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Endpoint boundedness of singular integrals: CMO space associated to Schrödinger operators

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ABSTRACT. Let $\mathcal{L} = -\Delta + V$ be a Schrödinger operator acting on $L^2(\mathbb{R}^n)$, where the nonnegative potential V belongs to the reverse Hölder class RH_q for some $q \geq n/2$. This article is primarily concerned with the study of endpoint boundedness for classical singular integral operators in the context of the space $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$, consisting of functions of vanishing mean oscillation associated with $\mathcal{L}$.

We establish the following main results: (i) the standard Hardy–Littlewood maximal operator is bounded on $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$; (ii) for each $j = 1, \dots, n$, the adjoint of the Riesz transform $\partial_j \mathcal{L}^{-1/2}$ is bounded from $C_0(\mathbb{R}^n)$ into $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$; and (iii) the approximation to the identity generated by the Poisson and heat semigroups associated with $\mathcal{L}$ characterizes $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$ appropriately.

These results recover the classical analogues corresponding to the Laplacian as a special case. However, the presence of the potential V introduces substantial analytical challenges, necessitating tools beyond the scope of classical Calderón–Zygmund theory. Our approach leverages precise heat kernel estimates and the structural properties of $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$ established by Song and the third author in [19].

CONTENTS

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After posting our manuscript on the arXiv (simultaneously with its submission), we learned that the boundedness of the Hardy–Littlewood maximal operator on $\mathrm{CMO}_{\mathcal{L}}$ had also been established independently in the dissertation of Wanjun Li (Sun Yat-sen University). We thank the referees for their careful reading and valuable suggestions, which have greatly improved the paper. Li is supported by ARC DP220100285. Wu is supported by NNSF of China # 12201002 and Anhui NSF of China # 2208085QA03.

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1. Introduction and main results

Let us consider the Schrödinger operator

$$\mathcal{L} = -\Delta + V(x) \text{ on } L^2(\mathbb{R}^n), \quad n \geq 3,$$

where the nonnegative potential V is not identically zero, and $V \in RH_q$ for some $q \geq n/2$, which by definition means that $V \in L^q_{\mathrm{loc}}(\mathbb{R}^n), V \geq 0$, and there exists a constant $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V(y)^q dy\right)^{1/q} \leq \frac{C}{|B|}\int_B V(y)\,dy \tag{1.1}$$

holds for all balls B in $\mathbb{R}^n$. Following [9], a locally integrable function f belongs to $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ if

$$\|f\|_{\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)} := \sup_{B=B(x_B,r_B):\, r_B<\rho(x_B)} \frac{1}{|B|}\int_B |f(y) - f_B|\,dy + \sup_{B=B(x_B,r_B):\, r_B\geq\rho(x_B)} \frac{1}{|B|}\int_B |f(y)|\,dy < \infty. \tag{1.2}$$

The critical radii above are determined by the function $\rho(x;V) = \rho(x)$, which was first introduced by Shen [18, Definition 1.3] and takes the explicit form

$$\rho(x) = \sup\left\{r > 0 : \frac{1}{r^{n-2}}\int_{B(x,r)} V(y)\,dy \leq 1\right\}. \tag{1.3}$$

This article focuses on $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$, the space of vanishing mean oscillation associated to $\mathcal{L}$, which is the closure of $C_c^\infty(\mathbb{R}^n)$ (the space of smooth functions with compact support) in the $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ norm. As a crucial subspace of $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$, it satisfies the duality relations

$$\left(\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)\right)^* = H^1_{\mathcal{L}}(\mathbb{R}^n) \quad \text{and} \quad (H^1_{\mathcal{L}}(\mathbb{R}^n))^* = \mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n), \tag{1.4}$$

where the Hardy-type space $H^1_{\mathcal{L}}(\mathbb{R}^n)$ is defined by

$$H^1_{\mathcal{L}}(\mathbb{R}^n) = \left\{f \in L^1(\mathbb{R}^n) : \mathcal{P}^* f(x) = \sup_{t>0}\left|e^{-t\sqrt{\mathcal{L}}} f(x)\right| \in L^1(\mathbb{R}^n)\right\}$$

with norm $\|f\|_{H^1_{\mathcal{L}}(\mathbb{R}^n)} = \left\|\mathcal{P}^* f\right\|_{L^1(\mathbb{R}^n)}$. See [7, 9, 14] for details. Additional equivalent characterizations of $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$ via mean oscillation and tent spaces, respectively, can be found in [19] by L. Song and the third author.

The space $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$ shares key similarities with the classical vanishing mean oscillation space: when $V \equiv 0$, $\mathrm{CMO}_{\Delta}(\mathbb{R}^n)$ (resp. $\mathrm{BMO}_{\Delta}(\mathbb{R}^n)$) coincides exactly with the standard $\mathrm{CMO}(\mathbb{R}^n)$ (resp. $\mathrm{BMO}(\mathbb{R}^n)$), and the dualities (1.4) reduce to their classical counterparts. However, $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$ demonstrates certain properties distinct from the classical setting. For instance, the convolution of a compactly supported bump function with a function of $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$ may fail to remain in $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$; see [19, Lemma 4.1].

The aim of this paper is to study endpoint boundedness for classical singular integral operators in the context of the space $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$. Central to this pursuit are the boundedness of cornerstone operators such as the Hardy–Littlewood maximal operator and the Riesz transforms on this space. Additionally, the development of suitable approximations to the identity compatible with the structure of this space requires careful consideration.

Part I. The (uncentered) Hardy–Littlewood maximal function M on $\mathbb{R}^n$ is a well-known operator and plays a fundamental role in harmonic analysis. However, its behaviour on $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$ remains unclarified.

Recall that for the classical case with $V \equiv 0$, it's known that for a function $f \in \mathrm{BMO}(\mathbb{R}^n)$, it may occur that $Mf \equiv +\infty$, and a typical example is $f(x) = \log|x|$ (in contrast, for any $f \in \mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$, we have $Mf(x) < +\infty$ for a.e. $x \in \mathbb{R}^n$). Nevertheless, there exists a constant C depending only on n such that for any $f \in \mathrm{BMO}(\mathbb{R}^n)$ for which Mf is not identically equal to infinity, we have

$$\|Mf\|_{\mathrm{BMO}(\mathbb{R}^n)} \le C\|f\|_{\mathrm{BMO}(\mathbb{R}^n)};$$

see [1, Theorem 4.2] by Bennett, DeVore and Sharpley. The further boundedness of M and its fractional counterpart on VMO were investigated in [17] and [11], respectively, where VMO is the BMO-closure of UC $\cap$ BMO, and UC is the class of all uniformly continuous functions. Alternatively, $f \in \mathrm{VMO}(\mathbb{R}^n)$ if and only if $f \in \mathrm{BMO}(\mathbb{R}^n)$ and

$$\lim_{a\to 0} \sup_{B:\, r_B \le a} |B|^{-1} \int_B |f(x) - f_B|\, dx = 0.$$

Very recently, the boundedness of M on the classical $\mathrm{CMO}(\mathbb{R}^n)$ was established in [15]. Since the nonnegative potential V is assumed not to be identically zero, we have

$$\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n) \subsetneqq \mathrm{CMO}(\mathbb{R}^n) \subsetneqq \mathrm{VMO}(\mathbb{R}^n).$$

Our first result is to characterize the Hardy–Littlewood maximal operator M on $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$, which also clarifies the boundedness of M on $\mathrm{CMO}(\mathbb{R}^n)$.

Theorem 1.1. *Suppose $V \in RH_q$ for some $q \ge n/2$ and let $\mathcal{L} = -\Delta + V$. For each $f \in \mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$, the Hardy–Littlewood maximal function Mf belongs to $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$ as well, with*

$$\|Mf\|_{\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)} \le C\|f\|_{\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)}, \tag{1.5}$$

where the constant $C > 0$ is independent of f.

Recall that the boundedness of M on $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$, namely,

$$\|Mf\|_{\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)} \le C\|f\|_{\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)}, \tag{1.6}$$

was given in [9, Theorem 5]. To prove Theorem 1.1 based on this result, we will apply the characterization of $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$ in terms of the behaviour of mean oscillation given in [19] (see (vi) of Theorem 2.1 below), and give a more refined modification of the argument for [1, Theorem 4.2]. Note that $\mathrm{CMO}(\mathbb{R}^n)$ can also be characterized via mean oscillation, which coincides with $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$ whenever taking $V \equiv 0$, hence our proof also reveals the behaviour of M on the classical $\mathrm{CMO}(\mathbb{R}^n)$ (Remarkably, functions $f \in \mathrm{CMO}(\mathbb{R}^n)$ for which Mf is identically infinite must be ruled out, such as $f(x) = \ln\ln|x| \cdot 1_{\{|\cdot|\geq e\}}(x)$). See Remark 3.2 for details.

Part II. Consider the jth Riesz transform $R_j = \dfrac{\partial}{\partial x_j}\mathcal{L}^{-1/2}$ associated to $\mathcal{L}$ on $\mathbb{R}^n$, $j = 1, \dots, n$. Shen [17] established that when $V \in RH_q$ for $n/2 \leq q < n$, then

$$\|R_j f\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)} \quad \text{for } 1 < p \leq p_0,$$

where $\dfrac{1}{p_0} = \dfrac{1}{q} - \dfrac{1}{n}$. When $V \in RH_n$, R_j is a Calderón–Zygmund operator for each j. Hence it suffices to consider the case $V \in RH_q$ with $n/2 \leq q < n$. Let $R_j(x, y)$ be the kernel of the Riesz transform R_j. Then the adjoint of R_j is given by

$$R_j^* g(x) = \lim_{\varepsilon\to 0}\int_{|y-x|>\varepsilon} R_j(y, x) g(y)dy.$$

By duality, the above boundedness of R_j deduces that R_j^* is bounded on $L^{p'}(\mathbb{R}^n)$ with $p_0' \leq p' < \infty$, where $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. Moreover, R_j^* is bounded from $L^\infty(\mathbb{R}^n)$ to $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$, which is useful to give a characterization of $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ via R_j^*. Concretely, for each $f \in \mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$, we can write

$$f = \phi_0 + \sum_{j=1}^{n} R_j^* \phi_j, \quad \phi_j \in L^\infty(\mathbb{R}^n),\ 0 \leq j \leq n.$$

See [22, Theorem 1.3] by the third author and L.X. Yan.

To continue this line, our second result is as follows. Let $C_0(\mathbb{R}^n)$ be the space of all continuous functions on $\mathbb{R}^n$ which vanish at infinity.

Theorem 1.2. *Suppose $V \in RH_q$ for some $q \geq n/2$ and let $\mathcal{L} = -\Delta + V$. The adjoint Riesz transform R_j^* associated to $\mathcal{L}$ is bounded from $C_0(\mathbb{R}^n)$ to $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$ for $j = 1, \dots, n$.*

When $V \equiv 0$, the boundedness above is known, based on the Fourier transform of the classical Riesz transform

$$\frac{\partial}{\partial x_j}(-\Delta)^{-1/2}, \qquad j = 1, \dots, n;$$

see [6, Lemma 1] for details. For any generic potential $V \in RH_q$, techniques from Fourier transform are not workable, and we will show Theorem 1.2 by exploiting estimates for the kernels of Riesz transforms and applying preliminaries in [18].

