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Duality for outer $L^{p,\infty}$ spaces

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ABSTRACT. The $L^{p,\infty}$ quasi-norm of functions on a measure space can be characterized in terms of their pairing with normalized characteristic functions. We generalize this result to the case of the outer $L^{p,\infty}$ quasi-norms for appropriate sizes. This characterization provides an explicit form for certain implicitly defined quasi-norms appearing in an article of Di Plinio and Fragkos.

CONTENTS

1. Introduction

The characterization of the classical $L^{p,\infty}$ quasi-norm (or weak L^p quasi-norm) of functions on a measure space (X, ω) in terms of their pairing with appropriately normalized characteristic functions is a standard result in analysis. On one hand, for every $p \in (1, \infty]$ it exhibits a Köthe duality between the $L^{p,\infty}$ spaces and the collection of characteristic functions endowed with the $L^{p',1}$ quasi-norm analogous to the Köthe duality between the L^p and $L^{p'}$ spaces. On the other hand, for every $p \in (0, 1]$ the pairing is more subtle.

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To state both of the characterizations, for every measure space (X, ω) we introduce the following auxiliary notations. First, we define Σ to be the collection of ω-measurable subsets of X and $\mathcal{M}$ to be the collection of Σ-measurable functions on X. Next, we define the collection $\Sigma_\omega \subseteq \Sigma$ by

$$\Sigma_\omega \coloneqq \Big\{A \in \Sigma :\ \omega(A) \notin \{0, \infty\}\Big\},$$

and for every measurable subset $A \in \Sigma_\omega$ we define the collection $\Sigma''_\omega(A) \subseteq \Sigma$ by

$$\Sigma''_\omega(A) \coloneqq \Big\{B \in \Sigma :\ B \subseteq A, \omega(B) \geq \frac{\omega(A)}{2}\Big\}.$$

We have the following characterization of the $L^{p,\infty}$ quasi-norms, where the notation $\alpha \sim_C \beta$ for $\alpha, \beta, C > 0$ means that $\alpha \leq C\beta$ and $\beta \leq C\alpha$.

Theorem 1.1 (e.g. Lemma 2.6 in [MS13])**.** *For every $p \in (0, \infty]$ there exists a constant $C = C(p)$ such that for every measure space (X, ω) the following properties hold true.*

(i) For every $p > 1$ and every function $f \in L^{p,\infty}(X, \omega)$ we have

$$\|f\|_{L^{p,\infty}(X,\omega)} \sim_C \sup\Big\{\omega(A)^{\frac{1}{p}-1}\|f 1_A\|_{L^1(X,\omega)} :\ A \in \Sigma_\omega\Big\}.$$

(ii) For every $p \leq 1$ and every function $f \in L^{p,\infty}(X, \omega)$ we have

$$\|f\|_{L^{p,\infty}(X,\omega)} \sim_C \sup\Big\{\inf\Big\{\omega(A)^{\frac{1}{p}-1}\|f 1_B\|_{L^1(X,\omega)} :\ B \in \Sigma''_\omega(A)\Big\} :\ A \in \Sigma_\omega\Big\}.$$

In fact, the restriction to major subsets in the pairing is necessary in property (ii), but it would still provide the desired characterization even in property (i).

In this article, we are interested in proving an analogous result in the case of the outer $L^{p,\infty}$ quasi-norm of functions on an outer measure space introduced by Do and Thiele in [DT15]. We refer to [DT15], the articles [Fr21, Fr23], and the Ph.D. thesis of the author [Fr22] for a detailed introduction to the theory of outer L^p spaces. However, to make this article self-contained, we recall the necessary definitions in Section 2.

We also briefly point out that the framework provided by the outer L^p spaces has found many applications within harmonic analysis, in particular in the context of the time-frequency analysis of modulation invariant operators such as the Carleson maximal operator and the bilinear Hilbert transform. We refer to [DT15], the articles of Di Plinio and Ou [DO18a, DO18b], of Culiuc, Di Plinio, and Ou [CDO18], of Di Plinio, Do, and Uraltsev [DDU18], of Amenta and Uraltsev [AU20, AU22], of Di Plinio and Fragkos [DF25], of Uraltsev and Warchalski [UW22], as well as the Ph.D. thesis of Uraltsev [Ur17] for more details.

In particular, in this article we are interested in the case of the outer $L^{p,\infty}$ quasi-norms on a *σ-finite setting* (X, μ, ω), where X is a set, μ a σ-finite *outer measure* on X, and ω a σ-finite measure on X. For example, for every σ-finite measure space (X, ω) we have the σ-finite setting (X, μ, ω) where μ is the outer measure on X generated via *minimal coverings* by the collection Σ of measurable subsets and the function $\omega\colon\ \Sigma\to[0,\infty]$. Another example is given by the set $X=\mathbb{R}^n\times(0,\infty)$, the measure $\mathrm{d}\omega(y,t)=\mathrm{d}y\,\mathrm{d}t/t$, and the outer measure μ on X generated via minimal coverings by the collection $\mathcal{E}$ of tents, namely

$$E(x,s)\coloneqq\Big\{(y,t)\in X\colon\ |x-y|<s-t,t<s\Big\},\qquad \mathcal{E}\coloneqq\Big\{E(x,s)\colon\ (x,s)\in X\Big\},$$

and the function $\sigma\colon\ \mathcal{E}\to(0,\infty)$ defined for every $(x,s)\in X$ by

$$\sigma(E(x,s))=s.$$

Next, let $\mathcal{A}\subseteq\Sigma$ be a collection of measurable subsets and let $S=(S,\mathcal{A})$ be a *size*. In particular, every size $(S,\mathcal{A})$ satisfies the following property.

(iv) There exists a constant $K\in[1,\infty)$ such that for every measurable function f on X, every subset $A\in\mathcal{A}$, and every measurable subset $B\in\Sigma$ we have

$$S(f)(A)\le K[S(f1_B)(A)+S(f1_{B^c})(A)]. \tag{1.1}$$

For example, we have the sizes ℓ^∞_ω and ℓ^r_ω with $r\in(0,\infty)$ on the collection $\mathcal{A}=\{A\in\Sigma\colon\ \mu(A)\notin\{0,\infty\}\}$ defined by

$$\begin{aligned}\ell^\infty_\omega(f)(A)&\coloneqq\|f1_A\|_{L^\infty(X,\omega)},\\ \ell^r_\omega(f)(A)&\coloneqq\mu(A)^{-\frac{1}{r}}\|f1_A\|_{L^r(X,\omega)}.\end{aligned} \tag{1.2}$$

It is easy to observe that the size $(\ell^1_\omega,\mathcal{A})$ satisfies the inequality in (1.1) with constant $K=1$. In fact, it satisfies the equality.

For every size $S=(S,\mathcal{A})$ we define the outer $L^\infty_\mu(S)$ spaces by the following quasi-norms on functions $f\in\mathcal{M}$

$$\|f\|_{L^\infty_\mu(S)}=\|f\|_{L^{\infty,\infty}_\mu(S)}\coloneqq\sup\Big\{S(f)(A)\colon\ A\in\mathcal{A}\Big\}.$$

Next, for every $\lambda\in(0,\infty)$ and every function $f\in\mathcal{M}$ we define

$$\mu(S(f)>\lambda)\coloneqq\inf\Big\{\mu(A)\colon\ A\in\Sigma,\|f1_{A^c}\|_{L^\infty_\mu(S)}\le\lambda\Big\}. \tag{1.3}$$

Finally, for all $p\in(0,\infty)$, $q\in(0,\infty]$ we define the outer Lorentz $L^{p,q}_\mu(S)$ spaces by the following quasi-norms on functions $f\in\mathcal{M}$

$$\|f\|_{L^{p,q}_\mu(S)}\coloneqq p^{\frac{1}{q}}\Big\|\lambda\mu(S(f)>\lambda)^{\frac{1}{p}}\Big\|_{L^q((0,\infty),\mathrm{d}\lambda/\lambda)},$$

where the exponent q^{-1} for $q=\infty$ is understood to be 0. In fact, we can define the same quasi-norms via the decreasing rearrangement function f^* on $[0,\infty)$ associated with f defined by

$$f^*(t)\coloneqq\inf\Big\{\alpha\in[0,\infty)\colon\ \mu(S(f)>\alpha)\le t\Big\},$$

where the infimum over an empty collection is understood to be ∞. Then, we have

$$\|f\|_{L^{p,q}_\mu(S)}=\Big\|t^{\frac{1}{p}}f^*(t)\Big\|_{L^q((0,\infty),\mathrm{d}t/t)}.$$

To state the main result of this article, for every σ-finite setting (X,μ,ω) we introduce the following auxiliary notations. We define the collection $\Sigma_\mu \subseteq \Sigma$ by

$$\Sigma_\mu := \Big\{A \in \Sigma :\ \mu(A) \notin \{0,\infty\}\Big\},$$

and for every $r \in (0,\infty)$ and every size $(S,\mathcal{A})$ we define the sizes $(S_r,\mathcal{A})$, $(S_\infty,\mathcal{A})$ as follows. For every $A \in \mathcal{A}$ and every measurable function f on X we define

$$S_r(f)(A) := (S(f^r)(A))^{\frac{1}{r}}, \tag{1.4}$$

$$S_\infty(f)(A) := \|f 1_A\|_{L^\infty(X,\omega)}. \tag{1.5}$$

It is easy to observe that for every $r \in (0,\infty]$ we have

$$(\ell^r_\omega, \Sigma_\mu) = ((\ell^1_\omega)_r, \Sigma_\mu).$$

Note that in general we have

$$S_\infty(f)(A) \neq \lim_{r\to\infty} S_r(f)(A),$$

for example for $X = \mathbb{R}$, ω the Lebesgue measure, $\mu \equiv 1$, $S = \ell^1_\omega$, $f = 1_\mathbb{R}$, and $A = \mathbb{R}$. In particular, we have $\omega(\mathbb{R}) = \infty$ and $\mu(\mathbb{R}) = 1$, yielding

$$S_\infty(f)(\mathbb{R}) = 1, \qquad \lim_{r\to\infty} S_r(f)(\mathbb{R}) = \infty.$$

We are ready to state the main theorem of this article, which is a generalization of property (i) in Theorem 1.1 to the case of the outer $L^{p,\infty}$ quasi-norms with respect to arbitrary sizes. To improve its readability, we omit explicit expressions for the constants. However, in the proofs we keep track of their dependence on the parameters.

Theorem 1.2. *For all $a,q,r \in (0,\infty]$, every $p \in (0,\infty)$, and every $K \in [1,\infty)$ there exists a constant $C = C(a,p,q,r,K)$ such that for every σ-finite setting (X,μ,ω) and every size $(S,\mathcal{A})$ on X satisfying the condition in* (1.1) *with constant K the following property holds true.*

(i) For every $p > a$, $p \neq \infty$ and every function $f \in L^{p,\infty}_\mu(S_r)$ we have

$$\|f\|_{L^{p,\infty}_\mu(S_r)} \sim_C \sup\Big\{\mu(A)^{\frac{1}{p}-\frac{1}{a}} \|f 1_A\|_{L^{a,q}_\mu(S_r)} :\ A \in \Sigma_\mu\Big\}.$$

We point out some comments about the statement of Theorem 1.2. First, for $p = \infty$, we exhibit a sufficient condition on the size to obtain a generalization of property (i) too, see Remark 4.3. In particular, the sizes appearing in the context of time-frequency analysis and considered in [DF25], as well as those of iterated nature considered in [DDU18, Ur17, UW22] satisfy this sufficient condition, see Section 5.

It is also worth noting that the arguments used to prove Theorem 1.2 are different from those appearing in [Fr21], where we characterized the outer $L^p_\mu(\ell^r_\omega)$ quasi-norm in terms of the outer $L^1_\mu(\ell^1_\omega)$ pairing with arbitrary functions with normalized outer $L^{p'}_\mu(\ell^{r'}_\omega)$ quasi-norm. The difference is due to the fact that in Theorem 1.2 we have the same size S both in the outer $L^{p,\infty}(S)$ quasi-norm to be characterized and the outer $L^{a,q}(S)$ quasi-norm used to measure the pairing. Therefore, we rely only on the quasi-subadditivity of sizes, namely property (iv) of sizes, to compare the super level measures

$$\mu(S(f) > \rho), \qquad \mu(S(f1_A) > \lambda),$$

where $A \in \Sigma$ is an appropriate measurable subset and $\rho, \lambda \in (0, \infty)$.

