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Kähler-Einstein submanifolds of Cartan-Hartogs domains

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ABSTRACT. In this short paper, we show that any Kähler-Ricci soliton on a complex manifold which admits a holomorphic isometric embedding into a Hartogs domain over an irreducible bounded symmetric domain equipped with the Bergman metric must be a Kähler-Einstein metric.

CONTENTS

1. Introduction

Holomorphic isometric embedding is an important topic in complex geometry. Calabi [2] obtained the celebrated results on the existence, global extendability and rigidity of a local holomorphic isometric embedding into a complex space form. In particular, Calabi proved that any complex space form cannot be locally holomorphically isometrically embedded into another complex space form with a different curvature sign with respect to the canonical Kähler metrics. Calabi's orginal idea is to reduce the metric tensor equation to the functional identity involving the diastasis functions he introduced. On the other hand, Umehara [14] studied a more general question whether two complex space forms can share a common complex submanifold with the induced metrics and proved that two complex space forms with different curvature signs cannot share a common Kähler submanifold by using Calabi's diastasis function. When two complex manifolds share a common Kähler submanifolds with induced metrics, they are called relatives by Di Scala and Loi [7]. Furthermore, Di Scala and Loi proved that a bounded domain equipped with its canonical Bergman metric cannot be a relative to a Hermitian symmetric space of compact type equipped with the canonical metric [7]. Using the method of Nash algebraic functions developed in [8], Huang and Yuan showed that a complex

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Euclidean space and a Hermitian symmetric space of noncompact type cannot be relatives [9]. This method introduced by Huang and Yuan turns out to be very powerful and has been used by many authors in the study of relativity problem (cf. [3], [17], [4], [19], [5] and references therein).

More recently, in a series of papers, Loi and Mossa considered the very interesting rigidity problem of Kähler-Ricci solitons that are holomophically isometrically embedded in Kähler manifold with canonical metrics, such as (indefinite) complex space forms, bounded homogeneous domains, flag manifolds of special type [11], [12], [13]. In particular, they showed that these Kähler-Ricci solitons must be trivial, i.e. Kähler-Einstein metrics. Recall that a Kähler metric ω_M on a complex manifold M is called a Kähler-Ricci soliton if there exists a holomorphic vector field X such that $Ric(\omega_M) = \lambda\omega_M + L_X\omega_M$. When $X = 0$ or X is Killing with respect to ω_M, then the Kähler-Ricci soliton becomes a Kähler-Einstein metric. In Loi and Mossa's proofs, one crucial ingredient is the method of Nash algebraic functions developed in [8], [9] by Huang and Yuan.

One interesting object in the study of the relativity problem is the Cartan-Hartogs domain, which differs from the aforementioned objects because its Bergman kernel function is much more complicated than being the product of Nash algebraic functions to some powers. It is suspected that the Cartan-Hartogs domain and the complex Euclidean space cannot be relatives. Only partial results were obtained in [6], [4] before the full solution in [18] by Ji and the author, where the automorphism group is used to handle the part of non-Nash algebraic functions in the Bergman kernel. In this paper, along the study of [11], [12], [13], we consider the rigidity problem of Kähler-Ricci solitons that are holomophically isometrically embedded in the Cartan-Hartogs domain. Note that the case of the general Cartan-Hartogs domain is not covered in the studies of Loi and Mossa, where the potential functions of canonical metrics of the ambient spaces are the product of Nash algebraic functions to some powers.

The Cartan-Hartogs domain is defined as follows. Let $D \subset \mathbb{C}^d$ be a domain and φ be a continuous positive function on D. The domain

$$\Omega = \left\{(\xi, Z) \in \mathbb{C}^{d_0} \times D \,:\, |\xi|^2 < \varphi(Z)\right\} \tag{1}$$

is called a Hartogs domain over D with d_0-dimensional fibers. When D is a bounded homogeneous domain and $\varphi(Z) = K_D(Z, \overline{Z})^{-s}$, the Bergman kernel $K((Z,\xi), \overline{(W,\eta)})$ of Ω is obtained by Ishi, Park and Yamamori [10]. In particular, when D is a bounded symmetric domain, Ω is called a Cartan-Hartogs domain, whose Bergman kernel was obtained in [16] earlier. Denote the Bergman metric on Ω by ω_Ω, given by $\omega_\Omega = \sqrt{-1}\partial\bar\partial \log K((Z,\xi), \overline{(Z,\xi)})$.

