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On Schur's irreducibility results and extended Hermite polynomials

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ABSTRACT. Fix $c \in \{0,2\}$, and let $n \geq 2$ be an integer such that, if $c = 2$, then $2n+1 \neq 3^u$ for any $u \geq 2$. Let $\phi(x) \in \mathbb{Z}[x]$ be a monic polynomial that is irreducible modulo every prime less than $2n+c$. Let $a_i(x) \in \mathbb{Z}[x]$ for $0 \leq i \leq n-1$ be polynomials with degree less than $\deg \phi(x)$. Let $a_n \in \mathbb{Z}$, and assume the content of $(a_n a_0(x))$ is not divisible by any prime less than $2n+c$. For a positive integer j, let u_j denote the product of the odd numbers less than or equal to j. We show that the polynomial

$$\frac{a_n}{u_{2n+c}}\phi(x)^{2n} + \sum_{j=0}^{n-1} a_j(x)\frac{\phi(x)^{2j}}{u_{2j+c}}$$

is irreducible over the field $\mathbb{Q}$ of rational numbers. This generalizes a well-known result of Schur, which states that the polynomial $\sum_{j=0}^{n} a_j \frac{x^{2j}}{u_{2j+c}}$ with $a_j \in \mathbb{Z}$ and $|a_0| = |a_n| = 1$ is irreducible over $\mathbb{Q}$. To establish our results, we use a beautiful result of Jindal and Khanduja [19, Theorem 1.3] and several number-theoretic results related to prime numbers. We illustrate our results with examples.

CONTENTS

1. INTRODUCTION AND STATEMENT OF RESULTS

For each non-negative integer j, we define u_j as the product of the odd numbers less than or equal to j. In particular, we have $u_0 = u_2 = 1$, $u_4 = 3$, $u_6 = 15$, and so on.

In 1929, Schur [20] proved the following result:

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Theorem 1.1. Fix $c \in \{0, 2\}$, and let n be a positive integer such that, if $c = 2$, then $2n + 1 \neq 3^u$ for any $u \geq 2$. Let $a_j \in \mathbb{Z}$ for $0 \leq j \leq n$ with $|a_0| = |a_n| = 1$. Then the polynomial

$$\sum_{j=0}^{n} a_j \frac{x^{2j}}{u_{2j+c}}$$

is irreducible over $\mathbb{Q}$.

As a consequence of the above theorem, Schur also proved that the m-th classical Hermite polynomial, given by

$$H_m(x) = \sum_{j=0}^{\lfloor m/2 \rfloor} (-1)^j \binom{m}{2j} u_{2j} x^{m-2j},$$

is irreducible if m is even and is x times an irreducible polynomial if m is odd.

In this paper, we extend Theorem 1.1 by employing ϕ-Newton polygons (see Definition 2.1) together with several number-theoretic properties of prime numbers. More precisely, our main result is the following.

Theorem 1.2. Fix $c \in \{0, 2\}$, and let $n \geq 2$ be an integer such that, if $c = 2$, then $2n + 1 \neq 3^u$ for any $u \geq 2$. Let $\phi(x) \in \mathbb{Z}[x]$ be a monic polynomial that is irreducible modulo every prime less than $2n + c$. Suppose $a_n \in \mathbb{Z}$ and $a_0(x), \ldots, a_{n-1}(x) \in \mathbb{Z}[x]$ satisfy:

(i) $\deg a_i(x) < \deg \phi(x)$ for $0 \leq i \leq n - 1$;
(ii) the content of $a_n a_0(x)$ is not divisible by any prime less than $2n + c$.

Then the polynomial

$$f_c(x) = \frac{a_n}{u_{2n+c}} \phi(x)^{2n} + \sum_{j=0}^{n-1} a_j(x) \frac{\phi(x)^{2j}}{u_{2j+c}}$$

is irreducible over $\mathbb{Q}$.

For any integer $m > 2$, we wish to point out here that we can always construct a polynomial $\phi(x)$ with degree greater than 1 which is irreducible modulo all primes below m. This can be seen as follows.

Remark 1.3. For any given prime p, it can be seen from quadratic reciprocity that there always exists an integer b such that $x^2 - x - b$ is irreducible modulo p. An application of the Chinese Remainder Theorem will give us an integer c such that the polynomial $x^2 - x - c$ is irreducible modulo all primes below a number m.