Next, Theorem 1.2 does not contain a generalization of property (ii) in Theorem 1.1 to the case of the outer $L^{p,\infty}$ quasi-norms with respect to arbitrary sizes. However, we are not able to show a counterexample to such a generalization either. In fact, even in the case of the sizes appearing in [DF25] and described in Section 5, we can neither prove a generalization nor exhibit a counterexample. It would be interesting to answer whether such a generalization holds true or not, but the question is beyond the scope of this article.

Nevertheless, in the particular case of the sizes defined in (1.2), we can generalize also property (ii) in Theorem 1.1 to the case of the outer $L^{p,\infty}$ quasi-norms. To state the second result of this article, for every σ-finite setting (X, μ, ω) we introduce the following auxiliary notation. For every measurable subset $A \in \Sigma_\mu$ we define the collection $\Sigma'_\mu(A) \subseteq \Sigma$ by

$$\Sigma'_\mu(A) \coloneqq \Big\{ B \in \Sigma \colon \, B \subseteq A, \mu(A \setminus B) \leq \frac{\mu(A)}{2} \Big\}.$$

Again, to improve the readability of the theorem, we omit explicit expressions for the constants but in the proofs we keep track of their dependence on the parameters.

Theorem 1.3. *For all $a, p, q, r \in (0, \infty]$ there exists a constant $C = C(a, p, q, r)$ such that for every σ-finite setting (X, μ, ω) the following properties hold true.*

(i) For every $p > a$ and every function $f \in L^{p,\infty}_\mu(\ell^r_\omega)$ we have

$$\|f\|_{L^{p,\infty}_\mu(\ell^r_\omega)} \sim_C \sup \Big\{ \mu(A)^{\frac{1}{p} - \frac{1}{a}} \|f1_A\|_{L^{a,q}_\mu(\ell^r_\omega)} \colon \, A \in \Sigma_\mu \Big\}.$$

(ii) For every $p \leq a$, $p \neq \infty$ and every function $f \in L^{p,\infty}_\mu(\ell^r_\omega)$ we have

$$\|f\|_{L^{p,\infty}_\mu(\ell^r_\omega)} \sim_C \sup \Big\{ \inf \Big\{ \mu(A)^{\frac{1}{p} - \frac{1}{a}} \|f1_B\|_{L^{a,q}_\mu(\ell^r_\omega)} \colon \, B \in \Sigma'_\mu(A) \Big\} \colon \, A \in \Sigma_\mu \Big\}.$$

As for Theorem 1.1, the restriction to major subsets in the pairing is necessary in property (ii), but it would still provide the desired characterization even in property (i), see Remark 4.4. In fact, given a σ-finite measure space (X, ω), let μ be the outer measure on X generated via minimal coverings by Σ and ω described above. Then Theorem 1.3 for the σ-finite setting (X, μ, ω), $a = q = 1$, and $r = \infty$ recovers Theorem 1.1 for (X, ω), since in this case we have $L^{p,q}_\mu(\ell^\infty_\omega) = L^{p,q}(X, \omega)$.

Moreover, we can compare the collections $\Sigma''_\omega(A)$ and $\Sigma'_\mu(A)$ appearing in the statement of property (ii) in Theorem 1.1 and Theorem 1.3 respectively. By the additivity on disjoint sets of the measure ω, in the case of the collection $\Sigma''_\omega(A)$ we have the equality

$$\Sigma''_\omega(A) = \Sigma'_\omega(A) \coloneqq \Big\{B \in \Sigma \colon B \subseteq A, \omega(A \setminus B) \le \frac{\omega(A)}{2}\Big\}.$$

However, in the case of the collection $\Sigma'_\mu(A)$ in general we only have the inclusion

$$\Sigma'_\mu(A) \subseteq \Sigma''_\mu(A) \coloneqq \Big\{B \in \Sigma \colon B \subseteq A, \mu(B) \ge \frac{\mu(A)}{2}\Big\}.$$

In fact, in the characterization of the outer $L^{p,\infty}$ quasi-norm stated in property (ii) in Theorem 1.3 we cannot replace $\Sigma'_\mu(A)$ with $\Sigma''_\mu(A)$. We clarify this claim in Remark 4.5.

Finally, the interest in the characterization of outer $L^{p,\infty}$ quasi-norms by pairing with normalized characteristic functions was sparked by the function spaces appearing in Subsection 3.8 in the article of Di Plinio and Fragkos [DF25]. The latter ones, denoted by $X^{p,q}_a(J, \varkappa, \mathsf{size}_{2,\star})$, are quasi-normed spaces of functions on the collection $\mathcal{H}$ of dyadic Heisenberg tiles in $\mathbb{R}^2$. In particular, the set $\mathcal{H}$ is the collection of dyadic rectangles of area 1 in $\mathbb{R}^2$. The spaces introduced in [DF25] are associated with new implicitly defined quasi-norms. More specifically, the new quasi-norm of a function f is defined in terms of the pairing of f with appropriately normalized characteristic functions. In particular, both the pairing and the renormalization of the characteristic functions are measured in terms of outer Lorentz $L^{p,q}$ spaces with respect to an appropriate size. As a corollary of Theorem 1.2, we show that the $X^{p,q}_a(J, \varkappa, \mathsf{size}_{2,\star})$ quasi-norms are equivalent to explicit outer $L^{p,\infty}$ ones with respect to an appropriate size, see Corollary 5.2. We provide all the definitions to state this equivalence in Section 5. Here we briefly point out that the sizes $(S^r, \mathcal{A})$ with $r \in [1, \infty)$ appearing in [DF25] differ from those defined in (1.2) in the following way, namely for every subset $A \in \mathcal{A}$ we have

$$S^r(f)(A) \coloneqq \sup\Big\{\ell^r_\omega(f)(\widetilde{A}) \colon \widetilde{A} \in \widetilde{\mathcal{A}}\Big\},$$

where $\widetilde{\mathcal{A}}$ is an appropriate collection of strict subsets of A. The size $(S^1, \mathcal{A})$ satisfies the inequality in (1.1) with constant $K = 1$, but it does not satisfy the equality. Because of this difference, we cannot use decomposition arguments akin to those for the size ℓ^1_ω appearing in [Fr21]. In fact, the question of Köthe duality for the outer L^p spaces with respect to these sizes remains open and, in the case of the sizes appearing in [DF25], it will be addressed in future work.

We briefly comment on the use of the spaces $X^{p,q}_a(J, \varkappa, \mathsf{size}_{2,\star})$ in [DF25]. In [DF25] the main focus of the authors is about improving the L^p estimates for the Carleson maximal operator $\mathcal{C}$ defined for every Schwartz function $f \in \mathcal{S}(\mathbb{R})$ and every $x \in \mathbb{R}$ by

$$\mathcal{C}f(x) \coloneqq \sup\Big\{\Big|\int_{\xi \le N} \widehat{f}(\xi) e^{ix\xi}\,\mathrm{d}\xi\Big| \colon N \in \mathbb{R}\Big\}.$$

In particular, Di Plinio and Fragkos are interested in the case when the exponent is close to the endpoint case $p = 1$. The previously known best estimate was a restricted weak-type L^p one of the following form. For every $p \in (1, \infty)$, every subset $|F| \subseteq \mathbb{R}$, and every measurable function f such that $|f| \leq 1_F$ we have

$$\|\mathcal{C}f\|_{L^{p,\infty}(\mathbb{R})} \leq \frac{Cp^2}{p-1}|F|^{\frac{1}{p}},$$

where the constant C is independent of p. This bound was due to the work of Hunt [Hu68] building on the seminal article of Carleson [Ca66]. An equivalent estimate was later proved by Lacey and Thiele in [LT00] in the language of generalized restricted weak-type L^p bounds.

The improvement obtained in [DF25] is a proof of a weak-type L^p estimate of the following form. For every $p \in (1, 2]$ and every Schwartz function $f \in \mathcal{S}(\mathbb{R})$ we have

$$\|\mathcal{C}f\|_{L^{p,\infty}(\mathbb{R})} \leq \frac{C}{p-1}\|f\|_{L^p(\mathbb{R})},$$

where the constant C is independent of p. This result (Theorem A in [DF25]) follows from a sparse norm bound for the Carleson maximal operator (Theorem B in [DF25]). In fact, Di Plinio and Fragkos prove a sparse norm bound for the Carleson maximal operator associated with Hörmander-Mihlin multipliers. We refer to [DF25] for both the definition of a sparse norm bound and the description of the family of weighted estimates for the operator implied by the sparse norm bound.

The proof of the sparse norm bound follows from a standard stopping time argument and two inequalities involving the spaces $X_a^{p,q}(J, \varkappa, \mathsf{size}_{2,\star})$. The first inequality is a version of Hölder's inequality for outer L^p spaces, the second is an estimate for the Carleson embedding map from classical to outer L^p spaces. The role played by the two inequalities is analogous to the two-step programme outlined in [DT15].

More in details, the result concerning the first inequality (Proposition 3.9 in [DF25]) states that for all $a, p_1 \in (1, \infty)$, $a \leq p_1$ and $p_2, \dots, p_m \in [1, \infty)$ such that

$$\varepsilon := \Big(\sum_{j=1}^{m} \frac{1}{p_j}\Big) - 1 > 0,$$

we have

$$\Big\|\prod_{j=1}^{m} F_j\Big\|_{L^1(E(J),\omega_\varkappa)} \leq \frac{Ca}{\varepsilon(a-1)}\|F_1\|_{X_a^{p_1,\infty}(J,\varkappa,\mathsf{size}_{2,\star})}\times \\ \times \prod_{j=2}^{m} \max\{\|F\|_{L_{\mu_\varkappa}^{p_j,\infty}(s_j)}, \|F\|_{L_{\mu_\varkappa}^{\infty}(s_j)}\}, \tag{1.6}$$

where the sizes $\mathsf{size}_{2,\star}, s_2, \dots, s_m$ satisfy an appropriate condition. We provide all the necessary definitions in Section 5. In view of the equivalence between $X_a^{p,q}$ quasi-norms and explicit outer $L^{p,\infty}$ ones stated in Corollary 5.2, we obtain

Proposition 3.9 in [DF25] as corollary of the outer Hölder's inequality for outer Lorentz $L^{p,q}$ spaces, see Appendix A.

The second inequality is the boundedness of the Carleson embedding map W measured in the $X_2^{tp',\infty}(J,\kappa,\mathsf{size}_{2,\star})$ spaces with a constant independent of $p \in (1,2]$ but dependent of $t \in (1,\infty)$ (Theorem D in [DF25]). For every Schwartz function $f \in \mathcal{S}(\mathbb{R})$ the embedded function $W[f]\colon \mathcal{H} \to [0,\infty)$ is defined on every $H = I_H \times J_H \in \mathcal{H}$ by

$$W[f](H) \coloneqq \sup\Big\{|\langle f,\phi\rangle| \colon \phi \in \Phi(H)\Big\},$$

where $\Phi(H)$ is an appropriate collection of L^1-normalized smooth enough functions adapted to I_H and such that $\operatorname{supp}\widehat{\phi} \subseteq J_H$. Then, for all $t \in (1,\infty)$, $p \in (1,2]$, every dyadic interval J, every collection $\mathcal{P} \subseteq \mathcal{H}$ of dyadic Heisenberg tiles, and every Schwartz function $f \in \mathcal{S}(\mathbb{R})$ we have

$$\|W[f]1_{\mathcal{P}}\|_{X_2^{tp',\infty}(J,\kappa,\mathsf{size}_{2,\star})} \leq C(t,\kappa)[f]_{p,\mathcal{P}}. \tag{1.7}$$

In the previous display, the quasi-norm $[f]_{p,\mathcal{P}}$ measures the minimal value on I_H for every $H \in \mathcal{P}$ of the maximal L^p average of f. In view of the equivalence stated in Corollary 5.2 the estimate in (1.7) is a weak version of the boundedness of the Carleson embedding map W, namely

$$\|W[f]1_{\mathcal{P}}\|_{L^{tp',\infty}_{\mu_\kappa}(s_\kappa)} \leq C(t,\kappa)[f]_{p,\mathcal{P}},$$

for an appropriate size s_κ. It should be compared with two previously known results. First, the weak-type estimate for the Carleson embedding map W proved by Do and Thiele in [DT15], namely

$$\|W[f]1_{\mathcal{P}}\|_{L^{2,\infty}_{\mu_\kappa}(s_\kappa)} \leq C(\kappa)[f]_{2,\mathcal{P}}.$$

Next, the strong-type estimate for the Carleson embedding map W applied to the localization of the function f proved by Di Plinio and Ou in [DO18a], namely

$$\|W[f1_{3J}]1_{\mathcal{P}}\|_{L^{tp'}_{\mu_\kappa}(s_\kappa)} \leq C(t,\kappa,p)[f]_{p,\mathcal{P}}.$$

We conclude the Introduction with two observations. First, spaces similar to those in [DF25] can be defined in the setting (X,μ,ω) with $X = \mathbb{R}^n \times (0,\infty)$ and the outer measure μ on X generated via minimal coverings by tents described above. This σ-finite setting is associated with the Calderón–Zygmund theory of translation and dilation invariant operators, for which endpoint estimates are well understood. In fact, the sizes used in this σ-finite setting are of the form defined in (1.2), hence Theorem 1.3 holds true for these outer $L^{p,\infty}$ spaces. It would be interesting to understand whether arguments analogous to those in [DF25] have useful applications also in the context of Calderón–Zygmund theory, but this question is beyond the scope of this article.