If D is a bounded homogeneous domain, by the formula in [10], the Bergman kernel $K((Z,\xi), \overline{(W,\eta)})$ is a rational function, for $s \in \mathbb{Z}$. Then the rigidity problem of Kähler-Ricci solitons in this case is covered in [12]. However, in general, for $s \notin \mathbb{Q}$, $K((Z,\xi),\overline{(W,\eta)})$ becomes much more complicated and the general argument in [9] is inapplicable. The result of this paper is:

Theorem 1.1. *Let Ω be a Hartogs domain over an irreducible bounded symmetric domain D defined by (1). Let ω_M be a Kähler-Ricci soliton on a complex manifold M and $F : (M,\omega_M) \to (\Omega,\omega_\Omega)$ be a local holomorphic isometric embedding. Then ω_M must be a Kähler-Einstein metric.*

The proof of Theorem 1.1 is based on the general strategy in [8] [9], the reduction to functional equation introduced in [11], and the new input developed in [18] to treat the functional equation.

We conclude the paper with a remark. One notes that Theorem 1.1 also holds for the Hartogs domain Ω over a bounded, complete circular, homogeneous, Lu Qi-Keng domain equipped with the Bergman metric. All the argument works the same provided that Lemma 2.1 holds for such Hartogs domain and we explain there why this is the case.

2. Proof of Theorem 1.1

Let us now collect some results that will be used in the proof of Theorem 1.1. The first lemma is a consequence of Proposition 2 in [15]. Although in [15] a more general result was merely stated for the Hartogs domain over an irreducible bounded symmetric domain, if the reader applies the argument in [16], [1] carefully, it is not difficulty to find out Lemma 2.1 also holds for the Hartogs domain Ω over a bounded, complete circular, homogeneous, Lu Qi-Keng domain. As in the solution of relativity problem [18], this lemma is crucial. Let Ω be the Hartogs domain over a bounded symmetric domain defined in (1).

Lemma 2.1 ([15]). *For any point $(\xi_0, Z_0) \in \Omega$, there exists $\Phi \in Aut(\Omega)$, such that $\Phi(\xi_0, Z_0) = (\xi_0', 0)$.*

The second lemma is an explicit description of the Bergman kernel functions over Ω. Note that if D is a bounded homogeneous domain, its Bergman kernel function $K_D(Z,\overline{W})$ is a rational function on $D \times \mathrm{conj}(D)$

Lemma 2.2 ([10]). *When D is a bounded homogeneous domain and let $\varphi(Z) = K_D(Z,\overline{Z})^{-s}$, the Bergman kernel of Ω is*

$$K((Z,\xi),\overline{(W,\eta)}) = K_D(Z,\overline{W})^{d_0 s+1} R(t), \tag{2}$$

where $R(t) = \pi^{-d_0}\sum_{j=0}^{d} \frac{c(s,j)(j+d_0)!}{(1-t)^{j+d_0+1}}$ is a rational function in $t = K_D(Z,\overline{W})^s\langle\xi,\eta\rangle$, with constants $c(s,j)$ satisfying that $F(ks) = \sum_{j=0}^{d} c(s,j)(k+1)_j$, for the polynomial function F given by (18) in [10].

Recall that with the Bochner coordinates z with $z(p) = 0$ at p in a Kähler manifold M with a real analytic Kähler metric ω_M, the diastasis function can be locally written as

$$D^M(z,0) = |z|^2 + \sum_{|j|,|k|\geq 2} b_{jk} z^j \bar{z}^k$$

near p. Then the following result follows from Calabi's classical result and the definition of Bochner coordinates (cf. Proposition 3.1 in [11]).

Lemma 2.3. *Let (M, ω_M) be a Kähler manifold, and z be the Bochner coordinate near $p \in M$ with $z(p) = 0$. Then neither the diastasis function $D^M(z,0)$ nor $\det\left[\frac{\partial^2 D^M(z,0)}{\partial z_i \partial \bar{z}_j}\right]$ has pluriharmonic terms in its Taylor expansion.*

The last lemma is contained in the proof of Theorem 1.1 in [9] and this version is used in the proof of Theorem 2.1 in [11].

