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On pinned distance problem for Cartesian product sets: the parabolic method

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ABSTRACT. The Falconer distance problem for Cartesian product sets was introduced and studied by Iosevich and Liu ([13]). In this paper, by implementing a new observation on Cartesian product sets associated with a particular parabolic structure, we study the pinned version of Falconer distance problem for Cartesian product sets, and improve the threshold for the Falconer distance set in [13] in certain cases.

CONTENTS

1. Introduction and statement of main results

The Falconer distance conjecture ([8]) says that if the Hausdorff dimension of $E \subset \mathbb{R}^d$, $d \geq 2$, is greater than $\frac{d}{2}$, then the Lebesgue measure of the distance set $\Delta(E) = \{|x-y| \,:\, x, y \in E\}$ is positive. Recent celebrated results [12, 4, 5] show that for every compact subset E of $\mathbb{R}^d$ with $d \geq 2$, the Lebesgue measure of the distance set $\Delta(E) = \{|x-y| \,:\, x, y \in E\}$ is positive if the Hausdorff dimension of E satisfies $\dim_{\mathcal{H}}(E) > \frac{d}{2} + \frac{1}{4}$ when d is even, and $\dim_{\mathcal{H}}(E) > \frac{d}{2} + \frac{1}{4} + \frac{1}{8d-4}$ when d is odd. This improved the well-known result by Wolff [22] in two dimensions and Erdogan [7] in higher dimensions.

Recall that Falconer distance problem on Cartesian product sets was studied by Iosevich and Liu ([13]) via Mattila integral, which states as follows.

Theorem A ([13]). *Let $E = A \times B$, where A and B are compact subsets of $\mathbb{R}$ with positive s_A, s_B-dimensional Hausdorff measure, respectively. If $s_A + s_B + \max(s_A, s_B) > 2$, the Lebesgue measure of $\Delta(E)$ is positive.*

Theorem B ([13]). *Suppose that E is a compact subset of $\mathbb{R}^d$ of the form $A_1 \times A_2 \times \cdots \times A_d$, where $A_j \subset \mathbb{R}$ has positive s_j-dimensional Hausdorff measure for*

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all $1 \leq j \leq d$. *Suppose that* $\sum_{j=1}^{d} s_j > \frac{d^2}{2d-1}$. *Then the Lebesgue measure of* $\Delta(E)$ *is positive.*

Their result improved Erdogan's $\frac{d}{2} + \frac{1}{3}$ exponent in higher dimension for Cartesian products. Note also that they studied the pinned version of Falconer distance problem [15].

In this paper, we study the pinned version of Falconer distance problem for Cartesian product sets by implementing a new observation on Cartesian product sets associated with a particular parabolic structure. We improve the threshold for the Falconer distance set of product sets $A_1 \times A_2 \times \cdots \times A_d$ in Theorem **B** in a certain case which has been widely studied in numerous contexts ([14, 3, 2]).

We now state our main results in detail. In what follows, for any set $E \subset \mathbb{R}^d$ and let $x \in E$, we define $\triangle_x(E)$ to be the pinned version of the distance set, that is

$$\triangle_x(E) := \{|x-y| : y \in E\}.$$

Moreover, for a set $A \subset \mathbb{R}$, we denote $A^d = A \times \cdots \times A \subset \mathbb{R}^d$.

Theorem 1.1. *Let* $A, B \subset \mathbb{R}$ *be compact subsets and* $d \geq 3$.

1. If one of the following conditions holds:

$$\begin{cases} 1 < \dim_{\mathcal{H}}((A \cap B)^2) \leq \dfrac{14}{13} \\ \dim_{\mathcal{H}}(A) + \dfrac{(d-2)}{d} \dim_{\mathcal{H}}(B) + \dfrac{d-1}{d(d-2)} \dim_{\mathcal{H}}(A \cap B) \geq \dfrac{d^2 - 3d + \beta \cdot d + 1}{d(d-2)}, \end{cases}$$

$$\begin{cases} \dfrac{14}{13} < \dim_{\mathcal{H}}((A \cap B)^2) \leq \dfrac{5}{4} \\ \left[\dfrac{(d-2)}{d} \dim_{\mathcal{H}}(B) + \dim_{\mathcal{H}}(A)\right] + \dfrac{16d-16}{3d(d-2)} \cdot \dim_{\mathcal{H}}(A \cap B) > 1 + \dfrac{3d\beta + 4d - 4}{3d(d-2)}, \end{cases}$$

$$\begin{cases} \dfrac{5}{4} < \dim_{\mathcal{H}}((A \cap B)^2) \\ \dim_{\mathcal{H}}(A) + \dfrac{(d-2)}{d} \dim_{\mathcal{H}}(B) > 1 + \dfrac{\beta d - 2d + 2}{d(d-2)}, \end{cases}$$

then there is a point $(b_1, \ldots, b_d) \in B^d$ *such that the Hausdorff dimension of pinned distance set of the product* $\triangle_{(b_1,\ldots,b_d)}(A^d)$ *is no less than* β, *where* A^d *is the product set of* A.

2. If one of the following conditions holds:

$$\begin{cases} 1 < \dim_{\mathcal{H}}((A\cap B)^2) \le \dfrac{14}{13} \\ \dim_{\mathcal{H}}(A) + \dfrac{(d-2)}{d}\dim_{\mathcal{H}}(B) + \dfrac{d-1}{d(d-2)}\dim_{\mathcal{H}}(A\cap B) \ge \dfrac{d^2-d+1}{d(d-2)}, \end{cases}$$

$$\begin{cases} \dfrac{14}{13} < \dim_{\mathcal{H}}((A\cap B)^2) \le \dfrac{5}{4} \\ \left[\dfrac{(d-2)}{d}\dim_{\mathcal{H}}(B) + \dim_{\mathcal{H}}(A)\right] + \dfrac{16d-16}{3d(d-2)}\cdot \dim_{\mathcal{H}}(A\cap B) > 1 + \dfrac{10d-4}{3d(d-2)}, \end{cases}$$

$$\begin{cases} \dfrac{5}{4} < \dim_{\mathcal{H}}((A\cap B)^2) \\ \dim_{\mathcal{H}}(A) + \dfrac{(d-2)}{d}\dim_{\mathcal{H}}(B) > 1 + \dfrac{2}{d(d-2)}, \end{cases}$$

then there is $(b_1, \dots, b_d) \in B^d$ such that $\triangle_{(b_1,\dots,b_d)}(A^d)$ contains an non-empty interval, where A^d is the product set of A.