It should also be noted that in Theorem 1.2, the assumption "the content of $a_0(x)$ is not divisible by any prime less than $2n + c$" cannot be omitted, as illustrated below:

c	$\phi(x)$	$f(x)$ (reducible over $\mathbb{Q}$)
0	$x^2 - x + 5$	$f(x) = \dfrac{\phi(x)^4}{3} - 3 = \dfrac{1}{3}\big(\phi(x)^2 + 3\big)\big(\phi(x)^2 - 3\big)$
2	$x^2 - x + 11$	$f(x) = \dfrac{\phi(x)^4}{15} + 6\dfrac{\phi(x)^2}{3} + 15 = \dfrac{1}{15}\big(\phi(x)^2 + 15\big)^2$

The following examples illustrate that Theorem 1.2 may not hold if a_n is replaced by a monic polynomial $a_n(x)$ with integer coefficients having degree less than $\deg \phi(x)$.

c	$\phi(x)$	Polynomial $f(x)$ with 0 as a root
0	$x^2 - x + 5$	$f(x) = (x-3)\dfrac{\phi(x)^4}{3} + (x+26)\phi(x)^2 + 5(x-5)$
2	$x^2 - x + 11$	$f(x) = (x-15)\dfrac{\phi(x)^4}{15} + (x+366)\dfrac{\phi(x)^2}{3} + (x-121)$

In both cases the constructed polynomial $f(x)$ has 0 as a root, and hence is reducible over $\mathbb{Q}$.

We also point out that the analogue of Theorem 1.2 need not hold for $n = 1$. For instance, if $c = 0$ and $\phi(x) \in \mathbb{Z}[x]$ is a monic polynomial of degree ≥ 3, then the polynomial $\phi(x)^2 - x^2 = (\phi(x) + x)(\phi(x) - x)$ is reducible over $\mathbb{Q}$.

As an application of Theorem 1.2, we prove the following result.

Theorem 1.4. Let $m \geq 3$ be an integer, and let $\phi(x) \in \mathbb{Z}[x]$ be a monic polynomial irreducible modulo all primes $\leq m$. Suppose

$$a_{\left[\frac{m}{2}\right]} \in \mathbb{Z} \quad \text{and} \quad a_0(x), \dots, a_{\left[\frac{m}{2}\right]-1}(x) \in \mathbb{Z}[x]$$

satisfy the following conditions:

(i) $\deg a_i(x) < \deg \phi(x)$ for $0 \leq i \leq \left[\frac{m}{2}\right] - 1$,

(ii) the content of $(a_{\left[\frac{m}{2}\right]} a_0(x))$ is not divisible by any prime $\leq m$.

Then the polynomial

$$H_m^{\phi}(x) = a_{\left[\frac{m}{2}\right]}\phi(x)^m + \sum_{j=1}^{\left[\frac{m}{2}\right]} \binom{m}{2j} u_{2j} a_{\left[\frac{m}{2}\right]-j}(x)\phi(x)^{m-2j}$$

is irreducible over $\mathbb{Q}$ if m is even, and is $\phi(x)$ times an irreducible polynomial if m is odd, provided $m \neq 3^u$ for any integer $u \geq 2$.

In the above theorem, if we take $\phi(x) = x$, $a_{\left[\frac{m}{2}\right]} = (-1)^{\left[\frac{m}{2}\right]}$, and $a_i(x) = (-1)^i$ for $0 \leq i \leq \left[\frac{m}{2}\right] - 1$, then $H_m^{\phi}(x)$ becomes the m-th classical Hermite polynomial.

The following corollary is an immediate consequence of the above theorem.

Corollary 1.5. Let $m \geq 3$ be an integer, and let $\phi(x) \in \mathbb{Z}[x]$ be a monic polynomial irreducible modulo all primes less than m. Then the polynomial

$$H_m(\phi(x)) = \sum_{j=0}^{\left[\frac{m}{2}\right]} (-1)^j \binom{m}{2j} u_{2j} \phi(x)^{m-2j}$$

is irreducible over $\mathbb{Q}$ if m is even, and is $\phi(x)$ times an irreducible polynomial if m is odd, provided $m \neq 3^u$ for any integer $u \geq 2$.

We note that several authors have employed the concept of x-Newton polygons along with results on primes from analytic number theory to establish the irreducibility of various families of polynomials. Notable examples include Bessel polynomials [5, 7], Laguerre polynomials [8, 9], truncated exponential Taylor polynomials [2], Schur polynomials [3, 4], and truncated binomial expansions [6, 17].