Lemma 2.4. *Let $S = \{\phi_1, \cdots, \phi_l\}$ be a finite set of holomorphic functions on an open neighborhood U of 0 in $\mathbb{C}^n$. Then there exists a maximal algebraic independent subset $\{\phi_1, \cdots, \phi_k\} \subset S$ over the field $\mathcal{R}$ of rational functions on U, and holomorphic Nash algebraic functions $\hat{\phi}_j(t, X_1, \cdots, X_k)$, such that $\phi_j(t) = \hat{\phi}_j(t, \phi_1(t), \cdots, \phi_k(t))$ for all $1 \leq j \leq l$ after shrinking U toward the origin if needed.*

We are now in the position to prove the theorem.

Proof of Theorem 1.1. Let p be a point on M and $\{z = (z_1, \cdots, z_m)\}$ be the Bochner coordinate in an open neighborhood U of p such that $z(p) = 0$, where the Calabi's diastasis function is denoted by $D^M(z, w)$. By the reduction of the Kähler-Ricci soliton equation

$$Ric(\omega_M) = \lambda\omega_M + L_X\omega_M$$

in [11], we have

$$\exp\left\{-\frac{\lambda}{2}D^M(z,0) - \phi_M(z) + f_{m+1}(z) + \overline{f_{m+1}(z)}\right\} = \det\left[\frac{\partial^2 D^M(z,0)}{\partial z_i \partial \bar{z}_j}\right],$$

where f_{m+1} is a germ of holomorphic function at p and

$$\phi_M = \sum_{j=1}^{m} f_j \frac{\partial D^M(z,0)}{\partial z_j} + \overline{f_j}\frac{\partial D^M(z,0)}{\partial \bar{z}_j}$$

is the soliton potential function, for germs of holomorphic functions $f_1, \cdots, f_m$ at p, satisfying

$$L_X\omega_M = \frac{\sqrt{-1}}{2}\partial\bar{\partial}\phi_M.$$

It follows from Lemma 2.3 that

$$\exp\{-\phi_M(z,\bar{z}) + f_{m+1}(z) + \overline{f_{m+1}(z)}\} = \det\left[\frac{\partial^2 D^M(z,0)}{\partial z_i \partial \bar{z}_j}\right]\exp\left\{\frac{\lambda}{2}D^M(z,0)\right\} \quad (3)$$

does not have pluriharmonic terms. Denote

$$\tilde{\phi}_M(z,\bar{z}) = -\phi_M(z,\bar{z}) + f_{m+1}(z) + \overline{f_{m+1}(z)}. \tag{4}$$

Since $\exp\{\tilde{\phi}_M\}$ does not have pluriharmonic terms, so is $\tilde{\phi}_M$.

On the other hand, by Lemma 2.2, Calabi's diastasis function for the Bergman metric ω_Ω is given by

$$\begin{aligned} D^\Omega(\xi, Z, \eta, W) = (sd_0+1)\log \frac{K_D(Z,\bar{Z})K_D(W,\bar{W})}{K_D(Z,\bar{W})K_D(W,\bar{Z})} \\ + \log \frac{R(K_D(Z,\bar{Z})^s\langle \xi,\bar{\xi}\rangle)R(K_D(W,\bar{W})^s\langle \eta,\bar{\eta}\rangle)}{R(K_D(Z,\bar{W})^s\langle \xi,\bar{\eta}\rangle)R(K_D(W,\bar{Z})^s\langle \eta,\bar{\xi}\rangle)}. \end{aligned}$$

By Calabi's fundamental result, $G = (g,h) : M \to \Omega$ is a holomorphic isometric embedding in the sense $F^*\omega_\Omega = \mu\omega_M$ if and only if

$$D^M(z,0) = \mu D^\Omega(g(z), h(z), g(0), h(0)) \tag{5}$$

for any z near $z(p) = 0$.

