chi')$ be a morphism in $G\mathbf{UBC}^{\mathcal{R}}_{\mathrm{pc}}$ and note $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(f)(C, \rho, \mu) = (C, \rho, f_*\mu)$. We let $f_*(\phi) : C_0(X') \to \mathrm{End}_{\mathbf{C}}(C)$ be the homomorphism defined with $f_*\mu$. Then we have the relation

$$f_*\phi(\theta') = \phi(f^*\theta')$$

for all θ' in $C_0(X')$. In particular, $(f_*\phi)(\chi') = \phi(\chi)$. This relation implies that $p_\chi = p_{\chi'}$ and $\Lambda_{(X',\chi')}(\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(f)(A)) = \Lambda_{(X,\chi)}(A)$ (note Definition 3.3.2b). These equalities imply the assertion.

We finally verify Assertion 14.3.3. If $A : (C, \rho, \mu) \to (C', \rho', \mu')$ is a morphism in $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{Z} \subseteq \mathcal{O}(X) \otimes G_{can,max})$, then $A\phi(\chi)$ is in $\mathbf{C}$ by Lemma 5.9. This implies that $\Lambda_{(X,\chi)}(A)$ is a morphism in the ideal $\mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)$. □

We now consider the cone sequence (4.5) for $E = K\mathbf{C}\mathcal{X}^G_{G_{can,max}}$ whose boundary is the natural transformation

$$\partial^{\mathrm{Cone}} : K\mathbf{C}\mathcal{X}^G_{G_{can,max}}(\mathcal{O}^\infty(-)) \to \Sigma K\mathbf{C}\mathcal{X}^G_{G_{can,max}}(-) \tag{14.7}$$

of functors from $G\mathbf{UBC}$ to $\mathbf{Sp}$. The canonical inclusions $\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X \otimes G_{can,min}) \to \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{Z} \subseteq \mathcal{O}(X) \otimes G_{can,min})$ give a further transformation

$$\Sigma K\mathbf{C}\mathcal{X}^G_{G_{can,max}}(-) \xrightarrow{\simeq} \Sigma K^{C^*\mathrm{Cat}}(\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{Z} \subseteq \mathcal{O}(-) \otimes G_{can,min})) \tag{14.8}$$

which is actually an equivalence (see the argument for the left vertical equivalence in (14.13) applied to the case $Y = G_{can,min}$). The composition of the transformations (14.7) with the equivalence (14.8) will also be called the cone boundary transformation

$$\hat{\partial}^{\mathrm{Cone}} : K\mathbf{C}\mathcal{X}^G_{G_{can,max}}(\mathcal{O}^\infty(-)) \to \Sigma K^{C^*\mathrm{Cat}}(\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{Z} \subseteq \mathcal{O}(-) \otimes G_{can,max}))$$

of functors from $G\mathbf{UBC}$ to $\mathbf{Sp}$, but we add the $\hat{}$ in order to distinguish it from (14.7).

Definition 14.5. *We define the natural transformation*

$$\mathrm{Asmbl}^{\Lambda} := K^{C^*\mathrm{Cat}}(\Lambda) \circ \hat{\partial}^{\mathrm{Cone}} : K\mathbf{C}\mathcal{X}^G_{G_{can,max}}(\mathcal{O}^\infty(-)) \to \underline{\Sigma \mathrm{KK}(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)} \tag{14.9}$$

of functors from $G\mathbf{UBC}^{\mathcal{R}}_{\mathrm{pc}}$ to $\mathbf{Sp}$.

If X is in $G\mathbf{UBC}_{\mathrm{pc}}$ (see Definition 14.1), then it is G-bounded, but not necessarily bounded. We let $X_{\mathcal{B}_{max}}$ denote the object of $G\mathbf{UBC}_{\mathrm{bd}}$ (see Definition 13.10) obtained from X by replacing the bornology of X by the maximal bornology.

Proposition 14.6. *There is a canonical equivalence of functors*

$$K\mathbf{C}\mathcal{X}_{G_{can,min}}(\mathcal{O}^{\infty}((-)_{\mathcal{B}_{max}})) \simeq K\mathbf{C}\mathcal{X}_{G_{can,max}}(\mathcal{O}^{\infty}(-))) \tag{14.10}$$

from $G\mathbf{UBC}_{\mathrm{pc}}$ *to* $\mathbf{Sp}$.

Proof. We employ the notion of continuous equivalence introduced in [11, Def. 3.21]. Recall the Definition 2.11 of a locally finite subset of a G-bornological space. In the present situation, we have a G-coarse space Z with two G-bornologies. We denote the two objects in $G\mathbf{BC}$ by Z_0 and Z_1. The identity map of Z is a continuous equivalence between Z_0 and Z_1 if the following conditions on every G-invariant subset L of Z are equivalent:

(1) L is locally finite in Z_0.
(2) L is locally finite in Z_1.

In this case, we have an obvious equality in $C^*\mathbf{Cat}^{\mathrm{nu}}$

$$\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(Z_0) = \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(Z_1)\,. \tag{14.11}$$

Lemma 14.7. *If X in $G\mathbf{UBC}$ is G-bounded and such that G acts properly, then the bornological coarse spaces*

$$X \otimes G_{can,max} \quad \textit{and} \quad X_{\mathcal{B}_{max}} \otimes G_{can,min}$$

are continuously equivalent, and

$$\mathcal{O}(X) \otimes G_{can,max} \quad \textit{and} \quad \mathcal{O}(X_{\mathcal{B}_{max}}) \otimes G_{can,min}$$

are continuously equivalent (in both cases by the identity map of the underlying sets).

Proof. We consider the second case. The first is similar and simpler. Let L be a G-invariant subset of $[0,\infty)\times X\times G$. Since X is G-bounded we can choose a bounded subset B of X such that $GB = X$. For n in $\mathbb{N}$ and subset A of X, we consider the intersections $L_{n,e} := L \cap ([0,n] \times X \times \{e\})$ and $L_{n,A} := L \cap ([0,n] \times A \times G)$.

(1) L is locally finite in $\mathcal{O}(X) \otimes G_{can,max}$ if and only if $L_{n,A}$ is finite for every n in $\mathbb{N}$ and bounded subset A of X. In particular, $L_{n,B}$ is finite. Hence, L is locally finite in $\mathcal{O}(X) \otimes G_{can,max}$ if and only if $L_{n,X}$ consists of finitely many G-orbits for every n in $\mathbb{N}$. Here we use that every G-orbit is locally finite in $\mathcal{O}(X) \otimes G_{can,max}$ since G acts properly on X.
(2) If L is locally finite in $\mathcal{O}(X_{\mathcal{B}_{\max}}) \otimes G_{can,min}$, if and only if $L_{n,e}$ is finite for every n in $\mathbb{N}$. This is the case exactly if $L_{n,X}$ consists of finitely many G-orbits. $\square$

Let Y be any object in $G\mathbf{BC}$ and X be in $G\mathbf{UBC}$. Then we have a diagram in $C^*\mathbf{Cat}^{\mathrm{nu}}$

$$\begin{tikzcd}
& \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(X\otimes Y) \arrow[r] \arrow[d] & \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(X)\otimes Y) \arrow[r] \arrow[d, equal] & \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}^\infty(X)\otimes Y) & \\
0 \arrow[r] & \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{Z}\subseteq\mathcal{O}(X)\otimes Y) \arrow[r] & \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(X)\otimes Y) \arrow[r] & \frac{\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(X)\otimes Y)}{\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{Z}\subseteq\mathcal{O}(X)\otimes Y)} \arrow[r] & 0
\end{tikzcd} \tag{14.12}$$

which is natural in X, where the lower sequence is exact, and where the square commutes. If we apply $K^{C^*\mathrm{Cat}}$ and use Definition 3.4, then we get the (natural in X) commutative diagram

$$\begin{tikzcd}
{} \arrow[r] & K\mathbf{C}\mathcal{X}^G_Y(X) \arrow[r] \arrow[d, "\simeq"] & K\mathbf{C}\mathcal{X}^G_Y(\mathcal{O}(X)) \arrow[r] \arrow[d, equal] & K\mathbf{C}\mathcal{X}^G_Y(\mathcal{O}^\infty(X)\otimes Y) \arrow[r] \arrow[d, dotted, "\simeq"] & {} \\
{} \arrow[r] & K^{C^*\mathrm{Cat}}(\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{Z}\subseteq\mathcal{O}(X)\otimes Y)) \arrow[r] & K^{C^*\mathrm{Cat}}(\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(X)\otimes Y)) \arrow[r] & K^{C^*\mathrm{Cat}}\left(\frac{\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(X)\otimes Y)}{\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{Z}\subseteq\mathcal{O}(X)\otimes Y)}\right) \arrow[r] & {}
\end{tikzcd} \tag{14.13}$$

The lower sequence is a fibre sequence by the exactness of $K^{C^*\mathrm{Cat}}$ ([12, Thm. 1.32.5] or [9, Prop.14.7]), and the upper sequence is an instance of the cone sequence (4.5). We now argue that the left vertical morphism is an equivalence (essentially the same argument as for the left vertical arrow in (6.8)). First of all, for every n in $\mathbb{N}$ the inclusion

$$\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(Z_n)\simeq\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(Z_n\subseteq\mathcal{O}(X)\otimes Y)$$

is a unitary equivalence by [7, Lem. 6.10(2)], where $Z_n := [0,n]\times X\times Y$ has the structures induced from $\mathcal{O}(X)\otimes Y$. The inclusion $X\otimes Y\to Z_n$ given by $(x,y)\mapsto(0,x,y)$ is a coarse equivalence. Hence, the induced map

$$K\mathbf{C}\mathcal{X}^G(X\otimes Y)\to K\mathbf{C}\mathcal{X}^G(\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(Z_n))\xrightarrow{\simeq}K^{C^*\mathrm{Cat}}(\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(Z_n\subseteq\mathcal{O}(X)\otimes Y))$$

is an equivalence for every n in $\mathbb{N}$. We now use that by definition

$$\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{Z}\subseteq\mathcal{O}(X)\otimes Y)\cong\mathrm{colim}_{n\in\mathbb{N}}\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(Z_n\subseteq\mathcal{O}(X)\otimes Y)$$

and that $K^{C^*\mathrm{Cat}}$ commutes with filtered colimits by [9, Thm. 14.4]. Hence, we get an equivalence

$$K\mathbf{C}\mathcal{X}^G(X\otimes Y)\xrightarrow{\simeq}K^{C^*\mathrm{Cat}}(\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{Z}\subseteq\mathcal{O}(X)\otimes Y))$$

induced by the canonical inclusion. This is exactly the left vertical arrow in (14.13).

We now assume that X is in $G\mathbf{UBC}_{\mathrm{pc}}$ (see Definition 14.1) and note that X is then G-bounded. Using two instances of the the diagram (14.13), one for X and $Y = G_{can,max}$, and one for $X_{\mathcal{B}_{\max}}$ and $Y = G_{can,min}$, and the equalities of C^*-categories resulting from Lemma 14.7 and (14.11) saying that the corresponding lower fibre sequences of the two diagrams are equivalent we get the desired equivalence (14.10). □

Let (X,χ) be in $G\mathbf{UBC}^{\mathcal{R}}_{\mathrm{pc}}$ (see the text before Proposition 14.3). Recall Definition 13.11 of Asmbl^{Θ} and Definition 14.5 of Asmbl^{Λ}.

Proposition 14.8. *We have a commutative square*

$$\begin{tikzcd}
K\mathbf{C}\mathcal{X}^{G}_{G_{can,min}}(\mathcal{O}^{\infty}(X_{\mathcal{B}_{\max}})) \arrow[r, "\mathrm{Asmbl}^{\Theta}_{X_{\mathcal{B}_{max}}}"] \arrow[d, dash, "(14.10)"', "\simeq"] & \Sigma\mathrm{KK}(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G) \arrow[d, equal] \\
K\mathbf{C}\mathcal{X}^{G}_{G_{can,max}}(\mathcal{O}^{\infty}(X)) \arrow[r, "\mathrm{Asmbl}^{\Lambda}_{(X,\chi)}"] & \Sigma\mathrm{KK}(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)
\end{tikzcd} \tag{14.14}$$

which depends naturally on the coefficient category $\mathbf{C}$ *in* $\mathrm{Fun}(BG, C^{*}\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg,eadd},\omega\mathrm{add}})$.

Proof. Recall the construction of the functor Θ in Definition 13.1 (see also (13.9)) and of Λ in Definition 14.2. We get the following morphism of exact sequences of C^{*}-categories.

$$\begin{tikzcd}
0 \arrow[r] & \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{Z} \subseteq \mathcal{O}(X) \otimes G_{can,max}) \arrow[r] \arrow[d, "\Lambda_{(X,\chi)}"] \arrow[dd, equal, bend right] & \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(X) \otimes G_{can,max}) \arrow[r] \arrow[d, "\Lambda_{(X,\chi)}"] \arrow[dd, equal, bend right] & \frac{\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(X) \otimes G_{can,max})}{\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{Z} \subseteq \mathcal{O}(X) \otimes G_{can,max})} \arrow[r] \arrow[d] \arrow[dd, equal, bend right] & 0 \\
0 \arrow[r] & \mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G) \arrow[r] & \mathrm{Idem}(\mathbf{U}) \arrow[r] & \frac{\mathrm{Idem}(\mathbf{U})}{\mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)} \arrow[r] & 0 \\
0 \arrow[r] & \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{Z} \subseteq \mathcal{O}(X_{\mathcal{B}_{max}}) \otimes G_{can,min}) \arrow[r] \arrow[u, "\Theta_{X_{\mathcal{B}_{max}}}"] & \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(X_{\mathcal{B}_{max}}) \otimes G_{can,min}) \arrow[r] \arrow[u, "\Theta_{\mathcal{O}(X_{\mathcal{B}_{\max}})}"] & \frac{\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(X_{\mathcal{B}_{max}}) \otimes G_{can,min})}{\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{Z} \subseteq \mathcal{O}(X_{\mathcal{B}_{max}}) \otimes G_{can,min})} \arrow[r] \arrow[u] & 0
\end{tikzcd} . \tag{14.15}$$

The right vertical maps are induced from the universal property of quotients. The round equalities are consequences of Lemma 14.7 and (14.11). The right equality is responsible for the left vertical equivalence in (14.14) up to identifications, see the proof of Proposition 14.6. We apply $K^{C^{*}\mathrm{Cat}}$ and consider the segment of the long exact sequences which involve the boundary map. We use the identification given by the right vertical equivalences in the two instances of (14.13) with X and $G_{can,max}$ and $X_{\mathcal{B}_{max}}$ and $Y = G_{can,min}$ in order to express the K-theory of the quotient categories in terms of coarse K-homology.

$$\begin{tikzcd}
K\mathbf{C}\mathcal{X}^{G}_{c,G_{can,max}}(\mathcal{O}^{\infty}(X)) \arrow[r, "\partial^{\mathrm{Cone}}"] \arrow[dd, dash, "\simeq"'] & \Sigma K^{C^{*}\mathrm{Alg}}(\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{Z} \subseteq \mathcal{O}(X) \otimes G_{can,max})) \arrow[dd, dash, "\simeq"'] \arrow[dr, "K^{C^{*}\mathrm{Alg}}(\Lambda_{(X,\chi)})"] & \\
& & \mathrm{KK}(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G) \\
K\mathbf{C}\mathcal{X}^{G}_{c,G_{can,min}}(\mathcal{O}^{\infty}(X_{\mathcal{B}_{max}})) \arrow[r, "\partial^{\mathrm{Cone}}"] & \Sigma K^{C^{*}\mathrm{Alg}}(\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{Z} \subseteq \mathcal{O}(X_{\mathcal{B}_{max}}) \otimes G_{can,min})) \arrow[ur, "K^{C^{*}\mathrm{Alg}}(\Theta_{X_{\mathcal{B}_{max}}})"'] &
\end{tikzcd} . \tag{14.16}$$

The left square commutes since it is induced by an equality of exact sequences of C^{*}-categories. We must provide the filler of the right triangle.