3. If one of the following conditions holds:

$$\begin{cases} 1 < \dim_{\mathcal{H}}((A\cap B)^2) \le \dfrac{14}{13} \\ \dim_{\mathcal{H}}(A) + \dfrac{(d-2)}{d}\dim_{\mathcal{H}}(B) + \dfrac{d-1}{d(d-2)}\dim_{\mathcal{H}}(A\cap B) \ge \dfrac{(d-1)^2}{d(d-2)}, \end{cases}$$

$$\begin{cases} \dfrac{14}{13} < \dim_{\mathcal{H}}((A\cap B)^2) \le \dfrac{5}{4} \\ \left[\dfrac{(d-2)}{d}\dim_{\mathcal{H}}(B) + \dim_{\mathcal{H}}(A)\right] + \dfrac{16d-16}{3d(d-2)}\cdot \dim_{\mathcal{H}}(A\cap B) > 1 + \dfrac{7d-4}{3d(d-2)}, \end{cases}$$

$$\begin{cases} \dfrac{5}{4} < \dim_{\mathcal{H}}((A\cap B)^2) \\ \dim_{\mathcal{H}}(A) + \dfrac{(d-2)}{d}\dim_{\mathcal{H}}(B) > 1 - \dfrac{1}{d}, \end{cases}$$

then there is $(b_1, \dots, b_d) \in B^d$ such that $\triangle_{(b_1,\dots,b_d)}(A^d)$ has positive Lebesgue measure, where A^d is the product set of A.

Note that our main theorem implies the following special case.

Corollary 1.2. *Let $A \subset \mathbb{R}$ be a compact subset and $d \ge 3$.*

1. If one of the following conditions holds:

$$\begin{cases} 1 < \dim_{\mathcal{H}}(A^2) \le \dfrac{14}{13} \\ \dim_{\mathcal{H}}(A) \ge \dfrac{d^2 - 3d + \beta \cdot d + 1}{(d-1)(2d-3)}, \end{cases}$$

$$\begin{cases} \dfrac{14}{13} < \dim_{\mathcal{H}}(A^2) \le \dfrac{5}{4} \\ \dim_{\mathcal{H}}(A) > \dfrac{3d(d-2+\beta)}{(d-1)(6d+4)} + \dfrac{2}{3d+2}, \end{cases}$$

$$\begin{cases} \dfrac{5}{4} < \dim_{\mathcal{H}}(A^2) \\ \dim_{\mathcal{H}}(A) > \dfrac{d^2 - 4d + \beta d + 2}{(d-2)(2d-2)}, \end{cases}$$

then there is $(y_1, \dots, y_d) \in A^d$ *such that the Hausdorff dimension of pinned distance set of the product* $\triangle_{(y_1,\dots,y_d)}(A^d)$ *is no less than* β*, where* A^d *is the product set of* A*.*

2. If one of the following conditions holds:

$$\begin{cases} 1 < \dim_{\mathcal{H}}(A^2) \le \dfrac{14}{13} \\ \dim_{\mathcal{H}}(A) \ge \dfrac{d^2 - d + 1}{(d-1)(2d-3)}, \end{cases}$$

$$\begin{cases} \dfrac{14}{13} < \dim_{\mathcal{H}}(A^2) \le \dfrac{5}{4} \\ \dim_{\mathcal{H}}(A) > \dfrac{1}{2} + \dfrac{5d-2}{(d-1)(6d+4)}, \end{cases}$$

$$\begin{cases} \dfrac{5}{4} < \dim_{\mathcal{H}}(A^2) \\ \dim_{\mathcal{H}}(A) > \dfrac{1}{2} + \dfrac{d}{(d-2)(2d-2)}, \end{cases}$$

then there is $(y_1, \dots, y_d) \in A^d$ *such that* $\triangle_{(y_1,\dots,y_d)}(A^d)$ *contains an non-empty interval, where* A^d *is the product set of* A*.*

3. If one of the following conditions holds:

$$\begin{cases} 1 < \dim_{\mathcal{H}}(A^2) \leq \dfrac{14}{13} \\ \dim_{\mathcal{H}}(A) \geq \dfrac{(d-1)^2}{(d-1)(2d-3)}, \end{cases}$$

$$\begin{cases} \dfrac{14}{13} < \dim_{\mathcal{H}}(A^2) \leq \dfrac{5}{4} \\ \dim_{\mathcal{H}}(A) > \dfrac{1}{2} + \dfrac{1}{3d+2}, \end{cases}$$

$$\begin{cases} \dfrac{5}{4} < \dim_{\mathcal{H}}(A^2) \\ \dim_{\mathcal{H}}(A) > \dfrac{1}{2}, \end{cases}$$

then there is $(y_1, \dots, y_d) \in A^d$ such that $\triangle_{(y_1,\dots,y_d)}(A^d)$ has positive Lebesgue measure, where A^d is the product set of A.

Based on part 3 in the Corollary 1.2 on the pinned version of the distance set, we improve the threshold for the Falconer distance set of product sets $A_1 \times A_2 \times \cdots \times A_d$ in [13] in certain case: the Hausdorff dimensions s_{d-1} and s_d of A_{d-1} and A_d, respectively, satisfying $s_{d-1} + s_d > \frac{5}{4}$. We state this in details in below.