By Lemma 2.1, there exists a holomorphic isometry σ of Ω such that $\sigma(F(p)) = (\xi_0, 0)$. Moreover, $\sigma\circ F : M \to \Omega$ is again a local holomorphic isometric embedding, still denoted by F. Now, (5) reads

$$\begin{aligned} D^M(z,0) = \mu(sd_0+1)\log \frac{K_D(h(z),\overline{h(z)})K_D(0,\bar{0})}{K_D(h(z),\bar{0})K_D(0,\overline{h(z)})} \\ + \mu\log \frac{R(K_D(h(z),\overline{h(z)})^s\langle g(z),\overline{g(z)}\rangle)R(K_D(0,\bar{0})^s\langle \xi_0,\bar{\xi}_0\rangle)}{R(K_D(h(z),\bar{0})^s\langle g(z),\bar{\xi}_0\rangle)R(K_D(0,\overline{h(z)})^s\langle \xi_0,\overline{g(z)}\rangle)}. \end{aligned} \tag{6}$$

Since D is a bounded complete circular domain, $K_D(0,\cdot) = K_D(\cdot,\bar{0}) = c_0$ is a constant. This reduces (6) to

$$D^M(z,0) = c_1 \log K_D(h(z),\overline{h(z)}) + \mu\log \frac{R(K_D(h(z),\overline{h(z)})^s\langle g(z),\overline{g(z)}\rangle)}{R(c_0^s\langle g(z),\bar{\xi}_0\rangle)R(c_0^s\langle \xi_0,\overline{g(z)}\rangle)} + c_2, \tag{7}$$

where $c_1 = \mu(sd_0+1)$, $c_2 = -c_1\log c_0 + \mu\log(c_0^s|\xi_0|^2)$ are real constants. Denote

$$H(z,\bar{z}) = K_D(h(z),\overline{h(z)})^s\langle g(z),\overline{g(z)}\rangle,$$

where $K_D(h(z),\overline{h(z)})^s$ is a rational function in h and $\bar{h}$ to a power of s. Write

$$\frac{\lambda}{2}D^M(z,0) = c_3\log K_D(h(z),\overline{h(z)}) + c_4\log \frac{R(K_D(h(z),\overline{h(z)})^s\langle g(z),\overline{g(z)}\rangle)}{R(c_0^s\langle g(z),\bar{\xi}_0\rangle)R(c_0^s\langle \xi_0,\overline{g(z)}\rangle)} + c_5, \tag{8}$$

for real constants c_3, c_4, c_5. It thus follows from (7) that

$$\begin{aligned}\frac{\partial D^M(z,0)}{\partial z_j} &= c_1 \frac{\frac{\partial K_D(h(z),\overline{h(z)})}{\partial Z_\alpha}\frac{\partial h_\alpha}{\partial z_j}(z)}{K_D(h(z),\overline{h(z)})} \\ &- \mu c_0^s \frac{R'(c_0^s\langle g(z),\bar{\xi}_0\rangle)}{R(c_0^s\langle g(z),\bar{\xi}_0\rangle)}\langle \frac{\partial g}{\partial z_j}(z),\bar{\xi}_0\rangle + \mu \frac{R'(H(z,\bar{z})}{R(H(z,\bar{z}))}\frac{\partial H(z,\bar{z})}{\partial z_j},\end{aligned} \tag{9}$$

where the straightforward calculation yields

$$\begin{aligned}\frac{\partial H(z,\bar{z})}{\partial z_j} &= sK_D(h(z),\overline{h(z)})^{s-1}\frac{\partial K_D(h(z),\overline{h(z)})}{\partial Z_\alpha}\frac{\partial h_\alpha}{\partial z_j}(z)\langle g(z),\overline{g(z)}\rangle \\ &+ K_D(h(z),\overline{h(z)})^s\langle \frac{\partial g}{\partial z_j}(z),\overline{g(z)}\rangle.\end{aligned} \tag{10}$$