As a consequence, we will show (see Lemma 4.1 below) a Riesz-type representation that for every continuous linear functional ℓ on $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$, there exists a uniquely finite Borel measure μ_0 such that ℓ can be realized by

$$\ell(g) = \int_{\mathbb{R}^n} g(x)\, d\mu_0(x), \quad \forall\, g \in \mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n),$$

where μ_0 satifies that its Riesz transforms $R_j(d\mu_0)(x) = \int R_j(x,y)\, d\mu_0(y)$ associated to $\mathcal{L}$ for $j = 1, 2, \ldots, n$, are all finite Borel measures.

Part III. Consider the approximation to the identity on $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$. As aforementioned, the standard approximation to the identity can not match $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$ well due to the potential V. Even for a radial bump function ϕ satisfying

$$\operatorname{supp}\phi \subseteq B(0,1), \quad 0 \le \phi \le 1 \quad \text{and} \quad \int \phi(x)\, dx = 1,$$

the convolution $A_t f = t^{-n}\phi(t^{-1}\cdot) * f$ for $f \in \mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$ may not belong to $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$, unless assuming additional conditions such as $f \in C_c^\infty(\mathbb{R}^n)$. In this article, we consider the approximation to the identity arising from semigroups associated to $\mathcal{L}$.

Theorem 1.3. *Suppose $V \in RH_q$ for some $q \ge n/2$ and let $\mathcal{L} = -\Delta + V$. For any $f \in \mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$, we have $e^{-t\sqrt{\mathcal{L}}} f \in \mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$ for each $t > 0$, and*

$$\lim_{t\to 0} e^{-t\sqrt{\mathcal{L}}} f = f \quad \text{in } \mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n). \tag{1.7}$$

In particular, if $f \in C_c^\infty(\mathbb{R}^n)$, then we also have $\lim_{t\to 0} e^{-t\sqrt{\mathcal{L}}} f(x) = f(x)$ uniformly for all $x \in \mathbb{R}^n$.

The analogous conclusion remains valid when replacing the Poisson semigroup $e^{-t\sqrt{\mathcal{L}}}$ by the heat semigroup $e^{-t\mathcal{L}}$.

To establish this, we first show that for any $f \in \mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ and $t > 0$, the function $e^{-t\sqrt{\mathcal{L}}} f$ also belongs to $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$, and a corresponding result holds in $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$ (see Lemma 5.1). Our main ingredient is the characterization of $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$ via the theory of tent spaces established in [19]. Consequently, it suffices to verify (1.7) for functions in $C_c^\infty(\mathbb{R}^n)$, and we can utilize the classical Poisson semigroup $e^{-t\sqrt{-\Delta}}$ to streamline the argument.

This paper is organized as follows. In Section 2 we introduce the necessary preliminaries in characterizations of $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$ and the auxiliary function ρ. In Section 3 we provide the proof of Theorem 1.1. In Section 4 we present the proof of Theorem 1.2. The argument for Theorem 1.3 will be discussed in the last section.

2. Preliminaries

We recall some preliminaries on $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$ and the auxiliary function ρ defined in (1.3).

A remarkable fact is the self-improvement property: if $V \in RH_q$ with $q > 1$, then there exists $\varepsilon > 0$ depending only on the constant C in (1.1) and the dimension n such that $V \in RH_{q+\varepsilon}$. Consequently, the assumption "$V \in RH_q$ for some $q \geq n/2$" can be rewritten as "$V \in RH_q$ for some $q > n/2$". This fact is useful for dealing with some critical indices that appear in our article below.

Combining works in [7, 14, 19], we have the following characterizations of $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$.

Theorem 2.1. *Suppose $V \in RH_q$ for some $q \geq n/2$ and let $\mathcal{L} = -\Delta + V$. The following statements are equivalent.*

(i) *f is in $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$.*

(ii) *f is in the closure in the $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ norm of $C_c^\infty(\mathbb{R}^n)$.*

(iii) *f is in the closure in the $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ norm of $C_0(\mathbb{R}^n)$.*

(iv) *f is in the pre-dual space of the Hardy space $H^1_{\mathcal{L}}(\mathbb{R}^n)$.*

(v) *f is in $\mathcal{B}_{\mathcal{L}}$, where $\mathcal{B}_{\mathcal{L}}$ is the subspace of $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ satisfying $\widetilde{\gamma}_i(f) = 0$ for $1 \leq i \leq 5$, where*

$$\widetilde{\gamma}_1(f) = \lim_{a\to 0} \sup_{B:\, r_B \leq a} \left(|B|^{-1} \int_B |f(x) - f_B|^2 \, dx \right)^{1/2};$$

$$\widetilde{\gamma}_2(f) = \lim_{a\to \infty} \sup_{B:\, r_B \geq a} \left(|B|^{-1} \int_B |f(x) - f_B|^2 \, dx \right)^{1/2};$$

$$\widetilde{\gamma}_3(f) = \lim_{a\to \infty} \sup_{B:\, B\subseteq (B(0,a))^c} \left(|B|^{-1} \int_B |f(x) - f_B|^2 \, dx \right)^{1/2};$$

$$\widetilde{\gamma}_4(f) = \lim_{a\to \infty} \sup_{B:\, r_B \geq \max\{a, \rho(x_B)\}} \left(|B|^{-1} \int_B |f(x)|^2 \, dx \right)^{1/2};$$

$$\widetilde{\gamma}_5(f) = \lim_{a\to \infty} \sup_{\substack{B:\, B\subseteq (B(0,a))^c \\ r_B \geq \rho(x_B)}} \left(|B|^{-1} \int_B |f(x)|^2 \, dx \right)^{1/2}.$$

Here x_B denotes the center of B, and the function ρ is defined in (1.3).

(vi) *f is in $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ and satisfies $\widetilde{\gamma}_1(f) = \widetilde{\gamma}_3(f) = \widetilde{\gamma}_5(f) = 0$.*

Next we review the slowly varying property of the critical radii function $\rho(x)$.

Lemma 2.2. ([18, Lemma 1.4].) *Suppose $V \in RH_q$ for some $q \geq n/2$. There exist $c > 1$ and $k_0 \geq 1$ such that for all $x, y \in \mathbb{R}^n$,*

$$c^{-1}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-k_0}\rho(x) \leq \rho(y) \leq c\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{k_0+1}}\rho(x). \tag{2.1}$$

In particular, $\rho(x) \approx \rho(y)$ when $y \in B(x,r)$ and $r \lesssim \rho(x)$.

Hence $0 < \rho(x) < \infty$ for each $x \in \mathbb{R}^n$, and ρ is locally bounded from above and below. This fact will be used frequently in our article.

3. Hardy–Littlewood maximal operator on $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$: proof of Theorem 1.1

Let M denote the uncentered Hardy–Littlewood maximal function. The aim of this section is to explore the boundedness of M on $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$.

Proof of Theorem 1.1. *Step I.* We begin by showing that for any given $f \in \mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$, we have $Mf < +\infty$ for a.e. $x \in \mathbb{R}^n$.

This fact has been proven in [9] by splitting the function f into a local part and a nonlocal part. Alternatively, here we present an alternative proof by directly applying the definition (1.2) of the $\mathrm{BMO}_{\mathcal{L}}$ norm.

Indeed, for any $f \in \mathrm{BMO}_{\mathcal{L}}$, it follows from the definition of the $\mathrm{BMO}_{\mathcal{L}}$ norm that

$$\begin{aligned} Mf(x) &\leq C_n \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|dy \\ &\leq C_n\|f\|_{\mathrm{BMO}_{\mathcal{L}}} \sup_{r>0} \max\left\{\Big(\frac{\rho(x)}{r}\Big)^n, 1\right\}, \end{aligned}$$

and the $\sup_{r>0}$ can be improved to $\sup_{r>\delta}$ for some $\delta > 0$ due to the Lebesgue differentiation theorem. Specifically, for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, x) > 0$ such that

$$Mf(x) \leq \max\left\{|f(x)| + \varepsilon, C_n\|f\|_{\mathrm{BMO}_{\mathcal{L}}} \sup_{r>\delta} \max\left\{\Big(\frac{\rho(x)}{r}\Big)^n, 1\right\}\right\},$$

Since $0 < \rho(x) < \infty$ for each $x \in \mathbb{R}^n$, we obtain $Mf(x) < +\infty$, a.e. $x \in \mathbb{R}^n$.

Step II. Now we show that Mf belongs to $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$ when $f \in \mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$.

For any given $f \in \mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$, it follows from Theorem 2.1 that it's equivalent to $f \in \mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ and $\widetilde{\gamma}_1(f) = \widetilde{\gamma}_3(f) = \widetilde{\gamma}_5(f) = 0$. By (1.6), we have $Mf \in \mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$, hence it suffices to verify that $\widetilde{\gamma}_1(Mf) = \widetilde{\gamma}_3(Mf) = \widetilde{\gamma}_5(Mf) = 0$.

The following argument refines and modifies the approach in [1, Theorem 4.2]; see also [17].

Note that for any B,

$$\frac{1}{|B|}\int_B \left|f(x)-f_B\right|^2 dx \le \frac{1}{|B|^2}\iint_{B\times B} \left|f(x)-f(y)\right|^2 dydx$$
$$= \frac{1}{|B|^2}\iint_{B\times B} \left|f(x)-f_B+f_B-f(y)\right|^2 dydx$$
$$\le \frac{2}{|B|}\int_B \left|f(x)-f_B\right|^2 dx, \tag{3.1}$$

and

$$\frac{1}{|B|^2}\iint_{B\times B} \Big||f(x)|-|f(y)|\Big|^2 dydx \le \frac{1}{|B|^2}\iint_{B\times B} \left|f(x)-f(y)\right|^2 dydx,$$

from which it follows readily that $|f| \in \mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$. Besides, $M|f| = Mf$. Thus we may assume without loss of generality that f is nonnegative.

Now, for any $0 \le f \in \mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$, note that Mf and $\widetilde{\gamma}_i(Mf)$ for $i = 1, 3, 5$ can be defined using cubes with sides parallel to the coordinate axes instead of balls. In the following proof, "cube" always refers to such cubes. For any given cube Q, denote its sidelength by $\ell(Q)$. Let $\kappa > 0$ be a constant to be chosen later. For each $x \in Q$, define

$$M_1 f(x) := \sup_{Q'\ni x:\, \ell(Q')<\kappa\ell(Q)} f_{Q'} \quad \text{and} \quad M_2 f(x) := \sup_{Q'\ni x:\, \ell(Q')\ge\kappa\ell(Q)} f_{Q'}.$$

Clearly, $Mf = \max\{M_1 f, M_2 f\}$ on Q. Set

$$\Omega = \left\{x \in Q : Mf(x) > (Mf)_Q\right\}, \quad \Omega_1 = \{x \in \Omega : M_1 f(x) \ge M_2 f(x)\}$$

and $\Omega_2 = \Omega \setminus \Omega_1$. Then

$$\frac{1}{|Q|}\int_Q \left|Mf(x)-(Mf)_Q\right| dx = \frac{2}{|Q|}\int_\Omega \Big(Mf(x)-(Mf)_Q\Big)\, dx$$
$$= 2\sum_{i=1}^{2} \frac{1}{|Q|}\int_{\Omega_i} \Big(M_i f(x)-(Mf)_Q\Big)\, dx. \tag{3.2}$$

We begin by considering the term involving M_1 in (3.2). For the above cube Q, let $Q^* = (2\kappa+1)Q$ denote the cube with the same center as Q and sidelength

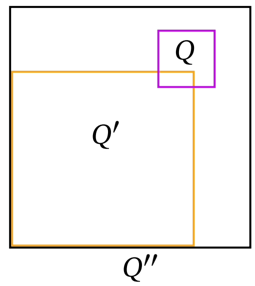

FIGURE 1. Q'' contains Q and Q', and shares a common vertex with Q'.