This filler will be given by a unitary equivalence (see [9, Def. 17.9] for the definition of this notion in the non-unital case) of functors on the level of C^{*}-categories which will be induced from the equivalence provided by the following lemma.

Lemma 14.9. *The following triangle is filled by a natural unitary equivalence:*

$$\begin{tikzcd}
\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(X)\otimes G_{can,max}) \arrow[dd, equal] \arrow[dr, "\Lambda_{(X,\chi)}"] & \\
& \mathrm{Idem}(\mathbf{U}) \\
\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(X_{\mathcal{B}_{max}})\otimes G_{can,min}) \arrow[ur, "\Theta_{X_{\mathcal{B}_{max}}}"'] &
\end{tikzcd} .$$

Proof. We consider an object (C,ρ,μ) on the common domain of the functors. We define

$$U := uv^*$$

in $\mathbf{U}$ with u as in (13.6) and v as in (14.4). By (13.8) and (14.6) we have

$$UU^* = p\,, \quad U^*U = p_\chi\,,$$

where $\tilde{p}$ and p_χ are as in (13.2) and (14.1), respectively. We conclude that $Up_\chi = pU$ and that we therefore have a unitary isomorphism $U\colon (C,\rho,p_\chi)\to(C,\rho,p)$ in $\mathrm{Idem}(\mathbf{U})$ as desired.

In order to verify that U implements a natural transformation we must check the compatibility with morphisms. Let $A\colon (C,\rho,\mu)\to(C',\rho',\mu')$ be a morphism in the domain of the functors. We let U' be defined as above for (C',ρ',μ'). Then by (13.7) and (14.5) we have

$$U'\Lambda_{(X,\chi)}(A) = \Theta_X(A)U\,.$$

□

In view of [9, Rem. 17.10], the unitary equivalence from Lemma 14.9 implements a unitary equivalence filling

$$\begin{tikzcd}
\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{Z}\subseteq\mathcal{O}(X)\otimes G_{can,max}) \arrow[dd, equal] \arrow[dr, "\Lambda_{(X,\chi)}"] & \\
& \mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}}\rtimes_r G) \\
\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{Z}\subseteq\mathcal{O}(X_{\mathcal{B}_{max}})\otimes G_{can,min}) \arrow[ur, "\Theta_{X_{\mathcal{B}_{max}}}"'] &
\end{tikzcd}$$

We now use [9, Lem. 17.11] which provides the desired filler of the right triangle in (14.16). □

Remark 14.10. In Proposition 14.8, we could state a stronger assertion saying that there is an equivalence of natural transformations from $G\mathbf{UBC}^{\mathcal{R}}_{\mathrm{pc}}$. The constructions on the C^*-category level done in the proof are sufficiently natural. But writing out the details would amount to write out large higher coherence diagrams. Since we do not really need this naturality, we refrain from doing so. □

We consider (X,χ) in $G\mathbf{UBC}^{\mathcal{R}}_{\mathrm{pc}}$. Recall the Paschke morphism p_X from (1.18). We use Definition 4.9 in order to rewrite the domain of $\mathrm{Asmbl}^{\Lambda}_{(X,\chi)}$ introduced in Definition 14.5. Recall the Definition 12.11 of $\mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C}}$.

Proposition 14.11. *We have a commutative square*

$$\begin{tikzcd} K^{G,\mathcal{X}}_{\mathbf{C}}(X) \arrow[r, "\mathrm{Asmbl}^{\Lambda}_{(X,\chi)}"] \arrow[d, "p_X"'] & \Sigma\mathrm{KK}(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G) \arrow[d, equal] \\ K^{G,\mathrm{An}}_{\mathbf{C}}(\iota^{\mathrm{top}}(X)) \arrow[r, "\mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C},\iota^{\mathrm{top}}(X)}"] & \Sigma\mathrm{KK}(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G) \end{tikzcd} \tag{14.17}$$

which depends naturally on the coefficient category $\mathbf{C}$ *in* $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg,eadd},\omega\mathrm{add}})$.

Proof. We consider the following commutative diagram of exact sequences in $C^*\mathbf{Cat}^{\mathrm{nu}}$

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{C}(X) \arrow[r] \arrow[d, equal, "(6.4)"] & \mathbf{D}(X) \arrow[r] \arrow[d, equal, "(6.3)"] & \mathbf{Q}(X) \arrow[r] \arrow[d, equal, "(6.5)"] & 0 \, . \\ 0 \arrow[r] & \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{Z} \subseteq \mathcal{O}(X) \otimes G_{can,max}) \arrow[r] \arrow[d, "\Lambda_{(X,\chi)}"] & \bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(X) \otimes G_{can,max}) \arrow[r] \arrow[d, "\Lambda_{(X,\chi)}"] & \frac{\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(X)\otimes G_{can,max})}{\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{Z}\subseteq\mathcal{O}(X)\otimes G_{can,max})} \arrow[r] \arrow[d, "\bar{\Lambda}_{(X,\chi)}"] & 0 \\ 0 \arrow[r] & \mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G) \arrow[r] & \mathrm{Idem}(\mathbf{U}) \arrow[r] & \frac{\mathrm{Idem}(\mathbf{U})}{\mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)} \arrow[r] & 0 \end{tikzcd} \tag{14.18}$$

We use the right vertical equivalence of (14.13) for X and $Y = G_{can,max}$ and Definition 4.9 in order to get the equivalence

$$K^{G,\mathcal{X}}_{\mathbf{C}}(X) \simeq K^{C^*\mathrm{Cat}}\left(\frac{\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{O}(X) \otimes G_{can,max})}{\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{Z} \subseteq \mathcal{O}(X) \otimes G_{can,max})}\right) \simeq K^{C^*\mathrm{Cat}}(\mathbf{Q}(X)) \, .$$

We now expand the square (14.17) as follows:

$$\begin{tikzcd}
K^{G,\mathcal{X}}_{\mathbf{C}}(X) \arrow[r, "\hat{\partial}^{\mathrm{Cone}}"] \arrow[rr, bend left, "\mathrm{Asmbl}^{\Lambda}_{(X,\chi)}"] & \Sigma K^{C^*\mathrm{Cat}}(\bar{\mathbf{C}}^{G,\mathrm{ctr}}_{\mathrm{lf}}(\mathcal{Z} \subseteq \mathcal{O}(X) \otimes G_{can,max})) \arrow[r, "K^{C^*\mathrm{Cat}}(\Lambda_{(X,\chi)})"] & \Sigma \mathrm{KK}(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G) \arrow[d, equal] \\
K^{C^*\mathrm{Cat}}(\mathbf{Q}(X)) \arrow[u, "\simeq"'] \arrow[r, "K^{C^*\mathrm{Cat}}(\bar{\Lambda}_{(X,\chi)})"] \arrow[d, "p_X"'] & K^{C^*\mathrm{Cat}}(\frac{\mathrm{Idem}(\mathbf{U})}{\mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)}) \arrow[r, "\partial"] \arrow[d, equal] & \Sigma \mathrm{KK}(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G) \arrow[d, equal] \\
K^{G,\mathrm{An}}_{\mathbf{C}}(\iota^{\mathrm{top}}(X)) \arrow[r, "\mathrm{ctc} \circ \epsilon^* \circ (- \rtimes G)"] \arrow[rr, bend right, "\mathrm{Asmbl}^{\mathrm{an}}_{\iota^{\mathrm{top}}(X)}"'] & K^{C^*\mathrm{Cat}}(\frac{\mathrm{Idem}(\mathbf{U})}{\mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)}) \arrow[r, "\partial"] & \Sigma \mathrm{KK}(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)
\end{tikzcd} \tag{14.19}$$

where ctc is the change-of-target functor (12.19) and ϵ^* is as in (12.16). The commutativity of the upper triangle reflects the definition of $\mathrm{Asmbl}^{\Lambda}_{(X,\chi)}$ in Definition 14.5. The filler of the middle hexagon is obtained from the naturality of boundary operators for the morphism of fibre sequences obtained by applying $K^{C^*\mathrm{Cat}}$ to (14.18). The lower triangle reflects the Definition 12.11 of $\mathrm{Asmbl}^{\mathrm{an}}_{\iota^{\mathrm{top}}(X)}$ where also the notation appearing on the lower left horizontal arrow is explained.

So in order to produce a filler of the square (14.17) we must provide a filler of the lower left square in (14.19). This is the assertion of the following lemma.

Lemma 14.12. *We have a commutative square*

$$\begin{tikzcd}
K^{C^*\mathrm{Cat}}(\mathbf{Q}(X)) \arrow[r, "K^{C^*\mathrm{Cat}}(\bar{\Lambda}_{(X,\chi)})"] \arrow[d, "p_X"'] & K^{C^*\mathrm{Cat}}(\frac{\mathrm{Idem}(\mathbf{U})}{\mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)}) \arrow[d, equal] \\
K^{G,\mathrm{An}}_{\mathbf{C}}(\iota^{\mathrm{top}}(X)) \arrow[r, "\mathrm{ctc} \circ \epsilon^* \circ (- \rtimes G)"] & K^{C^*\mathrm{Cat}}(\frac{\mathrm{Idem}(\mathbf{U})}{\mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)})
\end{tikzcd}$$

Proof. We start with the following diagram:

$$\begin{tikzcd}
\mathrm{KK}^G(C_0(X), C_0(X) \otimes \mathbf{Q}(X)) \arrow[r, "\mu_X, (6.12)"] \arrow[d, "- \rtimes G"] & \mathrm{KK}^G(C_0(X), \mathbf{Q}^{(G)}_{\mathrm{std}}) \arrow[d, "- \rtimes G"] \\
\mathrm{KK}(C_0(X) \rtimes G, (C_0(X) \otimes \mathbf{Q}(X)) \rtimes G) \arrow[r, "\mu_X \rtimes G"] \arrow[d, "\epsilon^*"] & \mathrm{KK}(C_0(X) \rtimes G, \mathbf{Q}^{(G)}_{\mathrm{std}} \rtimes G) \arrow[d, "\epsilon^*"] \\
\mathrm{KK}(\mathbb{C}, (C_0(X) \otimes \mathbf{Q}(X)) \rtimes G) \arrow[r, "\mu_X \rtimes G"] & \mathrm{KK}(\mathbb{C}, \mathbf{Q}^{(G)}_{\mathrm{std}} \rtimes G)
\end{tikzcd} \tag{14.20}$$

where ϵ^* is given by pre-composition in KK with the morphism described in (12.13). The first square commutes since $- \rtimes G$ is a functor. The second square commutes since KK is a bifunctor.

The next diagram extends (14.20) to the left:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}})}(C_0(X), C_0(X)) \times \mathrm{KK}(\mathbb{C}, \mathbf{Q}(X)) \arrow[r, "\hat{\otimes}"] \arrow[d, "(-\rtimes G)\times \mathrm{id}"] & \mathrm{KK}^G(C_0(X), C_0(X) \otimes \mathbf{Q}(X)) \arrow[d, "-\rtimes G"] \\
\mathrm{Hom}_{C^*\mathbf{Alg}^{\mathrm{nu}}}(C_0(X) \rtimes G, C_0(X) \rtimes G) \otimes \mathrm{KK}(\mathbb{C}, \mathbf{Q}(X)) \arrow[r, "\hat{\otimes}"] \arrow[d, "\epsilon^* \times \mathrm{id}"] & \mathrm{KK}(C_0(X) \rtimes G, (C_0(X) \otimes \mathbf{Q}(X)) \rtimes G) \arrow[d, "\epsilon^*"] \\
\mathrm{Hom}_{C^*\mathbf{Alg}^{\mathrm{nu}}}(\mathbb{C}, C_0(X) \rtimes G) \otimes \mathrm{KK}(\mathbb{C}, \mathbf{Q}(X)) \arrow[r, "\hat{\otimes}"] & \mathrm{KK}(\mathbb{C}, (C_0(X) \otimes \mathbf{Q}(X)) \rtimes G)
\end{tikzcd} \tag{14.21}$$

The second square commutes since $\hat{\otimes}$ in (6.11) is a bifunctor. The argument for the commutativity of the first square is the same as for the third square in (9.10). We finally specialize (14.21) at $\mathrm{id}_{C_0(X)}$ in $\mathrm{Hom}_{\mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}})}(C_0(X), C_0(X))$ and get

$$\begin{tikzcd}
\mathrm{KK}(\mathbb{C}, \mathbf{Q}(X)) \arrow[r, "\mathrm{id}_{C_0(X)} \hat{\otimes}"] \arrow[d, equal] & \mathrm{KK}^G(C_0(X), C_0(X) \otimes \mathbf{Q}(X)) \arrow[d, "-\rtimes G"] \\
\mathrm{KK}(\mathbb{C}, \mathbf{Q}(X)) \arrow[r, "\mathrm{id}_{C_0(X)\rtimes G} \hat{\otimes}"] \arrow[d, equal] & \mathrm{KK}(C_0(X) \rtimes G, (C_0(X) \otimes \mathbf{Q}(X)) \rtimes G) \arrow[d, "\epsilon^*"] \\
\mathrm{KK}(\mathbb{C}, \mathbf{Q}(X)) \arrow[r, "\epsilon \hat{\otimes} \mathrm{id}_{\mathbf{Q}(X)\rtimes G}"] & \mathrm{KK}(\mathbb{C}, (C_0(X) \otimes \mathbf{Q}(X)) \rtimes G)
\end{tikzcd} \tag{14.22}$$

Forming the horizontal composition of (14.22) and (14.20) and using Definition 6.14 of p_X yields the bold part of the commutative diagram

$$\begin{tikzcd}
\mathrm{KK}(\mathbb{C}, \mathbf{Q}(X) \arrow[r, "p_X"] \arrow[d, equal] & \mathrm{KK}^G(C_0(X), \mathbf{Q}^{(G)}_{\mathrm{std}}) \arrow[d, "\epsilon^* \circ (-\rtimes G)"] \\
\mathrm{KK}(\mathbb{C}, \mathbf{Q}(X) \arrow[r] \arrow[dr, dotted, "K^{C^*\mathrm{Cat}}(\Gamma_{(X,\chi)})"'] & \mathrm{KK}(\mathbb{C}, \mathbf{Q}^{(G)}_{\mathrm{std}} \rtimes G) \arrow[d, "\mathrm{ctc}"] \\
 & K^{C^*\mathrm{Cat}}\left(\frac{\mathrm{Idem}(\mathbf{U})}{\mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)}\right)
\end{tikzcd} \tag{14.23}$$

Unfolding the definitions we see that the dotted morphism is induced by a functor

$$\Gamma_{(X,\chi)} : \mathbf{Q}(X) \to \frac{\mathrm{Idem}(\mathbf{U})}{\mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G)} \tag{14.24}$$

which has the following description:

(1) objects: The functor $\Gamma_{(X,\chi)}$ sends the object (C,ρ,μ) in $\mathbf{Q}(X)$ to the object (C,ρ,id_C) in $\frac{\mathrm{Idem}(\mathbf{U})}{\mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}}\rtimes_r G)}$.

(2) morphisms: The functor $\Gamma_{(X,\chi)}$ sends a morphism $[A]\colon (C,\rho,\mu)\to(C',\rho',\mu')$ in $\mathbf{Q}(X)$ to the morphism

$$[\sum_{g\in G}\sigma(\phi'(\chi)\phi'(g^*\chi)A,g)]\colon (C,\rho,\mathrm{id}_C)\to(C',\rho,\mathrm{id}_{C'}) \tag{14.25}$$

in $\frac{\mathrm{Idem}(\mathbf{U})}{\mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}}\rtimes_r G)}$. Here we use the formula (12.11) for p_χ which enters the definition of ϵ^*, and σ is as in (2.10).