Note that the right side of (10) is a rational function in $g, \bar{g}, h, \bar{h}$, the first order holomorphic derivatives of g, h, and $K_D(h(z),\overline{h(z)})^s$. So is the right side of (9). Denote $f = (f_1, \cdots, f_{m+1})$, and the first order holomorphic derivative of g, h along any z_j direction by g', h'. Therefore, by (4), we may write

$$\begin{aligned}&\tilde{\phi}_M(z,\bar{z}) = \\ &R_1\left(g(z),h(z),f(z),g'(z),h'(z),\overline{g(z)},\overline{h(z)},\overline{f(z)},\overline{g'(z)},\overline{h'(z)},K_D(h(z),\overline{h(z)})^s\right),\end{aligned} \tag{11}$$

where R_1 is a rational function in $g, h, f, g', h', \bar{g}, \bar{h}, \bar{f}, \bar{g}', \bar{h}'$, and $K_D(h,\bar{h})^s$. Also a straightforward calculation yields

$$\begin{aligned}\frac{\partial^2 H(z,\bar{z})}{\partial z_j \partial \bar{z}_k} &= s(s-1)K_D(h(z),\overline{h(z)})^{s-2}\frac{\partial K_D(h(z),\overline{h(z)})}{\partial \bar{Z}_\beta}\frac{\partial K_D(h(z),\overline{h(z)})}{\partial Z_\alpha} \\ &\quad \frac{\partial h_\alpha}{\partial z_j}(z)\overline{\frac{\partial h_\beta}{\partial z_k}}(z)\langle g(z),\overline{g(z)}\rangle \\ &+ sK_D(h(z),\overline{h(z)})^{s-1}\frac{\partial^2 K_D(h(z),\overline{h(z)})}{\partial Z_\alpha \partial \bar{Z}_\beta}\frac{\partial h_\alpha}{\partial z_j}(z)\overline{\frac{\partial h_\beta}{\partial z_k}}(z)\langle g(z),\overline{g(z)}\rangle \\ &+ sK_D(h(z),\overline{h(z)})^{s-1}\frac{\partial K_D(h(z),\overline{h(z)})}{\partial Z_\alpha}\frac{\partial h_\alpha}{\partial z_j}(z)\langle g(z),\overline{\frac{\partial g}{\partial z_k}}(z)\rangle \\ &+ sK_D(h(z),\overline{h(z)})^{s-1}\frac{\partial K_D(h(z),\overline{h(z)})}{\partial \bar{Z}_\beta}\overline{\frac{\partial h_\beta}{\partial z_k}}(z)\langle \frac{\partial g}{\partial z_j}(z),\overline{g(z)}\rangle \\ &+ K_D(h(z),\overline{h(z)})^s\langle \frac{\partial g}{\partial z_j}(z),\overline{\frac{\partial g}{\partial z_k}}(z)\rangle,\end{aligned} \tag{12}$$

and the right side is a rational function in $g, h, g', h', \bar{g}, \bar{h}, \bar{g}', \bar{h}'$ and $K_D(h, \bar{h})^s$. It follows $\det\left[\frac{\partial^2 D^M(z,0)}{\partial z_i \partial \bar{z}_j}\right]$ is also a rational function in $g, h, g', h', \bar{g}, \bar{h}, \bar{g}', \bar{h}'$ and $K_D(h, \bar{h})^s$ and thus we may write

$$\det\left[\frac{\partial^2 D^M(z,0)}{\partial z_i \partial \bar{z}_j}\right] = R_2\left(g(z), h(z), g'(z), h'(z), \overline{g(z)}, \overline{h(z)}, \overline{g'(z)}, \overline{h'(z)}, K_D(h(z), \overline{h(z)})^s\right),$$

where R_2 is a rational function in $g, h, g', h', \bar{g}, \bar{h}, \bar{g}', \bar{h}'$, and $K_D(h, \bar{h})^s$. By taking logarithmic, (3) reads

$$\begin{aligned} R_1 & \left(g(z), h(z), f(z), g'(z), h'(z), \overline{g(z)}, \overline{h(z)}, \overline{f(z)}, \overline{g'(z)}, \overline{h'(z)}, K_D(h(z), \overline{h(z)})^s\right) \\ & = \log R_2\left(g(z), h(z), g'(z), h'(z), \overline{g(z)}, \overline{h(z)}, \overline{g'(z)}, \overline{h'(z)}, K_D(h(z), \overline{h(z)})^s\right) \\ & \quad + c_3 \log K_D(h(z), \overline{h(z)}) + c_4 \log \frac{R(K_D(h(z), \overline{h(z)})^s \langle g(z), \overline{g(z)} \rangle)}{R(c_0^s \langle g(z), \bar{\xi}_0 \rangle) R(c_0^s \langle \xi_0, \overline{g(z)} \rangle)} + c_5. \end{aligned}$$