$(2\kappa+1)\ell(Q)$. Then $M_1 f(x) = M_1(f 1_{Q^*})(x)$ for any $x \in Q$, and

$$\begin{aligned}
&\frac{1}{|Q|}\int_{\Omega_1}\Big(M_1 f(x) - (Mf)_Q\Big)\,dx\\
\le&\frac{1}{|Q|}\int_{\Omega_1}\Big(M_1 f(x) - (M_1 f)_Q\Big)\,dx\\
\le&\frac{1}{|Q|}\int_{Q}\Big|M_1 f(x) - f_{Q^*} + f_{Q^*} - (M_1 f)_Q\Big|\,dx\\
\le&\frac{2}{|Q|}\int_{Q}\Big|M_1 f(x) - f_{Q^*}\Big|\,dx\\
\le&2\left(\frac{1}{|Q|}\int_{Q}\Big|M_1\left(f 1_{Q^*} - f_{Q^*}\right)(x)\Big|^2\,dx\right)^{1/2}\\
\le&2\left(\frac{1}{|Q|}\int_{Q}\Big|M\left[\left(f - f_{Q^*}\right)1_{Q^*}\right](x)\Big|^2\,dx\right)^{1/2}\\
\le&C\kappa^{n/2}\left(\frac{1}{|Q^*|}\int_{Q^*}\Big|f(x) - f_{Q^*}\Big|^2\,dx\right)^{1/2}.
\end{aligned} \tag{3.3}$$

It remains to consider the other term (i.e., $i = 2$) on the right-hand side of (3.2). For any fixed $x \in \Omega_2$, we have $Mf(x) = M_2 f(x) > (Mf)_Q$. Let Q' be any cube containing x with $\ell(Q') \ge \kappa\ell(Q)$. Let Q'' be a cube with $\ell(Q'') = \ell(Q) + \ell(Q')$ which contains both Q and Q' and shares some common faces (i.e., they have a common vertex); see Figure 1.

Then $Mf(y) \ge f_{Q''}$ for any $y \in Q$, so $f_{Q''} \le (Mf)_Q$. Without loss of generality, one may assume that $\ell(Q')/\ell(Q) \in \mathbb{N}_+$, thus $Q''\backslash Q'$ can be partitioned into $\frac{|Q''|-|Q'|}{|Q|} = (1+\frac{\ell(Q')}{\ell(Q)})^n - (\frac{\ell(Q')}{\ell(Q)})^n$ mutually disjoint cubes of side length $\ell(Q)$. Hence, by the pigeonhole principle, there exists a cube $P \subset Q'' \setminus Q$ with $\ell(P) = \ell(Q)$ such that $f_P \le f_{Q''\backslash Q'}$. As a consequence and by the nonnegative assumption of f,

$$\begin{aligned} f_{Q'} - (Mf)_Q &\le f_{Q'} - f_{Q''} \\ &= f_{Q'} - \frac{|Q'|}{|Q''|} f_{Q'} - \frac{1}{|Q''|}\int_{Q''\setminus Q'} f(y)dy \\ &\le \frac{|Q''|-|Q'|}{|Q''|}\left[f_{Q'} - f_{Q''\setminus Q'}\right] \\ &\le \frac{|Q''|-|Q'|}{|Q''|}\left[f_{Q'} - f_P\right] \\ &\lesssim \frac{(\ell(Q')+\ell(Q))^n - \ell(Q')^n}{(\ell(Q')+\ell(Q))^n}\log\left(\frac{\ell(Q')}{\ell(Q)}\right)\|f\|_{\mathrm{BMO}} \\ &\lesssim \frac{\log\kappa}{\kappa}\|f\|_{\mathrm{BMO}}. \end{aligned}$$

Taking the supremum over all such cubes Q', we obtain

$$M_2 f(x) - (Mf)_Q \le C\frac{\log\kappa}{\kappa}\|f\|_{\mathrm{BMO}} \quad \text{for } x \in \Omega_2.$$

Consequently,

$$\frac{1}{|Q|}\int_{\Omega_2}\Big(M_2 f(x) - (Mf)_Q\Big)\,dx \le C\frac{\log\kappa}{\kappa}\|f\|_{\mathrm{BMO}}.$$

Therefore, for any $\kappa > 1$,

$$\begin{aligned} &\frac{1}{|Q|}\int_Q \Big|Mf(x) - (Mf)_Q\Big|\,dx \\ &\le C\left[\kappa^{n/2}\bigg(\frac{1}{|Q^*|}\int_{Q^*}\Big|f(x) - f_{Q^*}\Big|^2\,dx\bigg)^{1/2} + \frac{\log\kappa}{\kappa}\|f\|_{\mathrm{BMO}}\right]. \end{aligned} \tag{3.4}$$

Now we verify $\widetilde{\gamma}_i(Mf) = 0$ for $i = 1, 3, 5$. For any given $\varepsilon > 0$, since $f \in \mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$ satisfies $\widetilde{\gamma}_i(f) = 0$ for $i = 1, 3, 5$, there exist two integers $I_\varepsilon >> 1$ and $J_\varepsilon >> 1$ such that

$$\sup_{P:\,\ell(P)\le 2^{-I_\varepsilon}}\left(\frac{1}{|P|}\int_P |f(x) - f_P|^2 dx\right)^{1/2} < \varepsilon, \tag{3.5a}$$

$$\sup_{P:\,P\subseteq(Q(0,2^{J_\varepsilon}))^c}\left(\frac{1}{|P|}\int_P |f(x) - f_P|^2 dx\right)^{1/2} < \varepsilon, \tag{3.5b}$$

$$\sup_{P\subseteq(Q(0,2^{J_\varepsilon}))^c,\,\ell(P)\ge\rho(c_P)}\left(\frac{1}{|P|}\int_P |f(x)|^2 dx\right)^{1/2} < \varepsilon, \tag{3.5c}$$

where c_P dentoes the center of the cube P, and $Q(0, 2^{J_\varepsilon})$ denotes the cube with $c_Q = 0$ and $\ell(Q) = 2^{J_\varepsilon}$.

Hence, taking $\kappa = \varepsilon^{-1/n}$,

- one may combine (3.4) and (3.5a) to see when $(2\kappa+1)\ell(Q) \le 2^{-I_\varepsilon}$, we have

$$\frac{1}{|Q|}\int_Q \left|Mf(x)-(Mf)_Q\right| dx \le C\left[\varepsilon^{1/2} + \|f\|_{\mathrm{BMO}}\, \varepsilon^{1/(2n)}\right], \tag{3.6}$$

  so

$$\lim_{a\to 0} \sup_{Q:\, \ell_Q \le a} \left(|Q|^{-1}\int_Q \left|Mf(x)-(Mf)_Q\right| dx\right) = 0.$$

  Applying a John–Nirenberg type inequality associated to small cubes with $(2\kappa+1)\ell(Q) \le 2^{-I_\varepsilon}$, we conclude that $\widetilde{\gamma}_1(Mf) = 0$. This approach can be used to verify $\widetilde{\gamma}_3(Mf) = \widetilde{\gamma}_5(Mf) = 0$, that is, the L^2 integrals involved therein can be replaced by the corresponding L^1 integrals.

- Similarly, one may combine (3.4) and (3.5b) to see when $Q^* = (2\kappa+1)Q \subseteq (Q(0,2^{J_\varepsilon}))^c$, (3.6) still holds. Hence, $\widetilde{\gamma}_3(Mf) = 0$.

It remains to show $\widetilde{\gamma}_5(Mf) = 0$. Note that for any cube Q with $\ell(Q) \ge \rho(c_Q)$, there exists a constant $C > 0$ independent of Q such that there exists a sequence $\{Q(x_k, \rho(x_k))\}_k$ such that

$$Q \subseteq \bigcup_k Q(x_k,\rho(x_k)) \quad \text{and} \quad \sum_k |Q(x_k,\rho(x_k))| \le C|Q|. \tag{3.7}$$

Hence, to verify $\widetilde{\gamma}_5(Mf) = 0$ for any given $0 \le f \in \mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$, it suffices to show

$$\lim_{a\to\infty} \sup_{\substack{Q:\, Q\subseteq (Q(0,a))^c \\ \ell(Q)=\rho(c_Q)}} \frac{1}{|Q|}\int_Q Mf(x)dx = 0. \tag{3.8}$$

For any given Q with $\ell(Q) = \rho(c_Q)$ and for any $x \in Q$, define

$$M_1'f(x) := \sup_{Q'\ni x:\, \ell(Q')<\rho(c_Q)} f_{Q'} \quad \text{and} \quad M_2'f(x) := \sup_{Q'\ni x:\, \ell(Q')\ge\rho(c_Q)} f_{Q'}.$$

Clearly, $Mf = \max\{M_1'f, M_2'f\}$ on Q, and so

$$\frac{1}{|Q|}\int_Q Mf(x)dx \le \sum_{i=1}^{2} \frac{1}{|Q|}\int_Q M_i'f(x)dx.$$

Note that

$$\begin{aligned}\frac{1}{|Q|}\int_Q M_1'f(x)dx &= \frac{1}{|Q|}\int_Q M_1'(f1_{3Q})(x)dx \\ &= \left(\frac{1}{|Q|}\int_Q \left|M'(f1_{3Q})(x)\right|^2 dx\right)^{1/2} \\ &\le C\left(\frac{1}{|Q|}\int_{\mathbb{R}^n} \left|f1_{3Q}(x)\right|^2 dx\right)^{1/2} \\ &\le C\left(\frac{1}{|3Q|}\int_{3Q} \left|f(x)\right|^2 dx\right)^{1/2}. \qquad (3.9)\end{aligned}$$

On the other hand, there exists a constant $C > 0$ such that for any cube $Q' \ni x$ with $\ell(Q') \ge \rho(c_Q)$, $f_{Q'} \le Cf_{Q''} \le CM_2'f(c_Q)$, where Q'' is the smallest cube containg Q and Q'. Hence,

$$\frac{1}{|Q|}\int_Q M_2'f(x)dx \le CM_2'f(c_Q).$$

For any given $\varepsilon > 0$, let J_ε be the positive integer as in (3.5c). For any $Q(y,\rho(y)) \cap Q(0,2^{J_\varepsilon}) \neq \emptyset$, it follows from Lemma 2.2 that $\rho(y) \approx \rho(z)$ for any $z \in Q(y,\rho(y)) \cap Q(0,2^{J_\varepsilon})$, thus $Q(y,\rho(y)) \subset Q(z, C\rho(z))$ for some $C > 1$ independent of y and z. Without loss of generality, we may assume that

$$\rho(0) \le 2^{J_\varepsilon}, \qquad (3.10)$$

which can be achieved by taking J_ε sufficiently large. Then by Lemma 2.2 again,

$$\sup_{x\in Q(0,2^{J_\varepsilon})} \rho(x) \le c\rho(x_0)^{\frac{1}{k_0+1}} 2^{J_\varepsilon\cdot\frac{k_0}{k_0+1}} \le c2^{J_\varepsilon}$$

for some $c > 1$. Hence,

$$\bigcup_{Q(y,\rho(y)):\, Q(y,\rho(y))\cap Q(0,2^{J_\varepsilon})\neq\emptyset} Q(y,\rho(y)) \subseteq Q(0, C'2^{J_\varepsilon}) \qquad (3.11)$$

for some $C' > 1$.