Corollary 1.3. *Let $d \geq 3$. Suppose that E is a compact subset of $\mathbb{R}^d$ of the form $A_1 \times A_2 \times \cdots \times A_d$, where $A_j \subset \mathbb{R}$ has positive s_j-dimensional Hausdorff measure for all $1 \leq j \leq d$. Suppose that $\sum_{j=1}^{d-2} s_j > \frac{d}{2} - 1$ and $s_{d-1} + s_d > \frac{5}{4}$. Then there is a point $(y_1, \dots, y_d) \in E$ such that the Lebesgue measure of the pinned distance set $\Delta_{(y_1,\dots,y_d)}(E)$ is positive.*

Remark 1.4. *Note that Iosevich and Liu ([15]) studied the pinned version of distance set for general set $E, F \subset \mathbb{R}^d$. Our results improve the threshold obtained by [15] when applying their results to $E = A_1 \times \cdots \times A_d$ and $F = B_1 \times \cdots \times B_d$. In fact, one can see from our proofs below that as long as one can prove a variant of distance result for the particular parabolic distance, one can improve the result for the classical distance for product sets.*

Besides, there are other variant distance problems related to the classical Falconer distance set. (see for example [9, 10, 16, 6, 18, 11, 1, 19]) Our technique can be also applied to improve the threshold for those distance problems in the case of Cartesian product sets. Finally, before we proceed to prove our results, we note here that the key observation is to use parabolic distance which allows us to get improvement when we change to the usual Euclidean distance. In other words, the proofs in different theorems are similar.

2. Proofs of Theorem 1.1 and Corollary 1.2: parabolic attack

Let $\Phi : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}$ be a smooth function that satisfies *Phong–Stein rotation curvature condition* and *Sogge's cinematic curvature condition*, that is Φ has a nonzero Monge–Ampere determinant

$$\det\begin{pmatrix} 0 & \nabla_x\Phi \\ \nabla_y\Phi & \frac{\partial^2\Phi}{\partial x\partial y}\end{pmatrix} \neq 0$$

and for any $t > 0, x \in \mathbb{R}^d, \{\nabla_y\Phi : \Phi(x,y) = t\}$ has nonzero Gaussian curvature. In particular, the parabolic distance $\Phi(x,y) := (x_1-y_1)^2+\cdots+(x_{d-1}-y_{d-1})^2+(x_d-y_d)$ satisfies both curvature conditions.

Theorem 2.1 ([15]). *Let $\Phi \in C^\infty(\mathbb{R}^d\times\mathbb{R}^d)$, $E, F \subset \mathbb{R}^d$. Suppose that Φ satisfies the Phong–Stein rotation curvature condition and the cinematic curvature condition. Then there is a probability measure μ_F on F such that for μ_F-a.e. $y \in F$,*

(1) $\dim_{\mathcal{H}}(\triangle_y^\Phi(E)) \geq \beta$, *if* $\dim_{\mathcal{H}}(E) + \dfrac{d-1}{d+1}\dim_{\mathcal{H}}(F) > d-1+\beta$,

(2) $\left|\triangle_y^\Phi(E)\right| > 0$, *if* $\dim_{\mathcal{H}}(E) + \dfrac{d-1}{d+1}\dim_{\mathcal{H}}(F) > d$,

(3) $\left(\triangle_y^\Phi(E)\right)^0 \neq \emptyset$, *if* $\dim_{\mathcal{H}}(E) + \dfrac{d-1}{d+1}\dim_{\mathcal{H}}(F) > d+1$.

We then have a direct corollary from Theorem 2.1 as follows when applying their E and F to product sets.

Corollary 2.2. *In particular, if $E = F := A\times\cdots\times A \subset \mathbb{R}^d$, where $A \subset \mathbb{R}$ is a compact subset. Then there is a probability measure μ_F on F such that for μ_F-a.e $y \in F$,*

(1) $\dim_{\mathcal{H}}(A) > \dfrac{(d-1+\beta)(d+1)}{2d^2} \implies \dim_{\mathcal{H}}(\triangle_y^\Phi(E)) \geq \beta$,

(2) $\dim_{\mathcal{H}}(A) > \dfrac{d+1}{2d} \implies \left|\triangle_y^\Phi(E)\right| > 0$,

(3) $\dim_{\mathcal{H}}(A) > \dfrac{(d+1)^2}{2d^2} \implies \left(\triangle_y^\Phi(E)\right)^0 \neq \emptyset$.

Remark 2.3. *The threshold in Corollary* 1.2 *is better than the threshold in Corollary* 2.2 *for all large d.*

Next, we recall the following two auxiliary lemmas.

Lemma 2.4 ([17]). *For any compact set $\Omega \subset \mathbb{R}^2$ with $\dim_{\mathcal{H}}(\Omega) > 1$, there exists a point $x \in \Omega$ such that*

$$\dim_{\mathcal{H}}(\triangle_x(\Omega)) \geq \min\left\{\frac{4}{3}\cdot\dim_{\mathcal{H}}(\Omega) - \frac{2}{3},\ 1\right\}.$$

Shmerkin [20] improved this bound for small values of $\dim_{\mathcal{H}}(\Omega) \in (1,\ 1.04)$, $\dim_{\mathcal{H}}(\triangle_x(\Omega)) > \frac{2}{3}+\frac{1}{42}$ for many x. Very recently, D.M. Stull further improves the lower bound of the dimension of the pinned planar distance set for the small value of $\dim_{\mathcal{H}}(\Omega)$ by using the effective dimension.