Furthermore, we point out that the proofs of our results share certain similarities with the arguments presented in the non-ϕ-expansion setting in [3, 4], as well as in the ϕ-expansion setting in [15, 19].

We now provide some examples related to Theorem 1.2.

Example 1.6. Consider $\phi(x) = x^3 - x + 37$. It can be verified that $\phi(x)$ is irreducible modulo 2, 3, 5, and 7. Let $j \geq 2$, and let a_j be integers. Assume that the polynomials $a_i(x) \in \mathbb{Z}[x]$ for $0 \leq i \leq j-1$ each have degree less than 3, and the content of $(a_j a_0(x))$ is not divisible by any prime less than $2j$. Then, applying Theorem 1.2 with $c = 0$, the polynomial

$$f_j(x) = \frac{a_j}{u_{2j}} \phi(x)^{2j} + \sum_{i=0}^{j-1} a_i(x) \frac{\phi(x)^{2i}}{u_{2i}}$$

is irreducible over $\mathbb{Q}$ for $j \in \{2, 3, 4, 5\}$.

Example 1.7. Consider $\phi(x) = x^2 - x + 17$. It can be verified that $\phi(x)$ is irreducible modulo 2, 3, 5, and 7. Let $j \geq 2$, and let a_j be integers. Assume that the polynomials $a_i(x) \in \mathbb{Z}[x]$ for $0 \leq i \leq j-1$ each have degree less than 2, and the content of $(a_j a_0(x))$ is not divisible by any prime less than $2j+2$. Then, using Theorem 1.2 with $c = 2$, the polynomial

$$g_j(x) = \frac{a_j}{u_{2j+2}} \phi(x)^{2j} + \sum_{i=0}^{j-1} a_i(x) \frac{\phi(x)^{2i}}{u_{2i+2}}$$

is irreducible over $\mathbb{Q}$ for $j \in \{2, 3, 4\}$.

It should be noted that the irreducibility of the polynomials $f_5(x)$ and $g_4(x)$ does not seem to follow from any known irreducibility criteria (cf. [1], [10], [11], [12], [13], [16], [18]).

2. Preliminary results.

We first introduce the notions of Gauss valuation and ϕ-Newton polygon. For a prime p, let v_p denote the p-adic valuation on $\mathbb{Q}$, defined for any non-zero integer b as the highest power of p dividing b.

We denote by v_p^x the Gaussian valuation, which extends v_p and is defined on the polynomial ring $\mathbb{Z}[x]$ by

$$v_p^x\left(\sum_i b_i x^i\right) = \min_i\{v_p(b_i)\}, \quad b_i \in \mathbb{Z}.$$

Definition 2.1. Let p be a prime number and $\phi(x) \in \mathbf{Z}[x]$ be a monic polynomial which is irreducible modulo p. Let $f(x)$ belonging to $\mathbf{Z}[x]$ be a polynomial having ϕ-expansion[1] $\sum\limits_{i=0}^{n} b_i(x)\phi(x)^i$ with $b_0(x)b_n(x) \neq 0$. Let P_i stand for the point in the plane having coordinates $(i, v_p^x(b_{n-i}(x)))$ when $b_{n-i}(x) \neq 0$, $0 \leq i \leq n$. Let μ_{ij} denote the slope of the line joining the points P_i and P_j if $b_{n-i}(x)b_{n-j}(x) \neq 0$. Let i_1 be the largest index $0 < i_1 \leq n$ such that

$$\mu_{0i_1} = \min\{\mu_{0j} \mid 0 < j \leq n,\ b_{n-j}(x) \neq 0\}.$$

If $i_1 < n$, let i_2 be the largest index $i_1 < i_2 \leq n$ such that

$$\mu_{i_1 i_2} = \min\{\mu_{i_1 j} \mid i_1 < j \leq n,\ b_{n-j}(x) \neq 0\}$$

and so on. The ϕ-Newton polygon of $f(x)$ with respect to p is the polygonal path having segments $P_0P_{i_1}, P_{i_1}P_{i_2}, \ldots, P_{i_{k-1}}P_{i_k}$ with $i_k = n$. These segments are called the edges of the ϕ-Newton polygon of $f(x)$ and their slopes from left to right form a strictly increasing sequence. The ϕ-Newton polygon minus the horizontal part (if any) is called its principal part.