Note that M_2' is bounded by its corresponding centered maximal counterpart, thus when c_Q is far away from the origin,

$$\begin{aligned}M_2'f(c_Q) &\le C \sup_{Q'=Q'(c_Q,\ell(Q')):\, \ell(Q')\ge\rho(c_Q)} f_{Q'} \\ &= C\max\Bigg\{\sup_{Q'(c_Q,\ell(Q'))\subseteq(Q(0,2^{J_\varepsilon}))^c:\, \ell(Q')\ge\rho(c_Q)} f_{Q'}, \\ &\qquad \sup_{Q'(c_Q,\ell(Q'))\cap Q(0,2^{J_\varepsilon})\neq\emptyset:\, \ell(Q')\ge\rho(c_Q)} f_{Q'}\Bigg\}.\end{aligned}$$

For any $Q'(c_Q, \ell(Q'))$ with $\ell(Q') \geq \rho(c_Q)$, if $Q'(c_Q, \ell(Q')) \subseteq (Q(0, 2^{J_\varepsilon}))^c$, it follows from (3.5c) to see $f_{Q'} < \varepsilon$, as desired.

On the other hand, $Q'(c_Q, \ell(Q')) \cap Q(0, 2^{J_\varepsilon}) \neq \emptyset$ and $\ell(Q') \geq \rho(c_Q)$. Similar to (3.7), there exists a sequence $\{Q(x_k, \rho(x_k))\}_k$ such that

$$Q'(c_Q, \ell(Q')) \subseteq \bigcup_k Q(x_k, \rho(x_k)) \quad \text{and} \quad \sum_k |Q(x_k, \rho(x_k))| \leq C|Q'(c_Q, \ell(Q'))|.$$

Denote

$$\Lambda_1 = \left\{k : \ Q(x_k, \rho(x_k)) \subseteq (Q(0, 2^{J_\varepsilon}))^c\right\}$$

and

$$\Lambda_2 = \left\{k : \ Q(x_k, \rho(x_k)) \cap Q(0, 2^{J_\varepsilon}) \neq \emptyset\right\}.$$

Combining the definition (1.2), (3.5c) and (3.11), whenever

$$Q'(c_Q, \ell(Q')) \cap Q(0, 2^{J_\varepsilon}) \neq \emptyset,$$

we have

$$\begin{aligned} f_{Q'(c_Q,\ell(Q'))} &\leq \frac{1}{|Q'(c_Q, \ell(Q'))|} \sum_{k\in\Lambda_1} \int_{Q(x_k,\rho(x_k))} f(x)dx \\ &\quad + \frac{1}{|Q'(c_Q, \ell(Q'))|} \int_{\bigcup_{k\in\Lambda_2} Q(x_k,\rho(x_k))} f(x)dx \\ &\leq C\varepsilon \frac{\sum_{k\in\Lambda_1} |Q(x_k, \rho(x_k))|}{|Q'(c_Q, \ell(Q'))|} + C\frac{|Q(0, C'2^{J_\varepsilon})|}{|Q'(c_Q, \ell(Q'))|} f_{Q(0,C'2^{J_\varepsilon})} \\ &\leq C\varepsilon + C\|f\|_{\mathrm{BMO}_{\mathcal{L}}} \frac{|Q(0, C'2^{J_\varepsilon})|}{|Q'(c_Q, \ell(Q'))|}, \end{aligned}$$

where the last inequality follows from (3.10) and $C' > 1$.

To continue, observe that there exists a positive $J'_\varepsilon > J_\varepsilon$ such that for any $Q(c_Q, \rho(c_Q)) \subseteq Q(0, 2^{J'_\varepsilon})^c$, we have

$$\frac{|Q(0, C'2^{J_\varepsilon})|}{|Q'(c_Q, \ell(Q'))|} < \varepsilon \quad \text{whenever } \ Q'(c_Q, \ell(Q')) \cap Q(0, 2^{J_\varepsilon}) \neq \emptyset, \ \ell(Q') \geq \rho(c_Q). \tag{3.12}$$

Indeed, since $|c_Q| \geq 2^{J'_\varepsilon}$, for any cube $Q'(c_Q, \ell(Q'))$ having nonempty intersection with $Q(0, 2^{J_\varepsilon})$, we have

$$\ell(Q') \gtrsim |c_Q| - 2^{J_\varepsilon} \geq 2^{J'_\varepsilon - 1},$$

which ensures (3.12) by taking $J'_\varepsilon \gtrsim J_\varepsilon - (\log_2 \varepsilon)/n$. Note that the condition "$\ell(Q') \geq \rho(c_Q)$" for such Q' is also compatible: since $Q(c_Q, \rho(c_Q)) \subseteq Q(0, 2^{J'_\varepsilon})^c$, there exists some integer $j \geq 0$ such that $|c_Q| \approx 2^{J'_\varepsilon + j}$, then by Lemma 2.2 and (3.10),

$$\rho(c_Q) \lesssim 2^{(J'_\varepsilon + j)\cdot \frac{k_0}{k_0+1}},$$

which implies that if $Q'(c_Q, \ell(Q')) \cap Q(0, 2^{J_\varepsilon}) \neq \emptyset$, we must have $\ell(Q') \geq 2^{J'_\varepsilon + j - 1} - 2^{J_\varepsilon} \geq \rho(c_Q)$ by taking $J'_\varepsilon \gtrsim (k_0 + 1)J_\varepsilon$ sufficently large.

Therefore, for any $\varepsilon > 0$, there exists a positive integer $J'_\varepsilon > J_\varepsilon$ such that for any cube $Q(c_Q, \rho(Q)) \subseteq (Q(0, 2^{J'_\varepsilon}))^c$,

$$M'_2 f(c_Q) \lesssim \varepsilon.$$

From the above, (3.8) holds, and $\widetilde{\gamma}_5(Mf) = 0$ follows readily.

We complete the proof of Theorem 1.1. □

Remark 3.1. *Let R_j^* be the adjoint of the Riesz transform R_j associated to $\mathcal{L}$ for $j = 1, 2, \dots, n$. Recall that (see [18, (5.5)]) for $V \in RH_q$ with $q > n/2$,*

$$\|R_j^* f\|_{L^p(\mathbb{R}^n)} \le C_p \|f\|_{L^p(\mathbb{R}^n)} \quad \text{for } p'_0 \le p < \infty,$$

where $p'_0 = p_0/(p_0 - 1)$ and $1/(p_0) = (1/q) - (1/n)$.

For the endpoint case $p = \infty$, we have

$$\|R_j^* f\|_{\mathrm{BMO}_\mathcal{L}} \le C\|f\|_{L^\infty(\mathbb{R}^n)}.$$

More precisely, for each $g \in \mathrm{BMO}_\mathcal{L}(\mathbb{R}^n)$, we can represent g as $g = \varphi_0 + \sum_{j=0}^{n} R_j^* \varphi_j$, where $\varphi_j \in L^\infty(\mathbb{R}^n)$ for $0 \le j \le n$, and $\|g\|_{\mathrm{BMO}_\mathcal{L}} \approx \sum_{j=0}^{n} \|\varphi_j\|_{L^\infty}$; see [22, Theorem 1.3].

Furthermore, one may combine (1.6) and the John–Nirenberg type inequality to see for $1 \le p < \infty$ and $f \in L^\infty(\mathbb{R}^n)$,

$$\sup_{B = B(x_B, r_B):\, r_B \ge \rho(x_B)} \left(\frac{1}{|B|} \int_B |R_j^* f(x)|^p dx \right)^{1/p} \le C_p \|f\|_{L^\infty}.$$

Remark 3.2. *As a straightforward consequence, when $V \equiv 0$ (i.e., $\rho(x) \equiv +\infty$), our Theorem 1.1 implies that the Hardy–Littlewood maximal operator is bounded on the classical $\mathrm{CMO}(\mathbb{R}^n)$, which has been established in [15]. Specifically, for any f belongs to $\mathrm{CMO}(\mathbb{R}^n)$ for which Mf is not identically equal to infinity, then Mf also belongs to $\mathrm{CMO}(\mathbb{R}^n)$, and*

$$\|Mf\|_{\mathrm{BMO}(\mathbb{R}^n)} \le C\|f\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

This result is a straightforward consequence of our argument by noting that $f \in \mathrm{CMO}(\mathbb{R}^n)$ if and only if $f \in \mathrm{BMO}(\mathbb{R}^n)$ and $\widetilde{\gamma}_1(f) = \widetilde{\gamma}_2(f) = \widetilde{\gamma}_3(f) = 0$, hence it suffices to prove $\widetilde{\gamma}_1(Mf) = \widetilde{\gamma}_2(Mf) = \widetilde{\gamma}_3(Mf) = 0$. Among them, $\widetilde{\gamma}_1(Mf) = \widetilde{\gamma}_3(Mf) = 0$ has been proved, and one may combine (3.4) and $\widetilde{\gamma}_2(f) = 0$ to deduce (3.6) also holds for any cube whose sidelength is sufficiently large, that is, we obtain the remaining $\widetilde{\gamma}_2(Mf) = 0$.

Now we address the fact that there exists $f \in \mathrm{CMO}(\mathbb{R}^n)$ such that $Mf \equiv +\infty$. An interesting example illustrating this phenomenon was constructed in [15], and we present a distinct one here.

We consider $n = 1$ for simplicity. Define

$$f(x) = \begin{cases} \ln\ln|x|, & |x| \geq e, \\ 0, & |x| < e. \end{cases}$$

It's clear f is a uniformly continuous function on $\mathbb{R}$. As a consequence, $\widetilde{\gamma}_1(f) = 0$.