Lemma 2.5 ([21]). *For any analytic set* $\Omega \subset \mathbb{R}^2$ *with* $\dim_{\mathcal{H}}(\Omega) > 1$*, there exists a point* $x \in \Omega$ *such that*

$$\dim_{\mathcal{H}}(\triangle_x(\Omega)) \geq \frac{\dim_{\mathcal{H}}(\Omega)}{4} + \frac{1}{2}.$$

Remark 2.6. *When* $\dim_{\mathcal{H}}(\Omega)$ *is close to* 1*, then the lower bound obtained by Stull is bigger than the lower bound* $\min\left\{\frac{4}{3}\dim_{\mathcal{H}}(\Omega) - \frac{2}{3}, 1\right\}$*. To compare these two lemmas, one can actually have that there is a point* $x \in \Omega$ *such that*

$$\dim_{\mathcal{H}}(\triangle_x(\Omega)) \geq \frac{\dim_{\mathcal{H}}(\Omega)}{4} + \frac{1}{2}, \ \textit{if} \ 1 < \dim_{\mathcal{H}}(\Omega) \leq \frac{14}{13}; \textit{ and}$$

$$\dim_{\mathcal{H}}(\triangle_x(\Omega)) \geq \begin{cases} \frac{4}{3}\dim_{\mathcal{H}}(\Omega) - \frac{2}{3}, \ \textit{if} \ \frac{14}{13} < \dim_{\mathcal{H}}(\Omega) \leq \frac{5}{4}, \\ 1, \ \textit{if} \ \frac{5}{4} < \dim_{\mathcal{H}}(\Omega). \end{cases} \tag{2.1}$$

2.1. Proof of Theorem 1.1. Let $d \geq 3$ and $y_{d-1}, y_d \in A \cap B$ and $x_0, x_1 \in B$. Consider the sets in $\mathbb{R}^{d-1}$.

$$E := (A \times \cdots \times A) \times \triangle^2_{(y_{d-1}, y_d)}(A^2) \subset \mathbb{R}^{d-2} \times \mathbb{R}^1; \text{ and}$$
$$F := (B \times \cdots \times B) \times - \triangle^2_{(x_0, x_1)}(B^2) \subset \mathbb{R}^{d-2} \times \mathbb{R}^1.$$

If $\Phi_{d-1} : \mathbb{R}^{d-1} \times \mathbb{R}^{d-1} \longrightarrow \mathbb{R}$ is the parabolic distance, then there is a probability measure μ_F on F such that for μ_F-a.e. $y \in F$ which is of the form

$$\left(b_1, \dots, b_{d-2}, -\left(|x_0 - b_{d-1}|^2 + |x_1 - b_d|^2\right)\right),$$

satisfying

(1) $\dim_{\mathcal{H}}(\triangle^{\Phi}_y(E)) \geq \beta$, if $\dim_{\mathcal{H}}(E) + \dfrac{d-2}{d}\dim_{\mathcal{H}}(F) > d - 2 + \beta$,

(2) $\left|\triangle^{\Phi}_y(E)\right| > 0$, if $\dim_{\mathcal{H}}(E) + \dfrac{d-2}{d}\dim_{\mathcal{H}}(F) > d - 1$,

(3) $\left(\triangle^{\Phi}_y(E)\right)^0 \neq \emptyset$, if $\dim_{\mathcal{H}}(E) + \dfrac{d-2}{d}\dim_{\mathcal{H}}(F) > d$.

However note that

$$\triangle^{\Phi}_y(E) = \triangle^2_{(b_1, \dots, b_{d-2}, y_{d-1}, y_d)}(A^d) + |x_0 - b_{d-1}|^2 + |x_1 - b_d|^2,$$

which says the set $\triangle^2_{(y_1, \dots, y_d)}(A^d)$ is a translation of $\triangle^{\Phi}_y(E)$ so that we have

(1) $\dim_{\mathcal{H}}(\triangle^{\Phi}_y(E)) = \dim_{\mathcal{H}}(\triangle^2_{(b_1, \dots, b_{d-2}, y_{d-1}, y_d)}(A^d))$
$= \dim_{\mathcal{H}}(\triangle_{(b_1, \dots, b_{d-2}, y_{d-1}, y_d)}(A^d))$, and

(2) $\left|\triangle^{\Phi}_y(E)\right| = \left|\triangle^2_{(b_1, \dots, b_{d-2}, y_{d-1}, y_d)}(A^d)\right| \leq C_A \cdot \left|\triangle_{(b_1, \dots, b_{d-2}, y_{d-1}, y_d)}(A^d)\right|$, and

(3) $\left(\triangle_y^{\Phi}(E)\right)^0 \neq \emptyset \implies \left(\triangle_{(b_1,\ldots,b_{d-2},y_{d-1},y_d)}(A^d)\right)^0 \neq \emptyset.$

Then one has

(1) $\dim_{\mathcal{H}}(\triangle_{(b_1,\ldots,b_{d-2},y_{d-1},y_d)}(A^d)) \geq \beta,$
if $\dim_{\mathcal{H}}(E) + \dfrac{d-2}{d}\dim_{\mathcal{H}}(F) > d-2+\beta,$

(2) $\left|\triangle_{(b_1,\ldots,b_{d-2},y_{d-1},y_d)}(A^d)\right| > 0,$ if $\dim_{\mathcal{H}}(E) + \dfrac{d-2}{d}\dim_{\mathcal{H}}(F) > d-1,$

(3) $\left(\triangle_{(b_1,\ldots,b_{d-2},y_{d-1},y_d)}(A^d)\right)^0 \neq \emptyset,$ if $\dim_{\mathcal{H}}(E) + \dfrac{d-2}{d}\dim_{\mathcal{H}}(F) > d.$