Note that the sum in (14.25) has finitely many non-zero terms. In order to show Lemma 14.12, we must provide an equivalence

$$K^{C^*\mathrm{Cat}}(\Gamma_{(X,\chi)})\simeq K^{C^*\mathrm{Cat}}(\bar\Lambda_{(X,\chi)}), \tag{14.26}$$

where

$$\bar\Lambda_{(X,\chi)}\colon \mathbf{Q}(X)\to\frac{\mathrm{Idem}(\mathbf{U})}{\mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}}\rtimes_r G)} \tag{14.27}$$

is as in (14.18). It has the following explicit description derived from Definition 14.2:

(1) objects: The functor $\bar\Lambda_{(X,\chi)}$ sends the object (C,ρ,μ) in $\mathbf{Q}(X)$ to the object (C,ρ,p_χ) in $\frac{\mathrm{Idem}(\mathbf{U})}{\mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}}\rtimes_r G)}$.

(2) morphisms: The functor $\bar\Lambda_{(X,\chi)}$ sends a morphism $[A]\colon (C,\rho,\mu)\to(C',\rho',\mu')$ in $\mathbf{Q}(X)$ to the morphism

$$[\sum_{g\in G}\sigma(\phi'(g^*\chi)A\phi(\chi),g)]\colon (C,\rho,p_\chi)\to(C',\rho',p'_\chi) \tag{14.28}$$

in $\frac{\mathrm{Idem}(\mathbf{U})}{\mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}}\rtimes_r G)}$, see (14.2)

Recall the notion of a MvN equivalence of functors from [9, Def. 17.12]. We claim that the functors $\bar\Lambda_{(X,\chi)}$ and $\Gamma_{(X,\chi)}$ are MvN equivalent. The claim implies the equivalence (14.26) by [9, Prop. 16.18 & 17.14].

The MvN equivalence $v\colon\bar\Lambda_{(X,\chi)}\to\Gamma_{(X,\chi)}$ is given by the family of partial isometries $v=([v_{(C,\rho,\mu)}])_{(C,\rho,\mu)\in\mathbf{Q}(X)}$, where $v_{(C,\rho,\mu)}\colon (C,\rho,p_\chi)\to(C,\rho,\mathrm{id}_C)$ is the canonical inclusion. This inclusion is given by the morphism $p_\chi\colon C\to C$ which indeed belongs to $\mathbf{U}$. Note that in the summands in (14.25), we can replace $A\phi(\chi)$ by $\phi'(\chi)A$ since A is pseudo-local by Lemma 5.8 and we take the quotient by $\mathrm{Idem}(\mathbf{C}^{(G)}_{\mathrm{std}}\rtimes_r G)$. Naturality of v is now obvious since the formulas (14.25) and (14.28) for the action of the functors on morphisms coincide after this replacement. This finishes the proof of Lemma 14.12. □

To complete the proof of Proposition 14.11 we observe by an inspection of the constructions that they depend naturally on the coefficient category $\mathbf{C}$ in $\mathrm{Fun}(BG,C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg,eadd,}\omega\mathrm{add}})$. □

By equipping a G-simplicial complex X with the structures induced by the metric we obtain an object $m(X)$ of $G\mathbf{UBC}$. We further use the notation introduced in the diagram (13.15) in order to interpret X in $G\mathbf{UBC}_{\mathrm{bd}}$ or $G\mathbf{Top}$. In the following statement and its proof, we must be very precise about this interpretation.

Proposition 14.13. *If X is a G-finite G-simplicial complex with finite stabilizers, then we have a commutative square*

$$\begin{tikzcd} \Sigma K\mathbf{C}^G(t(X)) \arrow[r, "\Sigma\mathrm{Asmbl}^h_{\mathbf{C},t(X)}"] \arrow[d, "\simeq"] & \Sigma K\mathbf{C}^G(*) \arrow[d, "\simeq"] \\ K^{G,\mathrm{An}}_{\mathbf{C}}(\iota^{\mathrm{top}}(m(X))) \arrow[r, "\mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C},\iota^{\mathrm{top}}(m(X))}"] & \Sigma\mathrm{KK}(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G) \end{tikzcd} \tag{14.29}$$

which depends naturally on $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg,eadd},\omega\mathrm{add}})$.

Proof. Note that $s(X) = m(X)_{\mathcal{B}_{\max}}$ in the notation introduced before Proposition 14.6. Note further that $m(X)$ actually belongs to the subcategory $G\mathbf{UBC}_{\mathrm{pc}}$ described in Definition 14.1. We can therefore choose χ in $\mathcal{R}(m(X))$. We consider the diagram

$$\begin{tikzcd} \Sigma K\mathbf{C}^G(t(X)) \arrow[r, "\Sigma\mathrm{Asmbl}^h_{\mathbf{C},t(X)}"] \arrow[d, "\simeq", "13.12"'] & \Sigma K\mathbf{C}^G(*) \arrow[d, "13.5"', "\simeq"] \\ K\mathbf{C}\mathcal{X}^G_{G_{can,min}}(\mathcal{O}^\infty(s(X)) \arrow[r, "\mathrm{Asmbl}^{\Theta}_{s(X)}"] \arrow[d, "\simeq", "14.6"'] & \Sigma\mathrm{KK}(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G) \arrow[d, equal] \\ K\mathbf{C}\mathcal{X}^G_{G_{can,max}}(\mathcal{O}^\infty(m(X))) \arrow[r, "\mathrm{Asmbl}^{\Lambda}_{(m(X),\chi)}"] \arrow[d, equal, "\mathrm{def}"] & \Sigma\mathrm{KK}(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G) \arrow[d, equal] \\ K^{G,\mathcal{X}}_{\mathbf{C}}(m(X)) \arrow[r, "\mathrm{Asmbl}^{\Lambda}_{(m(X),\chi)}"] \arrow[d, "\simeq", "p_{m(X)}"'] & \Sigma\mathrm{KK}(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G) \arrow[d, equal] \\ K^{G,\mathrm{An}}_{\mathbf{C}}(\iota^{\mathrm{top}}(m(X))) \arrow[r, "\mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C},\iota^{\mathrm{top}}(m(X))}"] & \Sigma\mathrm{KK}(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G) \end{tikzcd}$$

The lowest left vertical map is an equivalence by an application of our main Theorem 1.5.2. The statement that each of the above squares commute is proven, from top to bottom, in Proposition 13.12.3, Proposition 14.8, the definitions, and Proposition 14.11. All squares depend naturally on the coefficient category $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg,eadd},\omega\mathrm{add}})$. This shows the proposition. □

Proof of Theorem 1.9. We choose a model for $E_{\mathcal{F}}G^{\mathrm{CW}}$ which is a G-simplicial complex. Then we apply π_* to the square (14.29) and form the colimit of the resulting squares of homotopy groups for X running over the G-finite subcomplexes of $E_{\mathcal{F}}G$. This yields (1.22). □

Remark 14.14. In the proof of Theorem 1.9, we must apply π_* before taking the colimit over the subcomplexes. The reason is that we have only constructed the boundary of the square (14.29) naturally in X. For the fillers, we just have shown existence for every X separately. □

15. Davis–Lück functors and the argument of Kranz

In this section, we review the argument of Kranz [26] for the comparison of the Davis–Lück assembly map with the Kasparov assembly map which involves the Meyer–Nest assembly map as an intermediate step. In more detail, Kranz compares the Davis–Lück assembly map with the Meyer–Nest assembly map, which is known to coincide with the analytical assembly map. We will review these comparisons below. In fact, Kranz' paper has two separate parts. On the one hand, he shows that the Davis–Lück assembly map associated to a functor

$$K^G : \mathrm{KK}^G_{\mathrm{sep}} \to \mathrm{Fun}(G\mathbf{Orb}, \mathbf{Sp})$$

satisfying certain axioms (stated in Assumption 15.6) is equivalent to the Meyer–Nest assembly map. On the other hand, he provides a concrete construction of such a functor K^G. We recall this construction in detail with the goal of showing that it only involves formal manipulations using the calculus of equivariant KK-theory as developed in [12].

We first recall the Meyer–Nest approach to the Baum–Connes assembly map [30]. Given the results of [12] and the present paper, we will give an almost self-contained treatment, the only exception is the usage of [30, Prop. 4.6] in the proof of Proposition 15.2 below. We interpret the terminology introduced in [30] in the stable ∞-category $\mathrm{KK}^G_{\mathrm{sep}}$ introduced in [12, Def. 1.8] instead of the triangulated homotopy category of $\mathrm{KK}^G_{\mathrm{sep}}$ as considered by Meyer–Nest. We call a subcategory of $\mathrm{KK}^G_{\mathrm{sep}}$ localizing[8] if it is thick and closed under countable direct sums. In the following, we use the restriction, induction and crossed-product functors on the level of stable ∞-categories as introduced in [12, Sec. 1.5].

Definition 15.1.

(1) *We define $\mathcal{CI}$ as the localizing subcategory of $\mathrm{KK}^G_{\mathrm{sep}}$ generated by the objects of the form $\mathrm{Ind}^G_{H,s}(A)$ for all finite subgroups H of G and objects A in $\mathrm{KK}^H_{\mathrm{sep}}$. The objects of $\mathcal{CI}$ will be called compactly induced.*

(2) *We define $\mathcal{CC}$ as the localizing subcategory of $\mathrm{KK}^G_{\mathrm{sep}}$ given by all objects A with $\mathrm{Res}^G_{H,s}(A) = 0$ for all finite subgroups H of G.*

[8] Usually, localizing subcategories are stable, cocomplete subcategories of stable, cocomplete ∞-categories. Since $\mathrm{KK}^G_{\mathrm{sep}}$ is only known to admit countable colimits, we must use this ad-hoc definition.

We note here that $\mathcal{CC}$ is localising because the restriction functors commute with countable sums [12, Lem. 4.3]. The proof of the following proposition is based on a general adjoint functor theorem applicable in this situation.

Proposition 15.2. *There exists an adjunction*

$$\mathrm{incl} : \mathcal{CI} \leftrightarrows \mathrm{KK}^G_{\mathrm{sep}} : C \tag{15.1}$$

Proof. For any object A in $\mathrm{KK}^G_{\mathrm{sep}}$, by [30, Prop. 4.6], there is an object $\tilde{A}$ in $\mathcal{CI}$ with a morphism $\tilde{A} \to A$ (called the Dirac morphism) inducing an equivalence of functors $\mathrm{KK}^G_{\mathrm{sep}}(-,\tilde{A}) \to \mathrm{KK}^G_{\mathrm{sep}}(-,A)$ from $\mathcal{CI}^{\mathrm{op}}$ to $\mathbf{Sp}$. Hence, for any A in $\mathrm{KK}^G_{\mathrm{sep}}$, the functor $\mathrm{KK}^G_{\mathrm{sep}}(-,A)_{|\mathcal{CI}^{\mathrm{op}}} : \mathcal{CI}^{\mathrm{op}} \to \mathbf{Sp}$ is representable by an object of $\mathcal{CI}$. This implies the existence of the right adjoint C to incl as follows for instance from [27, Prop. 5.1.10]. □

Let $\mathcal{C}$ be a stable ∞-category. Recall that a semi-orthogonal decomposition of $\mathcal{C}$ is a pair $(\mathcal{A},\mathcal{B})$ of full stable subcategories such that $\mathrm{map}_{\mathcal{C}}(A,B) \simeq 0$ for all A in $\mathcal{A}$ and B in $\mathcal{B}$, and such that for every object C of $\mathcal{C}$ there exists a fibre sequence $A \to C \to B$ with A in $\mathcal{A}$ and B in $\mathcal{B}$. For the sake of completeness of the presentation, we give the following list of equivalent conditions on a pair $(\mathcal{A},\mathcal{B})$ of stable subcategories, and refer for more details to [29, Sec. 7.2.1]:

(1) The pair $(\mathcal{A},\mathcal{B})$ is a semi-orthogonal decomposition of $\mathcal{C}$.
(2) The pair $(\mathcal{A},\mathcal{B})$ is a t-structure on $\mathcal{C}$.
(3) The inclusion $\mathcal{A} \to \mathcal{C}$ has a right adjoint and $\mathcal{B}$ is the right orthogonal complement of $\mathcal{A}$.
(4) The inclusion $\mathcal{B} \to \mathcal{C}$ has a left adjoint and $\mathcal{A}$ is the left orthogonal complement of $\mathcal{B}$.

Proposition 15.3. *The pair $(\mathcal{CI},\mathcal{CC})$ is a semi-orthogonal decomposition of* $\mathrm{KK}^G_{\mathrm{sep}}$.

Proof. For every subgroup H of G, we have an adjunction

$$\mathrm{Ind}^G_{H,s} : \mathrm{KK}^H_{\mathrm{sep}} \leftrightarrows \mathrm{KK}^G_{\mathrm{sep}} : \mathrm{Res}^G_{H,s}$$

which can be obtained from [12, Thm. 1.23.1] by restriction to the separable subcategories. It is an immediate consequence of the existence of these adjunctions that $\mathrm{KK}^G_{\mathrm{sep}}(A,B) \simeq 0$ for all A in $\mathcal{CI}$ and B in $\mathcal{CC}$. We get in fact the following stronger assertion that $\mathcal{CC}$ consists precisely of the objects B of $\mathrm{KK}^G_{\mathrm{sep}}$ with $\mathrm{KK}^G_{\mathrm{sep}}(A,B) \simeq 0$ for all A in $\mathcal{CI}$, i.e. that $\mathcal{CC}$ is the right orthogonal complement to $\mathcal{CI}$.

In view of Proposition 15.2, the following is precisely a specialization of the argument that Condition 3 above implies Condition 1. We must show that for any object A of $\mathrm{KK}^G_{\mathrm{sep}}$, there is a fibre sequence

$$C(A) \longrightarrow A \longrightarrow N(A) \tag{15.2}$$

with $C(A)$ in $\mathcal{CI}$ and $N(A)$ in $\mathcal{CC}$. By Proposition 15.2 we have a fibre sequence of functors $C \to \mathrm{id}_{\mathrm{KK}^G_{\mathrm{sep}}} \to N$, where $N : \mathrm{KK}^G_{\mathrm{sep}} \to \mathrm{KK}^G_{\mathrm{sep}}$ is defined as the cofibre of the counit of the adjunction in (15.1). It suffices to show that N takes values in $\mathcal{CC}$. Let A be in $\mathrm{KK}^G_{\mathrm{sep}}$. Then for every B in $\mathcal{CI}$ we have $\mathrm{KK}^G_{\mathrm{sep}}(B, N(A)) \simeq \mathrm{cofib}(\mathrm{KK}^G_{\mathrm{sep}}(B, C(A)) \to \mathrm{KK}^G_{\mathrm{sep}}(B, A))$. But $\mathrm{KK}^G_{\mathrm{sep}}(B, C(A)) \to \mathrm{KK}^G_{\mathrm{sep}}(B, A)$ is an equivalence by the construction of C so that $\mathrm{KK}^G_{\mathrm{sep}}(B, N(A)) \simeq 0$. Since, as seen above, $\mathcal{CC}$ is precisely the right-orthogonal complement of $\mathcal{CI}$ this implies that $N(A)$ belongs to $\mathcal{CC}$. □

Let A be in $\mathrm{KK}^G_{\mathrm{sep}}$.

Definition 15.4. *The Meyer–Nest assembly map for G is the map*

$$\mu^{\mathrm{MN}}_* : \mathrm{KK}_{\mathrm{sep}}(\mathbb{C}, C(A) \rtimes_r G) \to \mathrm{KK}_{\mathrm{sep}}(\mathbb{C}, A \rtimes_r G)$$

induced by $C(A) \to A$ *in* $\mathrm{KK}^G_{\mathrm{sep}}$.

The following theorem is an immediate consequence of [30, Prop. 5.2] which yields the comparison of the Meyer–Nest assembly map and Kasparov's assembly map.