By polarization, it is equivalent to

$$\begin{aligned} R_1 & \left(g(z), h(z), f(z), g'(z), h'(z), \bar{g}(w), \bar{h}(w), \bar{f}(w), \bar{g}'(w), \bar{h}'(w), K_D(h(z), \bar{h}(w))^s\right) \\ & = \log R_2\left(g(z), h(z), g'(z), h'(z), \bar{g}(w), \bar{h}(w), \bar{g}'(w), \bar{h}'(w), K_D(h(z), \bar{h}(w))^s\right) \\ & \quad + c_3 \log K_D(h(z), \bar{h}(w)) + c_4 \log \frac{R(K_D(h(z), \bar{h}(w))^s \langle g(z), \bar{g}(w) \rangle)}{R(c_0^s \langle g(z), \bar{\xi}_0 \rangle) R(c_0^s \langle \xi_0, \bar{g}(w) \rangle)} + c_5, \end{aligned} \tag{13}$$

where $\bar{\chi}(w) = \overline{\chi(\bar{w})}$ for $\bar{w} \in U$. Applying Lemma 2.4, there exists a subset of all components of $g(z), h(z), f(z), g'(z), h'(z)$, denoted by $\phi_1(z), \cdots, \phi_k(z)$, such that any $\chi \in \{g, h, f, g', h'\}$ can be written as

$$\chi(z) = \hat{\chi}(z, \phi_1(z), \cdots, \phi_k(z)),$$

for a holomorphic Nash algebraic function $\hat{\chi}(z, X)$ in z and $X = (X_1, \cdots, X_k)$. Define

$$\begin{aligned} \Psi(z, X, w) = R_1 & \Big(\hat{g}(z, X), \hat{h}(z, X), \hat{f}(z, X), \hat{g}'(z, X), \hat{h}'(z, X), \\ & \quad \bar{g}(w), \bar{h}(w), \bar{f}(w), \bar{g}'(w), \bar{h}'(w), K_D(\hat{h}(z, X), \bar{h}(w))^s\Big) \\ & - \log R_2\Big(\hat{g}(z, X), \hat{h}(z, X), \hat{g}'(z, X), \hat{h}'(z, X), \\ & \quad \bar{g}(w), \bar{h}(w), \bar{g}'(w), \bar{h}'(w), K_D(\hat{h}(z, X), \bar{h}(w))^s\Big) \\ & - c_3 \log K_D(\hat{h}(z, X), \bar{h}(w)) \end{aligned} \tag{14}$$

$$+ c_4 \log \frac{R(K_D(\hat{h}(z,X), \bar{h}(w))^s \langle \hat{g}(z,X), \bar{g}(w)\rangle)}{R(c_0^s \langle \hat{g}(z,X), \bar{\xi}_0\rangle) R(c_0^s \langle \xi_0, \bar{g}(w)\rangle)} - c_5$$

and

$$\Psi^\beta(z,X,w) = \left(\partial_w^\beta \Psi\right)(z,X,w)$$

stands for the holomorphic mixed derivative of $\Psi(z,X,w)$ along w of order $|\beta|$.

Lemma 2.5. *For any w near 0 and any (z,X), $\Psi(z,X,w) \equiv \Psi(z,X,0)$.*

Proof. It suffices to show that $\Psi^\beta(z,X,0) \equiv 0$ for all $|\beta| > 0$. Note that $\bar{g}(0) = \bar{\xi}_0, \bar{h}(0) = 0$. It follows that