We begin by verifying that $f \in \mathrm{BMO}(\mathbb{R})$. To see it, we will use

$$\|f\|_{\mathrm{BMO}(\mathbb{R})} \approx \sup_{I:=[a,b]\subseteq\mathbb{R}} \inf_{\mathsf{Avg}_I\in\mathbb{R}} \frac{1}{b-a}\int_a^b \left|f(x) - \mathsf{Avg}_I\right| dx.$$

Let M be a sufficiently large integer. For any interval $I = [a, b]$,

- *Case I. $I \subseteq [-10M, 10M]$. Then*

$$\frac{1}{b-a}\int_a^b \left|f(x) - f_{[a,b]}\right| dx \leq 2\|f\|_{L^\infty([-10M,10M])}.$$

- *Case II. $I \cap (\mathbb{R} \setminus [-10M, 10M]) \neq \emptyset$.*
  Write $[a, b] = [c - R, c + R]$ with $c = \frac{a+b}{2}$ and $R = \frac{b-a}{2}$.
  - *Subcase 1. $I \cap [-M, M] = \emptyset$.*
    Since f is an even function, one may assume that $[a, b] \subseteq (M, +\infty)$.
    * *Subsubcase 1-1. If $c > 2R$, let $\mathsf{Avg}_I = \ln\ln c$. Combining $a = c - R > \max\{M, R\}$, we have*

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b \left|f(x) - \mathsf{Avg}_I\right| dx \\
\leq &\max\Big\{\ln\ln c - \ln\ln(c-R), \ln\ln(c+R) - \ln\ln c\Big\} \\
\leq &\max\left\{\ln\left(1 + \frac{\ln(1+\frac{R}{a})}{\ln a}\right), \ln\left(1 + \frac{\ln(1+\frac{R}{c})}{\ln c}\right)\right\} \\
\leq &\ln\left(1 + \frac{\ln 2}{\ln M}\right),
\end{aligned}$$

      which is sufficiently small.

* *Subsubcase 1-2. Otherwise, if* $c < 2R$*, let* $\mathsf{Avg}_I = \ln\ln R$.

$$\begin{aligned}&\frac{1}{b-a}\int_a^b \left|f(x)-\mathsf{Avg}_I\right|dx\\ =&\frac{1}{2R}\int_{M<x<3R}\left|\ln\left(1+\frac{\ln\frac{x}{R}}{\ln R}\right)\right|dx\\ \lesssim&\int_0^3\left|\ln\left(1+\frac{\ln x}{\ln M}\right)\right|dx \le C_M,\end{aligned}$$

and C_M *is sufficiently small since* $M >> 1$.

Note that the argument in Subcase 1 also implies that $\widetilde{\gamma}_3(f) = 0$ *(in fact we do not use the assumption "*$I \cap (\mathbb{R} \setminus [-10M, 10M]) \neq \emptyset$*").*

– *Subcase 2.* $I \cap [-M, M] \neq \emptyset$*. Then* $R > 9M/2$ *and it's obvious that* $|c| < 2R$*, thus* $[a, b] \subseteq [-3R, 3R]$*. Similar to Subsubcase 1-2, let* $\mathsf{Avg}_I = \ln\ln R$ *and*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b \left|f(x)-\mathsf{Avg}_I\right|dx\\ \le&\frac{1}{2R}\int_{e<|x|<3R}\left|\ln\left(1+\frac{\ln\frac{|x|}{R}}{\ln R}\right)\right|dx\\ \lesssim&\int_{|x|<3}\left|\ln\left(1+\frac{\ln|x|}{\ln M}\right)\right|dx\\ \le&2C_M.\end{aligned}$$

Notably, we also complete the proof of $\widetilde{\gamma}_2(f) = 0$.

From the above, we obtain $f \in \mathrm{BMO}(\mathbb{R})$ *and* $\widetilde{\gamma}_1(f) = \widetilde{\gamma}_2(f) = \widetilde{\gamma}_3(f) = 0$. *Hence* $f \in \mathrm{CMO}(\mathbb{R})$*, as desired.*

For any $x \in \mathbb{R}$*, we may assume* $x \geq 0$ *since* f *is even. Consider the interval* $I_R = [x, x + 2R]$ *for some* $R >> 1$,

$$\frac{1}{2R}\int_{I_R}|f(y)|\,dy \ge \frac{1}{2R}\int_{x+R}^{x+2R}\ln\ln y\,dy \ge \frac{\ln\ln R}{2},$$

which deduces that $Mf(x) = +\infty$ *for every* $x \in \mathbb{R}$*. This is a remarkable phenomenon that* $f \in \mathrm{CMO}$ *can not ensure that* $Mf < +\infty$*. However, for any* f *that belongs to* CMO *for which* Mf *is not identically equal to infinity, then* Mf *is bounded on* CMO.

4. Riesz transforms and $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$: proof of Theorem 1.2

For each $j = 1, \dots, n$, let $R_j(x, y)$ be kernels of Riesz transforms $R_j = \frac{\partial}{\partial x_j}\mathcal{L}^{-1/2}$. Then the adjoint of R_j is given by

$$R_j^* g(x) = \lim_{\varepsilon\to 0}\int_{|y-x|>\varepsilon} R_j(y, x)g(y)dy.$$

Proof of Theorem 1.2. For each given $j = 1, \dots, n$, recall that R_j^* is a bounded linear operator from $L^\infty(\mathbb{R}^n)$ to $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$. This, combined with the fact that $C_0(\mathbb{R}^n)$ is the closure of $C_c^\infty(\mathbb{R}^n)$ in $L^\infty(\mathbb{R}^n)$, deduces that

$$R_j^*(C_0(\mathbb{R})) \subseteq \overline{R_j^*\,(C_c^\infty(\mathbb{R}^n))}^{\mathrm{BMO}_{\mathcal{L}}},$$

where $C_0(\mathbb{R}^n)$ is the space of all continuous functions on $\mathbb{R}^n$ which vanish at infinity. Meanwhile, it follows from Theorem C in [19] that the spaces $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$ is the closure in the $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ norm of $C_0(\mathbb{R}^n)$ (i.e., (iii) of Theorem 2.1). Therefore, to prove Theorem 1.2, it suffices to show

$$R_j^*\,(C_c^\infty(\mathbb{R}^n)) \subseteq C_0(\mathbb{R}^n). \tag{4.1}$$

Step I. Let $R_j^0(x, y)$ be kernels of the classical Riesz transforms $R_j^0 = \frac{\partial}{\partial x_j}(-\Delta)^{-1/2}$, $j = 1, \dots, n$. Since $V \in RH_q$ for some $q \ge n/2$, we may assume that $n/2 \le q < n$ (the case for $q \ge n$ is simpler because the relevant kernels then exhibit better regularity). Notably, one remarkable fact is the self-improvement of the RH_q class that for any $V \in RH_q$, there exists $\varepsilon > 0$ depending only on C in (1.1) and the dimension n such that $V \in RH_{q+\varepsilon}$. Therefore, $V \in B_{q_1}$ for some $n/2 \le q < q_1 < n$.

Applying Lemmas 5.7 and 5.8 in [18], we have for each $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned}\left|R_j^*(\varphi)(x)\right| &\le \left|\int_{|y-x|>\rho(x)} R_j(y, x)\,\varphi(y)\,dy\right| \\ &\quad+ \left|\int_{|y-x|\le\rho(x)} \left[R_j(y, x) - R_j^0(y, x)\right]\varphi(y)\,dy\right| \\ &\quad+ \left|\int_{|y-x|\le\rho(x)} R_j^0(y, x)\,\varphi(y)\,dy\right| \\ &\le C\left[M\left(|\varphi|^{p_1'}\right)(x)\right]^{1/p_1'} + 2\sup_{\tau>0}\left|\int_{|y-x|>\tau} R_j^0(y, x)\,\varphi(y)\,dy\right|, \end{aligned} \tag{4.2}$$

where $\frac{1}{p_1'} = 1 - \frac{1}{q_1} + \frac{1}{n}$, and M denotes the uncentered Hardy–Littlewood maximal operator. It's clear that

$$\left\| \left[M\left(|\varphi|^{p_1'}\right)\right]^{1/p_1'} \right\|_{L^\infty} \le \|\varphi\|_{L^\infty} \quad \text{and} \quad \left[M\left(|\varphi|^{p_1'}\right)(x)\right]^{1/p_1'} \to 0 \ \text{ as } \ |x| \to \infty,$$

provided by $\varphi \in C_c^\infty(\mathbb{R}^n)$. Moreover, by Cotlar's inequality,

$$\sup_{\tau>0} \left| \int_{|y-x|>\tau} R_j^0(y,x)\,\varphi(y)dy \right| \le C\left[M\left(R_j^0(\varphi)\right)(x) + M(\varphi)(x)\right]. \tag{4.3}$$

It's clear that $R_j^0(\varphi) \in C_0(\mathbb{R}^n)$ since the Fourier transform of $R_j^0(\varphi)$ belongs to $L^1(\mathbb{R}^n)$. This allows us to verify readily that the left-hand side of (4.3) is an L^∞ function and vanishes at infinity, as desired.

From the above, $R_j^*(\varphi) \in L^\infty$ and $\lim_{|x|\to\infty} R_j^*(\varphi) = 0$ for any given $\varphi \in C_c^\infty(\mathbb{R}^n)$.

Step II. It remains to show $R_j^*(\varphi)$ is continuous. To this end, it suffices to show for any given $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$, there exists a positive constant $\theta = \theta(x_0,\varepsilon)$ sufficiently small, such that for any $x_1 \in B(x_0,\theta)$,

$$\left|R_j^*(\varphi)(x_1) - R_j^*(\varphi)(x_0)\right| \lesssim \varepsilon \quad \text{for } \ x_1 \in B(x_0,\theta). \tag{4.4}$$

Firstly, we fix a positive integer $\kappa_0 >> 1$ such that

$$2^{-\kappa_0(2-n/q_1)} < \frac{\varepsilon}{\left\| \left[M\left(|\varphi|^{p_1'}\right)\right]^{1/p_1'} \right\|_{L^\infty}}. \tag{4.5}$$

For every $j = 1,\dots,n$ and $x \in B(x_0, 2^{-\kappa_0}\rho(x_0))$, rewrite

$$\begin{aligned} R_j^*(\varphi)(x) = & \int_{|y-x|>2^{-\kappa_0}\rho(x_0)} R_j(y,x)\,\varphi(y)\,dy \\ & + \lim_{\tau\to 0} \int_{\tau<|y-x|\le 2^{-\kappa_0}\rho(x_0)} \left[R_j(y,x) - R_j^0(y,x)\right]\varphi(y)\,dy \\ & + \left[(R_j^0)^*(\varphi)(x) - \int_{|y-x|>2^{-\kappa_0}\rho(x_0)} R_j^0(y,x)\,\varphi(y)\,dy\right] \\ =: & \ T_1(\varphi)(x) + T_2(\varphi)(x) + T_3(\varphi)(x), \end{aligned}$$

where $(R_j^0)^*(\varphi)(x) = -R_j^0(\varphi)(x)$ due to the anti-symmetric property of the kernel of R_j^0.