Next, spaces similar to those in [DF25], namely defined via weak-type testing conditions, have appeared crucially also in [UW22], see the definition of the local sizes $X^{q,r,+}_{\mu^1_\Theta,\nu_\beta}\mathbb{S}$ in Subsection 3.4. As in [DF25], the introduction of such weaker sizes allows to prove more flexible versions of the outer Hölder's inequality, as well as estimates for the embedding maps similar to that in (1.7). Again, it would be interesting to understand whether these local sizes fit in the results of this article and, if so, whether this provides any simplification or insight in the arguments in [UW22]. However, also this question is beyond the scope of this article.

Guide to the article. In Section 2 we recall the definitions for the objects listed in the Introduction, we introduce additional ones, and we make some auxiliary observations. In Section 3 we prove Theorem 1.2. After that, in Section 4 we prove the remaining properties stated in Theorem 1.3. Finally, in Section 5 we introduce the setting on the collection $\mathcal{H}$ of dyadic Heisenberg tiles in $\mathbb{R}^2$ and all the relevant notions. Moreover, we state and prove the result about the spaces introduced by Di Plinio and Fragkos in Corollary 5.2. In view of this result, we detail the alternative proof of Proposition 3.9 in [DF25] in Appendix A. In Appendix B we provide the proofs of the auxiliary results stated in Section 4.

2. Preliminaries

2.1. Auxiliary definitions. First, we recall that an *outer measure* μ on a set X is a function from $\mathcal{P}(X)$, the collection of subsets of X, to $[0,\infty]$ satisfying the following properties:

(i) $\mu(\emptyset) = 0$.
(ii) For all subsets $A, B \subseteq X$ such that $A \subseteq B$ we have $\mu(A) \le \mu(B)$.
(iii) For every countable collection $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{P}(X)$ we have

$$\mu\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) \le \sum_{n\in\mathbb{N}} \mu(A_n).$$

A standard way to generate outer measures is via *minimal coverings* as follows. Let $\mathcal{E} \subseteq \Sigma$ be a collection of measurable sets and $\sigma \colon \mathcal{E} \to [0,\infty]$ be a function. For every subset $A \subseteq X$ we define $\mu(A) \in [0,\infty]$ by

$$\mu(A) := \inf\Big\{ \sum_{B\in\mathcal{F}} \omega(B) \colon \mathcal{F} \subseteq \mathcal{E}, A \subseteq \bigcup_{B\in\mathcal{F}} B \Big\},$$

where the infimum over an empty collection is understood to be 0.

Then, we define a *σ-finite setting* (X, μ, ω) as follows. Let X be a set, let ω be a σ-finite measure on X, and let μ be a σ-finite outer measure on X. Namely, we assume there exist two countable collections $\{A_n : n \in \mathbb{N}\}, \{B_n : n \in \mathbb{N}\} \subseteq \mathcal{P}(X)$ such that

$$X = \bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} B_n,$$
$$\forall n \in \mathbb{N}, \qquad \omega(A_n) < \infty, \qquad \mu(B_n) < \infty.$$

We also assume that for every subset $A \subseteq X$ we have

$$\mu(A) = 0 \Rightarrow \omega(A) = 0.$$

Next, let $\mathcal{A} \subseteq \Sigma$ be a collection of measurable subsets. We recall that a *size* $S = (S, \mathcal{A})$ on the collection $\mathcal{M}$ of Σ-measurable functions on X is a function from $\mathcal{M} \times \mathcal{A}$ to $[0, \infty]$ satisfying the following properties:

(i) For every $\lambda \in \mathbb{R}$, every function $f \in \mathcal{M}$, and every subset $A \in \mathcal{A}$ we have

$$S(\lambda f)(A) = |\lambda| S(f)(A).$$

(ii) For all functions $f, g \in \mathcal{M}$ such that $|f| \leq |g|$ ω-almost everywhere and every subset $A \in \mathcal{A}$ we have

$$S(f)(A) \leq S(g)(A).$$

(iii) There exists a constant $C \in [1, \infty)$ such that for all functions $f, g \in \mathcal{M}$ and every subset $A \in \mathcal{A}$ we have

$$S(f + g)(A) \leq C[S(f)(A) + S(g)(A)].$$

In particular, for every size $(S, \mathcal{A})$ the constant K appearing in property (iv) stated in the Introduction is smaller than or equal to the constant C appearing in property (iii).

Finally, let (X, μ, ω) be a σ-finite setting. For every measurable subset $A \in \Sigma$ we define the collection $\Sigma_\mu(A) \subseteq \Sigma$ by

$$\Sigma_\mu(A) \coloneqq \Big\{ B \in \Sigma : \ B \subseteq A, \mu(B) \notin \{0, \infty\} \Big\},$$

the collection $\Sigma^\circ_\omega(A) \subseteq \Sigma$ by

$$\Sigma^\circ_\omega(A) \coloneqq \Big\{ B \in \Sigma : \ B \subseteq A, \omega(A \setminus B) = 0 \Big\},$$

and the value $\mu^\circ(A) \in [0, \infty]$ by

$$\mu^\circ(A) \coloneqq \inf \Big\{ \mu(B) : \ B \in \Sigma^\circ_\omega(A) \Big\}.$$

2.2. Auxiliary observations. Let (X, μ, ω) be a σ-finite setting. First, we observe that for all $p \in (0, \infty)$, $q \in (0, \infty]$ and every measurable subset $A \in \Sigma$ we have

$$\|1_A\|_{L^{p,q}_\mu(\ell^\infty_\omega)} = p^{\frac{1}{q}} \Big\| \lambda \mu(\ell^\infty_\omega(1_A) > \lambda)^{\frac{1}{p}} \Big\|_{L^q((0,\infty), \mathrm{d}\lambda/\lambda)} = \Big(\frac{p}{q}\Big)^{\frac{1}{q}} \mu^\circ(A)^{\frac{1}{p}}, \tag{2.1}$$

where $(p/q)^{\frac{1}{q}}$ for $q = \infty$ is understood to be 1.

Next, we state and prove the following auxiliary result.

Lemma 2.1. *Let (X, μ, ω) be a σ-finite setting, and let $(S, \mathcal{A})$ be a size on X satisfying the condition in* (1.1) *with constant $K \in [1, \infty)$.*

For every $r \in (0, \infty)$, every $\rho \in (0, \infty)$, every $\delta \in [0, 1]$, every measurable function f on X, and every measurable subset $\Omega \in \Sigma$ such that

$$\|f 1_{\Omega^c}\|_{L^\infty_\mu(S_r)} \leq K^{-\frac{1}{r}} (1 - \delta^r)^{\frac{1}{r}} \rho, \tag{2.2}$$

we have

$$\mu(S_r(f) > \rho) \leq \mu\Big(S_r(f 1_\Omega) > K^{-\frac{1}{r}} \delta \rho \Big).$$

Proof. We argue by contradiction and we assume that

$$\mu(S_r(f) > \rho) > \mu\Big(S_r(f 1_{\Omega}) > K^{-\frac{1}{r}} \delta\rho\Big).$$

In particular, by the definition in (1.3), there exists a measurable subset $\Theta \in \Sigma$, $\Theta \subseteq \Omega$ such that

$$\|f 1_{\Omega} 1_{\Theta^c}\|_{L^\infty_\mu(S_r)} \leq K^{-\frac{1}{r}} \delta\rho, \qquad \mu(\Theta) < \mu(S_r(f) > \rho).$$

Therefore, there exists a subset $A \in \mathcal{A}$ such that

$$\begin{aligned} S(f^r 1_{\Omega} 1_{\Theta^c})(A) &= S_r(f 1_{\Omega} 1_{\Theta^c})(A)^r \leq K^{-1}\delta^r \rho^r, \\ S(f^r 1_{\Theta^c})(A) &= S_r(f 1_{\Theta^c})(A)^r > \rho^r. \end{aligned}$$

By the inequality in (1.1), we have

$$S_r(f 1_{\Omega^c})(A)^r = S(f^r 1_{\Omega^c})(A) > K^{-1}(1-\delta^r)\rho^r.$$

Together with the condition in (2.2), the inequality in the previous display yields a contradiction. □

3. Proof of Theorem 1.2

We prove Theorem 1.2 under the assumption $K = 1$. For arbitrary $K \in [1,\infty)$ we apply the same arguments only obtaining a different constant C.

Moreover, we make some additional assumptions. At the end of the proof we comment on the modifications needed to drop them.

Let $p \in (0,\infty)$, $f \in L^{p,\infty}_\mu(S_r)$, $\lambda \in (0,\infty)$, and $A \in \Sigma$.

- We assume that there exists $\rho \in (0,\infty)$ such that
$$\|f\|_{L^{p,\infty}_\mu(S_r)} = \rho\mu(S_r(f) > \rho)^{\frac{1}{p}} \in (0,\infty). \tag{3.1}$$
- We assume that there exists a measurable subset $\Omega_\lambda \in \Sigma$ such that
$$\|f 1_{\Omega_\lambda^c}\|_{L^\infty_\mu(S_r)} \leq \lambda, \qquad \mu(\Omega_\lambda) = \mu(S_r(f) > \lambda) \in [0,\infty). \tag{3.2}$$
- We assume that there exists a measurable subset $B \in \Sigma^\circ_\omega(A)$ such that
$$\mu(B) = \mu^\circ(A) = \inf\Big\{\mu(D) : \, D \in \Sigma^\circ_\omega(A)\Big\}. \tag{3.3}$$
- We assume that $a = 1$ and we define $p' \in (-\infty,0) \cup [1,\infty]$ by
$$1 = \frac{1}{p} + \frac{1}{p'}.$$
Therefore, for $p \in [1,\infty]$ we have $p' \in [1,\infty]$ and for $p \in (0,1)$ we have $p' \in (-\infty,0)$.

Proof of Theorem 1.2. Without loss of generality we assume $f \not\equiv 0$, otherwise the property is trivially satisfied.

For every $p \in (1,\infty]$ we have $p' \neq \infty$. Therefore, for every $q \in (0,\infty]$ the inequality

$$\sup\Big\{\mu(A)^{-\frac{1}{p'}}\|f1_A\|_{L^{1,q}_\mu(S_r)} \colon A \in \Sigma_\mu\Big\} \leq 2\Big(\frac{p'}{q}\Big)^{\frac{1}{q}}\|f\|_{L^{p,\infty}_\mu(S_r)},$$

where $(p'/q)^{\frac{1}{q}}$ for $q = \infty$ is understood to be 1, follows from outer Hölder's inequality for outer Lorentz $L^{p,q}$ spaces (Theorem A.2) and the equality in (2.1).

We turn to the proof of the inequality

$$\|f\|_{L^{p,\infty}_\mu(S_r)} \leq C\sup\Big\{\mu(A)^{-\frac{1}{p'}}\|f1_A\|_{L^{1,q}_\mu(S_r)} \colon A \in \Sigma_\mu\Big\}.$$

We distinguish two cases.

Case I: $r = \infty$**.** Without loss of generality we assume

$$\Omega_\rho \subseteq \Big\{x \in X \colon |f(x)| > \rho\Big\}.$$

By the inclusion in the previous display and the equality in (2.1), for every $q \in (0,\infty]$ we have

$$\|f1_{\Omega_\rho}\|_{L^{1,q}_\mu(\ell^\infty_\omega)} \geq \|\rho 1_{\Omega_\rho}\|_{L^{1,q}_\mu(\ell^\infty_\omega)} = q^{-\frac{1}{q}}\rho\mu(\Omega_\rho),$$

where $q^{\frac{1}{q}}$ for $q = \infty$ is understood to be 1. Together with the assumptions in (3.1)–(3.3), the previous inequality yields the desired inequality.