$$K_D(\hat{h}(z,X), \bar{h}(0))^s = c_0^s.$$

It thus follows from the expression in (14) that $\Psi^\beta(z,X,0)$ is a holomorphic Nash algebraic function in (z,X). Assume it is not constant. Then there exists a holomorphic polynomial $P(z,X,y) = A_d(z,X)y^d + \cdots + A_0(z,X)$ of degree d in y, with $A_0(z,X) \not\equiv 0$ such that $P(z,X,\Psi^\beta(z,X,0)) \equiv 0$, where all $A_j(z,X)$ are polynomials in z, X for $0 \le j \le d$. As $\Psi(z,\phi_1(z),\cdots,\phi_k(z),w) \equiv 0$ for $z \in U, \bar{w} \in U$ by (13), it follows that $\Psi^\beta(z,\phi_1(z),\cdots,\phi_k(z),0) \equiv 0$ and therefore $A_0(z,\phi_1(z),\cdots,\phi_k(z)) \equiv 0$. This means that $\phi_1(z),\cdots,\phi_k(z)$ are algebraic dependent over $\mathcal{R}$. This is a contradiction and it follows that $\Psi^\beta(z,X,0)$ is a constant. Therefore, $\Psi^\beta(z,X,0) = \Psi^\beta(z,\phi_1(z),\cdots,\phi_k(z),0) \equiv 0$. □

Lemma 2.6. *Assume (M,ω_M) is a non-trivial Kähler-Ricci soliton, i.e. $L_X\omega_M \neq 0$. Then there exists some $w \neq 0$, such that*

$$\begin{aligned} &R_1\Big(\hat{g}(z,X), \hat{h}(z,X), \hat{f}(z,X), \hat{g}'(z,X), \hat{h}'(z,X),\\ &\qquad \bar{g}(w), \bar{h}(w), \bar{f}(w), \bar{g}'(w), \bar{h}'(w), K_D(\hat{h}(z,X), \bar{h}(w))^s\Big)\\ &- R_1\Big(\hat{g}(z,X), \hat{h}(z,X), \hat{f}(z,X), \hat{g}'(z,X), \hat{h}'(z,X),\\ &\qquad \bar{g}(0), \bar{h}(0), \bar{f}(0), \bar{g}'(0), \bar{h}'(0), K_D(\hat{h}(z,X), \bar{h}(0))^s\Big)\end{aligned}$$

must depend on (z,X).

Proof. We argue by contradiction. Suppose not. Namely, for any $\bar{w} \in U$,

$$\begin{aligned} &R_1\Big(\hat{g}(z,X), \hat{h}(z,X), \hat{f}(z,X), \hat{g}'(z,X), \hat{h}'(z,X),\\ &\qquad \bar{g}(w), \bar{h}(w), \bar{f}(w), \bar{g}'(w), \bar{h}'(w), K_D(\hat{h}(z,X), \bar{h}(w))^s\Big)\\ &- R_1\Big(\hat{g}(z,X), \hat{h}(z,X), \hat{f}(z,X), \hat{g}'(z,X), \hat{h}'(z,X),\\ &\qquad \bar{g}(0), \bar{h}(0), \bar{f}(0), \bar{g}'(0), \bar{h}'(0), K_D(\hat{h}(z,X), \bar{h}(0))^s\Big)\end{aligned}$$

does not depend on (z, X). It thus only depends on w holomorphically. Let $w = \bar{z}, X = \phi = (\phi_1(z), \cdots, \phi_k(z))$. By the polarization of (11),

$$\begin{aligned}
&R_1\Big(\hat{g}(z,\phi), \hat{h}(z,\phi), \hat{f}(z,\phi), \hat{g}'(z,\phi), \hat{h}'(z,\phi),\\
&\qquad \bar{g}(\bar{z}), \bar{h}(\bar{z}), \bar{f}(\bar{z}), \bar{g}'(\bar{z}), \bar{h}'(\bar{z}), K_D(\hat{h}(z,\phi), \bar{h}(\bar{z}))^s\Big)\\
&- R_1\Big(\hat{g}(z,\phi), \hat{h}(z,\phi), \hat{f}(z,\phi), \hat{g}'(z,\phi), \hat{h}'(z,\phi),\\
&\qquad \bar{g}(0), \bar{h}(0), \bar{f}(0), \bar{g}'(0), \bar{h}'(0), K_D(\hat{h}(z,\phi), \bar{h}(0))^s\Big)\\
&= \tilde{\phi}_M(z,\bar{z}) - \tilde{\phi}_M(z,0) = \tilde{\phi}_M(z,\bar{z}) - \text{constant},
\end{aligned}$$

because $\tilde{\phi}_M(z,0) \equiv$ a constant as $\tilde{\phi}_M$ does not have non-constant pluriharmonic terms in its Taylor expansion. However, by the contradiction hypothesis, $\tilde{\phi}_M(z,\bar{z})$ must depend on $\bar{z}$ holomorphically. Therefore $L_X\omega_M = \frac{\sqrt{-1}}{2}\partial\bar{\partial}\tilde{\phi}_M = 0$. This is a contradiction. □