Theorem 15.5. *There is a commutative square*

$$\begin{tikzcd} RK^{G,\mathrm{an}}_{C(A)}(E_{\mathbf{Fin}}G^{\mathrm{CW}}) \arrow[r, "\simeq"] \arrow[d, "\simeq"', "\mu^{\mathrm{Kasp}}_{C(A)}"] & RK^{G,\mathrm{an}}_{A}(E_{\mathbf{Fin}}G^{\mathrm{CW}}) \arrow[d, "\mu^{\mathrm{Kasp}}_{A}"] \\ \mathrm{KK}_{\mathrm{sep}}(\mathbb{C}, C(A) \rtimes_r G) \arrow[r, "\mu^{\mathrm{MN}}_*"] & \mathrm{KK}_{\mathrm{sep}}(\mathbb{C}, A \rtimes_r G) \end{tikzcd}$$

where the vertical maps are instances of Kasparov's assembly map of Definition 12.8 *for the family of finite subgroups, and the horizontal maps are induced by the morphism* $C(A) \to A$.

Proof. First we note that the square commutes by the naturality of the Kasparov assembly map with respect to morphisms between coefficients. Using Definition 12.7 the upper horizontal map is equivalent to the map

$$\mathrm{colim}_{W \subseteq E_{\mathbf{Fin}}G^{\mathrm{CW}}} \mathrm{KK}^G_{\mathrm{sep}}(C_0(W), C(A)) \to \mathrm{colim}_{W \subseteq E_{\mathbf{Fin}}G^{\mathrm{CW}}} \mathrm{KK}^G_{\mathrm{sep}}(C_0(W), A),$$

where the colimits run over the G-finite sub-complexes of $E_{\mathbf{Fin}}G^{\mathrm{CW}}$. It is an equivalence by the definition of $C(A) \to A$, since $C_0(W)$ belongs to $\mathcal{CI}$ for every W appearing in the colimit.

The verification of the fact that $\mu^{\mathrm{Kasp}}_{C(A)}$ is an equivalence is more complicated. The reference [30] employs the work of [31] (isomorphism of the induction map) and [15, Prop. 2.3] (compatibility of induction with the Kasparov assembly map). Using the results of the present paper, Theorem 16.1 gives an independent proof of this fact in the case of discrete groups. Note that [30] considers the more general case of locally compact groups. □

We now consider a family $(K^H)_{H \subseteq G}$ of functors

$$K^H : \mathrm{KK}^G_{\mathrm{sep}} \to \mathrm{Fun}(H\mathbf{Orb}, \mathbf{Sp}), \quad A \mapsto K^H_A$$

indexed by the subgroups H of G. In order to formulate the properties of this family required for Kranz' argument, we consider the functor

$$i_H^G\colon H\mathbf{Orb} \to G\mathbf{Orb}\,, \quad S \mapsto G \times_H S$$

and let $i_{H,!}^G$ denote the left Kan extension functor along i_H^G. We assume $(K^H)_{H\subseteq G}$ has the following properties:

Assumption 15.6.

(1) *K^G preserves countable colimits.*

(2) *For every A in $\mathrm{KK}_{\mathrm{sep}}^G$ and subgroup H of G, we have an equivalence*[9]

$$K_A^G(G/H) \simeq \mathrm{KK}_{\mathrm{sep}}(\mathbb{C}, (\mathrm{Res}_{H,s}^G(A) \rtimes_r H)_s)\,. \tag{15.3}$$

(3) *For any subgroup H of G, we have a commutative square*

$$\begin{tikzcd} \mathrm{KK}_{\mathrm{sep}}^H \arrow[r, "K^H"] \arrow[d, "\mathrm{Ind}_{H,s}^G"] & \mathrm{Fun}(H\mathbf{Orb}, \mathbf{Sp}) \arrow[d, "i_{H,!}^G"] \\ \mathrm{KK}_{\mathrm{sep}}^G \arrow[r, "K^G"] & \mathrm{Fun}(G\mathbf{Orb}, \mathbf{Sp}) \end{tikzcd} \tag{15.4}$$

Note that we are mainly interested in the member K^G of the family $(K^H)_{H\subseteq G}$. The other members are only used to formulate Assumption 15.6.3. In the example of the family $(K^H)_{H\subseteq G}$ used below, the functors K^H are constructed by applying Definition 15.10 to H in place of G. In this case, the members K^H have analoguous properties as K^G.

In view of Definition 10.1, we consider K^G as a functor from $\mathrm{KK}_{\mathrm{sep}}^G$ to the stable ∞-category of $\mathbf{Sp}$-valued equivariant homology theories. In particular, for A in $\mathrm{KK}_{\mathrm{sep}}^G$ and X in $G\mathbf{Top}$ we have a well-defined evaluation $K_A^G(X)$ in $\mathbf{Sp}$.

The argument of Kranz is then based on the following commutative diagram

$$\begin{tikzcd} K_{C(A)}^G(E_{\mathbf{Fin}}G^{\mathrm{CW}}) \arrow[r, "\mu^{\mathrm{MN}}_{A,E_{\mathbf{Fin}}G^{\mathrm{CW}}}"] \arrow[d, "\mu^{\mathrm{DL}}_{C(A),E_{\mathbf{Fin}}G^{\mathrm{CW}}}"] & K_A^G(E_{\mathbf{Fin}}G^{\mathrm{CW}}) \arrow[d, "\mu^{\mathrm{DL}}_{A},E_{\mathbf{Fin}}G^{\mathrm{CW}}"] \\ K_{C(A)}^G(*) \arrow[r, "\mu^{\mathrm{MN}}_{A,*}"] & K_A^G(*) \end{tikzcd} \tag{15.5}$$

Here the vertical Davis–Lück assembly maps (12.4) are induced by the map $E_{\mathbf{Fin}}G^{\mathrm{CW}} \to *$. Moreover, the horizontal Mayer–Nest assembly maps are induced by the map $C(A) \to A$. By Assumption 15.6.2 the map $\mu^{\mathrm{MN}}_{A,*}$ is indeed the map from Definition 15.4.

Theorem 15.7 (Kranz)**.** *We have an equivalence $\mu^{\mathrm{DL}}_{A,E_{\mathbf{Fin}}G^{\mathrm{CW}}} \simeq \mu^{\mathrm{MN}}_{A,*}$.*

[9] The subscript s at various functors indicates their restriction to the subcategory of separable algebras.

Proof. The square in (15.5) yields an equivalence of $\mu^{\mathrm{DL}}_{A,E_{\mathbf{Fin}}G^{\mathrm{CW}}}$ with $\mu^{\mathrm{MN}}_{A,*}$ provided one can show that $\mu^{\mathrm{DL}}_{C(A),E_{\mathbf{Fin}}G^{\mathrm{CW}}}$ and $\mu^{\mathrm{MN}}_{A,E_{\mathbf{Fin}}G^{\mathrm{CW}}}$ are equivalences. This is the content of the following two lemmas.

Let A be in $\mathrm{KK}^G_{\mathrm{sep}}$.

Lemma 15.8. *The Meyer–Nest assembly map $\mu^{MN}_{A,E_{\mathbf{Fin}}G^{\mathrm{CW}}}$ is an equivalence.*

Proof. Since K^G is exact by Assumption 15.6.1, using (15.2) we see that it suffices to show that

$$K^G_{N(A)}(E_{\mathbf{Fin}}G^{\mathrm{CW}}) \simeq 0\,. \tag{15.6}$$

Since $N(A)$ belongs to $\mathcal{CC}$ we have $\mathrm{Res}^G_{H,s}(N(A)) \simeq 0$ for all H in $\mathbf{Fin}$. As a consequence of (15.3) we conclude $K^G_{N(A)}(G/H) \simeq 0$ for every H in $\mathbf{Fin}$. On the other hand, by the characterization (12.2) of the homotopy type of $E_{\mathbf{Fin}}G^{\mathrm{CW}}$ we have $Y^G(E_{\mathbf{Fin}}G^{\mathrm{CW}})(G/H) \simeq 0$ (see (10.2) for Y^G) provided $H \notin \mathbf{Fin}$. As an immediate consequence of the formula (10.4) for the evaluation of a homology theory on a G-topological space we get the desired equivalence (15.6). □

Let A be in $\mathrm{KK}^G_{\mathrm{sep}}$.

Lemma 15.9. *If A is in $\mathcal{CI}$, then the Davis–Lück assembly map $\mu^{\mathrm{DL}}_{A,E_{\mathbf{Fin}}G^{\mathrm{CW}}}$ is an equivalence.*

Proof. Since K^G preserves countable colimits and $\mathcal{CI}$ is generated by $\mathrm{Ind}^G_{H,s}(B)$ for all B in $\mathrm{KK}^G_{\mathrm{sep}}$ and all finite subgroups H of G it suffices to show that $\mu^{\mathrm{DL}}_{\mathrm{Ind}^G_{H,s}(B),E_{\mathbf{Fin}}G^{\mathrm{CW}}}$ is an equivalence for such data. By Diagram (15.4) we have an equivalence $K^G_{\mathrm{Ind}^G_{H,s}(B)} \simeq i^G_{H,!}K^G_B$. It is now a general fact (see e.g. [9, Lem. 19.25] for an argument) that for a functor $E : H\mathbf{Orb} \to \mathbf{M}$ with cocomplete stable target $\mathbf{M}$ we have a natural equivalence of functors

$$i^G_{H,!}E \simeq E\circ\mathrm{Res}^G_H : G\mathbf{Top} \to \mathbf{M}\,.$$

We therefore get the commutative square

$$\begin{tikzcd}
K^G_{\mathrm{Ind}^G_{H,s}(B)}(E_{\mathbf{Fin}}G^{\mathrm{CW}}) \arrow[r, "\mu^{\mathrm{DL}}_{\mathrm{Ind}^G_{H,s}(B),E_{\mathbf{Fin}}G^{\mathrm{CW}}}"] \arrow[d, "\simeq"] & K^G_{\mathrm{Ind}^G_{H,s}(B)}(*) \arrow[d, "\simeq"] \\
K^H_B(\mathrm{Res}^G_H(E_{\mathbf{Fin}}G^{\mathrm{CW}})) \arrow[r, "!"] & K^H_B(\mathrm{Res}^G_H(*))
\end{tikzcd}$$

Since $\mathrm{Res}^G_H(E_{\mathbf{Fin}}G^{\mathrm{CW}}) \to \mathrm{Res}^G_H(*)$ is a homotopy equivalence in $H\mathbf{Top}$ we conclude that the map marked by ! is an equivalence. This implies that the map $\mu^{\mathrm{DL}}_{\mathrm{Ind}^G_{H,s}(B),E_{\mathbf{Fin}}G^{\mathrm{CW}}}$ is an equivalence. □

This finishes the proof of Theorem 15.7. □

We now discuss the construction of the functor K^G. It is based on the ideas of Kranz [26], but we reformulate the construction such that it only uses the formal aspects of the calculus of equivariant KK-theory as developed in [12]. We give full details since we use them crucially in the argument for Proposition 16.2, which in turn is used in Theorem 16.1.

We start with the adjunction

$$\mathbb{C}[-] : G\mathbf{Set} \leftrightarrows \mathrm{Fun}(BG, C^*\mathbf{Cat}) : \mathrm{Ob} \tag{15.7}$$

whose left adjoint sends a G-set S to the G-C^*-category $\mathbb{C}[S]$ with the G-set S of objects and morphisms generated by the identities [3, Lem. 3.8]. By [3, Lem. 3.7] the inclusion $\mathrm{Fun}(BG, C^*\mathbf{Cat}) \to \mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ is again a left-adjoint. By post-composition with this inclusion we therefore get a left-adjoint functor

$$\mathbb{C}[-] : G\mathbf{Set} \to \mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$$

which we denote by the same symbol for simplicity.

Recall the functor $y^G : \mathrm{KK}^G_{\mathrm{sep}} \to \mathrm{KK}^G$ from [12, Def. 1.8].

Definition 15.10. *We define the functors*

$$\hat{K}^G : \mathrm{KK}^G \to \mathrm{Fun}(G\mathbf{Orb}, \mathbf{Sp}), \quad A \mapsto K^{C^*\mathbf{Cat}}((A \otimes_{\max} \mathrm{kk}^G_{C^*\mathbf{Cat}}(\mathbb{C}[-])) \rtimes_r G)$$

and

$$K^G := \mathrm{KK}^G_{\mathrm{sep}} \xrightarrow{y^G} \mathrm{KK}^G \xrightarrow{\hat{K}^G} \mathbf{Sp}.$$

In order to verify that K^G satisfies the Assumption 15.6, we analyse the construction of these functors through various intermediate constructs. The most difficult part is thereby Assumption 15.4.3. If one is not interested in the details of the argument one could skip the material until Theorem 15.18 and just accept its statement.

We start with the functor

$$\begin{aligned} &\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}) \times G\mathbf{Set} \\ &\xrightarrow{\mathrm{id}_{\mathrm{Fun}(BG,C^*\mathbf{Cat}^{\mathrm{nu}})} \times \mathbb{C}[-]} \mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}) \times \mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}) \\ &\xrightarrow{-\otimes_{\max}-} \mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}) \\ &\xrightarrow{\mathrm{kk}^G_{C^*\mathbf{Cat}}} \mathrm{KK}^G. \end{aligned} \tag{15.8}$$

Using the exponential law, the above defines a functor

$$R^G : \mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}) \to \mathrm{Fun}(G\mathbf{Set}, \mathrm{KK}^G), \quad \mathbf{C} \mapsto R^G_{\mathbf{C}}.$$

Let $i_\omega : G\mathbf{Set}_\omega \to G\mathbf{Set}$ denote the inclusion of the full subcategory of countable G-sets, and let $\mathrm{Fun}^{\coprod_\omega}$ denote the full subcategory of a functor category of countable coproduct preserving functors.

Lemma 15.11.

(1) R^G *is s-finitary.*

(2) *The restriction of* R^G *to* $\mathrm{Fun}(BG, C^*\mathbf{Cat})$ *sends unitary equivalences to equivalences.*
(3) *The functor* R^G *sends weak Morita equivalences to equivalences.*
(4) *We have a canonical factorization*

$$\begin{tikzcd}
\mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}}) \arrow[r, "\mathrm{kk}^G"] \arrow[d, "\mathrm{incl}"] & \mathrm{KK}^G \arrow[d, dotted, "F^G"] \\
\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}) \arrow[r, "R^G"] \arrow[ur, "\mathrm{kk}^G_{C^*\mathbf{Cat}}"] & \mathrm{Fun}(G\mathbf{Set}, \mathrm{KK}^G)
\end{tikzcd}$$

(5) *The functor* F^G *preserves colimits.*
(6) *We have a factorization*

$$\begin{tikzcd}
\mathrm{KK}^G_{\mathrm{sep}} \arrow[r, "y^G"] \arrow[dd, dotted, "F^G_s"] & \mathrm{KK}^G \arrow[d, "F^G"] \\
 & \mathrm{Fun}(G\mathbf{Set}, \mathrm{KK}^G) \arrow[d, "i^*_\omega"] \\
\mathrm{Fun}^{\amalg_\omega}(G\mathbf{Set}_\omega, \mathrm{KK}^G_{\mathrm{sep}}) \arrow[r, "y^G"] & \mathrm{Fun}(G\mathbf{Set}_\omega, \mathrm{KK}^G)
\end{tikzcd} \tag{15.9}$$

such that F^G_s *preserves countable colimits.*

Proof. Using the fact that $\mathrm{kk}^G_{C^*\mathbf{Cat}}$ is symmetric monoidal [12, Thm. 1.35] we can rewrite the functor in (15.8) as

$$\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}) \times G\mathbf{Set} \stackrel{\mathrm{kk}^G_{C^*\mathbf{Cat}} \times \mathrm{kk}^G_{C^*\mathbf{Cat}}(\mathbb{C}[-])}{\to} \mathrm{KK}^G \times \mathrm{KK}^G \stackrel{-\otimes_{\max}-}{\to} \mathrm{KK}^G . \tag{15.10}$$

The Assertions 1, 2 and 3 now follow from the corresponding properties of the functor $\mathrm{kk}^G_{C^*\mathbf{Cat}}$ stated in [12, Thm. 1.32], where for 1 we also use that the tensor structure on KK^G preserves colimits in each variable. In order to show Assertion 4, we again use the Formula (15.10). It is then clear that we must define F^G by the composition

$$\begin{aligned} F^G : \mathrm{KK}^G \stackrel{\mathrm{id}_{\mathrm{KK}^G} \times \mathrm{kk}^G_{C^*\mathbf{Cat}}(\mathbb{C}[-])}{\to} & \mathrm{KK}^G \times \mathrm{Fun}(G\mathbf{Set}, \mathrm{KK}^G) \\ \stackrel{-\otimes_{\max}-}{\to} & \mathrm{Fun}(G\mathbf{Set}, \mathrm{KK}^G) , \\ A \mapsto & F^G_A \end{aligned} \tag{15.11}$$

Since $- \otimes_{\max} -$ preserves colimits in each argument we conclude Assertion 5.