To guarantee the condition $\dim_{\mathcal{H}}(E)+\frac{d-2}{d}\dim_{\mathcal{H}}(F) > d-2+\beta$ holds, we note that

$$\begin{aligned}
&\dim_{\mathcal{H}}(E) + \frac{d-2}{d}\dim_{\mathcal{H}}(F)\\
&\geq (d-2)\dim_{\mathcal{H}}(A) + \dim_{\mathcal{H}}(\triangle_{(y_{d-1},y_d)}(A^2))\\
&\quad + \frac{d-2}{d}\cdot\left((d-2)\dim_{\mathcal{H}}(B) + \dim_{\mathcal{H}}(\triangle_{(x_0,x_1)}(B^2))\right)\\
&= (d-2)\left[\frac{(d-2)}{d}\dim_{\mathcal{H}}(B) + \dim_{\mathcal{H}}(A)\right] + \dim_{\mathcal{H}}(\triangle_{(y_{d-1},y_d)}(A^2))\\
&\quad + \frac{d-2}{d}\dim_{\mathcal{H}}(\triangle_{(x_0,x_1)}(B^2)).
\end{aligned} \tag{2.2}$$

To apply Lemma 2.4 and Lemma 2.5, we need to assume that the Hausdorff dimension of the set $A\cap B$ is at least $\frac{1}{2}$, and hence we have the following three cases:

Case(1): Assume $1 < \dim_{\mathcal{H}}((A\cap B)^2) \leq \frac{14}{13}$, then (2.2) reveals that if we choose the point $(y_{d-1}, y_d) = (x_0, x_1) \in A\cap B$ such that Lemma 2.5 holds for the set $(A\cap B)^2$, then

$$\begin{aligned}
&\dim_{\mathcal{H}}(E) + \frac{d-2}{d}\dim_{\mathcal{H}}(F)\\
&\geq (d-2)\left[\frac{(d-2)}{d}\dim_{\mathcal{H}}(B) + \dim_{\mathcal{H}}(A)\right] + \dim_{\mathcal{H}}(\triangle_{(y_{d-1},y_d)}((A\cap B)^2))\\
&\quad + \frac{d-2}{d}\dim_{\mathcal{H}}(\triangle_{(x_0,x_1)}((A\cap B)^2))\\
&> (d-2)\left[\frac{(d-2)}{d}\dim_{\mathcal{H}}(B) + \dim_{\mathcal{H}}(A)\right] + \frac{2d-2}{d}\cdot\left[\frac{2\dim_{\mathcal{H}}(A\cap B)}{4} + \frac{1}{2}\right].
\end{aligned}$$

Suppose that

$$\begin{aligned}
&(d-2)\left[\frac{(d-2)}{d}\dim_{\mathcal{H}}(B) + \dim_{\mathcal{H}}(A)\right] + \frac{2d-2}{d}\cdot\left[\frac{2\dim_{\mathcal{H}}(A\cap B)}{4} + \frac{1}{2}\right]\\
&\geq d-2+\beta,
\end{aligned}$$

then we have

$$\dim_{\mathcal{H}}(A) + \frac{(d-2)}{d}\dim_{\mathcal{H}}(B) + \frac{d-1}{d(d-2)}\dim_{\mathcal{H}}(A\cap B) \geq \frac{d^2-3d+\beta\cdot d+1}{d(d-2)},$$

whenever $d \geq 3$.

Case(2): Assume $\frac{14}{13} < \dim_{\mathcal{H}}((A\cap B)^2) \leq \frac{5}{4}$, then (2.2) reveals that if we choose the point $(y_{d-1}, y_d) = (x_0, x_1)$ such that Lemma 2.4 holds for the set $(A\cap B)^2$, then

$$\begin{aligned}\dim_{\mathcal{H}}(E) + \frac{d-2}{d}\dim_{\mathcal{H}}(F) &\geq (d-2)\left[\frac{(d-2)}{d}\dim_{\mathcal{H}}(B) + \dim_{\mathcal{H}}(A)\right] \\ &\quad + \frac{2d-2}{d}\cdot\left(\frac{4}{3}\dim_{\mathcal{H}}((A\cap B)^2) - \frac{2}{3}\right) \\ &\geq (d-2)\left[\frac{(d-2)}{d}\dim_{\mathcal{H}}(B) + \dim_{\mathcal{H}}(A)\right] \\ &\quad + \frac{16d-16}{3d}\cdot\dim_{\mathcal{H}}(A\cap B) - \frac{4d-4}{3d}.\end{aligned}$$

Suppose that

$$(d-2)\left[\frac{(d-2)}{d}\dim_{\mathcal{H}}(B) + \dim_{\mathcal{H}}(A)\right] + \frac{16d-16}{3d}\cdot\dim_{\mathcal{H}}(A\cap B) - \frac{4d-4}{3d}$$
$$> d-2+\beta,$$

then we have

$$\left[\frac{(d-2)}{d}\dim_{\mathcal{H}}(B) + \dim_{\mathcal{H}}(A)\right] + \frac{16d-16}{3d(d-2)}\cdot\dim_{\mathcal{H}}(A\cap B)$$
$$> 1 + \frac{3d\beta+4d-4}{3d(d-2)},$$

whenever $d \geq 3$.