Case II: $r \neq \infty$**.** By Lemma 2.1, for every $\delta \in [0,1]$ we have

$$\mu(S_r(f) > \rho) \leq \mu\Big(S_r(f1_{\Omega_{(1-\delta^r)^{1/r}\rho}}) > \delta\rho\Big).$$

By the assumption on ρ in (3.1), we have

$$\mu(\Omega_{(1-\delta^r)^{1/r}\rho}) = \mu\Big(S_r(f) > (1-\delta^r)^{\frac{1}{r}}\rho\Big) \leq (1-\delta^r)^{-\frac{p}{r}}\mu(S_r(f) > \rho).$$

By the inequalities in the previous two displays, for every $q \in (0,\infty]$ we have

$$\|f1_{\Omega_{(1-\delta^r)^{1/r}\rho}}\|_{L^{1,q}_\mu(S_r)} \geq q^{-\frac{1}{q}}\delta(1-\delta^r)^{\frac{p}{rp'}}\mu(\Omega_{(1-\delta^r)^{1/r}\rho})^{\frac{1}{p'}}\rho\mu(S_r(f) > \rho)^{\frac{1}{p}},$$

where $q^{\frac{1}{q}}$ for $q = \infty$ is understood to be 1. Together with the assumptions in (3.1)–(3.3), the previous inequality yields the desired inequality for every $q \in (0,\infty]$

$$\|f\|_{L^{p,\infty}_\mu(S_r)} \leq C\sup\Big\{\mu(A)^{-\frac{1}{p'}}\|f1_A\|_{L^{1,q}_\mu(S_r)} \colon A \in \Sigma_\mu\Big\},$$

where the minimal value of the constant C is defined by

$$q^{\frac{1}{q}}\inf\Big\{\delta^{-1}(1-\delta^r)^{-\frac{p}{rp'}} \colon \delta \in [0,1]\Big\} = q^{\frac{1}{q}}p^{\frac{p}{r}}(p-1)^{\frac{1-p}{r}},$$

where $q^{\frac{1}{q}}$ for $q = \infty$ is understood to be 1. □

We comment on the modifications needed to drop the additional assumptions.

- For every function $f \in L^{p,\infty}_\mu(S_r)$ and every $\delta > 0$ there exists $\rho_\delta \in (0,\infty)$ such that
$$\rho_\delta \mu(S_r(f) > \rho_\delta)^{\frac{1}{p}} \le \|f\|_{L^{p,\infty}_\mu(S_r)} \le (1+\delta)\rho_\delta \mu(S_r(f) > \rho_\delta)^{\frac{1}{p}}.$$
- For every function $f \in L^{p,\infty}_\mu(S_r)$, every $\lambda \in (0,\infty)$, and every $\delta > 0$ there exists a measurable subset $\Omega_{\lambda,\delta} \in \Sigma$ such that
$$\|f 1_{\Omega^c_{\lambda,\delta}}\|_{L^\infty_\mu(S_r)} \le \lambda, \qquad \mu(\Omega_{\lambda,\delta}) \le (1+\delta)\mu(S_r(f) > \lambda).$$
- For every measurable subset $A \in \Sigma$ and every $\delta > 0$ there exists a measurable subset $B_\delta \in \Sigma^\circ_\omega(A)$ such that
$$\mu(B_\delta) \le (1+\delta)\mu^\circ(A).$$
In particular, for every measurable subset $A \in \Sigma$ we have
$$\mu(A) \ge \mu^\circ(A) = \|1_A\|_{L^{1,q}_\mu(\ell^\infty_\omega)}.$$
- For all $a, p, q, r \in (0,\infty]$, every setting (X,μ,ω), and every function $f \in L^{p,q}_\mu(S_r)$ we have
$$\|f\|_{L^{p,q}_\mu(S_r)} = \|f^a\|^{\frac{1}{a}}_{L^{p/a,q/a}_\mu(S_{r/a})}.$$
In particular, for every constant $C = C\Big(1, \frac{p}{a}, \frac{q}{a}, \frac{r}{a}\Big)$ such that
$$\|f\|_{L^{p/a,\infty}_\mu(S_{r/a})} \le C \sup\Big\{\mu(A)^{\frac{a}{p}-1}\|f1_A\|_{L^{1,q/a}_\mu(S_{r/a})} \colon A \in \Sigma_\mu\Big\},$$
we have
$$\|f\|_{L^{p,\infty}_\mu(S_r)} \le C^{\frac{1}{a}} \sup\Big\{\mu(A)^{\frac{1}{p}-\frac{1}{a}}\|f1_A\|_{L^{a,q}_\mu(S_r)} \colon A \in \Sigma_\mu\Big\}, \tag{3.4}$$
and analogously for an inequality in the opposite direction.

For every fixed $\delta > 0$ we can apply the arguments we described in the previous proofs. Taking $\delta > 0$ arbitrarily small we obtain a proof of Theorem 1.2 with the appropriate constants.

Remark 3.1. *For $r \neq \infty$, instead of assuming $a = 1$, we can assume $r = 1$. The same argument in the proof above yields the desired inequality for every $q \in (0,\infty]$*
$$\|f\|_{L^{p,\infty}_\mu(S)} \le \widetilde{C} \sup\Big\{\mu(A)^{\frac{1}{p}-\frac{1}{a}}\|f1_A\|_{L^{a,q}_\mu(S)} \colon A \in \Sigma_\mu\Big\},$$
where the minimal value of the constant $\widetilde{C}$ is defined by
$$\Big(\frac{q}{a}\Big)^{\frac{1}{q}} \inf\Big\{\delta^{-1}(1-\delta)^{-p(\frac{1}{a}-\frac{1}{p})} \colon \delta \in [0,1]\Big\} = \Big(\frac{q}{a}\Big)^{\frac{1}{q}} \frac{p}{a}\Big(\frac{p-a}{p}\Big)^{-p(\frac{1}{a}-\frac{1}{p})},$$

where $(\frac{q}{a})^{\frac{1}{q}}$ *for* $q=\infty$ *and* $a\neq\infty$ *is understood to be* 1. *Analogously to the last point of the list above, we obtain the inequality in* (3.4) *up to the constant*

$$\widetilde{C}\Big(\frac{a}{r},\frac{p}{r},\frac{q}{r},1\Big)^{\frac{1}{r}}=\Big(\frac{q}{a}\Big)^{\frac{1}{q}}\Big(\frac{p}{a}\Big)^{\frac{1}{r}}\Big(\frac{p-a}{p}\Big)^{-\frac{p}{r}(\frac{1}{a}-\frac{1}{p})}=C\Big(1,\frac{p}{a},\frac{q}{a},\frac{r}{a}\Big)^{\frac{1}{a}}.$$

4. Proof of Theorem 1.3

We prove Theorem 1.3 under the additional assumptions listed at the beginning of the previous section. Moreover, we make some other additional assumptions.

Let $g\in L^\infty_\mu(\ell^r_\omega)$, $p\in(0,\infty)$, $f\in L^{p,\infty}_\mu(\ell^r_\omega)$, $\lambda\in(0,\infty)$, and $B\in\Sigma$.

- We assume that there exists $A_\rho\in\mathcal{A}\subseteq\Sigma_\mu$ such that

$$\|g\|_{L^\infty_\mu(\ell^r_\omega)}=\ell^r_\omega(g)(A_\rho)=\rho\in(0,\infty). \tag{4.1}$$

- We assume that there exists a measurable subset $\Omega_\lambda(B)\in\Sigma$, $\Omega_\lambda(B)\subseteq B$ such that

$$\|f1_B1_{\Omega_\lambda(B)^c}\|_{L^\infty_\mu(\ell^r_\omega)}\leq\lambda,\qquad \mu(\Omega_\lambda(B))=\mu(\ell^r_\omega(f1_B)>\lambda)\in[0,\infty). \tag{4.2}$$

In fact, the second additional assumption in this list is analogous to the second one listed at the beginning of the previous section. At the end of the proof we comment on the modifications needed to drop the first.

Next, we state the following auxiliary results. We postpone their proofs to Appendix B.

Lemma 4.1. *Let* (X,μ,ω) *be a* σ*-finite setting. Let* $r\in(0,\infty)$. *Let* $f\in L^\infty_\mu(\ell^r_\omega)$. *Let* $A\in\Sigma_\mu$ *be a measurable subset such that*

$$\ell^r_\omega(f)(A)\geq\rho\in(0,\infty). \tag{4.3}$$

Then for every $\lambda\in(0,\rho)$ *we have*

$$\mu(A)\leq\frac{\|f\|^r_{L^\infty_\mu(\ell^r_\omega)}}{\rho^r-\lambda^r}\mu(\ell^r_\omega(f)>\lambda).$$

Lemma 4.2. *Let* (X,μ,ω) *be a* σ*-finite setting. Let* $p,r\in(0,\infty]$. *Let* $f\in L^\infty_\mu(\ell^r_\omega)\cap L^{p,\infty}_\mu(\ell^r_\omega)$. *For every* $\lambda\in(0,\infty)$ *such that*

$$\|f\|_{L^\infty_\mu(\ell^r_\omega)}>\lambda, \tag{4.4}$$

there exists a measurable subset $B\in\Sigma$ *such that*

$$\ell^r_\omega(f)(B)>\lambda,\qquad \|f1_{B^c}\|_{L^\infty_\mu(\ell^r_\omega)}\leq\lambda. \tag{4.5}$$

Proof of property (i) **in Theorem 1.3 for** $p=\infty$**.** Without loss of generality we assume $f\not\equiv0$, otherwise the property is trivially satisfied.

We turn to the proof of the remaining inequality

$$\|f\|_{L^\infty_\mu(\ell^r_\omega)}\leq C\sup\Big\{\mu(A)^{-1}\|f1_A\|_{L^{1,q}_\mu(\ell^r_\omega)}\colon\ A\in\Sigma_\mu\Big\}.$$

We distinguish two cases.

Case I: $r = \infty$**.** See **Case I** in the proof of Theorem 1.2.

Case II: $r \neq \infty$**.** By Lemma 4.1, for every $k \in (0,1)$ we have

$$\mu(A_\rho) \leq (1-k^r)^{-1}\mu(\Omega_{k\rho}(A_\rho)). \tag{4.6}$$

By the previous inequality, for every $q \in (0,\infty]$ and every $k \in (0,1)$ we have

$$\|f1_{A_\rho}\|_{L^{1,q}_\mu(\ell^r_\omega)} \geq q^{-\frac{1}{q}}(k-k^{r+1})\rho\mu(A_\rho).$$

where $q^{\frac{1}{q}}$ for $q = \infty$ is understood to be 1. Together with the assumptions in (3.1)–(3.3) and in (4.1)–(4.2), the previous inequality yields the desired inequality for every $q \in (0,\infty]$

$$\|f\|_{L^\infty_\mu(\ell^r_\omega)} \leq C \sup\Big\{\mu(A)^{-1}\|f1_A\|_{L^{1,q}_\mu(\ell^r_\omega)} \colon A \subseteq X, A \neq \varnothing\Big\},$$

where the minimal value of the constant C is defined by

$$q^{\frac{1}{q}} \inf\Big\{(k-k^{r+1})^{-1} \colon k \in (0,1)\Big\} = q^{\frac{1}{q}}(r+1)^{\frac{r+1}{r}} r^{-1},$$

where $q^{\frac{1}{q}}$ for $q = \infty$ is understood to be 1. □

Remark 4.3. *A sufficient condition on the size* $(S,\mathcal{A})$ *to generalize both Lemma* 4.1 *and property* (i) *in Theorem* 1.3 *for* $p = \infty$ *to the case of the sizes* $(S_r,\mathcal{A})$ *is the following.*

There exists a constant $\widetilde{K} \in [1,\infty)$ *such that for every measurable function* f *on* X*, every subset* $A \in \mathcal{A}$*, and every measurable subset* $B \in \Sigma$ *we have*

$$S(f)(A)\mu(A) \leq \widetilde{K}[\|f\|_{L^\infty_\mu(S)}\mu(B) + S(f1_{B^c})(A)\mu(A)]. \tag{4.7}$$

The conclusion of the generalized version of Lemma 4.1 *for a size* $(S_r,\mathcal{A})$ *where* $(S,\mathcal{A})$ *satisfies the inequality in* (4.7) *would be the following.*

For every $\lambda \in (0,\rho/\widetilde{K}^{1/r})$ *we have*

$$\mu(A) \leq \widetilde{K}\frac{\|f\|^r_{L^\infty_\mu(S_r)}}{\rho^r - \widetilde{K}\lambda^r}\mu(S_r(f) > \lambda).$$

The minimal value of the constant C *in the proof of property* (i) *in Theorem* 1.3 *for* $p = \infty$ *would change accordingly.*

It is easy to observe that the size $(\ell^1_\omega,\Sigma_\mu)$ *satisfies the inequality in* (4.7) *with constant* $\widetilde{K} = 1$.