We finally show Assertion 6. We let $\mathbb{C}_s[-]$ denote the restriction of $\mathbb{C}[-]$ to countable sets. We consider $\mathbb{C}_s[-]$ as a functor with values in the full subcategory $C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{sep}}$ of $C^*\mathbf{Cat}^{\mathrm{nu}}$ of C^*-categories with countably many objects and separable morphism spaces. The functor $\mathbb{C}_s[-]$ is still a left-adjoint. The restriction of the adjunction

$$A^f : C^*\mathbf{Cat}^{\mathrm{nu}} \leftrightarrows C^*\mathbf{Alg}^{\mathrm{nu}} : \mathrm{incl}$$

(see e.g. [3, Lem. 3.9]) to separable objects yields an adjunction

$$A^f_s : C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{sep}} \leftrightarrows C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} : \mathrm{incl}\,.$$

We define F^G_s by the formula

$$F^G_{s,(-)}(-) := (-) \otimes_{\max} \mathrm{kk}^G_{\mathrm{sep}}(A^f_s(\mathbb{C}_s[-]))\,. \tag{15.12}$$

The following chain of equivalences yields the commutative square (15.9), where for the moment we ignore the superscript $\coprod_\omega$ at the lower left corner

$$\begin{aligned} y^G \circ F^G_{s,(-)}(-) &\overset{\mathrm{def}}{\simeq} y^G((-) \otimes_{\max} \mathrm{kk}^G_{\mathrm{sep}}(A^f_s(\mathbb{C}_s[-]))) \\ &\overset{!}{\simeq} y^G(-) \otimes_{\max} \mathrm{kk}^G_{C^*\mathbf{Cat}}(\mathbb{C}[i_\omega(-)]))) \overset{\mathrm{def}}{\simeq} F^G_{y^G(-)}(i_\omega(-))\,. \end{aligned}$$

For the marked equivalence, we use that y^G is symmetric monoidal and the obvious equivalence $y^G(\mathrm{kk}^G_{\mathrm{sep}}(A^f_s(\mathbb{C}_s[-]))) \simeq \mathrm{kk}^G_{C^*\mathbf{Cat}}(\mathbb{C}[i_\omega(-)])$ of functors from $G\mathbf{Set}_\omega$ to KK^G.

It remains to show that for any A in $\mathrm{KK}^G_{\mathrm{sep}}$ the functor $F^G_{s,A}$ preserves countable coproducts. By definition, we have an equivalence

$$F^G_{s,A}(-) \overset{\mathrm{def}}{\simeq} A \otimes_{\max} \mathrm{kk}^G_{\mathrm{sep}}(A^f_s(\mathbb{C}_s[-]))$$

of functors from $G\mathbf{Set}_\omega$ to $\mathrm{KK}^G_{\mathrm{sep}}$. Because $\mathbb{C}_s[-]$ is a left-adjoint it preserves countable coproducts. The functor $\mathrm{kk}^G_{\mathrm{sep}} \circ A^f_s$ sends the relevant countable coproducts to sums by [12, Lem. 6.6]. Finally, by [12, Prop. 1.7] the tensor product $- \otimes_{\max} -$ on $\mathrm{KK}^G_{\mathrm{sep}}$ preserves countable sums in each argument. This finishes the construction of the factorization F^G_s asserted in 6.

It immediately follows from the definition in (15.12) that the functor F^G_s preserves countable colimits. Here we use again that $- \otimes_{\max} -$ on $\mathrm{KK}^G_{\mathrm{sep}}$ preserves countable colimits in each argument [12, Prop. 1.7]. This finishes the verification of Assertion 6. □

Let H be a subgroup of G and consider the object G/H in $G\mathbf{Set}$. We let $r^G_H : G\mathbf{Set} \to H\mathbf{Set}$ denote the functor which restricts the G-action on a set to an H-action. We consider the object G/H in $G\mathbf{Set}$.

Lemma 15.12.

(1) *We have a commutative square*

$$\begin{tikzcd} \mathrm{KK}^G \arrow[r, "F^G"] \arrow[d, "\mathrm{Res}^G_H"] & \mathrm{Fun}(G\mathbf{Set}, \mathrm{KK}^G) \arrow[d, "\mathrm{ev}_{G/H}"] \\ \mathrm{KK}^H \arrow[r, "\mathrm{Ind}^G_H"] & \mathrm{KK}^G \end{tikzcd} . \tag{15.13}$$

(2) *We have a commutative square*

$$\begin{tikzcd} \mathrm{KK}^H \arrow[r, "F^H"] \arrow[dd, "\mathrm{Ind}^G_H"] & \mathrm{Fun}(H\mathbf{Set}, \mathrm{KK}^H) \arrow[d, "r^{G,*}_H"] \\ & \mathrm{Fun}(G\mathbf{Set}, \mathrm{KK}^H) \arrow[d, "\mathrm{Ind}^G_H"] \\ \mathrm{KK}^G \arrow[r, "F^G"] & \mathrm{Fun}(G\mathbf{Set}, \mathrm{KK}^G) \end{tikzcd} . \tag{15.14}$$

Proof. We use the functor $A\colon\ C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{inj}} \to C^*\mathbf{Alg}^{\mathrm{nu}}$ (see e.g. [3, Def. 6.5]) and note that for S in $G\mathbf{Set}$ we get $A(\mathbb{C}[S]) \cong C_0(S)$ in $\mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}})$. Applying this to G/H in place of S und using the definition of the induction functor Ind^G_H from $\mathrm{Fun}(BH, C^*\mathbf{Alg}^{\mathrm{nu}})$ to $\mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}})$ applied to $\mathbb{C}$ with the trivial H-action we obtain the isomorphisms

$$A(\mathbb{C}[G/H]) \cong C_0(G/H) \cong \mathrm{Ind}^G_H(\mathbb{C})\,.$$

By [12, Prop. 6.9] for every $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ we have an equivalence

$$\mathrm{kk}^G_{C^*\mathbf{Cat}}(\mathbf{C}) \stackrel{\mathrm{def}}{=} \mathrm{kk}^G(A^f(\mathbf{C})) \xrightarrow{\simeq} \mathrm{kk}^G(A(\mathbf{C}))\,.$$

Hence,

$$\mathrm{kk}^G_{C^*\mathbf{Cat}}(\mathbb{C}[G/H]) \simeq \mathrm{kk}^G(A(\mathbb{C}[G/H])) \simeq \mathrm{kk}^G(\mathrm{Ind}^G_H(\mathbb{C})) \simeq \mathrm{Ind}^G_H(\mathrm{kk}^G(\mathbb{C}))\,,$$

where the symbol Ind^G_H on the right-hand side is the induction functor from KK^H to KK^G [12, Thm. 1.22]. Using (15.11), the following projection formula [12, Cor. 4.13]

$$\mathrm{Ind}^G_H(-) \otimes_{\max} (-) \simeq \mathrm{Ind}^G_H((-) \otimes_{\max} \mathrm{Res}^G_H(-)) \tag{15.15}$$

for functors $\mathrm{KK}^H \times \mathrm{KK}^G \to \mathrm{KK}^G$, and that $\mathrm{kk}^G(\mathbb{C})$ is the tensor unit of KK^G we get the following chain of equivalences of endofunctors

$$\begin{aligned} \mathrm{ev}_{G/H} \circ F^G_{(-)} &\simeq (-) \otimes_{\max} \mathrm{Ind}^G_H(\mathrm{kk}^G(\mathbb{C})) \\ &\simeq \mathrm{Ind}^G_H(\mathrm{Res}^G_H(-) \otimes_{\max} \mathrm{kk}^G(\mathbb{C})) \\ &\simeq \mathrm{Ind}^G_H(\mathrm{Res}^G_H(-)) \end{aligned}$$

of KK^G which provides the filler of the square in (15.13).

In order to construct the filler of the pentagon in (15.14), we note that we have obvious equivalences

$$r_H^{G,*}(\mathrm{kk}^H_{C^*\mathbf{Cat}}(\mathbb{C}[-])) \simeq \mathrm{kk}^H_{C^*\mathbf{Cat}}(r_H^{G,*}(\mathbb{C}[-])) \simeq \mathrm{kk}^H_{C^*\mathbf{Cat}}(\mathrm{Res}^G_H(\mathbb{C}[-])) \\ \simeq \mathrm{Res}^G_H(\mathrm{kk}^G_{C^*\mathbf{Cat}}(\mathbb{C}[-]))\,. \quad (15.16)$$

The chain of equivalences

$$\begin{aligned} \mathrm{Ind}_H^G \circ r_H^{G,*} \circ F^H_{(-)} &\overset{\mathrm{def}}{\simeq} \mathrm{Ind}_H^G((-) \otimes_{\max} r_H^{G,*}(\mathrm{kk}^H_{C^*\mathbf{Cat}}(\mathbb{C}[-]))) \\ &\overset{(15.16)}{\simeq} \mathrm{Ind}_H^G(- \otimes_{\max} \mathrm{Res}^G_H(\mathrm{kk}^G_{C^*\mathbf{Cat}}(\mathbb{C}[-]))) \\ &\overset{(15.15)}{\simeq} \mathrm{Ind}_H^G(-) \otimes_{\max} \mathrm{kk}^G_{C^*\mathbf{Cat}}(\mathbb{C}[-]) \\ &\overset{\mathrm{def}}{\simeq} F^G_{(-)} \circ \mathrm{Ind}_H^G \end{aligned}$$

provides the filler of the pentagon. □

We now consider the functor

$$L^G : \mathrm{KK}^G \xrightarrow{F^G} \mathrm{Fun}(G\mathbf{Set}, \mathrm{KK}^G) \xrightarrow{-\rtimes_r G} \mathrm{Fun}(G\mathbf{Set}, \mathrm{KK})\,.$$

By [12, Lem. 4.16] the restriction of $- \rtimes_r G$ to the subcategories of compact objects is a countable sum preserving functor

$$(- \rtimes_r G)_s : \mathrm{KK}^G_{\mathrm{sep}} \to \mathrm{KK}_{\mathrm{sep}}\,.$$

We can therefore also consider

$$L^G_s : \mathrm{KK}^G_{\mathrm{sep}} \xrightarrow{F^G_s} \mathrm{Fun}^{\coprod_\omega}(G\mathbf{Set}_\omega, \mathrm{KK}^G_{\mathrm{sep}}) \xrightarrow{-\rtimes_r G} \mathrm{Fun}^{\coprod_\omega}(G\mathbf{Set}_\omega, \mathrm{KK}_{\mathrm{sep}})\,.$$

Lemma 15.13.

(1) *L^G preserves colimits.*
(2) *For every subgroup H of G, we have a commutative square*

$$\begin{array}{ccc} \mathrm{KK}^H & \xrightarrow{L^H} & \mathrm{Fun}(H\mathbf{Set}, \mathrm{KK}) \\ \downarrow{\scriptstyle \mathrm{Ind}_H^G} & & \downarrow{\scriptstyle r_H^{G,*}} \\ \mathrm{KK}^G & \xrightarrow{L^G} & \mathrm{Fun}(G\mathbf{Set}, \mathrm{KK}) \end{array} \quad (15.17)$$

(3) *For every subgroup H of G, we have a commutative square*

$$\begin{array}{ccc} \mathrm{KK}^G & \xrightarrow{L^G} & \mathrm{Fun}(G\mathbf{Set}, \mathrm{KK}) \\ \downarrow{\scriptstyle \mathrm{Res}^G_H} & & \downarrow{\scriptstyle \mathrm{ev}_{G/H}} \\ \mathrm{KK}^H & \xrightarrow{-\rtimes_r H} & \mathrm{KK} \end{array} \quad (15.18)$$

(4) *We have a commutative diagram*

$$\begin{tikzcd} \mathrm{KK}^G_{\mathrm{sep}} \arrow[rr, "y^G"] \arrow[dd, "L^G_s"] & & \mathrm{KK}^G \arrow[d, "L^G"] \\ & & \mathrm{Fun}(G\mathbf{Set}, \mathrm{KK}) \arrow[d, "i^*_\omega"] \\ \mathrm{Fun}^{\coprod_\omega}(G\mathbf{Set}_\omega, \mathrm{KK}_{\mathrm{sep}}) \arrow[rr, "y"] & & \mathrm{Fun}(G\mathbf{Set}_\omega, \mathrm{KK}) \end{tikzcd} \tag{15.19}$$

(5) *For every subgroup H of G, we have a commutative square*

$$\begin{tikzcd} \mathrm{KK}^H_{\mathrm{sep}} \arrow[r, "L^H_s"] \arrow[d, "\mathrm{Ind}^G_{H,s}"] & \mathrm{Fun}^{\coprod_\omega}(H\mathbf{Set}_\omega, \mathrm{KK}_{\mathrm{sep}}) \arrow[d, "r^{G,*}_H"] \\ \mathrm{KK}^G_{\mathrm{sep}} \arrow[r, "L^G_s"] & \mathrm{Fun}^{\coprod_\omega}(G\mathbf{Set}_\omega, \mathrm{KK}_{\mathrm{sep}}) \end{tikzcd} \tag{15.20}$$

(6) *The functor L_s preserves countable colimits.*

Proof. Assertion 1 follows from 15.11.5 and the fact that $- \rtimes_r G \colon \mathrm{KK}^G \to \mathrm{KK}$ preserves colimits [12, Thm. 1.22].

For Assertion 2, we expand the square in (15.17) as follows:

$$\begin{tikzcd} \mathrm{KK}^H \arrow[r, "F^H"] \arrow[dd, "\mathrm{Ind}^G_H"] & \mathrm{Fun}(H\mathbf{Set}, \mathrm{KK}^H) \arrow[r, "\rtimes_r H"] \arrow[d, "r^{G,*}_H"] & \mathrm{Fun}(H\mathbf{Set}, \mathrm{KK}) \arrow[dd, "r^{G,*}_H"] \\ & \mathrm{Fun}(G\mathbf{Set}, \mathrm{KK}^H) \arrow[d, "\mathrm{Ind}^G_H"] \arrow[dr, "- \rtimes_r H"] & \\ \mathrm{KK}^G \arrow[r, "F^G"] & \mathrm{Fun}(G\mathbf{Set}, \mathrm{KK}^G) \arrow[r, "- \rtimes_r G"] & \mathrm{Fun}(G\mathbf{Set}, \mathrm{KK}) \end{tikzcd} \tag{15.21}$$

The left pentagon is precisely (15.14) and commutes by 15.12.2. The upper right square in (15.21) commutes by the associativity of composition of functors. Finally, the lower triangle commutes by the equivalence

$$(-) \rtimes_r H \simeq \mathrm{Ind}^G_H(-) \rtimes_r G \tag{15.22}$$

of functors from KK^H to KK [12, Thm. 1.23].

In order to show Assertion 3, we expand the square (15.18) as follows:

$$\begin{tikzcd}
\mathrm{KK}^G \arrow[r, "F^G"] \arrow[d, "\mathrm{Res}^G_H"] \arrow[rr, bend left, "L^G"] & \mathrm{Fun}(G\mathbf{Set}, \mathrm{KK}^G) \arrow[r, "-\rtimes_r G"] \arrow[d, "\mathrm{ev}_{G/H}"] & \mathrm{Fun}(G\mathbf{Set}, \mathrm{KK}) \arrow[d, "\mathrm{ev}_{G/H}"] \\
\mathrm{KK}^H \arrow[r, "\mathrm{Ind}^G_H"] \arrow[rr, bend right, "-\rtimes_r K"'] & \mathrm{KK}^G \arrow[r, "-\rtimes_r G"] & \mathrm{KK}
\end{tikzcd} \tag{15.23}$$

The right square commutes obviously, and the commutativity of the left square is considered in 15.12.1. The upper triangle reflects the definition of L^G, and the lower triangle commutes by (15.22).