Case(3): Assume $\frac{5}{4} < \dim_{\mathcal{H}}(A^2)$, then (2.2) reveals that if we choose the point $(y_{d-1}, y_d) = (x_0, x_1)$ such that Lemma 2.4 holds for the set $(A\cap B)^2$, then

$$\dim_{\mathcal{H}}(E) + \frac{d-2}{d}\dim_{\mathcal{H}}(F) \geq (d-2)\left[\frac{(d-2)}{d}\dim_{\mathcal{H}}(B) + \dim_{\mathcal{H}}(A)\right] + \frac{2d-2}{d}.$$

Suppose that

$$(d-2)\left[\frac{(d-2)}{d}\dim_{\mathcal{H}}(B) + \dim_{\mathcal{H}}(A)\right] + \frac{2d-2}{d} > d-2+\beta,$$

then we have

$$\dim_{\mathcal{H}}(A) + \frac{(d-2)}{d}\dim_{\mathcal{H}}(B) > 1 + \frac{\beta d - 2d + 2}{d(d-2)},$$

whenever $d \geq 3$. Therefore, we can conclude that for all $d \geq 3$, if one of the following holds,

$$\begin{cases} 1 < \dim_{\mathcal{H}}((A \cap B)^2) \leq \dfrac{14}{13} \\ \dim_{\mathcal{H}}(A) + \dfrac{(d-2)}{d} \dim_{\mathcal{H}}(B) + \dfrac{d-1}{d(d-2)} \dim_{\mathcal{H}}(A \cap B) \geq \dfrac{d^2 - 3d + \beta \cdot d + 1}{d(d-2)}, \end{cases}$$

$$\begin{cases} \dfrac{14}{13} < \dim_{\mathcal{H}}((A \cap B)^2) \leq \dfrac{5}{4} \\ \left[\dfrac{(d-2)}{d} \dim_{\mathcal{H}}(B) + \dim_{\mathcal{H}}(A)\right] + \dfrac{16d-16}{3d(d-2)} \cdot \dim_{\mathcal{H}}(A \cap B) > 1 + \dfrac{3d\beta + 4d - 4}{3d(d-2)}, \end{cases}$$

$$\begin{cases} \dfrac{5}{4} < \dim_{\mathcal{H}}((A \cap B)^2) \\ \dim_{\mathcal{H}}(A) + \dfrac{(d-2)}{d} \dim_{\mathcal{H}}(B) > 1 + \dfrac{\beta d - 2d + 2}{d(d-2)}, \end{cases}$$

then the Hausdorff dimension of pinned distance set of the product A^d is no less than β, which improve the threshold in the case of product sets. Similarly, we have for all $d \geq 3$, if one of the following holds,

$$\begin{cases} 1 < \dim_{\mathcal{H}}((A \cap B)^2) \leq \dfrac{14}{13} \\ \dim_{\mathcal{H}}(A) + \dfrac{(d-2)}{d} \dim_{\mathcal{H}}(B) + \dfrac{d-1}{d(d-2)} \dim_{\mathcal{H}}(A \cap B) \geq \dfrac{(d-1)^2}{d(d-2)}, \end{cases}$$

$$\begin{cases} \dfrac{14}{13} < \dim_{\mathcal{H}}((A \cap B)^2) \leq \dfrac{5}{4} \\ \left[\dfrac{(d-2)}{d} \dim_{\mathcal{H}}(B) + \dim_{\mathcal{H}}(A)\right] + \dfrac{16d-16}{3d(d-2)} \cdot \dim_{\mathcal{H}}(A \cap B) > 1 + \dfrac{7d-4}{3d(d-2)}, \end{cases}$$

$$\begin{cases} \dfrac{5}{4} < \dim_{\mathcal{H}}((A \cap B)^2) \\ \dim_{\mathcal{H}}(A) + \dfrac{(d-2)}{d} \dim_{\mathcal{H}}(B) > 1 - \dfrac{1}{d}, \end{cases}$$

then the pinned distance set of the product A^d has positive Lebesgue measure; and if one of the following holds,

$$\begin{cases} 1 < \dim_{\mathcal{H}}((A \cap B)^2) \leq \dfrac{14}{13} \\ \dim_{\mathcal{H}}(A) + \dfrac{(d-2)}{d} \dim_{\mathcal{H}}(B) + \dfrac{d-1}{d(d-2)} \dim_{\mathcal{H}}(A \cap B) \geq \dfrac{d^2 - d + 1}{d(d-2)}, \end{cases}$$

$$\begin{cases} \frac{14}{13} < \dim_{\mathcal{H}}((A\cap B)^2) \le \frac{5}{4} \\ \left[\frac{(d-2)}{d}\dim_{\mathcal{H}}(B) + \dim_{\mathcal{H}}(A)\right] + \frac{16d-16}{3d(d-2)}\cdot \dim_{\mathcal{H}}(A\cap B) > 1 + \frac{10d-4}{3d(d-2)}, \end{cases}$$

$$\begin{cases} \frac{5}{4} < \dim_{\mathcal{H}}((A\cap B)^2) \\ \dim_{\mathcal{H}}(A) + \frac{(d-2)}{d}\dim_{\mathcal{H}}(B) > 1 + \frac{2}{d(d-2)}, \end{cases}$$

then the pinned distance set of the product A^d contains an interval, which also improve the threshold in the case of product sets.

2.2. Proof of Corollary 1.2. Let $d \ge 3$ and $y_{d-1}, y_d, x_0, x_1 \in A$. Consider the sets in $\mathbb{R}^{d-1}$.

$$E := (A\times\cdots\times A)\times \triangle^2_{(y_{d-1},y_d)}(A^2) \subset \mathbb{R}^{d-2}\times\mathbb{R}^1; \text{ and}$$
$$F := (A\times\cdots\times A)\times -\triangle^2_{(x_0,x_1)}(A^2) \subset \mathbb{R}^{d-2}\times\mathbb{R}^1.$$

If $\Phi_{d-1} : \mathbb{R}^{d-1}\times\mathbb{R}^{d-1} \longrightarrow \mathbb{R}$ is the parabolic distance, then there is a probability measure μ_F on F such that for μ_F-a.e. $y\in F$ which is of the form

$$\left(y_1,\dots,y_{d-2}, -\left(|x_0-a_0|^2+|x_1-a_1|^2\right)\right),$$

satisfying

(1) $\dim_{\mathcal{H}}(\triangle^{\Phi}_y(E)) \ge \beta$, if $\dim_{\mathcal{H}}(E) + \frac{d-2}{d}\dim_{\mathcal{H}}(F) > d-2+\beta$,

(2) $\left|\triangle^{\Phi}_y(E)\right| > 0$, if $\dim_{\mathcal{H}}(E) + \frac{d-2}{d}\dim_{\mathcal{H}}(F) > d-1$,

(3) $\left(\triangle^{\Phi}_y(E)\right)^0 \ne \emptyset$, if $\dim_{\mathcal{H}}(E) + \frac{d-2}{d}\dim_{\mathcal{H}}(F) > d$.