Proof of property (ii) **in Theorem 1.3.** Without loss of generality we assume $f \not\equiv 0$, otherwise the property is trivially satisfied.

Let K' be defined by

$$K' \coloneqq \sup\Big\{\inf\Big\{\mu(A)^{-\frac{1}{p'}}\|f1_B\|_{L^{1,q}_\mu(\ell^r_\omega)} \colon B \in \Sigma'_\mu(A)\Big\} \colon A \in \Sigma_\mu\Big\}.$$

Case I: $r = \infty$**.** We start with the proof of the inequality

$$K' \leq C\|f\|_{L^{p,\infty}_\mu(\ell^\infty_\omega)}.$$

For every measurable subset $A \in \Sigma_\mu$ we define $\lambda_0 \in (0,\infty)$ by

$$\lambda_0 := 2^{\frac{1}{p}} \mu(A)^{-\frac{1}{p}} \|f\|_{L^{p,\infty}_\mu(\ell^\infty_\omega)}.$$

By the definition of the outer $L^{p,\infty}_\mu(\ell^\infty_\omega)$ quasi-norm we have

$$\mu(\Omega_{\lambda_0}) \leq \lambda_0^{-p} \|f\|^p_{L^{p,\infty}_\mu(\ell^\infty_\omega)} \leq \frac{\mu(A)}{2},$$

hence we define the measurable subset $B \in \Sigma'_\mu(A)$ by

$$B := A \setminus \Omega_{\lambda_0} \subseteq \Big\{x \in A \colon |f(x)| < \lambda_0\Big\}.$$

By the inclusion in the previous display and the equality in (2.1), for every $q \in (0,\infty]$ we have

$$\begin{aligned}\mu(A)^{-\frac{1}{p'}} \|f 1_B\|_{L^{1,q}_\mu(\ell^\infty_\omega)} &\leq 2^{\frac{1}{p}} \mu(A)^{-1} \|f\|_{L^{p,\infty}_\mu(\ell^\infty_\omega)} \|1_B\|_{L^{1,q}_\mu(\ell^\infty_\omega)} \\ &\leq 2^{\frac{1}{p}} q^{-\frac{1}{q}} \|f\|_{L^{p,\infty}_\mu(\ell^\infty_\omega)}.\end{aligned}$$

where $q^{\frac{1}{q}}$ for $q = \infty$ is understood to be 1. Together with the assumptions in (3.1)–(3.3), the previous inequality yields the desired inequality.

We turn to the proof of the inequality

$$\|f\|_{L^{p,\infty}_\mu(\ell^\infty_\omega)} \leq CK'.$$

Without loss of generality we assume

$$\Omega_\rho \subseteq \Big\{x \in X \colon |f(x)| > \rho\Big\}. \tag{4.8}$$

For every measurable subset $B_\rho \in \Sigma'_\mu(\Omega_\rho)$ we have

$$\mu(B_\rho) \geq \frac{\mu(\Omega_\rho)}{2}.$$

By the inclusion in (4.8), the equality in (2.1), and the previous inequality, for every $q \in (0,\infty]$ we have

$$\|f 1_{B_\rho}\|_{L^{1,q}(\ell^\infty_\omega)} \geq \|\rho 1_{B_\rho}\|_{L^{1,q}(\ell^\infty_\omega)} = q^{-\frac{1}{q}} \rho \mu(B_\rho),$$

where $q^{\frac{1}{q}}$ for $q = \infty$ is understood to be 1. Together with the assumptions in (3.1)–(3.3), the previous two inequalities yield the desired inequality.

Case II: $r \neq \infty$**.** We start with the proof of the inequality

$$K' \leq C\|f\|_{L^{p,\infty}_\mu(\ell^r_\omega)}.$$

For every measurable subset $A \in \Sigma_\mu$ we define $\lambda_0 \in (0,\infty)$ by

$$\lambda_0 := 2^{\frac{1}{p}} \mu(A)^{-\frac{1}{p}} \|f\|_{L^{p,\infty}_\mu(\ell^r_\omega)}.$$

By the definition of the outer $L^{p,\infty}_\mu(\ell^r_\omega)$ quasi-norm we have

$$\mu(\Omega_{\lambda_0}) \le \lambda_0^{-p} \|f\|^p_{L^{p,\infty}_\mu(\ell^r_\omega)} \le \frac{\mu(A)}{2}.$$

and we define the measurable subset $B \in \Sigma'_\mu(A)$ by

$$B \coloneqq A \setminus \Omega_{\lambda_0}.$$

By outer Hölder's inequality for outer Lorentz $L^{p,q}$ spaces (Theorem A.2 in Appendix A) and the equality in (2.1), for every $q \in (0, \infty]$ we have

$$\begin{aligned}\mu(A)^{-\frac{1}{p'}} \|f 1_B\|_{L^{1,q}_\mu(\ell^r_\omega)} &\le 2\mu(A)^{-\frac{1}{p'}} \|f 1_{\Omega^c_{\lambda_0}}\|_{L^\infty_\mu(\ell^r_\omega)} \|1_B\|_{L^{1,q}_\mu(\ell^\infty_\omega)} \\ &\le 2^{1+\frac{1}{p}} q^{-\frac{1}{q}} \|f\|_{L^{p,\infty}_\mu(\ell^r_\omega)},\end{aligned}$$

where $q^{\frac{1}{q}}$ for $q = \infty$ is understood to be 1. Together with the assumptions in (3.1)–(3.3), the previous inequality yields the desired inequality.

We turn to the proof of the inequality

$$\|f\|_{L^{p,\infty}_\mu(\ell^r_\omega)} \le CK'. \tag{4.9}$$

Without loss of generality we assume $f \in L^\infty_\mu(\ell^r_\omega) \cap L^{p,\infty}_\mu(\ell^r_\omega)$. In fact, for every $f \in L^{p,\infty}_\mu(\ell^r_\omega)$ and every $\varepsilon > 0$ there exists a measurable subset $A_\varepsilon \in \Sigma$ such that

$$\|f 1_{A_\varepsilon}\|_{L^\infty_\mu(\ell^r_\omega)} < \infty, \qquad \|f 1_{A_\varepsilon}\|_{L^{p,\infty}_\mu(\ell^r_\omega)} \le \|f\|_{L^{p,\infty}_\mu(\ell^r_\omega)} \le (1-\varepsilon^p)^{-\frac{1}{p}} \|f 1_{A_\varepsilon}\|_{L^{p,\infty}_\mu(\ell^r_\omega)}.$$

More in details, by the choice of ρ we have

$$\mu(\Omega_{\rho/\varepsilon}) \le \varepsilon^p \mu(\Omega_\rho).$$

For $A_\varepsilon \coloneqq \Omega^c_{\rho/\varepsilon}$ we have $f 1_{A_\varepsilon} \in L^\infty_\mu(\ell^r_\omega)$. Moreover, by the inequality in the previous display, the definition in (1.3), and the subadditivity of μ we have

$$(1-\varepsilon^p)\mu(\Omega_\rho) \le \mu(\Omega_\rho) - \mu(\Omega_{\rho/\varepsilon}) \le \mu(\Omega_\rho(A_\varepsilon) \cup \Omega_{\rho/\varepsilon}) - \mu(\Omega_{\rho/\varepsilon}) \le \mu(\Omega_\rho(A_\varepsilon)),$$

yielding

$$\|f\|_{L^{p,\infty}_\mu(\ell^r_\omega)} = \rho\mu(\Omega_\rho)^{\frac{1}{p}} \le (1-\varepsilon^p)^{-\frac{1}{p}} \rho\mu(\Omega_\rho(A_\varepsilon))^{\frac{1}{p}} \le (1-\varepsilon^p)^{-\frac{1}{p}} \|f 1_{A_\varepsilon}\|_{L^{p,\infty}_\mu(\ell^r_\omega)}.$$

Proving the inequality in (4.9) for $f 1_{A_\varepsilon} \in L^\infty_\mu(\ell^r_\omega) \cap L^{p,\infty}_\mu(\ell^r_\omega)$ and taking $\varepsilon > 0$ arbitrarily small yield the desired inequality for $f \in L^{p,\infty}_\mu(\ell^r_\omega)$.

To prove the inequality in (4.9) we argue as follows. By the definition of the outer $L^{p,\infty}_\mu(\ell^r_\omega)$ quasi-norm, the choice of ρ, and the subadditivity of μ, for every $M \in (1, \infty)$ we have

$$M\rho\mu(\Omega_{M\rho})^{\frac{1}{p}} \le \rho\mu(\Omega_\rho)^{\frac{1}{p}}, \qquad \mu(\Omega_\rho) \le \mu(\Omega_{M\rho}) + \mu(\Omega_\rho(\Omega^c_{M\rho})).$$

Hence, we have

$$0 < M^p\mu(\Omega_\rho) - \mu(\Omega_\rho) \le M^p[\mu(\Omega_\rho) - \mu(\Omega_{M\rho})] \le M^p\mu(\Omega_\rho(\Omega^c_{M\rho})). \tag{4.10}$$

The previous chain of inequalities yields

$$\|f\|_{L^{p,\infty}_\mu(\ell^r_\omega)} \leq M(M^p-1)^{-\frac{1}{p}}\rho\mu(\Omega_\rho(\Omega^c_{M\rho}))^{\frac{1}{p}}. \tag{4.11}$$

Moreover, by the chain of inequalities in (4.10) we have $\Omega_\rho(\Omega^c_{M\rho}) \neq \emptyset$. In particular, for every $\varepsilon > 0$ we have

$$\|f1_{\Omega^c_{M\rho}}\|_{L^\infty_\mu(\ell^r_\omega)} > \rho_\varepsilon,$$

where $\rho_\varepsilon \in (0,\rho)$ is defined by

$$\rho_\varepsilon \coloneqq (1-\varepsilon)\rho.$$

Therefore, by Lemma 4.2 there exists a measurable subset $B_{\rho_\varepsilon} \in \Sigma_\mu(\Omega^c_{M\rho})$ such that

$$\ell^r_\omega(f)(B_{\rho_\varepsilon}) > \rho_\varepsilon, \qquad \|f1_{\Omega^c_{M\rho}}1_{B^c_{\rho_\varepsilon}}\|_{L^\infty_\mu(\ell^r_\omega)} \leq \rho_\varepsilon. \tag{4.12}$$

By the inequality in (4.11) and the second inequality in (4.12), we have

$$\|f\|_{L^{p,\infty}_\mu(\ell^r_\omega)} \leq M(M^p-1)^{-\frac{1}{p}}\rho\mu(B_{\rho_\varepsilon})^{\frac{1}{p}}. \tag{4.13}$$

By the definition of K', there exists a subset $D_{\rho_\varepsilon} \in \Sigma'_\mu(B_{\rho_\varepsilon})$ such that

$$\|f1_{D_{\rho_\varepsilon}}\|_{L^{1,q}(\ell^r_\omega)} \leq K'\mu(B_{\rho_\varepsilon})^{\frac{1}{p'}}. \tag{4.14}$$

Furthermore, by the first inequality in (4.12) and the inclusion $B_{\rho_\varepsilon} \subseteq \Omega^c_{M\rho}$, for every subset $D \in \Sigma'_\mu(B_{\rho_\varepsilon})$ we have

$$\begin{aligned}\ell^r_\omega(f)(D) &\geq \mu(D)^{-\frac{1}{r}}[\rho^r_\varepsilon\mu(B_{\rho_\varepsilon}) - M^r\rho^r\mu(B_{\rho_\varepsilon}\setminus D)]^{\frac{1}{r}}\\ &\geq \mu(B_{\rho_\varepsilon})^{-\frac{1}{r}}\Big[\rho^r_\varepsilon\mu(B_{\rho_\varepsilon}) - M^r\rho^r\frac{\mu(B_{\rho_\varepsilon})}{2}\Big]^{\frac{1}{r}} \geq \Big(1-\frac{M^r_\varepsilon}{2}\Big)^{\frac{1}{r}}\rho_\varepsilon,\end{aligned}$$

where $M_\varepsilon \in (M,\infty)$ is defined by

$$M_\varepsilon \coloneqq \frac{M}{1-\varepsilon}.$$

In particular, for every $\lambda = k\Big(1-\frac{M^r_\varepsilon}{2}\Big)^{1/r}\rho_\varepsilon$ with $k \in (0,1)$ we have

$$\mu(B_{\rho_\varepsilon}) \leq 2\mu(D) \leq 2M^r_\varepsilon\Big(1-\frac{M^r_\varepsilon}{2}\Big)^{-1}(1-k^r)^{-1}\mu(\Omega_\lambda(D)), \tag{4.15}$$

where the second inequality follows by Lemma 4.1. By the second inequality in (4.15), for every $q \in (0,\infty)$ and every $k \in (0,1)$ we have

$$\|f1_D\|_{L^{1,\infty}_\mu(\ell^r_\omega)} \geq M^{-r}_\varepsilon\Big(1-\frac{M^r_\varepsilon}{2}\Big)^{\frac{r+1}{r}}(k-k^{r+1})\rho_\varepsilon\mu(D).$$