We observe the following facts:

- Note that (see [18, p. 540])

$$\begin{aligned} & \left(\int_{2^{j-1}\rho(x)<|y-x|\le 2^j\rho(x)} \left|R_j(y,x) - R_j^0(y,x)\right|^{p_1} dy \right)^{1/p_1} \\ & \le C(2^j)^{2-(n/q_1)}(2^j\rho(x))^{-n/p_1'} \quad \text{for } j \le 0, \end{aligned}$$

and $\rho(x) \approx \rho(x_0)$ for $x \in B(x_0, 2^{-\kappa_0}\rho(x_0))$, thus there exists a positive integer M such that

$$\begin{aligned}&\left(\int_{2^{j-1}\rho(x_0)<|y-x|\le 2^j\rho(x_0)} \left|R_j(y,x)-R_j^0(y,x)\right|^{p_1} dy\right)^{1/p_1}\\ &\le \left(\int_{2^{j-\mathsf{M}}\rho(x)<|y-x|\le 2^{j+\mathsf{M}}\rho(x)} \left|R_j(y,x)-R_j^0(y,x)\right|^{p_1} dy\right)^{1/p_1}\\ &\le C(2^j)^{2-(n/q_1)}(2^j\rho(x_0))^{-n/p_1'} \quad \text{for } j \le 0.\end{aligned}$$

Consequently, in combination with (4.5),

$$\begin{aligned}&|T_2(\varphi)(x)|\\ &\le \int_{|y-x|\le C2^{-\kappa_0}\rho(x)} \left|R_j(y,x)-R_j^0(y,x)\right| |\varphi(y)|\, dy\\ &\le \sum_{j=-\infty}^{-\kappa_0} \left(\int_{|y-x|\le 2^j\rho(x_0)} |\varphi(y)|^{p_1'} dy\right)^{1/p_1}\\ &\qquad \cdot \left(\int_{2^{j-1}\rho(x_0)<|y-x|\le 2^j\rho(x_0)} \left|R_j(y,x)-R_j^0(y,x)\right|^{p_1} dy\right)^{1/p_1}\\ &\le C2^{-\kappa_0(2-n/q_1)}\left[M\left(|\varphi|^{p_1'}\right)(x)\right]^{1/p_1'}\\ &\le C\varepsilon\end{aligned}$$

for any $x \in B(x_0, 2^{-\kappa_0}\rho(x_0))$.

- For the continuity of $T_3(\varphi)$, note that $(R_j^0)^*(\varphi) = -R_j^0(\varphi) \in C_0(\mathbb{R}^n)$ mentioned in *Step I*, it remains to show the continuity of

$$T_4(\varphi) := \int_{|y-x|>2^{-\kappa_0}\rho(x_0)} R_j^0(y,x)\,\varphi(y)\, dy,$$

which is a bounded function by combining (4.3) and the fact

$$R_j^0(C_c^\infty(\mathbb{R}^n)) \subseteq C_0(\mathbb{R}^n).$$

Denote $E_x := \{y : |y-x| \ge 2^{-\kappa_0}\rho(x)\}$ for $x \in B(x_0, 2^{-\kappa_0}\rho(x_0))$, then

$$\Delta_\mu : \quad x \mapsto |\Delta E(x)| := \left|\left(E_x \cap E_{x_0}\right) \setminus \left(E_x \cap E_{x_0}\right)\right|$$

is continuous on $B(x_0, 2^{-\kappa_0}\rho(x_0))$ and $\Delta_\mu(x_0) = 0$. This, combined with the explicit expression of $R_j^0(y,x)$, deduces that $T_4(\varphi)$ is continuous on x_0, whenever $\varphi \in C_c^\infty(\mathbb{R}^n)$.

From the above, (4.4) follows readily if one could prove that for any given $x_0 \in \mathbb{R}^n$, there exists $\theta < 2^{-\kappa_0}\rho(x_0)$ such that

$$\left|T_1(\varphi)(x_1) - T_1(\varphi)(x_0)\right| \lesssim \varepsilon \quad \text{for } x_1 \in B(x_0, \theta). \tag{4.6}$$

Step III. Now let's verify the continuity of $T_1(\varphi)$ by proving (4.6).

Given $x_0 \in \mathbb{R}^n$, for each $x_1 \in B(x_0, 2^{-\kappa_0}\rho(x_0))$,

$$\begin{aligned} &T_1(\varphi)(x_1) - T_1(\varphi)(x_0) \\ &= \int_{|y-x_0|>2^{-\kappa_0}\rho(x_0)} \left[R_j(y,x_1) - R_j(y,x_0)\right]\varphi(y)\,dy + \mathcal{E}(x_1), \end{aligned}$$

where

$$\left|\mathcal{E}(x_1)\right| \le \int_{\Delta E(x_1)} \left|R_j(y,x_1)\right| \left|\varphi(y)\right|\,dy \to 0 \quad \text{as} \quad x_1 \to x_0.$$

Hence it remains to show

$$\int_{|y-x_0|>2^{-\kappa_0}\rho(x_0)} \left[R_j(y,x_1) - R_j(y,x_0)\right]\varphi(y)\,dy \to 0 \quad \text{as} \quad x_1 \to x_0.$$

Let $\Gamma(x,y,\tau)$ denote the fundamental solution for the Schrödinger operator $-\Delta + (V + i\tau)$, $\tau \in \mathbb{R}$. Clearly,

$$\Gamma(x,y,\tau) = \Gamma(y,x,-\tau),$$

and $\nabla_y\Gamma(x,y,\tau)$ is a solution to the equation $-\Delta u + (V+i\tau)u = 0$ in $\mathbb{R}^n \setminus \{y\}$ as a function of x. Consequently, $\nabla_y\Gamma(y,x,\tau)$ is a solution to the equation $-\Delta u + (V+i(-\tau))u = 0$ in $B(x_0, 2^{-(\kappa_0+1)}\rho(x_0))$, whenever $|y-x_0| > 2^{-\kappa_0}\rho(x_0)$. Denote

$$\delta = 2 - n/q_1 > 0 \quad \text{and} \quad r_0 = 2^{-(\kappa_0+2)}\rho(x_0).$$

By the imbedding theorem of Morrey and [18, Lemma 4.6] (see also the last inequality in [18, page 534]), we have for any $x_1 \in B(x_0, 2^{-(\kappa_0+4)}\rho(x_0)) = B(x_0, r_0/4)$,

$$\begin{aligned} &\left|\nabla_y\Gamma(y,x_1,\tau) - \nabla_y\Gamma(y,x_0,\tau)\right| \\ &\le C|x_1 - x_0|^{1-n/p_1}\left(\int_{B(x_0,r_0)} \left|\nabla_x\nabla_y\Gamma(y,x,\tau)\right|^{p_1} dx\right)^{1/p_1} \\ &\le C\left(\frac{|x_1 - x_0|}{r_0}\right)^{\delta} \left\|\nabla_y\Gamma(y,x,\tau)\right\|_{L_x^\infty(B(x_0,2r_0))}\left[1 + \frac{1}{r_0^{n-2}}\int_{B(x_0,2r_0)} V(x)dx\right], \end{aligned}$$

where p_1 is the index in (4.2). Alternatively, one can apply a similar argument to that of [13, Proposition 2.12] to show the Hölder continuity.

To continue, we address the following facts.

- By Lemma 1.2 in [18],

$$\frac{1}{r_0^{n-2}}\int_{B(x_0,2r_0)} V\,dx \le C2^{-\kappa_0\delta} \lesssim 1.$$

- It's known that $\rho(x) \approx \rho(x_0)$ whenever $|x - x_0| \lesssim \rho(x_0)$. Now regard $\Gamma(y, x, \tau)$ is a solution to the equation $-\Delta u + (V + i\tau)u = 0$ in $\mathbb{R}^n \setminus \{x\}$ as a function of y, and $B(y, |y - x_0|/4) \cap B(x_0, 2r_0) = \emptyset$. Hence it follows from combining (4.8) and Theorem 2.7 in [18] that for each $m \in \mathbb{N}$, there exists a constant $C_m > 0$ such that

$$\begin{aligned}&\left\|\nabla_y \Gamma(y, x, \tau)\right\|_{L_x^\infty(B(x_0, 2r_0))}\\ \leq&\frac{C_m}{\left(1 + |\tau|^{1/2} R_y\right)^m \left(1 + R_y/\rho(x_0)\right)^m}\\ &\cdot\left[\frac{1}{R_y^{n-2}} \int_{B(y, R_y)} \frac{V(z)}{|z - y|^{n-1}} dz + \frac{1}{R_y^{n-1}}\right],\end{aligned}$$

where $R_y := |y - x_0|/4$.

Therefore, when $x_1 \in B(x_0, r_0/4)$ and $|y - x_0| > 2^{-\kappa_0}\rho(x_0) = 4r_0$, one may combine these estimates above to obtain

$$\begin{aligned}&\left|\nabla_y \Gamma(y, x_1, \tau) - \nabla_y \Gamma(y, x_0, \tau)\right|\\ \leq&\frac{C_m}{\left(1 + |\tau|^{1/2} R_y\right)^m \left(1 + R_y/\rho(x_0)\right)^m}\left(\frac{|x_1 - x_0|}{r_0}\right)^\delta\\ &\cdot\left[\frac{1}{R_y^{n-2}} \int_{B(y, R_y)} \frac{V(z)}{|z - y|^{n-1}} dz + \frac{1}{R_y^{n-1}}\right].\end{aligned}$$

Furthermore,

$$\begin{aligned}&\left|R_j(y, x_1) - R_j(y, x_0)\right|\\ =&\left|-\frac{1}{2\pi}\int_{\mathbb{R}} (-i\tau)^{-1/2}\left[\nabla_y \Gamma(y, x_1, \tau) - \nabla_y \Gamma(y, x_0, \tau)\right] d\tau\right|\\ \leq&\frac{C_m}{\left(1 + R_y/\rho(x_0)\right)^m}\left(\frac{|x_1 - x_0|}{r_0}\right)^\delta\left[\frac{1}{R_y^{n-1}} \int_{B(y, R_y)} \frac{V(z)}{|z - y|^{n-1}} dz + \frac{1}{R_y^{n}}\right].\end{aligned}$$

Note that

$$\begin{aligned}&\left\{y \in \mathbb{R}^n : \ |y - x_0| > 2^{-\kappa_0}\rho(x_0)\right\}\\ =&\bigcup_{k=-\kappa_0+1}^{\infty}\left\{y \in \mathbb{R}^n : \ 2^{k-1}\rho(x_0) < |y - x_0| \leq 2^k \rho(x_0)\right\},\end{aligned}$$

and for each $k \geq \kappa_0+1$, it follows from the Hardy–Littlewood–Sobolev inequality to obtain

$$\begin{aligned}
&\left|\int_{2^{k-1}\rho(x_0)<|y-x_0|\leq 2^k\rho(x_0)} \left[R_j(y,x_1)-R_j(y,x_0)\right]\varphi(y)\,dy\right| \\
&\leq \frac{C_m}{(1+2^k)^m}\left(\frac{|x_1-x_0|}{r_0}\right)^{\delta}\Bigg\{M\varphi(x_0)+\frac{1}{(2^k\rho(x_0))^{n-1}} \\
&\qquad \cdot\int_{2^{k-1}\rho(x_0)<|y-x_0|\leq 2^k\rho(x_0)}\left(\int_{\mathbb{R}^n}\frac{V(z)1_{B(y,R_y)}(z)}{|z-y|^{n-1}}dz\right)|\varphi(y)|dy\Bigg\} \\
&\leq \frac{C_m}{(1+2^k)^m}\left(\frac{|x_1-x_0|}{r_0}\right)^{\delta}\Bigg\{M\varphi(x_0)+\frac{1}{(2^k\rho(x_0))^{n-1}} \\
&\qquad \cdot\left(\int_{2^{k-1}\rho(x_0)<|y-x_0|\leq 2^k\rho(x_0)}\left|\int_{B(y,R_y)}\frac{V(z)}{|z-y|^{n-1}}dz\right|^{p_1}dy\right)^{1/p_1} \\
&\qquad \cdot\left(\int_{|y-x_0|\leq 2^k\rho(x_0)}|\varphi(y)|^{p_1'}dy\right)^{1/p_1'}\Bigg\} \\
&\leq \frac{C_m}{(1+2^k)^m}\left(\frac{|x_1-x_0|}{r_0}\right)^{\delta}M\varphi(x_0) \\
&\quad +\frac{C_m}{(1+2^k)^m}\left(\frac{|x_1-x_0|}{r_0}\right)^{\delta}\frac{1}{(2^k\rho(x_0))^{n-1}} \\
&\qquad \cdot\left(\int_{2^{k-2}\rho(x_0)<|y-x_0|\leq 2^{k+1}\rho(x_0)}V(z)^{q_1}dz\right)^{1/q_1} \\
&\qquad \cdot\left[M\left(|\varphi|^{p_1'}\right)(x_0)\right]^{1/p_1'}\left(2^k\rho(x_0)\right)^{n/p_1'}.
\end{aligned}$$