However note that

$$\triangle^{\Phi}_y(E) = \triangle^2_{(y_1,\dots,y_d)}(A^d) + |x_0-a_0|^2 + |x_1-a_1|^2,$$

which says the set $\triangle^2_{(y_1,\dots,y_d)}(A^d)$ is a translation of $\triangle^{\Phi}_y(E)$ so that we have

(1) $\dim_{\mathcal{H}}(\triangle^{\Phi}_y(E)) = \dim_{\mathcal{H}}(\triangle^2_{(y_1,\dots,y_d)}(A^d)) = \dim_{\mathcal{H}}(\triangle_{(y_1,\dots,y_d)}(A^d))$, and

(2) $\left|\triangle^{\Phi}_y(E)\right| = \left|\triangle^2_{(y_1,\dots,y_d)}(A^d)\right| \le C_A\cdot\left|\triangle_{(y_1,\dots,y_d)}(A^d)\right|$, and

(3) $\left(\triangle^{\Phi}_y(E)\right)^0 \ne \emptyset \implies \left(\triangle_{(y_1,\dots,y_d)}(A^d)\right)^0 \ne \emptyset$.

Then one has

(1) $\dim_{\mathcal{H}}(\triangle_{(y_1,\dots,y_d)}(A^d)) \ge \beta$, if $\dim_{\mathcal{H}}(E) + \frac{d-2}{d}\dim_{\mathcal{H}}(F) > d-2+\beta$,

(2) $\left|\triangle_{(y_1,\dots,y_d)}(A^d)\right| > 0$, if $\dim_{\mathcal{H}}(E) + \dfrac{d-2}{d}\dim_{\mathcal{H}}(F) > d-1$,

(3) $\left(\triangle_{(y_1,\dots,y_d)}(A^d)\right)^0 \neq \emptyset$, if $\dim_{\mathcal{H}}(E) + \dfrac{d-2}{d}\dim_{\mathcal{H}}(F) > d$.

To guarantee the condition $\dim_{\mathcal{H}}(E)+\frac{d-2}{d}\dim_{\mathcal{H}}(F) > d-2+\beta$ holds, we note that

$$\begin{aligned}
&\dim_{\mathcal{H}}(E) + \frac{d-2}{d}\dim_{\mathcal{H}}(F)\\
&\geq (d-2)\dim_{\mathcal{H}}(A) + \dim_{\mathcal{H}}(\triangle_{(y_{d-1},y_d)}(A^2))\\
&\quad + \frac{d-2}{d}\cdot\left((d-2)\dim_{\mathcal{H}}(A) + \dim_{\mathcal{H}}(\triangle_{(x_0,x_1)}(A^2))\right)\\
&= \frac{(2d-2)\cdot(d-2)}{d}\dim_{\mathcal{H}}(A) + \dim_{\mathcal{H}}(\triangle_{(y_{d-1},y_d)}(A^2))\\
&\quad + \frac{d-2}{d}\dim_{\mathcal{H}}(\triangle_{(x_0,x_1)}(A^2)).
\end{aligned} \tag{2.3}$$

To apply Lemma 2.4 and Lemma 2.5, we need to assume that the Hausdorff dimension of the set A is at least $\frac{1}{2}$, and hence we have the following three cases:

Case(1): Assume $1 < \dim_{\mathcal{H}}(A^2) \leq \frac{14}{13}$, then (2.3) reveals that if we choose the point $(y_{d-1}, y_d) = (x_0, x_1)$ such that Lemma 2.5 holds, then

$$\begin{aligned}
&\dim_{\mathcal{H}}(E) + \frac{d-2}{d}\dim_{\mathcal{H}}(F)\\
&\qquad > \frac{(2d-2)\cdot(d-2)}{d}\dim_{\mathcal{H}}(A) + \frac{2d-2}{d}\cdot\left[\frac{2\dim_{\mathcal{H}}(A)}{4} + \frac{1}{2}\right].
\end{aligned}$$

Suppose that

$$\frac{(2d-2)\cdot(d-2)}{d}\dim_{\mathcal{H}}(A) + \frac{2d-2}{d}\cdot\left[\frac{2\dim_{\mathcal{H}}(A)}{4} + \frac{1}{2}\right] \geq d-2+\beta,$$

then we have

$$\dim_{\mathcal{H}}(A) \geq \frac{d^2-3d+\beta\cdot d+1}{(d-1)(2d-3)},$$

whenever $d \geq 3$.

Case(2): Assume $\frac{14}{13} < \dim_{\mathcal{H}}(A^2) \leq \frac{5}{4}$, then (2.3) reveals that if we choose the point $(y_{d-1}, y_d) = (x_0, x_1)$ such that Lemma 2.4 holds, then

$$\begin{aligned}
&\dim_{\mathcal{H}}(E) + \frac{d-2}{d}\dim_{\mathcal{H}}(F)\\
&\geq \frac{(2d-2)\cdot(d-2)}{d}\dim_{\mathcal{H}}(A) + \frac{2d-2}{d}\cdot\left(\frac{8}{3}\dim_{\mathcal{H}}(A) - \frac{2}{3}\right)
\end{aligned}$$

$$= \frac{(d-1)(6d+4)}{3d} \dim_{\mathcal{H}}(A) - \frac{4(d-1)}{3d}.$$

Suppose that $\frac{(d-1)(6d+4)}{3d} \dim_{\mathcal{H}}(A) - \frac{4(d-1)}{3d} > d-2+\beta$, then we have

$$\dim_{\mathcal{H}}(A) > \frac{3d(d-2+\beta)}{(d-1)(6d+4)} + \frac{2}{3d+2},$$

whenever $d \geq 3$.