Taking $\varepsilon > 0$ arbitrarily small, together with the assumptions in (3.1)–(3.3) and in (4.2), the inequality in (4.13), the first inequality in (4.15), and the inequality in (4.14), the previous inequality yields the desired inequality for every $q \in (0,\infty]$

$$\|f\|_{L^{p,\infty}_\mu(\ell^r_\omega)} \leq 2CK',$$

where the minimal value of the constant C for every $q \in (0, \infty]$ is defined by

$$q^{\frac{1}{q}} \inf\Big\{(1 - M^{-p})^{-\frac{1}{p}}\Big(1 - \frac{M^r}{2}\Big)^{-\frac{r+1}{r}} M^r (k - k^{r+1})^{-1} : \, M \in (1, 2^{\frac{1}{r}}), k \in (0,1)\Big\} =$$

$$= q^{\frac{1}{q}} \frac{(r+1)^{\frac{r+1}{r}}}{r} \inf\Big\{M^r (1 - M^{-p})^{-\frac{1}{p}}\Big(1 - \frac{M^r}{2}\Big)^{-\frac{r+1}{r}} : \, M \in (1, 2^{\frac{1}{r}})\Big\},$$

where $q^{\frac{1}{q}}$ for $q = \infty$ is understood to be 1. □

We comment on the modifications needed to drop the additional assumption. For every function $g \in L^\infty_\mu(\ell^r_\omega)$ and every $\delta > 0$ there exists $A_{\rho,\delta} \in \mathcal{A}$ such that

$$(1 - \delta)\rho \leq \ell^r_\omega(g)(A_{\rho,\delta}) \leq \rho = \|g\|_{L^\infty_\mu(\ell^r_\omega)}.$$

For every fixed $\delta > 0$ we can apply the arguments we described in the previous proofs. Taking $\delta > 0$ arbitrarily small we obtain a proof of Theorem 1.3 with the appropriate constants.

Remark 4.4. *In the proof we never used the assumption $p \leq a$. The same argument yields the following alternative version of property* (*i*) *in Theorem* 1.3*, which is consistent with the case of classical $L^{p,\infty}$ quasi-norms on a measure space (X, ω).*

For every $p > a$ and every function $f \in L^{p,\infty}_\mu(\ell^r_\omega)$ we have

$$\|f\|_{L^{p,\infty}_\mu(\ell^r_\omega)} \sim_C \sup\Big\{\inf\Big\{\mu(A)^{\frac{1}{p}-\frac{1}{a}} \|f 1_B\|_{L^{a,q}_\mu(\ell^r_\omega)} : \, B \in \Sigma'_\mu(A)\Big\} : \, A \in \Sigma_\mu\Big\}.$$

Remark 4.5. *For every measurable subset $A \in \Sigma$, we recall we defined the collection $\Sigma''_\mu(A) \subseteq \Sigma$ by*

$$\Sigma''_\mu(A) := \Big\{B \in \Sigma : \, B \subseteq A, \mu(B) \geq \frac{\mu(A)}{2}\Big\}.$$

Let K'' be defined by

$$K'' := \sup\Big\{\inf\Big\{\mu(A)^{\frac{1}{p}-\frac{1}{a}} \|f 1_B\|_{L^{a,q}_\mu(\ell^r_\omega)} : \, B \in \Sigma''_\mu(A)\Big\} : \, A \in \Sigma_\mu\Big\}.$$

For every measurable subset $B \in \Sigma'_\mu(A)$ we have $\mu(A) \leq 2\mu(B)$, hence

$$K'' \leq K'.$$

In fact, for every $p \leq a$ and every function $f \in L^{p,\infty}_\mu(\ell^\infty_\omega)$ we have

$$\|f\|_{L^{p,\infty}_\mu(\ell^\infty_\omega)} \sim_C K''.$$

Moreover, for all $p \leq a$, $r \neq \infty$ and every function $f \in L^{p,\infty}_\mu(\ell^r_\omega)$ we have

$$K'' \leq C \|f\|_{L^{p,\infty}_\mu(\ell^r_\omega)}.$$

However, for every $M \in (1, \infty)$ there exists a σ-finite setting (X, μ, ω) and a function $f \in L^{p,\infty}_\mu(\ell^r_\omega)$ such that

$$\|f\|_{L^{p,\infty}_\mu(\ell^r_\omega)} \geq M K''.$$

In particular, let

$$\begin{aligned} X &= \{j \in \mathbb{N} \colon 1 \leq j \leq m\}, \\ \mu(A) &= 1, && \textit{for every subset } A \subseteq X,\ A \neq \varnothing, \\ \omega(A) &= |A|, && \textit{for every subset } A \subseteq X. \end{aligned}$$

Then, for the function $f = 1_X$ *we have*

$$\|f\|_{L^{p,\infty}_\mu(\ell^r_\omega)} = m^{\frac{1}{r}}, \qquad K'' \leq \sup\Big\{\mu(\{x_A\})^{\frac{1}{p}} \colon A \subseteq X, A \neq \varnothing\Big\} = 1,$$

where for every subset $A \subseteq X$, $A \neq \varnothing$ *we choose* $x_A \in A$ *arbitrarily. Taking* $m \in \mathbb{N}$ *big enough we obtain the desired inequality.*

5. Function spaces on the collection of Heisenberg tiles

5.1. σ-finite setting on the collection of Heisenberg tiles and lacunary sizes. For all $m, l \in \mathbb{Z}$ we define the *dyadic interval* $D(m,l)$ in $\mathbb{R}$ by

$$D(m,l) \coloneqq (2^l m, 2^l(m+1)],$$

and the collection $\mathcal{D}$ of dyadic intervals in $\mathbb{R}$ by

$$\mathcal{D} \coloneqq \Big\{D(m,l) \colon m, l \in \mathbb{Z}\Big\}.$$

Moreover, for all $m, n, l \in \mathbb{Z}$ we define the *dyadic Heisenberg tile* $H(m,n,l)$ in $\mathbb{R}^2$ by

$$H(m,n,l) \coloneqq D(m,l) \times D(n,-l),$$

and the collection $\mathcal{H}$ of dyadic Heisenberg tiles in $\mathbb{R}^2$ by

$$\mathcal{H} \coloneqq \Big\{H(m,n,l) \colon m,n,l \in \mathbb{Z}\Big\}.$$

Then, for all $M, L \in \mathbb{Z}$ we define the *stripe* $E(M,L) \subseteq \mathcal{H}$ by

$$E(M,L) = E(D(M,L)) \coloneqq \Big\{H(m,n,l) \in \mathcal{H} \colon D(M,L) \subseteq D(m,l)\Big\}.$$

Moreover, for all $M, N, L \in \mathbb{Z}$ and for every $\kappa \in \mathbb{Z}$, $\kappa \geq 0$ we define the *tree of 2^κ-tiles* $T_\kappa(M,N,L) \subseteq \mathcal{H}$ by

$$T_\kappa(M,N,L) \coloneqq \Big\{H(m,n,l) \in \mathcal{H} \colon D(m,l) \subseteq D(M,L), D(N,-L) \subseteq D(\lfloor n/2^\kappa \rfloor, \kappa - l)\Big\},$$

where for every $x \in \mathbb{R}$ we define $\lfloor x \rfloor \in \mathbb{Z}$ to be the biggest integer number smaller than or equal to x.

Next, for every $\kappa \in \mathbb{Z}$, $\kappa \geq 0$ we define $(X_\kappa, \mu_\kappa, \omega_\kappa)$ to be the σ-finite setting on the collection $\mathcal{H}$ of dyadic Heisenberg tiles as follows. Let

$$\begin{aligned} X_\kappa &= \mathcal{H}, \\ \mathcal{T}_\kappa &= \Big\{T_\kappa(m,n,l) \colon m,n,l \in \mathbb{Z}\Big\}, \\ \sigma_\kappa(T_\kappa(m,n,l)) &= 2^l, && \text{for all } m,n,l \in \mathbb{Z}, \\ \omega_\kappa(H(m,n,l)) &= 2^l, && \text{for all } m,n,l \in \mathbb{Z}, \end{aligned}$$

and let μ_κ be the outer measure on X_κ generated via minimal coverings by $\mathcal{T}_\kappa$ and σ_κ as described in Section 2.

After that, for every $\kappa \in \mathbb{Z}$, $\kappa \geq 0$ we define a collection of tiles $T \subseteq \mathcal{H}$ a *κ-lacunary tree* if it satisfies the following properties:

- There exists $M, N, L \in \mathbb{Z}$ such that
$$T \subseteq T_\kappa(M, N, L).$$
- For all $m, n, l, m', n', l' \in \mathbb{Z}$ such that
$$H(m, n, l), H(m', n', l') \in T, \qquad D(n, -l) \neq D(n', -l'),$$
we have
$$D(n, -l) \cap D(n', -l') = \varnothing.$$

Moreover, we define $\mathcal{T}_{\kappa,\mathrm{lac}}$ to be the collection of all κ-lacunary trees.

Finally, we define the *lacunary size* $(S_{\kappa,\mathrm{lac}}, \mathcal{T}_{\kappa,\mathrm{lac}})$ as follows. For every κ-lacunary tree $T_{\kappa,\mathrm{lac}} \in \mathcal{T}_{\kappa,\mathrm{lac}}$ we define

$$S_{\kappa,\mathrm{lac}}(f)(T_{\kappa,\mathrm{lac}}) \coloneqq \mu_\kappa(T_{\kappa,\mathrm{lac}})^{-1} \| f 1_{T_{\kappa,\mathrm{lac}}} \|_{L^1(X_\kappa, \omega_\kappa)},$$

and the outer Lorentz $L^{p,q}_{\mu_\kappa}(S_{\kappa,\mathrm{lac}})$ spaces as in the Introduction. In particular, for every $r \in (0, \infty]$ we denote by S^r_κ the size $(S_{\kappa,\mathrm{lac}})_r$ defined in (1.4) and in (1.5).

It is easy to observe that the size $(S_{\kappa,\mathrm{lac}}, \mathcal{T}_{\kappa,\mathrm{lac}})$ satisfies the inequality in (1.1) with constant $K = 1$ and the inequality in (4.7) with constant $\widetilde{K} = 1$.

Remark 5.1. *On the collection of Heisenberg tiles and its continuous counterpart, the upper half* 3*-space* $\mathbb{R}^2 \times (0, \infty)$*, further sizes of iterated nature were considered. We refer to* [DDU18, Ur17, UW22] *for more details. We briefly recall an instance of such sizes.*

We define $\mathcal{T} = \mathcal{T}_0$*,* $\mu = \mu_0$*, and* (X, ν, ω) *the* σ*-finite setting on the collection* $\mathcal{H}$ *of dyadic Heisenberg tiles as follows. Let* $X = \mathcal{H}$*,* $\omega = \omega_0$*,* $\mathcal{E} = \{E(M, L) : M, L \in \mathbb{Z}\}$*, and* $\tau(E(M, L)) = 2^L$ *for all* $M, L \in \mathbb{Z}$*. Let* ν *be the outer measure on* X *generated via minimal coverings by* $\mathcal{E}$ *and* τ *as described in Section* 2*.*

First, for every $t \in (0, \infty)$ *we define the size* $(S^t, \mathcal{T})$ *as follows. For every tree* $T \in \mathcal{T}$ *we define*

$$S^t(f)(T) \coloneqq \| f 1_T \|_{L^\infty_\mu(S^t_0)} + \| f 1_T \|_{L^\infty(X, \omega)}, \tag{5.1}$$

where $S^t_0 = (S_{0,\mathrm{lac}})_t$*, and the outer Lebesgue* $L^1_\mu(S^t)$ *spaces as in the Introduction.*

Next, we define the size $(S^{1,t}, \mathcal{E})$ *as follows. For every stripe* $E \in \mathcal{E}$ *we define*

$$S^{1,t}(f)(E) \coloneqq \nu(E)^{-1} \| f 1_E \|_{L^1_\mu(S^t)},$$

and the outer Lorentz $L^{p,q}_\nu(S^{1,t})$ *spaces as in the Introduction.*

For every $t \in (0, \infty)$ *the size* $(S^{1,t}, \mathcal{E})$ *satisfies the inequality in* (1.1) *with constant* $K = 2$ *by the definitions in* (1.3) *and in* (5.1)*. In particular, for all measurable functions* f, g *on* X *and every* $\lambda \in (0, \infty)$ *we have*

$$\mu(S^t(f + g) > \lambda) \leq \mu(S^t(f) > \lambda/2) + \mu(S^t(g) > \lambda/2),$$

yielding the quasi-triangle inequality for the quasi-norm $L^1_\mu(S^t)$.