Moreover, the reverse Hölder inequality possessed by $V \in RH_{q_1}$ deduces

$$\begin{aligned}
&\left(\int_{2^{k-2}\rho(x_0)<|y-x_0|\leq 2^{k+1}\rho(x_0)}V(z)^{q_1}dz\right)^{1/q_1} \\
\leq\, &C\left(2^k\rho(x_0)\right)^{n/q_1-2}\frac{1}{\left(2^k\rho(x_0)\right)^{n-2}}\int_{B(x_0,2^{k+1}\rho(x_0))}V\,dy.
\end{aligned}$$

Moreover, by using the doubling property (1.1) in [18], we have

$$\left(\int_{2^{k-1}\rho(x_0)<|y-x_0|\le 2^k\rho(x_0)} V(y)^{q_1}\,dy\right)^{1/q_1} \le \begin{cases} \left(2^k\rho(x_0)\right)^{n/q_1-2}, & \text{if } k<0; \\ \left(2^k\rho(x_0)\right)^{n/q_1-2} C_0^k, & \text{if } k\ge 0, \end{cases} \tag{4.7}$$

where $C_0 > 1$ is the doubling constant in (1.1) in [18]. Without loss of generality, assume that $C_0 > 2$ and take $m = 2\cdot\log_2 C_0$ such that for any $k \ge 0$,

$$\frac{C_m}{(1+2^k)^m} C_0^k \le \frac{C}{2^{k\log_2 C_0}},$$

where C is a positive constant independent of $k \ge 0$.

Therefore, for any given $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$,

$$\begin{aligned} &\left|\int_{|y-x_0|>2^{-\kappa_0}\rho(x_0)} \left[R_j(y,x_1) - R_j(y,x_0)\right]\varphi(y)\,dy\right| \\ &\le \sum_{k=-\kappa_0+1}^{\infty} \frac{C}{\max\{1, 2^{k\log_2 C_0}\}}\left(\frac{|x_1-x_0|}{2^{-(\kappa_0+1)}\rho(x_0)}\right)^{\delta}\left[M\left(|\varphi|^{p_1'}\right)(x_0)\right]^{1/p_1'} \\ &\quad + \sum_{k=-\kappa_0+1}^{\infty} \frac{C}{1+2^k}\left(\frac{|x_1-x_0|}{2^{-(\kappa_0+1)}\rho(x_0)}\right)^{\delta} M\varphi(x_0) \\ &\le C\frac{\kappa_0}{\left(2^{-(\kappa_0+1)}\rho(x_0)\right)^{\delta}}\left\{\left[M\left(|\varphi|^{p_1'}\right)(x_0)\right]^{1/p_1'} + M\varphi(x_0)\right\}|x_1-x_0|^{\delta} \to 0 \\ &\quad \text{as} \quad x_1 \to x_0. \end{aligned}$$

That is, (4.6) holds and $T_1(\varphi)$ is continuous on arbitrary given x_0.

This, combined with the result in *Step I*, deduces that $T_1(\varphi) \in C_0(\mathbb{R}^n)$, as desired.

Therefore, we complete the proof of Theorem 1.2. □

As a consequence, we have the following representation based on Theorem 1.2.

Lemma 4.1. *Suppose $V \in RH_q$ for some $q \ge n/2$. For every continuous linear functional ℓ on the $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$ space, there exists a uniquely finite Borel measure μ_0 whose Riesz transforms $R_j(d\mu_0)(x) = \int R_j(x,y)\,d\mu_0(y)$ associated to $\mathcal{L}$ for $j = 1, 2, \dots, n$ are all finite Borel measures, such that the functional ℓ can be realized by*

$$\ell(g) = \int_{\mathbb{R}^n} g(x)\,d\mu_0(x),$$

which is initially defined on the dense subspace $C_0(\mathbb{R}^n)$, and has a unique extension to $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$.

Proof. Given $\ell \in (\mathrm{CMO}_{\mathcal{L}})^*$, then there exists a constant $c > 0$ such that

$$|\ell(g)| \le c\|g\|_{\mathrm{BMO}_{\mathcal{L}}} \quad \text{for all } g \in \mathrm{CMO}_{\mathcal{L}}.$$

Notice that $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$ is the closure of $C_0(\mathbb{R}^n)$ in the $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ norm and the space $C_0(\mathbb{R}^n)$ is equipped with the supremum norm, clearly for each $\ell \in (\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n))^*$,

$$|\ell(g)| \le c\|g\|_{\mathrm{BMO}_{\mathcal{L}}} \le 2c\|g\|_{L^\infty} \quad \text{for all } g \in C_0(\mathbb{R}^n).$$

That is, ℓ is also a bounded linear functional on $C_0(\mathbb{R}^n)$. Hence it follows from the Riesz representation theorem (see [16, Section 6.19] for instance) that there exists a uniquely regular (complex-valued) Borel measure μ_0 whose total variation $|\mu_0|(\mathbb{R}^n) < \infty$, such that

$$\ell(g) = \int_{\mathbb{R}^n} g(x)\, d\mu_0(x) =: \mu_0(g) \quad \text{for all } g \in C_0(\mathbb{R}^n). \tag{4.8}$$

In turn, since $C_0(\mathbb{R}^n)$ is dense in $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$, the linear functional μ_0 given by (4.8) initially defined on $C_0(\mathbb{R}^n)$ has a unique extension to $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$. Thus every $\ell \in (\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n))^*$ can be realized by a uniquely finite Borel measure μ_0. In the sequel we fix such ℓ and μ_0.

Moreover, it follows from combining (4.8) and $\ell \in (\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n))^*$ that

$$|\mu_0(g)| = |\ell(g)| \le c\|g\|_{\mathrm{BMO}_{\mathcal{L}}} \quad \text{for all } g \in C_0(\mathbb{R}^n). \tag{4.9}$$

We aim to characterize properties of the measure μ_0 from the perspective of Riesz transforms, motivated by the analogous result for the Laplacian operator in place of $\mathcal{L}$.

To this end, note that the linear operator $R_j^* : C_0(\mathbb{R}^n) \to \mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$ is bounded by Theorem 1.2, and $C_0(\mathbb{R}^n)$ and $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$ are both Banach spaces, so the operator R_j, as the adjoint of R_j^*, satisfies

$$R_j\left((\mathrm{CMO}_{\mathcal{L}})^*\right) \subseteq (C_0)^*.$$

Alternatively, the above inclusion can be deduced by recalling that $R_j : H^1_{\mathcal{L}} = (\mathrm{CMO}_{\mathcal{L}})^* \to L^1$ is bounded (see [10] for instance). Hence $R_j(\ell)$ is a bounded linear functional on C_0 by means of

$$\langle R_j(\ell), g\rangle = \langle \ell, R_j^*(g)\rangle = \ell(R_j^*(g)) \quad \text{for all } g \in C_0(\mathbb{R}^n). \tag{4.10}$$

This, combined with $R_j^*(C_c^\infty) \subseteq C_0$ (i.e., (4.1) in the proof of Theorem 1.2) and the representation (4.8), implies that

$$\ell(R_j^*(\phi)) = \mu_0(R_j^*(\phi)) = \langle R_j(\mu_0), \phi\rangle \quad \text{for all} \quad \phi \in C_c^\infty(\mathbb{R}^n).$$

That is,

$$\int_{\mathbb{R}^n} R_j^*(\phi)(x)\, d\mu_0(x) = \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} R_j(x,y)\, d\mu_0(y)\right] \phi(x) \tag{4.11}$$

for all $\phi \in C_c^\infty(\mathbb{R}^n)$.

On the other hand, since $R_j(\ell)$ is a bounded linear functional on $C_0(\mathbb{R}^n)$, by the Riesz representation theorem again, there exists a finite Borel measure μ_j such that

$$\langle R_j(\ell), g\rangle = \int_{\mathbb{R}^n} g(x)\, d\mu_j(x) \quad \text{for all} \quad g \in C_0(\mathbb{R}^n).$$

In particular, using $R_j^*(C_c^\infty) \subseteq C_0$ again,

$$\int_{\mathbb{R}^n} \phi(x)\, d\mu_j(x) = \langle R_j(\ell), \phi\rangle = \ell(R_j^*(\phi)) = \int_{\mathbb{R}^n} R_j^*(\phi)(x)\, d\mu_0(x)$$

for all $\phi \in C_c^\infty(\mathbb{R}^n)$. This, together with (4.11), deduces that

$$\int_{\mathbb{R}^n} \phi(x) \left[\int_{\mathbb{R}^n} R_j(x,y)\, d\mu_0(y)\right] dx = \int_{\mathbb{R}^n} \phi(x)\, d\mu_j(x)$$

for all $\phi \in C_c^\infty(\mathbb{R}^n)$.

Then a standard argument by contradiction shows that

$$R_j(d\mu_0)(x)\, dx = d\mu_j(x), \ \text{ i.e., } \ \mu_j = R_j(d\mu_0). \tag{4.12}$$

Therefore, for any given $\ell \in (\mathrm{CMO}_{\mathcal{L}})^*$, it can be realized by a uniquely finite Borel measure μ_0, whose Riesz transforms $R_j(\mu_0)$ for $j = 1, 2, \dots, n$ are all finite Borel measures. The proof is complete. □

Remark 4.2. ***(Riesz transforms and subharmonicity)***
(i). When $V \equiv 0$, then $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n) = \mathrm{CMO}_{-\Delta}(\mathbb{R}^n) = \mathrm{CMO}(\mathbb{R}^n)$ is known by (ii) of Theorem 2.1 (since $\mathrm{CMO}(\mathbb{R}^n)$ is the closure of $C_c^\infty(\mathbb{R}^n)$ in the $\mathrm{BMO}(\mathbb{R}^n)$ norm). In this case, Lemma 4.1 says that every continuous linear functional on CMO can be realized by a finite measure μ_0 whose classical Riesz transforms $R_j^0(d\mu_0)$ for $j = 1, \dots, n$ are all finite measures.

Hence, it follows from the F. and M. Riesz theorem (see Corollary 1 in [20, p. 221] for instance) that there exists a function $f \in H^1(\mathbb{R}^n)$ such that $d\mu_0(x) = f(x)dx$, where $H^1(\mathbb{R}^n)$ is the classical Hardy space.