By composing 15.11.6 with $- \rtimes_r G$ and the equivalence

$$(- \rtimes_r G) \circ y^G \simeq y \circ (- \rtimes_r G)_s$$

we conclude Assertion 4.

In order to show Assertion 5, we precompose the square in (15.17) with y^H and y^G, respectively, and restrict the results to countable sets. We use that $\mathrm{Ind}^G_H \circ y^H \simeq y^G \circ \mathrm{Ind}^G_{H,s}$. This gives the outer square in

$$\begin{tikzcd}
\mathrm{KK}^H_{\mathrm{sep}} \arrow[r, "L^H_s"] \arrow[d, "\mathrm{Ind}^G_{H,s}"] \arrow[rr, bend left, "(L^H_{y^H})_{|H\mathbf{Set}_\omega}"] & \mathrm{Fun}^{\amalg_\omega}(H\mathbf{Set}_\omega, \mathrm{KK}_{\mathrm{sep}}) \arrow[r, "y"] \arrow[d, "r^{G,*}_H"] & \mathrm{Fun}(H\mathbf{Set}_\omega, \mathrm{KK}) \arrow[d, "r^{G,*}_H"] \\
\mathrm{KK}^G_{\mathrm{sep}} \arrow[r, "L^G_s"] \arrow[rr, bend right, "(L^G_{y^G})_{|G\mathbf{Set}_\omega}"'] & \mathrm{Fun}^{\amalg_\omega}(G\mathbf{Set}_\omega, \mathrm{KK}_{\mathrm{sep}}) \arrow[r, "y"] & \mathrm{Fun}(G\mathbf{Set}_\omega, \mathrm{KK})
\end{tikzcd} \tag{15.24}$$

We then use that r^G_H preserves countability and coproducts and therefore that $r^{G,*}_H$ preserves countable coproduct preserving functors. If we now employ the fact that y is fully faithful, then we get the filler of the left square.

Assertion 6 follows from 15.13.6 and the fact that $(-\rtimes_r G)_s$ preserves countable colimits [12, Lem. 4.16]. □

Let H be a subgroup of G. We have an adjunction

$$i^G_H : H\mathbf{Set} \leftrightarrows G\mathbf{Set} : r^G_H ,$$

where i^G_H sends the H-set S to the G-set $G \times_H S$. Consequently, we have an equivalence

$$r^{G,*}_H \simeq i^G_{H,!} : \mathrm{Fun}(H\mathbf{Set}, \mathcal{C}) \to \mathrm{Fun}(G\mathbf{Set}, \mathcal{C})$$

for any target category $\mathcal{C}$, where $i^G_{H,!}$ is the left Kan-extension functor. It restricts to an equivalence

$$r^{G,*}_H \simeq i^G_{H,!} : \mathrm{Fun}^{\coprod_\omega}(H\mathbf{Set}_\omega, \mathcal{C}) \to \mathrm{Fun}^{\coprod_\omega}(G\mathbf{Set}_\omega, \mathcal{C})$$

provided $\mathcal{C}$ has countable coproducts.

The functor i^G_H restricts to a functor $i^G_H : H\mathbf{Orb} \to G\mathbf{Orb}$. We note that the slice categories $H\mathbf{Orb}_{/S}$ for any S in $G\mathbf{Orb}$ are countable discrete. Therefore, the left Kan extension functor

$$i^G_{H,!} : \mathrm{Fun}(H\mathbf{Orb}, \mathcal{C}) \to \mathrm{Fun}(G\mathbf{Orb}, \mathcal{C})$$

exists provided $\mathcal{C}$ admits all countable coproducts. We let $i^G : G\mathbf{Orb} \to G\mathbf{Set}$ denote the inclusion. From now on we assume that $\mathcal{C}$ admits countable coproducts. We consider the square

$$\begin{tikzcd}
\mathrm{Fun}(H\mathbf{Set}_\omega, \mathcal{C}) \arrow[r, "i^{H,*}"] \arrow[d, "r^{G,*}_H"] & \mathrm{Fun}(H\mathbf{Orb}, \mathcal{C}) \arrow[d, "i^G_{H,!}"] \\
\mathrm{Fun}(G\mathbf{Set}_\omega, \mathcal{C}) \arrow[r, "i^{G,*}"] & \mathrm{Fun}(G\mathbf{Orb}, \mathcal{C})
\end{tikzcd}$$

In general, we do not expect that the square commutes.

Lemma 15.14. *The restriction of the square to countable coproduct preserving functors is a commutative square*

$$\begin{tikzcd}
\mathrm{Fun}^{\coprod_\omega}(H\mathbf{Set}_\omega, \mathcal{C}) \arrow[r, "i^{H,*}", "\simeq"'] \arrow[d, "r^{G,*}_H"] & \mathrm{Fun}(H\mathbf{Orb}, \mathcal{C}) \arrow[d, "i^G_{H,!}"] \\
\mathrm{Fun}^{\coprod_\omega}(G\mathbf{Set}_\omega, \mathcal{C}) \arrow[r, "i^{G,*}", "\simeq"'] & \mathrm{Fun}(G\mathbf{Orb}, \mathcal{C})
\end{tikzcd} \tag{15.25}$$

Proof. The inverse of the horizontal arrows are the left Kan-extension functors along i^H and i^G, respectively. Since we have a canonical isomorphism Since $i^G_H \circ i^H \cong i^G \circ i^G_H$ of functors from $H\mathbf{Orb}$ to $G\mathbf{Set}$ the square

$$\begin{tikzcd}
\mathrm{Fun}^{\coprod_\omega}(H\mathbf{Set}_\omega, \mathcal{C}) \arrow[d, "i^G_{H,!} \simeq r^{G,*}_H"] & \mathrm{Fun}(H\mathbf{Orb}, \mathcal{C}) \arrow[l, "i^H_!"', "\simeq"] \arrow[d, "i^G_{H,!}"] \\
\mathrm{Fun}^{\coprod_\omega}(G\mathbf{Set}_\omega, \mathcal{C}) & \mathrm{Fun}(G\mathbf{Orb}, \mathcal{C}) \arrow[l, "i^G_!"', "\simeq"]
\end{tikzcd} \tag{15.26}$$

commutes. We obtain (15.25) from (15.26) by inverting the horizontal arrows. □

Note that $\mathrm{KK}_{\mathrm{sep}}$ admits countable colimits [12, Thm. 1.4].

Proposition 15.15. *We have a commutative square*

$$\begin{tikzcd} \mathrm{KK}^H_{\mathrm{sep}} \arrow[r, "i^{H,*}L^H_s"] \arrow[d, "\mathrm{Ind}^G_{H,s}"] & \mathrm{Fun}(H\mathbf{Orb}, \mathrm{KK}_{\mathrm{sep}}) \arrow[d, "i^G_{H,!}"] \\ \mathrm{KK}^G_{\mathrm{sep}} \arrow[r, "i^{G,*}L^G_s"] & \mathrm{Fun}(G\mathbf{Orb}, \mathrm{KK}_{\mathrm{sep}}) \end{tikzcd} \tag{15.27}$$

Proof. We expand the square as

$$\begin{tikzcd} \mathrm{KK}^H_{\mathrm{sep}} \arrow[r, "L^H_s"] \arrow[d, "\mathrm{Ind}^G_{H,s}"] & \mathrm{Fun}^{\coprod_\omega}(H\mathbf{Set}_\omega, \mathrm{KK}_{\mathrm{sep}}) \arrow[r, "i^{H,*}"] \arrow[d, "r^{G,*}_H"] & \mathrm{Fun}(H\mathbf{Orb}, \mathrm{KK}_{\mathrm{sep}}) \arrow[d, "i^G_{H,!}"] \\ \mathrm{KK}^G_{\mathrm{sep}} \arrow[r, "L^G_s"] & \mathrm{Fun}^{\coprod_\omega}(G\mathbf{Set}_\omega, \mathrm{KK}_{\mathrm{sep}}) \arrow[r, "i^{G,*}"] & \mathrm{Fun}(G\mathbf{Orb}, \mathrm{KK}_{\mathrm{sep}}) \end{tikzcd}$$

The left square commutes by 15.13.5. The right square commutes by Lemma 15.14. □

We now observe by an inspection of the constructions:

Corollary 15.16. *We have a canonical equivalence of functors*

$$\hat{K}^G \simeq \mathrm{KK}^G(\mathbb{C}, -)\circ i^{G,*}L^G : \mathrm{KK}^G \to \mathrm{Fun}(G\mathbf{Orb}, \mathbf{Sp})\,.$$

Corollary 15.17.

(1) *The functor $\hat{K}^G$ preserves colimits.*

(2) *For every subgroup H of G, we have an equivalence*

$$\hat{K}^G_{(-)}(G/H) \simeq \mathrm{KK}(\mathbb{C}, \mathrm{Res}^G_H(-) \rtimes_r H)$$

of functors $\mathrm{KK}^G \to \mathbf{Sp}$.

(3) *The composition*

$$\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}) \xrightarrow{\mathrm{kk}^G_{C^*\mathbf{Cat}}} \mathrm{KK}^G \xrightarrow{\hat{K}^G} \mathrm{Fun}(G\mathbf{Orb}, \mathbf{Sp})$$

sends Morita equivalences to equivalences.

Proof. Assertion 1 follows from Lemma 15.13.1, the fact that $i^{G,*}$ obviously preserves colimits, and that $\mathrm{KK}^G(\mathbb{C}, -)$ preserves colimits since KK^G is stable and $\mathrm{kk}^G(\mathbb{C})$ in KK^G is compact.

Assertion 2 is a consequence of the commutativity of (15.18) and the definitions.

In order to show Assertion 3, note that the collection of evaluations at the orbits G/H for all subgroups H of G detects equivalences. In view of Assertion 2, it thus suffices to show that $\mathrm{KK}(\mathbb{C}, \mathrm{Res}^G_H(-) \rtimes_r H)$ sends Morita equivalences to equivalences. But this is true since $\mathrm{Res}^G_H(-)$ obviously preserves Morita equivalences, $- \rtimes_r G$ preserves Morita equivalences by [9, Prop. 16.11], and $\mathrm{KK}(\mathbb{C}, -) = K^{C^*\mathbf{Cat}}(-)$ sends Morita equivalences to equivalences by [9, Prop. 16.18]. □

Using the equivalence $\mathrm{KK}^G(\mathbb{C}, y^G(-)) \simeq \mathrm{KK}^G_{\mathrm{sep}}(\mathbb{C}, -)$ of functors from $\mathrm{KK}_{\mathrm{sep}}$ to **Sp** we get the formula

$$K^G \simeq \mathrm{KK}_{\mathrm{sep}}(\mathbb{C}, i^{G,*} L^G_s(-)) . \tag{15.28}$$

Theorem 15.18. *The functor K^G satisfies the Assumption* 15.6.

Proof. The functor K^G is exact since $\hat{K}^G$ is exact by Corollary 15.17.1 and y^G is exact.

In order to show that the functor K^G preserves countable colimits, we use (15.28), that L^G_s preserves countable colimits by 15.13.6, and that $\mathrm{KK}^G_{\mathrm{sep}}(\mathbb{C}, -)$ preserves countable colimits: Indeed, $\mathrm{KK}^G_{\mathrm{sep}}(\mathbb{C}, -)$ is exact by definition. To see that it preserves countable sums, we use the identification $\mathrm{KK}^G_{\mathrm{sep}}(\mathbb{C}, \mathrm{kk}_{\mathrm{sep}}(-)) \simeq K^{C^*\mathrm{Alg}}(-)$ of functors from $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to \mathbf{Sp}$, the fact that countable sums in $\mathrm{KK}_{\mathrm{sep}}$ are presented by countable sums in $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}$, and that $K^{C^*\mathrm{Alg}}$ sends countable sums to coproducts.

For A in $\mathrm{KK}^G_{\mathrm{sep}}$, we have a natural equivalence

$$\begin{aligned} K^G_A(G/H) &\simeq \mathrm{KK}(\mathbb{C}, L^G_{y^G(A)}(G/H)) \\ &\overset{15.13.3}{\simeq} \mathrm{KK}(\mathbb{C}, \mathrm{Res}^G_H(A) \rtimes_r H) . \end{aligned}$$

Finally the commutativity of the square in (15.4) is obtained by applying $\mathrm{KK}_{\mathrm{sep}}(\mathbb{C}, -)$ to the right part of the square in (15.27) and using that $\mathrm{KK}_{\mathrm{sep}}(\mathbb{C}, -) \colon \mathrm{KK}_{\mathrm{sep}} \to \mathbf{Sp}$ preserves countable colimits in order to commute $i^G_{H,!}$ with this functor. □

16. The generalized Green–Julg Theorem

In this section, we show a version of the generalized Green–Julg theorem, see [20, Thm. 13.1] stating that the Kasparov assembly map for the family **Fin** and proper G-C^*-algebras is an equivalence. In our statement, we replace the condition that the separable G-C^*-algebra A is proper by the weaker (see [30, Cor. 7.3]) homotopy theoretic condition that $\mathrm{kk}^G_{\mathrm{sep}}(A)$ belongs to the set $\mathcal{CI}$ generated by the compactly induced objects, see Definition 15.1.

In [15], it was shown more generally for locally compact groups G that the Kasparov assembly map is an equivalence for compactly induced coefficients. Our proof for discrete groups is logically independent of the results of [15] and also different from the one in [20]. In particular, it makes the proof of Theorem 15.5 independent of [15]. Our approach is based on the equivalence between the analytic and Davis–Lück assembly maps and that the analogous assertion for the latter is known.

We consider A be in $\mathrm{KK}^G_{\mathrm{sep}}$.

Theorem 16.1. *If A belongs to $\mathcal{CI}$, then the Kasparov assembly map*

$$\mu^{\mathrm{Kasp}}_{A,\mathbf{Fin}} \colon RK^{G,\mathrm{an}}_A(E_{\mathbf{Fin}}G^{\mathrm{CW}}) \to \mathrm{KK}(\mathbb{C}, A \rtimes_r G)$$

is an equivalence.

Proof. The proof of this theorem is based on a chain of comparison results of independent interest which eventually will be combined to provide an equivalence between $\mu^{\mathrm{Kasp}}_{A,\mathbf{Fin}}$ and $\mu^{\mathrm{DL}}_{A,\mathbf{Fin}}$. The latter is known to be an equivalence by Lemma 15.9.

Let $\mathbf{C}$ be in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg,eadd},\omega\mathrm{add}})$ so that $K\mathbf{C}^G : G\mathbf{Orb} \to \mathbf{Sp}$ is given by Definition 12.2. We then form $\mathbf{C}^u$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat})$ by Definition 2.4 and $\hat{K}^G_{\mathbf{C}^u} : G\mathbf{Orb} \to \mathbf{Sp}$ by Definition 15.10. Note that the latter only depends on the object $\mathrm{kk}^G_{C^*\mathbf{Cat}}(\mathbf{C}^u)$ in KK^G, but according to our general conventions we dopped the symbol $\mathrm{kk}^G_{C^*\mathbf{Cat}}$ from the notation.

Recall the Definition 12.3 of $\mathrm{Asmbl}^h_{\mathbf{C},\mathcal{F}}$ and $\mu^{\mathrm{DL}}_{\mathbf{C}^u,\mathcal{F}}$ from (12.4).