The size $(S^{1,t}, \mathcal{E})$ *satisfies the inequality in* (4.7) *with constant* $\widetilde{K} = 8$ *too, namely*

$$S^{1,t}(f)(A)\nu(A) \leq 8[\|f\|_{L^\infty_\nu(S^{1,t})}\nu(B) + S^{1,t}(f1_{B^c})(A)\nu(A)],$$

by the quasi-triangle inequality for the quasi-norm $L^1_\mu(S^t)$*, the definitions in* (1.3) *and in* (5.1)*, and the geometry of stripes, trees, lacunary trees, and their intersections, see Lemma 4.4 in* [Fr23]*. In particular, for every measurable function* f *on* X*, every* $\lambda \in (0,\infty)$*, every collection* $\mathcal{F} \subseteq \mathcal{E}$ *of pairwise disjoint stripes, every tree* $T \in \mathcal{T}$*, and every lacunary tree* $U \in \mathcal{T}_{0,\mathrm{lac}}$ *we have*

$$\begin{aligned}
&\mu(S^t(f) > \lambda) \leq \mu(S^t_0(f) > \lambda/2) + \mu(\ell^\infty_\omega(f) > \lambda/2),\\
&\mu(S^t_0(f) > \lambda) + \mu(\ell^\infty_\omega(f) > \lambda) \leq 2\mu(S^t(f) > \lambda),\\
&S^t_0(f1_{\widetilde{F}})(U) \leq \Big(\frac{1}{\mu(U \cap \widetilde{F})}\sum_{F \in \mathcal{F}} \mu(U \cap F) S^1_0(f^t 1_F)(U)\Big)^{\frac{1}{t}} \leq \sup_{F \in \mathcal{F}} S^t_0(f1_F)(U),\\
&\ell^\infty_\omega(f1_{\widetilde{F}})(T) \leq \sup_{F \in \mathcal{F}} \ell^\infty_\omega(f1_F)(T),
\end{aligned}$$

where $\widetilde{F} = \bigcup_{F \in \mathcal{F}} F$*. The last two inequalities in the previous display yield the following inequality both for* $S = S^t_0$ *and* $S = \ell^\infty_\omega$

$$\mu(S(f1_{\widetilde{F}}) > \lambda) \leq \sum_{F \in \mathcal{F}} \mu(S(f1_F) > \lambda).$$

5.2. The spaces $X^{p,q}_a(J, \varkappa, \mathrm{size}_{2,\star})$ and equivalence with the outer $L^{p,\infty}_{\mu_\varkappa}(S^2_\varkappa)$ spaces. The spaces $X^{p,q}_a(J, \varkappa, \mathrm{size}_{2,\star})$ appearing in Section 3.8 in [DF25] are associated with a triple of exponents $p, q, a \in [1,\infty]$, $p \geq a$, a dyadic interval $J \subseteq \mathbb{R}$, and a non-negative integer $\varkappa \in \mathbb{Z}$, $\varkappa \geq 0$. The size $(\mathrm{size}_{2,\star}, \mathcal{T}_{J,\varkappa})$ on the collection of functions with support in the stripe $E(J)$ is defined as follows. Let $\mathcal{T}_{J,\varkappa} \subseteq \mathcal{T}_\varkappa$ be the collection of trees of $2^\varkappa$-tiles contained in the stripe $E(J)$. For every tree $T_\varkappa \in \mathcal{T}_{J,\varkappa}$ and every function F supported in the stripe $E(J)$ we define

$$\mathrm{size}_{2,\star}(F)(T_\varkappa) := \|F1_{T_\varkappa}\|_{L^\infty_{\mu_\varkappa}(S^2_\varkappa)}. \tag{5.2}$$

The space $X^{p,q}_a(J, \varkappa, \mathrm{size}_{2,\star})$ is defined by the following quasi-norm on functions F supported in the stripe $E(J)$

$$\|F\|_{X^{p,q}_a(J,\varkappa,\mathrm{size}_{2,\star})} := \sup\Big\{\mu_\varkappa(A)^{\frac{1}{p}-\frac{1}{a}}\|F1_A\|_{L^{a,q}_{\mu_\varkappa}(\mathrm{size}_{2,\star})} \colon A \in \Sigma_{\mu_\varkappa}(E(J))\Big\}. \tag{5.3}$$

We are ready to state the equivalence between the $X^{p,q}_a(J, \varkappa, \mathrm{size}_{2,\star})$ spaces and the outer $L^{p,\infty}_{\mu_\varkappa}(S^2_\varkappa)$ ones.

Corollary 5.2. *For all* $a, p, q \in [1,\infty]$*,* $p > a$ *there exists a constant* $C = C(a,p,q)$ *such that the following properties hold true.*

For every $\varkappa \in \mathbb{Z}$*,* $\varkappa \geq 0$ *let* $(X_\varkappa, \mu_\varkappa, \omega_\varkappa)$ *be the setting and* $\mathcal{T}_\varkappa$ *be the collection of generators described above. For every dyadic interval* $J \in \mathcal{D}$*, and every function*

f on X_κ we have

$$\|f 1_{E(J)}\|_{L^{p,\infty}_{\mu_\kappa}(S^2_\kappa)} = \alpha \sim_C \beta = \|f\|_{X^{p,q}_a(J,\kappa,\mathsf{size}_{2,\star})}.$$

In particular, we have

$$\alpha \leq \Big(\frac{q}{a}\Big)^{\frac{1}{q}} \sqrt{\frac{p-a}{a}} \Big(\frac{p}{p-a}\Big)^{\frac{p}{2a}} \beta, \qquad \beta \leq 2^{\frac{1}{a}} \Big(\frac{p}{q(p-a)}\Big)^{\frac{1}{q}} \alpha.$$

In fact, we can define analogous quasi-norms and spaces and prove an analogous statement in the case of the settings on the upper half 3-space $\mathbb{R}^2 \times (0,\infty)$ described in [DT15, Fr23]. The settings on $\mathcal{H}$ described above are discrete models of those on the upper half 3-space.

Proof. The desired equivalence follows from Theorem 1.2, Remark 4.3 the definitions in (5.2) and in (5.3), and the following auxiliary observation. For every dyadic interval $J \in \mathcal{D}$ and every function F supported in $E(J)$ we have

$$\|F\|_{L^\infty_{\mu_\kappa}(S^2_\kappa)} = \|F\|_{L^\infty_{\mu_\kappa}(\mathsf{size}_{2,\star})},$$

hence for all $p, q \in (0,\infty]$ and every function F supported in $E(J)$ we have

$$\|F\|_{L^{p,q}_{\mu_\kappa}(S^2_\kappa)} = \|F\|_{L^{p,q}_{\mu_\kappa}(\mathsf{size}_{2,\star})}. \qquad \square$$

Appendix A. A proof of the inequality in (1.6)

We prove the inequality in (1.6) originally stated in Proposition 3.9 in [DF25] applying standard interpolation results in the theory of outer L^p spaces.

Proposition A.1 (Proposition 3.9 in [DF25]). *For every $m \in \mathbb{N}$, $m \geq 2$ there exists a constant $C = C(m)$ such that the following property holds true.*

Let $J \in \mathcal{D}$ be a dyadic interval in $\mathbb{R}$. Let $a, p_1 \in (1,\infty)$, $a \leq p_1$ and $p_2, \dots, p_m \in [1,\infty)$ satisfy

$$\varepsilon := \Big(\sum_{j=1}^m \frac{1}{p_j}\Big) - 1 > 0.$$

Let the sizes $(\mathsf{size}_{2,\star}, \mathcal{T}_\kappa)$, $(s_2, \mathcal{T}_\kappa), \dots, (s_m, \mathcal{T}_\kappa)$ satisfy for every tree $T_\kappa \in \mathcal{T}_\kappa$ and for all functions $F_1, \dots, F_m$ supported in $E(J)$

$$\ell^1_{\omega_\kappa}\Big(\prod_{j=1}^m F_j\Big)(T_\kappa) \leq \mathsf{size}_{2,\star}(F_1)(T_\kappa) \prod_{j=2}^m s_j(F_j)(T_\kappa).$$

Then for all functions $F_1, \dots, F_m$ supported in $E(J)$ we have

$$\Big\|\prod_{j=1}^m F_j\Big\|_{L^1(E(J),\omega_\kappa)} \leq \frac{Ca}{\varepsilon(a-1)} \|F_1\|_{X^{p_1,\infty}_a(J,\kappa,\mathsf{size}_{2,\star})} \times \\ \times \prod_{j=2}^m \max\{\|F\|_{L^{p_j,\infty}_{\mu_\kappa}(s_j)}, \|F\|_{L^\infty_{\mu_\kappa}(s_j)}\}.$$

We point out that in [DF25] the outer measure μ_κ is renormalized with the length of the dyadic interval J. Therefore, a normalizing factor $|J|^{-1}$ appears on the left hand side of the chain of inequalities in the statement of Proposition 3.9 in [DF25].

To improve the readability of the proof, we omit explicit mention of κ in the notation of μ_κ and ω_κ.

Proof. First, a standard inequality between classical and outer L^p quasi-norms (Proposition 3.6 in [DT15]) yields

$$\Big\|\prod_{j=1}^{m} F_j\Big\|_{L^1(E(J),\omega)} \leq \Big\|\prod_{j=1}^{m} F_j\Big\|_{L^{1,1}_\mu(\ell^1_\omega)}. \tag{A.1}$$

Next, we recall outer Hölder's inequality for outer Lorentz $L^{p,q}$ spaces (Proposition 3.5 in [DF25]).

Theorem A.2. *Let (X,μ,ω) be a σ-finite setting. Let $s, s_1, \dots, s_m$ be sizes on (X,μ,ω) such that for all measurable functions $F_1,\dots,F_m$ on X and every measurable subset $A \in \Sigma$ we have*

$$s\Big(\prod_{i=1}^{m} F_i\Big)(A) \leq \prod_{i=1}^{m} s_i(F_i)(A). \tag{A.2}$$

Then for all $p, p_1, \dots, p_m \in (0,\infty]$ and all $q, q_1, \dots, q_m \in (0,\infty]$ such that

$$\sum_{i=1}^{m} \frac{1}{p_i} = \frac{1}{p}, \qquad \sum_{i=1}^{m} \frac{1}{q_i} = \frac{1}{q},$$

we have

$$\Big\|\prod_{i=1}^{m} F_i\Big\|_{L^{p,q}_\mu(s)} \leq m^{\frac{1}{p}} \|F_i\|_{L^{p_i,q_i}_\mu(s_i)}.$$

We apply the previous theorem with the sizes s and s_1 defined by

$$s = \ell^1_\omega, \qquad s_1 = S^2_\kappa,$$

and the sizes $s_2,\dots,s_m$ satisfying the condition associated with the inequality in (A.2). Moreover, starting with the exponents $a, p_1, p_2, \dots, p_m$ satisfying the conditions in the statement of Proposition A.1 we define the exponents $r_2,\dots,r_m \in (1,\infty)$ and $q_2,\dots,q_m \in (1,\infty)$ by

$$P = \sum_{j=2}^{m} \frac{1}{p_j} = \frac{1}{p_1'} + \varepsilon, \qquad q_j = p_j P, \qquad r_j = a' q_j.$$

Therefore, we obtain the inequality

$$\Big\|\prod_{j=1}^{m} F_j\Big\|_{L^{1,1}_\mu(\ell^1_\omega)} \leq m \|F_1\|_{L^{a,\infty}_\mu(s_1)} \prod_{j=2}^{m} \|F_j\|_{L^{r_j,q_j}_\mu(s_j)}, \tag{A.3}$$