Case(3): Assume $\frac{5}{4} < \dim_{\mathcal{H}}(A^2)$, then (2.3) reveals that if we choose the point $(y_{d-1}, y_d) = (x_0, z_0)$ such that Lemma 2.4 holds, then

$$\dim_{\mathcal{H}}(E) + \frac{d-2}{d} \dim_{\mathcal{H}}(F) \geq \frac{(2d-2)\cdot(d-2)}{d} \dim_{\mathcal{H}}(A) + \frac{2d-2}{d}.$$

Suppose that $\frac{(2d-2)\cdot(d-2)}{d} \dim_{\mathcal{H}}(A) + \frac{2d-2}{d} > d-2+\beta$, then we have

$$\dim_{\mathcal{H}}(A) > \frac{d^2-4d+\beta d+2}{(d-2)(2d-2)},$$

whenever $d \geq 3$. Therefore, we can conclude that for all $d \geq 3$, if one of the following holds,

$$\begin{cases} 1 < \dim_{\mathcal{H}}(A^2) \leq \frac{14}{13} \\ \dim_{\mathcal{H}}(A) \geq \frac{d^2-3d+\beta\cdot d+1}{(d-2)(2d-3)}, \end{cases}$$

$$\begin{cases} \frac{14}{13} < \dim_{\mathcal{H}}(A^2) \leq \frac{5}{4} \\ \dim_{\mathcal{H}}(A) > \frac{3d(d-2+\beta)}{(d-1)(6d+4)} + \frac{2}{3d+2}, \end{cases}$$

$$\begin{cases} \frac{5}{4} < \dim_{\mathcal{H}}(A^2) \\ \dim_{\mathcal{H}}(A) > \frac{d^2-4d+\beta d+2}{(d-2)(2d-2)}, \end{cases}$$

then the Hausdorff dimension of pinned distance set of the product A^d is no less than β, which improve the threshold in the case of product sets. Similarly, one has the threshold for the product set such that the pinned distance set contains an interval and has positive Lebesgue measure. The proof is complete.

2.3. Proof of Corollary 1.3. Let $d \geq 3$ and $y_{d-1} \in A_{d-1}$, $y_d \in A_d$. Consider the sets in $\mathbb{R}^{d-1}$.

$$G := (A_1 \times \cdots \times A_{d-2}) \times \triangle^2_{(y_{d-1},y_d)}(A_{d-1} \times A_d) \subset \mathbb{R}^{d-2} \times \mathbb{R}^1; \text{ and}$$
$$F := (A_1 \times \cdots \times A_{d-2}) \times -\triangle^2_{(y_{d-1},y_d)}(A_{d-1} \times A_d) \subset \mathbb{R}^{d-2} \times \mathbb{R}^1.$$

If $\Phi_{d-1} : \mathbb{R}^{d-1} \times \mathbb{R}^{d-1} \longrightarrow \mathbb{R}$ is the parabolic distance, then there is a probability measure μ_F on F such that for μ_F-a.e. $y \in F$ which is of the form

$$\left(y_1, \ldots, y_{d-2}, -\left(|y_{d-1} - a_0|^2 + |y_d - a_1|^2\right)\right),$$

satisfying

$$\left|\triangle^\Phi_y(G)\right| > 0, \text{ if } \dim_{\mathcal{H}}(G) + \frac{d-2}{d} \dim_{\mathcal{H}}(F) > d-1.$$

However note that

$$\triangle^\Phi_y(G) = \triangle^2_{(y_1,\ldots,y_d)}(A_1 \times \cdots \times A_d) + |y_{d-1} - a_0|^2 + |y_d - a_1|^2,$$

which says the set $\triangle^2_{(y_1,\ldots,y_d)}(A_1 \times \cdots \times A_d)$ is a translation of $\triangle^\Phi_y(G)$ so that we have

$$\left|\triangle^\Phi_y(G)\right| = \left|\triangle^2_{(y_1,\ldots,y_d)}(A_1 \times \cdots \times A_d)\right| \leq C_A \cdot \left|\triangle_{(y_1,\ldots,y_d)}(A_1 \times \cdots \times A_d)\right|.$$

Then one has

$$\left|\triangle^2_{(y_1,\ldots,y_d)}(A_1 \times \cdots \times A_d)\right| > 0, \text{ if } \dim_{\mathcal{H}}(G) + \frac{d-2}{d} \dim_{\mathcal{H}}(F) > d-1,$$

To guarantee that the Hausdorff dimension of G and F fit the threshold, we may hope that

$$\frac{2d-2}{d}\left(\sum_{j=1}^{d-2} s_j + \dim_{\mathcal{H}}\left(\triangle^2_{(y_{d-1},y_y)}(A_{d-1} \times A_d)\right)\right) > d-1; \tag{2.4}$$

By Lemma2.4, we have if $s_{d-1} + s_d > \frac{5}{4}$, then $\dim_{\mathcal{H}}(A_{d-1} \times A_d) > \frac{5}{4}$ and hence

$$\dim_{\mathcal{H}}\left(\triangle^2_{(y_{d-1},y_y)}(A_{d-1} \times A_d)\right) = \dim_{\mathcal{H}}\left(\triangle_{(y_{d-1},y_y)}(A_{d-1} \times A_d)\right) \geq 1. \tag{2.5}$$

Combining (2.4) and (2.5), we have if $\sum_{j=1}^{d-2} s_j > \frac{d}{2} - 1$, then

$$\left|\triangle^2_{(y_1,\ldots,y_d)}(A_1 \times \cdots \times A_d)\right| > 0.$$

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