Proposition 16.2. *We have a canonical equivalence $K\mathbf{C}^G \simeq \hat{K}^G_{\mathbf{C}^u}$ and therefore for any family $\mathcal{F}$ of subgroups of G a commutative diagram*

$$\begin{tikzcd}
\hat{K}^G_{\mathbf{C}^u}(E_{\mathcal{F}}G^{\mathrm{CW}}) \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \mathrm{colim}_{G_{\mathcal{F}}\mathbf{Orb}}\hat{K}^G_{\mathbf{C}^u} \arrow[r, "\mu^{\mathrm{DL}}_{\mathbf{C}^u,\mathcal{F}}"] \arrow[d, "\simeq"] & \hat{K}^G_{\mathbf{C}^u}(*) \arrow[d, "\simeq"] \\
K\mathbf{C}^G(E_{\mathcal{F}}G^{\mathrm{CW}}) \arrow[r, "\simeq"] & \mathrm{colim}_{G_{\mathcal{F}}\mathbf{Orb}}K\mathbf{C}^G \arrow[r, "\mathrm{Asmbl}^h_{\mathbf{C},\mathcal{F}}"] & K\mathbf{C}^G(*)
\end{tikzcd} \tag{16.1}$$

which is natural for $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg,eadd},\omega\mathrm{add}})$.

Proof. For any effectively additive C^*-category $\mathbf{D}$, we define a functor

$$\mathbf{D}^u[-] : \mathbf{Set} \to C^*\mathbf{Cat}\,.$$

It sends a set X to the C^*-category $\mathbf{D}^u[X]$ whose objects are pairs $(D, (p_x)_{x\in X})$ of an object D of $\mathbf{D}^u$ and a family of mutually orthogonal effective projections on D such that $\{x \in X \mid p_x \neq 0\}$ is finite and $\sum_{x\in X} p_x = \mathrm{id}_D$. The morphisms $(D, (p_x)_{x\in X}) \to (D', (p'_x)_{x\in X})$ in $\mathbf{D}^u[X]$ are morphisms $a : D \to D'$ in $\mathbf{D}$ such that for all x, x' in X with $x \neq x'$ we have $p'_{x'} a p_x = 0$. A morphism $f : X \to X'$ of sets induces a unital functor $\mathbf{D}^u[X] \to \mathbf{D}^u[X']$ which sends $(D, (p_x)_{x\in X})$ to $(D, (\sum_{x\in f^{-1}(x')} p_x)_{x'\in X'})$ (here we use the assumption that $\mathbf{D}$ is effectively additive) and acts as identity on morphisms.

The construction of $\mathbf{D}^u[-]$ from $\mathbf{D}$ is functorial in $C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg,eadd}}$. If G acts on X and $\mathbf{D}$, then we get an induced action on $\mathbf{D}^u[X]$ by functoriality. We have thus defined a functor from $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg,eadd}})$ to $\mathrm{Fun}(G\mathbf{Set}, \mathrm{Fun}(BG, C^*\mathbf{Cat}))$.

For X in $G\mathbf{Set}$ and $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg,eadd}})$, we let $\tilde{\bar{\mathbf{C}}}^{\mathrm{ctr}}_{\mathrm{lf}}(X_{min,max})$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat})$ denote the object $\bar{C}^{\mathrm{ctr}}_{\mathrm{lf}}(X_{min,max})$ introduced in Definition 3.2 for the trivial group with the G-action induced by functoriality. In [7, Prop. 9.12 (1)], we have constructed an isomorphism

$$\tilde{\bar{\mathbf{C}}}^{\mathrm{ctr}}_{\mathrm{lf}}((-)_{min,max}) \cong \mathbf{C}^u[-]$$

of functors from $G\mathbf{Set}$ to $\mathrm{Fun}(BG, C^*\mathbf{Cat})$. For X in $G\mathbf{Set}$, it sends the object (C, μ) in $\tilde{\bar{\mathbf{C}}}^{\mathrm{ctr}}_{\mathrm{lf}}(X_{min,max})$ to the object $(C, (\mu(\{x\}))_{x\in X})$ in $\mathbf{C}^u[X]$ and acts as identity on morphisms. This isomorphism is clearly natural for $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg,eadd}})$. Restricting along $G\mathbf{Orb} \subseteq G\mathbf{Set}$ and applying $- \rtimes_r G$ we therefore get an equivalence

$$K^{C^*\mathbf{Cat}}(\tilde{\bar{\mathbf{C}}}^{\mathrm{ctr}}_{\mathrm{lf}}((-)_{min,max}) \rtimes_r G) \simeq K^{C^*\mathbf{Cat}}(\mathbf{C}^u[-] \rtimes_r G) \tag{16.2}$$

of functors from $G\mathbf{Orb}$ to $\mathbf{Sp}$ which is natural for $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg,eadd}})$.

We now use that $\mathbf{C}$ admits countable AV-sums. By (16.2) and [7, Prop. 9.12 (3)] we have a unitary equivalence

$$\phi : \tilde{\bar{\mathbf{C}}}^{\mathrm{ctr}}_{\mathrm{lf}}((-)_{min,max}) \rtimes_r G \xrightarrow{\simeq} \bar{\mathbf{C}}^{\mathrm{ctr}}_{\mathrm{lf}}((-)_{min,max} \otimes G_{can,min})$$

of functors from $G\mathbf{Set}$ to $C^*\mathbf{Cat}$. This construction is not natural in $\mathbf{C}$ since the first step in the proof of [7, Prop. p.1] going into [7, Prop. 9.12 (3)] involves the choice of an AV-sum $(\bigoplus_{g\in G} gC, (e^C_g)_{g\in G})$ for every object C of $\mathbf{C}$. But if $\kappa : \mathbf{C} \to \mathbf{C}'$ is a morphism in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg,eadd},\omega\mathrm{add}})$, then it preserves AV-sums and for every object C of $\mathbf{C}$ we have a unique multiplier unitary $u_C : \bigoplus_{g\in G} gC \to \bigoplus_{g\in G} g\kappa(C)$ such that $u_C e^C_g = e^{\kappa(C)}_g$ for every g in G. These unitaries induce a unitary filler of the square of $C^*\mathbf{Cat}$-valued functors

$$\begin{tikzcd}
\tilde{\bar{\mathbf{C}}}^{\mathrm{ctr}}_{\mathrm{lf}}((-)_{min,max}) \rtimes_r G \arrow[r, "\phi_C"] \arrow[d] & \bar{\mathbf{C}}^{\mathrm{ctr}}_{\mathrm{lf}}((-)_{min,max} \otimes G_{can,min}) \arrow[d] \\
\tilde{\bar{\mathbf{C}}}'^{\mathrm{ctr}}_{\mathrm{lf}}((-)_{min,max}) \rtimes_r G \arrow[r, "\phi_{C'}"] & \bar{\mathbf{C}}'^{\mathrm{ctr}}_{\mathrm{lf}}((-)_{min,max} \otimes G_{can,min})
\end{tikzcd}$$

whose vertical maps are induced by κ. We therefore get an equivalence of functors from $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg,eadd},\omega\mathrm{add}})$ to $\mathrm{Fun}(G\mathbf{Set}, C^*\mathbf{Cat}_{2,1})$. Since $K^{C^*\mathbf{Cat}}$ factorizes over the localization $C^*\mathbf{Cat} \to C^*\mathbf{Cat}_{2,1}$ at unitary equivalences, after applying $K^{C^*\mathbf{Cat}}$, restricting along $G\mathbf{Orb} \subseteq G\mathbf{Set}$, and using Definitions 12.2 and 3.4 we get an equivalence

$$K^{C^*\mathbf{Cat}}(\tilde{\bar{\mathbf{C}}}^{\mathrm{ctr}}_{\mathrm{lf}}((-)_{min,max}) \rtimes_r G) \xrightarrow{\simeq} K^{C^*\mathbf{Cat}}(\bar{\mathbf{C}}^{\mathrm{ctr}}_{\mathrm{lf}}((-)_{min,max} \otimes G_{can,min})) \simeq K\mathbf{C}^G \tag{16.3}$$

which is natural for $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg,eadd},\omega\mathrm{add}})$.

We have a natural transformation

$$v : \mathbf{C}^u \otimes_{\max} \mathbb{C}[-] \to \mathbf{C}^u[-]\,, \tag{16.4}$$

see (15.7) for $\mathbb{C}[-]$, of functors from $G\mathbf{Set}$ to $\mathrm{Fun}(BG, C^*\mathbf{Cat})$. Its component on X in $G\mathbf{Set}$ is the functor

$$v_X : \mathbf{C}^u \otimes_{\max} \mathbb{C}[X] \to \mathbf{C}^u[X]\,,$$

which sends the object (C, y) in $\mathbf{C}^u \otimes_{\max} \mathbb{C}[X]$ to the object $(C, (p^y)_{x\in X})$ with

$$p^y_x := \begin{cases} \mathrm{id}_C & x = y, \\ 0 & x \neq y, \end{cases}$$

and which acts by $a \otimes z \mapsto za$ on morphisms. The functor v_X is a Morita equivalence: It is fully faithful, and every object of $\mathbf{C}^u[X]$ is isomorphic to a finite sum of objects in the image of v_X. Since $K^{C^*\mathbf{Cat}}$ is Morita invariant and $- \rtimes_r G$ preserves Morita equivalences by [9, Prop. 16.11], after restriction along $G\mathbf{Orb} \subseteq G\mathbf{Set}$ we get a natural transformation of functors

$$\hat{K}^G_{\mathbf{C}^u} \simeq K^{C^*\mathbf{Cat}}((\mathbf{C}^u \otimes_{\max} \mathbb{C}[-]) \rtimes_r G) \simeq K^{C^*\mathbf{Cat}}(\mathbf{C}^u[-] \rtimes_r G) \tag{16.5}$$

from $G\mathbf{Orb}$ to $\mathbf{Sp}$ where we have used Definition 15.10 in order to see the first equivalence. Since the transformation (16.4) is clearly natural for $\mathbf{C}$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg,eadd},\omega\mathrm{add}})$, so is (16.5).

Combining (16.5), (16.3) and (16.2) we get the equivalence asserted in the proposition. □

Proposition 16.3. *If $\mathcal{F} \subseteq$ **Fin**, then have a commutative square*

$$\begin{tikzcd}
\Sigma RK^{G,\mathrm{an}}_{(\mathbf{C}^u)^{(G)}}(E_{\mathcal{F}}G^{\mathrm{CW}}) \arrow[r, "\Sigma\mu^{\mathrm{Kasp}}_{(\mathbf{C}^u)^{(G)},\mathcal{F}}"] \arrow[d, dash, "\simeq"] & \Sigma \mathrm{KK}(\mathbb{C}, (\mathbf{C}^u)^{(G)} \rtimes_r G) \\
RK^{G,\mathrm{An}}_{\mathbf{C}}(E_{\mathcal{F}}G^{\mathrm{CW}}) \arrow[r, "\mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C},\mathcal{F}}"'] & \Sigma \mathrm{KK}(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G) \arrow[u, "\simeq"']
\end{tikzcd} \tag{16.6}$$

which is natural in $\mathbf{C}$ *in* $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg,eadd},\omega\mathrm{add}})$.

Proof. We start with the construction of the square (16.6). Its left vertical morphism will be induced by a zig-zag and therefore does not have a preferred direction. We expand the square into the following commutative diagram:

$$\begin{tikzcd}
\Sigma RK^{G,\mathrm{an}}_{(\mathbf{C}^u)^G}(E_{\mathcal{F}}G^{\mathrm{CW}}) \arrow[r, "\Sigma\mu^{\mathrm{Kasp}}_{(\mathbf{C}^u)^{(G)},\mathcal{F},\max}"] \arrow[d, "\simeq"'] \arrow[rr, bend left, "\Sigma\mu^{\mathrm{Kasp}}_{(\mathbf{C}^u)^{(G)},\mathcal{F}}"] & \Sigma \mathrm{KK}(\mathbb{C}, (\mathbf{C}^u)^{(G)} \rtimes G) \arrow[r] \arrow[d, "\simeq"] & \Sigma \mathrm{KK}(\mathbb{C}, (\mathbf{C}^u)^{(G)} \rtimes_r G) \arrow[d, "\simeq"'] \\
\Sigma RK^{G,\mathrm{an}}_{\mathbf{C}^{(G)}_{\mathrm{std},+}}(E_{\mathcal{F}}G^{\mathrm{CW}}) \arrow[r, "\Sigma\mu^{\mathrm{Kasp}}_{\mathbf{C}^{(G)}_{\mathrm{std},+},\mathcal{F},\max}"] & \Sigma \mathrm{KK}(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std},+} \rtimes G) \arrow[r] & \Sigma \mathrm{KK}(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std},+} \rtimes_r G) \\
\Sigma RK^{G,\mathrm{an}}_{\mathbf{C}^{(G)}_{\mathrm{std}}}(E_{\mathcal{F}}G^{\mathrm{CW}}) \arrow[r, "\Sigma\mu^{\mathrm{Kasp}}_{\mathbf{C}^{(G)}_{\mathrm{std}},\mathcal{F},\max}"] \arrow[u, "\simeq"] & \Sigma \mathrm{KK}(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes G) \arrow[r] \arrow[u] & \Sigma \mathrm{KK}(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G) \arrow[u, "\simeq"] \\
RK^{G,\mathrm{an}}_{\mathbf{Q}^{(G)}_{\mathrm{std}}}(E_{\mathcal{F}}G^{\mathrm{CW}}) \arrow[r, "\mu^{\mathrm{Kasp}}_{\mathbf{Q}^{(G)}_{\mathrm{std}},\mathcal{F},\max}"] \arrow[u, "\simeq"] & \mathrm{KK}(\mathbb{C}, \mathbf{Q}^{(G)}_{\mathrm{std}} \rtimes G) \arrow[r] \arrow[u, "\simeq"] & \Sigma \mathrm{KK}(\mathbb{C}, \mathbf{C}^{(G)}_{\mathrm{std}} \rtimes_r G) \arrow[u, equal] \\
RK^{G,\mathrm{An}}_{\mathbf{C}}(E_{\mathcal{F}}G^{\mathrm{CW}}) \arrow[u, equal] \arrow[urr, "\simeq", "\mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C},\mathcal{F}}"'] & &
\end{tikzcd} . \tag{16.7}$$

The two upper rows of vertical maps are induced by the zig-zag

$$(\mathbf{C}^u)^{(G)} \to \mathbf{C}^{(G)}_{\mathrm{std},+} \leftarrow \mathbf{C}^{(G)}_{\mathrm{std}}$$

(see (10.10)), where the first map is a weak Morita equivalence and the second is a split relative Morita equivalence. We use (see below for details) that the functors $RK^{G,\mathrm{an}}_{-}(E_{\mathcal{F}}G^{\mathrm{CW}})$ and $\mathrm{KK}(\mathbb{C}, - \rtimes_r G)$ send weak Morita equivalences and split relative Morita equivalences to equivalences.

(1) Recall that $RK^{G,\mathrm{an}}_{\mathbf{D}}(E_{\mathcal{F}}G^{\mathrm{CW}}) \cong \mathrm{colim}_{W \subseteq E_{\mathcal{F}}G^{\mathrm{CW}}} K^{G,\mathrm{an}}_{\mathbf{D}}(W)$, where the colimit runs over the filtered poset of G-finite G-CW subcomplexes of $E_{\mathcal{F}}G$. For fixed W, the functor $\mathbf{D} \mapsto K^{G,\mathrm{an}}_{\mathbf{D}}(W)$ sends relative Morita equivalences to equivalences by Lemma 8.6.3. Its sends weak Morita equivalences to equivalences by [12, Thm. 1.32.3].

(2) Since we have the equivalence

$$\mathrm{KK}(\mathbb{C}, - \rtimes G) \simeq \mathrm{KK}(\mathbb{C}, -)\circ(- \rtimes G)\circ \mathrm{kk}^G_{C^*\mathbf{Cat}}$$

of functors from $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ to $\mathbf{Sp}$, the functor $\mathrm{KK}(\mathbb{C}, - \rtimes G)$ sends weak Morita equivalences to equivalences since already $\mathrm{kk}^G_{C^*\mathbf{Cat}}$ does so by [12, Thm. 1.32.3]. Hence, the middle upper vertical arrow is an equivalence. One could also show that the other vertical arrow in this column is an equivalence, but since this is not needed in our argument we will not go through the details here.