In particular, we observe that

$$Pa' \geq \Big(\frac{1}{p_1'} + \varepsilon\Big) p_1' > 1,$$

hence for every $j \in \{2, \dots, m\}$ we have

$$r_j > p_j,$$

hence the logarithmic convexity of the outer Lorentz $L^{p,q}$ spaces (Proposition 3.3 in [DT15]) yields the inequality

$$\|F_j\|_{L_\mu^{r_j,q_j}(s_j)} \leq \Big(\frac{P}{\varepsilon}\Big)^{\frac{1}{q_j}} \|F_j\|_{L_\mu^\infty(s_j)}^{\frac{r_j-p_j}{r_j}} \|F_j\|_{L_\mu^{p_j,\infty}(s_j)}^{\frac{p_j}{r_j}}. \tag{A.4}$$

Finally, the equivalence between the $X_a^{p,\infty}(J,\kappa,\mathsf{size}_{2,\star})$ spaces and the outer $L_\mu^{p,\infty}(S_\kappa^2)$ ones (Corollary 5.2) yields the inequality

$$\|F\|_{L_\mu^{p,\infty}(S_\kappa^2)} \leq \sqrt{\frac{p-a}{a}} \Big(\frac{p}{p-a}\Big)^{\frac{p}{2a}} \|F\|_{X_a^{p,\infty}(J,\kappa,\mathsf{size}_{2,\star})}. \tag{A.5}$$

The inequalities in (A.1) and in (A.3)–(A.5) yield the result stated in Proposition 3.9 in [DF25]. □

Appendix B. Proofs of Lemma 4.1 and Lemma 4.2

Proof of Lemma 4.1. Let $\varepsilon > 0$. By the existence of a measurable subset satisfying the inequality in (4.3), for every $\lambda \in (0,\rho)$ we have

$$\mu(\ell_\omega^r(f) > \lambda) > 0.$$

For every $\lambda \in (0,\rho)$ we choose a measurable subset $\Omega_\lambda \in \Sigma$ such that

$$\|f 1_{\Omega_\lambda^c}\|_{L_\mu^\infty(\ell_\omega^r)} \leq \lambda, \qquad \mu(\Omega_\lambda) \leq (1+\varepsilon)\mu(\ell_\omega^r(f) > \lambda).$$

We have

$$\begin{aligned}\|f\|_{L_\mu^\infty(\ell_\omega^r)}^r \mu(\Omega_\lambda) &\geq \|f 1_{\Omega_\lambda}\|_{L^r(X,\omega)}^r \\ &\geq \|f 1_A\|_{L^r(X,\omega)}^r - \|f 1_A 1_{\Omega_\lambda^c}\|_{L^r(X,\omega)}^r \geq \mu(A)(\rho^r - \lambda^r).\end{aligned}$$

Taking $\varepsilon > 0$ arbitrarily small we obtain the desired inequality. □

Remark B.1. *Under the condition on the size $(S,\mathcal{A})$ stated in (4.7), the chain of inequality in the previous display becomes*

$$\begin{aligned}\|f\|_{L_\mu^\infty(S_r)}^r \mu(\Omega_\lambda) = \|f^r\|_{L_\mu^\infty(S)} \mu(\Omega_\lambda) &\geq \frac{1}{\widetilde{K}} S(f^r)(A)\mu(A) - S(f^r 1_{\Omega_\lambda^c})(A)\mu(A) \\ &\geq \frac{\mu(A)}{\widetilde{K}}\Big((S_r(f)(A))^r - \widetilde{K}(S_r(f 1_{\Omega_\lambda^c})(A))^r\Big) \geq \frac{\mu(A)}{\widetilde{K}}(\rho^r - \widetilde{K}\lambda^r),\end{aligned}$$

yielding the desired inequality.

Proof of Lemma 4.2. We argue by contradiction and we assume that there exists no measurable subset $B \in \Sigma$ such that

$$\ell^r_\omega(f)(B) \geq \lambda,$$

hence

$$\lambda \geq \|f\|_{L^\infty_\mu(\ell^r_\omega)}.$$

This yields a contradiction with the inequality in (4.4).

Therefore, there exists a measurable subset $B \in \Sigma$ satisfying the first inequality in (4.5). We define the collection $\{B_n : n \in \mathbb{N}_\lambda\} \subseteq \Sigma$ of measurable subsets by a forward recursion on $n \in \mathbb{N}_\lambda$, where $\mathbb{N}_\lambda$ is a finite non-empty initial string of $\mathbb{N}$. We define the subset $A_0 = \varnothing$ and for all $n \in \mathbb{N}_\lambda$ we define the measurable subset $A_n \in \Sigma$ by

$$A_n = \bigcup_{m \leq n} B_m,$$

and the collection $\mathcal{B}_n \subseteq \Sigma$ of measurable subsets of X by

$$\mathcal{B}_n = \Big\{B \in \Sigma : \ \ell^r_\omega(f 1_{A^c_{n-1}})(B) \geq \lambda\Big\}.$$

In particular, we have $\mathcal{B}_1 \neq \varnothing$. If $\mathcal{B}_n$ is empty, we define $\mathbb{N}_\lambda \subseteq \mathbb{N}$ by

$$\mathbb{N}_\lambda = \{1, \dots, n-1\}.$$

If $\mathcal{B}_n$ is not empty, by Lemma 4.1 for every measurable subset $B \in \mathcal{B}_n$ we have

$$\mu(B) \leq C\|f\|_{L^\infty_\mu(\ell^r_\omega)} \mu(\ell^r_\omega(f) \geq \lambda/2) \leq C 2^p \lambda^{-p} \|f\|_{L^\infty_\mu(\ell^r_\omega)} \|f\|^p_{L^{p,\infty}_\mu(\ell^r_\omega)} < \infty,$$

hence there exists $j = j(n) \in \mathbb{Z}$ such that

$$\sup\Big\{\mu(B) : \ B \in \mathcal{B}_n\Big\} \in (2^j, 2^{j+1}].$$

We choose $B_n \in \mathcal{B}_n$ such that

$$\mu(B_n) \in (2^j, 2^{j+1}].$$

We claim that there exists $N = N(r, \lambda, \|f\|_{L^\infty_\mu(\ell^r_\omega)}) \in \mathbb{N}$ such that

$$j(n+N) \leq j(n) - 1, \tag{B.1}$$

We argue by contradiction and we assume there exists a collection $\{B_{n+i} : i \in \{0, \dots, N\}\} \subseteq \Sigma$ of pairwise disjoint measurable subsets such that for every $i \in \{0, 1, \dots, N\}$ we have

$$\ell^r_\omega(f)(B_{n+i}) \geq \lambda, \qquad \mu(B_{n+i}) \in (2^{j(n)}, 2^{j(n)+1}].$$

Then we have

$$\Big\|f 1_{\bigcup_{i=0}^N B_{n+i}}\Big\|^r_{L^r(X,\omega)} \geq \lambda^r \sum_{i=0}^N \mu(B_{n+i}) \geq \lambda^r \max\Big\{\mu\Big(\bigcup_{i=0}^N B_{n+i}\Big), (N+1)2^{j(n)}\Big\},$$

so that by the definition of $j(n)$ we have

$$\mu\Big(\bigcup_{i=0}^N B_{n+i}\Big) \leq 2^{j(n)+1},$$

hence

$$\ell^r_\omega(f)\Big(\bigcup_{i=0}^{N} B_{n+i}\Big) \geq \Big(\frac{N+1}{2}\Big)^{\frac{1}{r}}\lambda > \|f\|_{L^\infty_\mu(\ell^r_\omega)},$$

yielding a contradiction for N too big.

Therefore, the sequence $\{j(n):\ n \in \mathbb{N}_\lambda\}$ is non-increasing and we have that either $\mathbb{N}_\lambda$ is finite or it is infinite and, by the inequality in (B.1),

$$\inf\Big\{j(n):\ n \in \mathbb{N}_\lambda\Big\} = -\infty.$$

Defining the measurable subset $B \in \Sigma$ by

$$B = \bigcup_{n\in\mathbb{N}_\lambda} B_n,$$

we obtain the desired inequality. □

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References

[AU20] Amenta, Alex; Uraltsev, Gennady. The bilinear Hilbert transform in UMD spaces. *Math. Ann.* **378** (2020), no. 3-4, 1129–1221. MR4163524, Zbl 1471.42030, doi: 10.1007/s00208-020-02052-y. 322

[AU22] Amenta, Alex; Uraltsev, Gennady. Variational Carleson operators in UMD spaces. *J. Lond. Math. Soc. (2)* **105** (2022), no. 3, 1363–1417. MR4407103, Zbl 1520.42004, doi: 10.1112/jlms.12512. 322

[Ca66] Carleson, Lennart. On convergence and growth of partial sums of Fourier series. *Acta Math.* **116** (1966), 135–157. MR199631, Zbl 0144.06402, doi: 10.1007/BF02392815. 327

[CDO18] Culiuc, Amalia; Di Plinio, Francesco; Ou, Yumeng. Banach-valued multilinear singular integrals. *J. Lond. Math. Soc. (2)* **98** (2018), no. 2, 369–392. MR3873113, Zbl 1402.42013, doi: 10.1112/jlms.12139. 322

[DDU18] Di Plinio, Francesco; Do, Yen Q.; Uraltsev, Gennady N. Positive sparse domination of variational Carleson operators. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **18** (2018), no. 4, 1443–1458. MR3829751, Zbl 1403.42010, doi: 10.2422/2036-2145.201612_009. 322, 324, 341

[DF25] Di Plinio, Francesco; Fragkos, Anastasios. The weak-type Carleson theorem via wave packet estimates. *Trans. Amer. Math. Soc.* **378** (2025), no. 3, 1551–1592. MR4866344, Zbl 1559.42012, doi: 10.1090/tran/9303. 322, 324, 325, 326, 327, 328, 329, 342, 343, 344, 345

[DO18a] Di Plinio, Francesco; Ou, Yumeng. A modulation invariant Carleson embedding theorem outside local L^2. *J. Anal. Math.* **135** (2018), no. 2, 675–711. MR3829613, Zbl 1440.42053, doi: 10.1007/s11854-018-0049-4. 322, 328

[DO18b] Di Plinio, Francesco; Ou, Yumeng. Banach-valued multilinear singular integrals. *Indiana Univ. Math. J.* **67** (2018), no. 5, 1711–1763. MR3875242, Zbl 1416.42013, doi: 10.1512/iumj.2018.67.7466. 322

[DT15] Do, Yen; Thiele, Christoph. L^p theory for outer measures and two themes of Lennart Carleson united. *Bull. Amer. Math. Soc. (N.S.)* **52** (2015), no. 2, 249–296. MR3312633, Zbl 1318.42016, doi: 10.1090/S0273-0979-2014-01474-0. 322, 327, 328, 343, 344, 345

[Fr21] Fraccaroli, Marco. Duality for outer $L^p_\mu(\ell^r)$ spaces and relation to tent spaces. *J. Fourier Anal. Appl.* **27** (2021), no. 4, Paper No. 67, 48 pp. MR4292789, Zbl 1472.42037, doi: 10.1007/s00041-021-09869-4. 322, 324, 326

[Fr22] Fraccaroli, Marco. Topics in the L^p theory for outer measure spaces. *Ph.D. thesis, Universität Bonn*, 2022. 322

[Fr23] Fraccaroli, Marco. Duality for double iterated outer L^p spaces. *Studia Math.* **269** (2023), no. 3, 305–357. MR4566003, Zbl 1512.42033, doi: 10.4064/sm210423-19-6. 322, 342, 343

[Hu68] Hunt, Richard A. On the convergence of Fourier series. Orthogonal Expansions and their Continuous Analogues (Proc. Conf., Edwardsville, Ill., 1967), 235–255. *Southern Illinois Univ. Press, Carbondale, Ill.*, 1968. MR0238019, Zbl 0159.35701. 327

[LT00] Lacey, Michael; Thiele, Christoph. A proof of boundedness of the Carleson operator. *Math. Res. Lett.* **7** (2000), no. 4, 361–370. MR1783613, Zbl 0966.42009, doi: 10.4310/MRL.2000.v7.n4.a1. 327

[MS13] Muscalu, Camil; Schlag, Wilhelm. Classical and multilinear harmonic analysis. Vol. II. Cambridge Studies in Advanced Mathematics, 138. *Cambridge University Press, Cambridge*, 2013. xvi+324 pp. ISBN: 978-1-107-03182-1. MR3052499, Zbl 1282.42001, doi: 10.1017/CBO9781139410397. 322

[Ur17] Uraltsev, Gennady. Time-Frequency Analysis of the Variational Carleson Operator using outer-measure L^p spaces. *Ph.D. thesis, Universität Bonn*, 2017. 322, 324, 341

[UW22] Uraltsev, Gennady; Warchalski, Michał. The full range of uniform bounds for the bilinear Hilbert transform. *arXiv preprint* (2022). arXiv:2205.09851. 322, 324, 328, 329, 341

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