The following result is proved in [19] as Theorem 1.3 and will be used in the sequel. We omit its proof.

Proposition 2.2. Let n, k and ℓ be integers with $0 \leq \ell < k \leq \frac{n}{2}$ and p be a prime. Let $\phi(x) \in \mathbf{Z}[x]$ be a monic polynomial which is irreducible modulo p. Let $f(x)$ belonging to $\mathbf{Z}[x]$ be a monic polynomial not divisible by $\phi(x)$ having ϕ-expansion $\sum\limits_{i=0}^{n} f_i(x)\phi(x)^i$ with $f_n(x) \neq 0$. Assume that $v_p^x(f_i(x)) > 0$ for $0 \leq i \leq n-\ell-1$ and the right-most edge of the ϕ-Newton polygon of $f(x)$ with respect to p has slope less than $\frac{1}{k}$. Let $a_0(x), a_1(x), \ldots, a_n(x)$ be polynomials over $\mathbf{Z}$ satisfying the following conditions.

(i) $\deg a_i(x) < \deg \phi(x) - \deg f_i(x)$ for $0 \leq i \leq n$,
(ii) $v_p^x(a_0(x)) = 0$, i.e., the content of $a_0(x)$ is not divisible by p,

[1] If $\phi(x)$ is a fixed monic polynomial with coefficients in $\mathbf{Z}$, then any $f(x) \in \mathbf{Z}[x]$ can be uniquely written as a finite sum $\sum_i b_i(x)\phi(x)^i$ with $\deg b_i(x) < \deg \phi(x)$ for each i; this expansion will be referred to as the ϕ-expansion of $f(x)$.

(iii) the leading coefficient of $a_n(x)$ is not divisible by p.

Then the polynomial $\sum_{i=0}^{n} a_i(x) f_i(x)\phi(x)^i$ does not have a factor in $\mathbf{Z}[x]$ with degree lying in the interval $[(\ell+1)\deg\phi(x), (k+1)\deg\phi(x))$.

We now prove the following elementary result, which will be used in the proof of Theorem 1.2.

Lemma 2.3. Let n be a positive integer and let p be a prime number. Let $\phi(x) \in \mathbb{Z}[x]$ be a monic polynomial that is irreducible modulo p. Suppose $a_n \in \mathbb{Z}$ and $a_0(x), a_1(x), \dots, a_{n-1}(x) \in \mathbb{Z}[x]$ are polynomials, each having degree less than $\deg\phi(x)$. Let $p \nmid a_n$ and let $b_0, b_1, \dots, b_{n-1}$ be integers such that $p \mid b_j$ for each j with $0 \le j \le n-1$. Then the polynomial

$$F(x) = a_n\phi(x)^{2n} + \sum_{j=0}^{n-1} b_j a_j(x)\phi(x)^{2j}$$

cannot have any non-constant factor of degree less than $\deg\phi(x)$.

Proof. Let c denote the content of $F(x)$. Since $p \nmid a_n$, we have $p \nmid c$. Now, suppose for the sake of contradiction that there exists a primitive non-constant polynomial $h(x) \in \mathbb{Z}[x]$ dividing $F(x)$ with degree less than $\deg\phi(x)$. By Gauss's Lemma, there exists a polynomial $g(x) \in \mathbb{Z}[x]$ such that $\frac{F(x)}{c} = h(x)g(x)$.

The leading coefficients of $F(x)$, $h(x)$, and $g(x)$ are all coprime to p. Since p divides b_j for $0 \le j \le n-1$, it follows that when we reduce modulo p, we have

$$\overline{F}(x) = \overline{a_n}\overline{\phi}(x)^{2n} + \sum_{j=0}^{n-1} \overline{b_j}\overline{a_j}(x)\overline{\phi}(x)^{2j}.$$

Thus, on passing to $\mathbb{Z}/p\mathbb{Z}$, we see that the degree of $\bar{h}(x)$ is the same as that of $h(x)$. Hence, $\deg\bar{h}(x)$ is positive and less than $\deg\phi(x)$. This leads to a contradiction because $\bar{h}(x)$ is a divisor of $\frac{\overline{F}(x)}{\bar{c}} = \frac{\overline{a_n}}{\bar{c}}\overline{\phi}(x)^{2n}$, and $\overline{\phi}(x)$ is irreducible over $\mathbb{Z}/p\mathbb{Z}$. This contradiction completes the proof of the lemma. □

The following result, due to Schur [20], plays a significant role in the proof of Theorem 1.2.