(3) Since $\mathrm{KK}(\mathbb{C}, - \rtimes_r G) \simeq \mathrm{KK}(\mathbb{C}, -)\circ(- \rtimes_r G)\circ \mathrm{kk}^G_{C^*\mathbf{Cat}}$, as in the previous point, the functor $\mathrm{KK}(\mathbb{C}, - \rtimes_r G)$ sends weak Morita equivalences to equivalences. Since $- \rtimes_r G$ preserves Morita equivalences by [9, Prop. 16.11] and $\mathrm{KK}(\mathbb{C}, -) = K^{C^*\mathbf{Cat}}$ sends Morita equivalences to equivalences by [9, Prop. 16.18] we see that $\mathrm{KK}(\mathbb{C}, - \rtimes_r G)$ sends Morita equivalences to equivalences. In order to see that it also sends split relative Morita equivalences to equivalences, we apply $- \rtimes_r G$ to the diagram (2.6). In view of the existence of splits for p and q, exactness of the horizontal sequences is preserved. Because $- \rtimes_r G$ preserves Morita equivalences the resulting diagram shows that $\phi \rtimes_r G : \mathbf{D} \rtimes_r G \to \mathbf{E} \rtimes_r G$ is a relative Morita equivalence. Since $\mathrm{KK}(\mathbb{C}, -) = K^{C^*\mathbf{Cat}}$ is a Morita invariant homological functor, it sends relative Morita equivalences to equivalences by [9, Prop. 17.4].

The two upper right squares are provided by the natural transformation $- \rtimes G \to - \rtimes_r G$. The two lower left vertical arrows are induced by the boundary map of the fibre sequence associated to the exact sequence $0 \to \mathbf{C}^{(G)}_{\mathrm{std}} \to \mathbf{MC}^{(G)}_{\mathrm{std}} \to \mathbf{Q}^{(G)}_{\mathrm{std}} \to 0$ in $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$, see the proof of Proposition 10.15. This connecting map is an equivalence since $\mathbf{MC}^{(G)}_{\mathrm{std}}$ is flasque. The three left squares commute by the naturality of the Kasparov assembly map with respect to the coefficients in KK^G. The upper triangle and the lower triangle reflect the Definitions 12.8 and 12.12 of $\mu^{\mathrm{Kasp}}_{(\mathbf{C}^u)^{(G)},\mathcal{F}}$ and $\mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C},\mathcal{F}}$. □

Note that the statement of Theorem 16.1 depends on an object $\mathrm{kk}^G(A)$ in $\mathrm{KK}^G_{\mathrm{sep}}$ for a separable G-C^*-algebra A. In the proof, we want to relate the Kasparov assembly map $\mu^{\mathrm{Kasp}}_{A,\mathbf{Fin}}$ with the Davis-Lück assembly map $\mu^{\mathrm{DL}}_{A,\mathbf{Fin}}$ by comparing them with the analytic assembly maps $\mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C},\mathbf{Fin}}$ and $\mathrm{Asmbl}^{h}_{\mathbf{C},\mathbf{Fin}}$, respectively, for a suitably choice of G-C^*-category $\mathbf{C}$ and invoking Theorem 1.9. If A is a unital separable G-C^*-algebra, then we could take $\mathbf{C} = \mathbf{Hilb}_c(A)$. But not every class in $\mathrm{KK}^G_{\mathrm{sep}}$ is represented by a unital G-C^*-algebra. But every class is a fibre of a morphism between classes of unital algebras algebras. Indeed, if a class is represented by a G-C^*-algebra A, then it is equivalent to the fibre of $\mathrm{kk}^G(A^+) \to \mathrm{kk}^G(\mathbb{C})$. In order to apply this, we must model the unitalization map by a suitable essential functor between associated effectively additive G-C^*-categories. This is the contents of the following proposition.

Let A be in $\mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}})$ and consider the split unitalization sequence

$$0 \to A \to A^+ \xrightarrow{p} \mathbb{C} \to 0$$

whose split will be denoted by $e : \mathbb{C} \to A^+$.

Proposition 16.4. *There exists the following data:*

(1) $\mathbf{C}_+$, $\mathbf{C}_{\mathbb{C}}$ *in* $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg,eadd},\omega\mathrm{add}})$,
(2) $q : \mathbf{C}_+ \to \mathbf{C}_{\mathbb{C}}$ *in* $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg,eadd},\omega\mathrm{add}})$,
(3) $s : \mathbf{C}_{\mathbb{C}} \to \mathbf{C}_+$ *in* $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg,eadd},\omega\mathrm{add}})$,
(4) $i : A^+ \to (\mathbf{C}^u_+)^{(G)}$ *and* $j : \mathbb{C} \to (\mathbf{C}^u_{\mathbb{C}})^{(G)}$ *in* $\mathrm{Fun}(BG, C^*\mathbf{Cat})$,

with the following properties:

(1) *The squares*

$$\begin{tikzcd} A^+ \arrow[r, "p"] \arrow[d, "i"] & \mathbb{C} \arrow[d, "j"] \\ (\mathbf{C}^u_+)^{(G)} \arrow[r, "(q^u)^{(G)}"] & (\mathbf{C}^u_{\mathbb{C}})^{(G)} \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} A^+ \arrow[d, "i"] & \mathbb{C} \arrow[l, "e"'] \arrow[d, "j"] \\ (\mathbf{C}^u_+)^{(G)} & (\mathbf{C}^u_{\mathbb{C}})^{(G)} \arrow[l, "(s^u)^{(G)}"'] \end{tikzcd}$$

commute.
(2) *G weakly fixes the objects of $\mathbf{C}^u_+$ and $\mathbf{C}^u_{\mathbb{C}}$, see Definition* 2.9.
(3) *i and j are Morita equivalences.*
(4) *q is a quotient and $q \circ s = \mathrm{id}_{\mathbf{C}_{\mathbb{C}}}$.*

Proof. We let $\widehat{\mathbf{A}^+}$ be the full subcategory of $\mathbf{Hilb}_c(A^+)$ on the objects which are isomorphic to $\hat{A}^+$, see Example 2.18. Since the object $\hat{A}^+$ has an extension $(\hat{A}^+, \kappa)$ in $((\widehat{\mathbf{A}^+})^u)^{(G)}$ we have unitary isomorphisms $\kappa_g : \hat{A}^+ \to g\hat{A}^+$ in $\mathbf{Hilb}_c(A^+)$ for all g in G. It follows that $\widehat{\mathbf{A}^+}$ is G-invariant and therefore inherits a G-action from $\mathbf{Hilb}_c(A^+)$. Furthermore, we have $\widehat{\mathbf{A}^+} = (\widehat{\mathbf{A}^+})^u$ and G weakly fixes the objects of $(\widehat{\mathbf{A}^+})^u$.

We set

$$\mathbf{C}_+ := \widehat{\mathbf{A}^+} \otimes_{\max} \mathbf{Hilb}_c(\mathbb{C})$$

with the G-action induced from the first factor. We furthermore let $\mathbf{F}$ be the G-C^*-category with the same objects as $\widehat{\mathbf{A}^+}$ but morphism spaces isomorphic to $\mathbb{C}$ between any two objects. We have a canonical projection $q' : \widehat{\mathbf{A}^+} \to \mathbf{F}$ involving p and a split $s' : \mathbf{F} \to \widehat{\mathbf{A}^+}$ involving the units of A^+. We set

$$\mathbf{C}_{\mathbb{C}} := \mathbf{F} \otimes_{\max} \mathbf{Hilb}_c(\mathbb{C}) .$$

Then we have a quotient projection $q := q' \otimes \mathrm{id}_{\mathbf{Hilb}_c(\mathbb{C})} : \mathbf{C}_+ \to \mathbf{C}_{\mathbb{C}}$ and the split functor $s := s' \otimes \mathrm{id}_{\mathbf{Hilb}_c(\mathbb{C})} : \mathbf{C}_{\mathbb{C}} \to \mathbf{C}_+$ such that $q \circ s = \mathrm{id}_{\mathbf{C}_{\mathbb{C}}}$. Because of this equality the condition that q is a quotient simply means that it is bijective on objects.

We define $j : \mathbb{C} \to (\mathbf{C}^u_{\mathbb{C}})^{(G)}$ using the object $((\hat{A}^+, \mathbb{C}), \varkappa \otimes \mathrm{id}_{\mathbb{C}})$ and the canonical identification $\mathrm{End}_{(\mathbf{C}^u_{\mathbb{C}})^{(G)}}(((\hat{A}^+, \mathbb{C}), \varkappa \otimes \mathrm{id}_{\mathbb{C}})) \cong \mathbb{C}$. We further define $i : A^+ \to (\mathbf{C}^u_+)^{(G)}$ using the object $((\hat{A}^+, \mathbb{C}), \varkappa \otimes \mathrm{id}_{\mathbb{C}})$ and the canonical G-equivariant identification $\mathrm{End}_{(\mathbf{C}^u_+)^{(G)}}(((\hat{A}^+, \mathbb{C}), \varkappa \otimes \mathrm{id}_{\mathbb{C}})) \cong A^+$. Then the two squares commute.

If we forget the G-action, then $\mathbf{C}_+$ is isomorphic to $A^+ \otimes_{\max} \mathbf{Hilb}_c(\mathbb{C})$. We can conclude that $\mathbf{C}_+$ admits all AV-sums and is therefore effectively additive. A similar reasoning applies to $\mathbf{C}_{\mathbb{C}}$.

The functor q is full and hence non-degenerate. The split $s' : \mathbf{F} \to \widehat{\mathbf{A}^+}$ is unital and hence also non-degenerate. This implies that s is non-degenerate.

In order to show that i is a Morita equivalence, we note that any object in $(\mathbf{C}^u_+)^{(G)}$ is unitarily isomorphic to an object $((\hat{A}, H), \varkappa \otimes \mathrm{id}_H)$ for some finite-dimensional Hilbert space H. It is therefore unitarily isomorphic to a finite sum of copies of $i(A^+)$. The same reasoning applies to show that j is a Morita equivalence. □

We now finish the proof of the Theorem 16.1. The statement of the theorem depends on an object A of $\mathrm{KK}^G_{\mathrm{sep}}$ which is assumed to belong to $\mathcal{CI}$. We can choose an object of $\mathrm{Fun}(BG, C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}})$ which realizes A in $\mathrm{KK}^G_{\mathrm{sep}}$ upon applying $\mathrm{kk}^G_{\mathrm{sep}}$. So from now on A denotes this G-C^*-algebra.

We apply Proposition 16.4 to A in order to get the asserted data. For any functor F from $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}}_{\mathrm{ndeg,eadd},\omega\mathrm{add}})$ to an additive category, we get a decomposition

$$F(\mathbf{C}_+) \simeq F^{\mathrm{intrs}} \oplus F(\mathbf{C}_{\mathbb{C}}) ,$$

where the projection to and inclusion of the second summand are given by $F(q)$ and $F(s)$. We call F^{intrs} the interesting summand. A natural transformation $f : F \to F'$ of functors induces a map $f^{\mathrm{intrs}} : F^{\mathrm{intrs}} \to F'^{,\mathrm{intrs}}$ between the interesting summands. We call f^{intrs} the interesting summand of f. Finally, a natural equivalence $f \simeq f'$ between natural transformations induces a natural equivalence $f^{\mathrm{intrs}} \simeq f'^{,\mathrm{intrs}}$ between the interesting summands. We now have the following facts:

(1) The interesting summand of $\mu^{\mathrm{Kasp}}_{(\mathbf{C}^u_+)^{(G)},\mathbf{Fin}}$ is equivalent to the interesting summand of $\mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C}_+,\mathbf{Fin}}$ by Proposition 16.3.

(2) By Theorem 1.9 the interesting summand of $\mathrm{Asmbl}^{\mathrm{an}}_{\mathbf{C}_+,\mathbf{Fin}}$ is an equivalence if and only if the interesting summand of $\mathrm{Asmbl}^{h}_{\mathbf{C}_+,\mathbf{Fin}}$ is an equivalence.

(3) The interesting summand of $\mathrm{Asmbl}^{h}_{\mathbf{C}_+,\mathbf{Fin}}$ is equivalent to the interesting summand of $\mu^{\mathrm{DL}}_{\mathbf{C}^u_+,\mathbf{Fin}}$ by Proposition 16.2.

(4) The interesting summand of $\mu^{\mathrm{DL}}_{(\mathbf{C}^u_+)^{(G)},\mathbf{Fin}}$ is equivalent to the interesting summand of $\mu^{\mathrm{DL}}_{\mathbf{C}^u_+,\mathbf{Fin}}$ by Lemma 2.10. Here we use Property 2 of the data from Proposition 16.4.

(5) We note that the Davis–Lück assembly map $\mu^{\mathrm{DL}}_{-,\mathbf{Fin}}$ depends functorially on an object of KK^G. The pair of morphisms $p : A^+ \to \mathbb{C}$ and $e : \mathbb{C} \to A^+$ provides a decomposition $\mathrm{kk}^G(A^+) \simeq \mathrm{kk}^G(A) \oplus \mathrm{kk}^G(\mathbb{C})$. The commutative squares in Property 1 of the data from Proposition 16.4 provide a decomposition of the transformation $\mu^{\mathrm{DL}}_{i,\mathbf{Fin}}$ into a sum $(\mu^{\mathrm{DL}}_{i,\mathbf{Fin}})^{\mathrm{intrs}} \oplus \mu^{\mathrm{DL}}_{j,\mathbf{Fin}}$. Since i is a Morita equivalence and the transformation between the Davis–Lück assembly maps depends on $\hat{K}^G_i$, by Corollary 15.17.3 the transformations $\mu^{\mathrm{DL}}_{i,\mathbf{Fin}}$ and hence $(\mu^{\mathrm{DL}}_{i,\mathbf{Fin}})^{\mathrm{intrs}}$ are equivalences. We conclude that the interesting summand of $\mu^{\mathrm{DL}}_{(\mathbf{C}^u_+)^{(G)},\mathbf{Fin}}$ is equivalent to $\mu^{\mathrm{DL}}_{A,\mathbf{Fin}}$.

(6) By a completely analogous argument the summand of $\mu^{\mathrm{Kasp}}_{(\mathbf{C}_+)^{(G)},\mathbf{Fin}}$ is equivalent to $\mu^{\mathrm{Kasp}}_{A,\mathbf{Fin}}$. Here we use that the domain and target $RK^{G,\mathrm{an}}_{-}(E_{\mathbf{Fin}}G^{\mathrm{CW}})$ and $\mathrm{KK}(\mathbb{C}, -\rtimes_r G)$ of $\mu^{\mathrm{Kasp}}_{-,\mathbf{Fin}}$ considered as functors on $\mathrm{Fun}(BG, C^*\mathbf{Cat}^{\mathrm{nu}})$ via $\mathrm{kk}^G_{C^*\mathbf{Cat}}$ send Morita equivalences to equivalences. For $\mathrm{KK}(\mathbb{C}, -\rtimes_r G)$, this has been observed above in the proof of Corollary 15.17.3. For the other functor, we use the formula

$$RK^{G,an}_{-}(E_{\mathbf{Fin}}G^{\mathrm{CW}}) \simeq \mathrm{colim}_{W\subseteq E_{\mathbf{Fin}}G^{\mathrm{CW}}} \mathrm{KK}^G(C_0(W), -)\,,$$

where W runs over the G-finite subcomplexes of $E_{\mathbf{Fin}}G^{\mathrm{CW}}$, and Lemma 8.6.3 saying that $\mathrm{KK}^G(C_0(W), -)$ sends Morita equivalences to equivalences for every W.

By a combination of these facts we see that $\mu^{\mathrm{Kasp}}_{A,\mathbf{Fin}}$ is an equivalence if and only if $\mu^{\mathrm{DL}}_{A,\mathbf{Fin}}$ is an equivalence. Under the assumption that $\mathrm{kk}^G_{\mathrm{sep}}(A)$ belongs to $\mathcal{CI}$ we know that $\mu^{\mathrm{DL}}_{A,\mathbf{Fin}}$ is an equivalence by Lemma 15.9. □

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