We denote

$$\begin{aligned}
\tilde{H}_1(z,X,w) = {}&R_1\Big(\hat{g}(z,X), \hat{h}(z,X), \hat{f}(z,X), \hat{g}'(z,X), \hat{h}'(z,X),\\
&\qquad \bar{g}(w), \bar{h}(w), \bar{f}(w), \bar{g}'(w), \bar{h}'(w), K_D(\hat{h}(z,X), \bar{h}(w))^s\Big)\\
&- R_1\Big(\hat{g}(z,X), \hat{h}(z,X), \hat{f}(z,X), \hat{g}'(z,X), \hat{h}'(z,X),\\
&\qquad \bar{g}(0), \bar{h}(0), \bar{f}(0), \bar{g}'(0), \bar{h}'(0), K_D(\hat{h}(z,X), \bar{h}(0))^s\Big).
\end{aligned}$$

Then by Lemma 2.6, one may fix $w = w_0$ as in Lemma 2.6 and a complex plane in $\mathbb{C}^{m+k}$, whose holomorphic coordinate denoted by t, such that the restriction of $\tilde{H}_1(z,X,w_0)$ to t-plane, denoted by $\tilde{H}_1(t)$, is not constant. Denote the restriction of

$$\begin{aligned}
&\log R_2\Big(\hat{g}(z,X), \hat{h}(z,X), \hat{g}'(z,X), \hat{h}'(z,X),\\
&\qquad \bar{g}(w_0), \bar{h}(w_0), \bar{g}'(w_0), \bar{h}'(w_0), K_D(\hat{h}(z,X), \bar{h}(w_0))^s\Big)\\
&+ c_3 \log K_D(\hat{h}(z,X), \bar{h}(w_0)) - c_4 \log \frac{R(K_D(\hat{h}(z,X), \bar{h}(w_0))^s \langle \hat{g}(z,X), \bar{g}(w_0)\rangle)}{R(c_0^s\langle \hat{g}(z,X), \bar{\xi}_0\rangle) R(c_0^s \langle \xi_0, \bar{g}(w_0)\rangle)}\\
&- \log R_2\Big(\hat{g}(z,X), \hat{h}(z,X), \hat{g}'(z,X), \hat{h}'(z,X),\\
&\qquad \bar{g}(0), \bar{h}(0), \bar{g}'(0), \bar{h}'(0), K_D(\hat{h}(z,X), \bar{h}(0))^s\Big)
\end{aligned}$$

$$- c_3 \log K_D(\hat{h}(z,X), \bar{h}(0)) + c_4 \log \frac{R(K_D(\hat{h}(z,X), \bar{h}(0))^s \langle \hat{g}(z,X), \bar{g}(0) \rangle)}{R(c_0^s \langle \hat{g}(z,X), \bar{\xi}_0 \rangle) R(c_0^s \langle \xi_0, \bar{g}(0) \rangle)}$$

to t-plane by $\log \tilde{H}_2(t)$. Then it follows from Lemma 2.5 that

$$\exp\{\tilde{H}_1(t)\} = \tilde{H}_2(t). \tag{15}$$

Since $\tilde{H}_1(t)$ is not constant, then there exists some point $t_0 \in \mathbb{C} \cup \{\infty\}$, such that $\tilde{H}_1(t)$ blows up as $t \to t_0$. By the expression, $\tilde{H}_1$ blows up at a polynomial rate. By (15), $\tilde{H}_2(t)$ also blows up as $t \to t_0$. Moreover, By the expression, $\tilde{H}_2$ blows up at a polynomial rate as well. This contradicts to (15) and the theorem is proved. □

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