Lemma 2.4. For integers k and n with $n > k > 2$, at least one of the k numbers $2n+1, 2n+3, \dots, 2n+2k-1$ is divisible by a prime $p > 2k+1$. For $k = 2$, the same result holds unless $2n+1 = 25$. For $k = 1$, the same result holds unless $2n+1 = 3^u$ for some integer $u \ge 2$.

Remark 2.5. It is noteworthy that the first part of the above lemma can be rephrased to state that, for $k > 2$, the product of any k consecutive odd numbers each greater than $2k+1$ is divisible by a prime $p > 2k+1$.

3. Proof of Theorem 1.2

Proof of Theorem 1.2. By hypothesis, $n \geq 2$. It suffices to show that $F_c(x) = u_{2n+c} f_c(x)$ can not have a factor in $\mathbf{Z}[x]$ with degree lying in the interval $[1, (\frac{n}{2} + 1)\deg \phi(x))$. If we choose a prime p such that p divides $2n - 1 + c$, then we have p divides $\frac{u_{2n+c}}{u_{2j+c}}$ for each $0 \leq j \leq n-1$. Hence using Lemma 2.3, we see that $F_c(x)$ can not have any non-constant factor having degree less than $\deg \phi(x)$. Now assume that $F_c(x)$ has a factor in $\mathbf{Z}[x]$ with degree lying in the interval $[\deg \phi(x), (\frac{n}{2} + 1)\deg \phi(x))$. We make use of Proposition 2.2 to obtain a contradiction. We consider a new polynomial $g_c(x)$ with $a_n = 1 = a_j(x)$, $0 \leq j \leq n-1$, in $F_c(x)$ given by

$$g_c(x) = \phi(x)^{2n} + (2n-1+c)\phi(x)^{2n-2} + \cdots + (2n-1+c)\cdots(3+c)\phi(x)^2 + (2n-1+c)\cdots(3+c)(1+c).$$

We define integers c_i such that $c_{2n} = 1$, for an odd integer r we have $c_r = 0$ and $c_{2n-(r+1)} = (2n-1+c)(2n-3+c)\cdots(2n-r+2+c)(2n-r+c)$. Hence we can write $g_c(x) = \sum_{i=0}^{2n} c_i \phi(x)^i$. Observe that for every $i \in \{0, 1, \cdots, 2n-1\}$, c_i is divisible by the product of the odd numbers in the interval $(i+c, 2n-1+c]$. Also, for $0 \leq j \leq n$, $c_{2j} = \frac{u_{2n+c}}{u_{2j+c}}$. Let $\ell = k-1$ so that $\ell + 1 = k$. We now treat the proof in two cases depending on c.

Case (I) $c = 0$. In this situation, by Lemma 2.4, there is a prime factor $p \geq k+1$ that divides $c_{2n-\ell-1}$ and $c_{2n-\ell-2}$, which implies that $p | c_i$ for all $i \in \{0, 1, \cdots, 2n-\ell-1\}$. Clearly $p \nmid c_{2n}$. The slope of the right-most edge of the ϕ-Newton polygon of $g_c(x) (= g_0(x))$ with respect to p can be determined by

$$\max_{1 \leq j \leq n} \left\{ \frac{v_p(u_{2n}) - v_p(u_{2n}/u_{2j})}{2j} \right\}.$$

Using the fact that $v_p((2j-1)!) < (2j-1)/(p-1)$, for $1 \leq j \leq n$ we obtain

$$v_p(u_{2n}) - v_p(u_{2n}/u_{2j}) = v_p(u_{2j}) \leq v_p((2j-1)!) < \frac{2j-1}{p-1} < \frac{2j}{p-1}.$$

As $p \geq k+1$, we deduce that the slope of the right-most edge of the ϕ-Newton polygon of $g_0(x)$ with respect to p is $< \frac{1}{k}$. Hence, using Proposition 2.2, we have a contradiction. This completes the proof of the theorem in Case (I).

Case (II) $c = 2$. In this case, by Lemma 2.4, there is a prime factor $p \geq k+2$ that divides $c_{2n-\ell-1}$ and $c_{2n-\ell-2}$ unless either (i) $k = 2$ and $2n+1 = 3^u$ with some integer $u \geq 2$ or (ii) $k = 4$ and $n = 13$. For the moment, suppose we are not in either of the situations described by (i) and (ii), and fix $p \geq k+2$ dividing $c_{2n-\ell-1}$ and $c_{2n-\ell-2}$. Then $p \nmid c_{2n}$ and $p | c_i$ for all $i \in \{0, 1, \cdots, 2n-\ell-1\}$.

Next, we show that the slope of the right-most edge of the ϕ-Newton polygon of $g_c(x)(= g_2(x))$ with respect to p is $< \frac{1}{k}$, but we note here that our argument will only depend on p being a prime $\geq k+2$ and not on p dividing $c_{2n-\ell-1}$ and $c_{2n-\ell-2}$. The slope of the right-most edge of the ϕ-Newton polygon of $g_2(x)$ with respect to p can be determined by

$$\max_{1\leq j\leq n}\left\{\frac{v_p(u_{2n+2}) - v_p(u_{2n+2}/u_{2j+2})}{2j}\right\}.$$

For $1 \leq j \leq n$, we obtain

$$v_p(u_{2n+2}) - v_p(u_{2n+2}/u_{2j+2}) = v_p(u_{2j+2}) \leq v_p((2j+1)!).$$

If $p > 2j+1$, then $v_p((2j+1)!) = 0$. If $p \leq 2j+1$, then $k+1 \geq 2j$ and, from the fact that $v_p((2j+1)!) < (2j+1)/(p-1)$, we deduce that

$$v_p((2j+1)!) < \frac{2j+1}{p-1} \leq \frac{2j+1}{k+1} < \frac{2j}{k}.$$

It follows that the slope of the right-most edge of the ϕ-Newton polygon of $g_2(x)$ with respect to p is $< \frac{1}{k}$. Hence, using Proposition 2.2, we have a contradiction. Since by hypothesis, we have $2n+1 \neq 3^u$ for any integer $u \geq 2$, this contradiction completes the proof of Theorem 1.2. □

4. Proof of Theorem 1.4.

Proof of Theorem 1.4. To prove the result, it is sufficient to show that $H^{\phi}_{2n}(x)$ is irreducible for all integers $n \geq 2$ and $H^{\phi}_{2n+1}(x)$ is $\phi(x)$ times an irreducible polynomial for all non-negative integers n except in the case when $2n+1$ is of the form 3^u for some integer $u \geq 2$. Let $m = 2n$ or $2n+1$ according as m is even or odd. Let $b_i(x)$ with $0 \leq i \leq n-1$ belonging to $\mathbf{Z}[x]$ be polynomials having degree less than $\deg \phi(x)$. Let the content of $(a_n b_0(x))$ is not divisible by any prime less than or equal to m. Then, we define

$$f_0(x) = \frac{a_n}{u_{2n}}\phi(x)^{2n} + \sum_{j=0}^{n-1} b_j(x)\frac{\phi(x)^{2j}}{u_{2j}}, \text{ if } m = 2n,\, n \geq 2$$

and

$$f_2(x) = \frac{a_n}{u_{2n+2}}\phi(x)^{2n} + \sum_{j=0}^{n-1} b_j(x)\frac{\phi(x)^{2j}}{u_{2j+2}}, \text{ if } m = 2n+1 \text{ and } m \neq 3^u \text{ for } u \geq 2.$$

Using Theorem 1.2, we see that $f_0(x)$ and $f_2(x)$ are irreducible polynomials over $\mathbf{Q}$.

Observe that

$$u_{2j} = 1\cdot 3\cdots(2j-1) = \frac{(2j)!}{2\cdot 4\cdots 2j} = \frac{(2j)!}{2^j j!}.$$

Therefore we see that

$$\frac{H_{2n}^{\phi}(x)}{u_{2n}} = a_n \frac{\phi(x)^{2n}}{u_{2n}} + \sum_{j=0}^{n-1} \binom{n}{j} a_j(x) \frac{\phi(x)^{2j}}{u_{2j}},$$

and

$$\frac{H_{2n+1}^{\phi}(x)}{u_{2n+2}\phi(x)} = a_n \frac{\phi(x)^{2n}}{u_{2n+2}} + \sum_{j=0}^{n-1} \binom{n}{j} a_j(x) \frac{\phi(x)^{2j}}{u_{2j+2}}.$$

So, by taking $b_j(x) := \binom{n}{j} a_j(x)$ in $f_0(x)$ and $f_2(x)$, we have our result. □

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