That is, we obtain $(\mathrm{CMO}(\mathbb{R}^n))^* \subsetneq H^1(\mathbb{R}^n)$, as a straightforward consequence of Lemma 4.1 by taking $V \equiv 0$. Note that the reverse inclusion

$$H^1(\mathbb{R}^n) \subseteq (\mathrm{CMO}(\mathbb{R}^n))^*$$

is trivial by combining $(H^1(\mathbb{R}^n))^* = \mathrm{BMO}(\mathbb{R}^n)$ and $\mathrm{CMO}(\mathbb{R}^n) \subsetneqq \mathrm{BMO}(\mathbb{R}^n)$. Hence Lemma 4.1 implies the classical well-known result (see [5, Proposition 3.5] for instance)

$$(\mathrm{CMO}(\mathbb{R}^n))^* = H^1(\mathbb{R}^n).$$

Notably, we remind that a crucial ingredient to show the F. and M. Riesz theorem is the subharmonicity of $|F|^p$ for $p \geq (n-1)/n$, where

$$F(x,t) = \left(e^{-t\sqrt{-\Delta}}(d\mu_0)(x), e^{-t\sqrt{-\Delta}}\left(R_1^0(d\mu_0)\right)(x), \dots, e^{-t\sqrt{-\Delta}}\left(R_n^0(d\mu_0)\right)(x)\right)$$

for $(x,t) \in \mathbb{R}^{n+1}_+$, and the subharmonicity follows from the fact that $F(x,t)$ satisfies the generalized Cauchy-Riemann equations; see §3 in Chapter VII of [20] *or* §4 *in Chapter III of* [21] *for details.*

(ii). Let μ_0 be the finite measure in Lemma 4.1 *and $\mu_j = R_j(d\mu_0)$ for $j = 1, \dots, n$. Let*

$$u_j(x,t) := e^{-t\sqrt{\mathcal{L}}}(d\mu_j)(x) = \int_{\mathbb{R}^n} \mathcal{P}_t(x,y)\, d\mu_j(y), \;\; i = 0, 1, \dots, n,$$

be the Poisson-Stieltjes integral of the finite Borel measure μ_j. By using estimates for the Poisson kernels associated to the semigroup $e^{-t\sqrt{\mathcal{L}}}$ given in [19, Lemma 2.6], *it's clear that each u_j is continuous in $\mathbb{R}^{n+1}_+$ and*

$$\sup_{t>0} \int_{\mathbb{R}^n} |u_j(x,t)|\, dx \le C\, |\mu_j|(\mathbb{R}^n) < \infty.$$

Obviously, u_j is an $\mathbb{L}-$harmonic function associated to the operator $\mathbb{L} = -\partial_{tt} + \mathcal{L}$ in the sense of

$$\int_{\mathbb{R}^{n+1}_+} \nabla u_j \cdot \nabla \psi \, dY + \int_{\mathbb{R}^{n+1}_+} V u_j \, \psi \, dY = 0, \quad \forall \psi \in C^1_0(\mathbb{R}^{n+1}_+),$$

where $\nabla = (\nabla_x, \partial_t)$, and the capital letter $Y = (y,t)$ denotes a point in $\mathbb{R}^{n+1}_+$.

Moreover, we now give an extension of Lemma 2.6 in [8] *that the index $p \ge 1$ therein can be extended to $p > 0$: for any $B(Y, 4r) \subseteq \mathbb{R}^{n+1}_+$,*

$$\sup_{(x,t)\in B(Y,r/2)} |u_j(x,t)|^p \le \frac{c_p}{|B(Y,r)|} \int_{B(Y,r)} |u_j(x,t)|^p dx\, dt \quad \text{for } \; p > 0. \quad (4.13)$$

To this end, we claim that

(F) *for each $j = 0, 1, \dots, n$, $|u_j(x,t)|^2$ is a non-negative sub-harmonic function in $\mathbb{R}^{n+1}_+$.*

Let $\mathrm{Re}\, z$ *and* $\overline{z}$ *be the real part and the complex conjugate of $z \in \mathbb{C}$, respectively. Let $\langle \mathbf{z}, \mathbf{w} \rangle = \sum_{j=1}^{n+1} z_j \, \overline{w_j}$ for $\mathbf{z} = (z_1, \dots, z_{n+1})$, $\mathbf{w} = (w_1, \dots, w_{n+1}) \in \mathbb{C}^{n+1}$.*

For any given $Y \in \mathbb{R}^{n+1}_+$ and $B = B(Y, 4r) \subseteq \mathbb{R}^{n+1}_+$, let $\varphi \ge 0$ be a Lipschitz function satisfying $\operatorname{supp} \varphi \subseteq B$, *we have*

$$\begin{aligned}
&\iint_B \langle \nabla_{x,t} \, |u_j|^2, \nabla_{x,t}\, \varphi \rangle \, dxdt \\
=&2 \iint_B \langle \mathrm{Re}\, (\overline{u_j}\, \nabla_{x,t}\, u_j), \nabla_{x,t}\, \varphi \rangle \, dx\, dt \\
=&2\mathrm{Re} \iint_B \langle \nabla_{x,t}\, u_j, \nabla_{x,t}(u_j \varphi) \rangle \, dx\, dt - 2\mathrm{Re} \iint_B \langle \nabla_{x,t}\, u_j, \varphi \nabla_{x,t}\, u_j \rangle \, dx\, dt \\
=&-2\mathrm{Re} \iint_B (\Delta_{x,t}\, u_j) \overline{u_j}\, \varphi \, dx\, dt - 2 \iint_B |\nabla_{x,t} u_j|^2 \varphi \, dx\, dt
\end{aligned}$$

$$= -2\iint_B V|u_j|^2\varphi\,dx\,dt - 2\iint_B |\nabla_{x,t}u_j|^2\varphi\,dx\,dt$$
$$\le 0.$$

*Hence $|u_j(x,t)|^2$ is weakly subharmonic, and so the claim **(F)** holds by Problem 2.8 in* [12, p. 29]. *This allows us to apply Theorem 5.4 in* [2] *to see for every $p>0$,*

$$\sup_{(x,t)\in B(Y,r/2)} |u_j(x,t)|^2 \le c_p\left(\frac{1}{|B(Y,r)|}\int_{B(Y,r)} |u_j(x,t)|^{2p}dx\,dt\right)^{1/p},$$

where $c_p<\infty$ is a positive constant depending on p. As a consequence, (4.13) *follows readily. Indeed, one may verify that the function u_j in* (4.13) *can be replaced by any $\mathbb{L}-$harmonic function in the ball $B(Y,4r)$.*

Furthermore, let

$$F_{\mathcal{L}}(x,t) = \big(u_0(x,t), u_1(x,t), \dots, u_n(x,t)\big)$$

and $|F_{\mathcal{L}}(x,t)|^2 = \sum_{j=0}^n |u_j(x,t)|^2$. The argument above shows that $|F_{\mathcal{L}}(x,t)|^2$ is a non-negative sub-harmonic function in $\mathbb{R}^{n+1}_+$ and

$$\sup_{(x,t)\in B(Y,r/2)} |F_{\mathcal{L}}(x,t)|^p \le \frac{c_p}{|B(Y,r)|}\int_{B(Y,r)} |F_{\mathcal{L}}(x,t)|^p dx\,dt \quad \text{for } p>0.$$

It's natural to ask whether or not we can establish the subharmonicity of $|F_{\mathcal{L}}|^p$ for some $p\le 1$, by noticing the generalized Cauchy–Riemann equations are now no longer satisfied. Furthermore, it's interesting to consider the possibility of establishing an analogous version of the F. and M. Riesz theorem associated to $\mathcal{L}$ such that the finite measure μ_0 in Lemma 4.1 *must be absolutely continuous with Radon-Nikodym derivative in $H^1_{\mathcal{L}}(\mathbb{R}^n)$, that is, there exists $f\in H^1_{\mathcal{L}}(\mathbb{R}^n)$ such that $d\mu_0(x) = f(x)\,dx$.*

5. An approximation to the identity and $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$: proof of Theorem 1.3

In the end, we turn to consider an approximation to the identity arising from the semigroups associated to $\mathcal{L}$.

Actually, this is not a trivial fact, since the standard approximation to the identity can not match $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$ well due to the potential V. Even for a radial bump function ϕ satisfying

$$\operatorname{supp}\phi \subseteq B(0,1),\quad 0\le\phi\le 1 \quad\text{and}\quad \int \phi(x)\,dx = 1,$$

the convolution $A_tf = t^{-n}\phi(\cdot/t) * f$ for $f\in \mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$ satisfies $\widetilde{\gamma}_1(A_t(f)) = \widetilde{\gamma}_2(A_t(f)) = \widetilde{\gamma}_3(A_t(f)) = \widetilde{\gamma}_4(A_t(f)) = 0$, while the remaining $\widetilde{\gamma}_5(A_t(f)) = 0$ needs furthermore conditions on f such as compact support; see [19, Lemma 4.1]. This means that the usual average of a $\mathrm{CMO}_{\mathcal{L}}$ function may not fall into

$\mathrm{CMO}_{\mathcal{L}}$, which is quite different from the standard identity approximation in the classical CMO space and the $\mathrm{CMO}_{-\Delta+1}$ space (see [6]).

However, we will see that the limit behavior of the Poisson integral of $f \in \mathrm{CMO}_{\mathcal{L}}$ also possesses nice approximate properties, i.e., Theorem 1.3. The argument is also workable for the heat semigroups.

To show Theorem 1.3, we introduce the following auxiliary result first.

Lemma 5.1. *Suppose $V \in RH_q$ for some $q \geq n/2$ and let $\mathcal{L} = -\Delta + V$. There exists a constant $C > 0$ such that for any given $s > 0$ and $f \in \mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$, we have $e^{-s\sqrt{\mathcal{L}}} f \in \mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$, and*

$$\left\| e^{-s\sqrt{\mathcal{L}}} f \right\|_{\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)} \leq C\|f\|_{\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)}. \tag{5.1}$$

Additionally, if $f \in \mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$, then $e^{-s\sqrt{\mathcal{L}}} f$ belongs to $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$ as well.

For any given $s > 0$, it's clear that $\left| e^{-s\sqrt{\mathcal{L}}} f(x) \right| \leq CMf(x)$. However, this, combined with Theorem 1.1, can not be used to deduce the $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ norm of $e^{-s\sqrt{\mathcal{L}}} f$. To prove Lemma 5.1, we apply the characterization of $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$ in terms of tent spaces.

Theorem 5.2. *(see [19, Theorem B]) Suppose $V \in RH_q$ for some $q \geq n/2$. Then $f \in \mathrm{CMO}_{\mathcal{L}}$ if and only if $f \in L^2(\mathbb{R}^n, (1+|x|)^{-(n+\beta)}dx)$ for some $\beta > 0$ and $t\sqrt{\mathcal{L}}e^{-t\sqrt{\mathcal{L}}} f \in T^{\infty}_{2,C}$, with*

$$\|f\|_{\mathrm{BMO}_{\mathcal{L}}} \approx \left\| t\sqrt{\mathcal{L}}e^{-t\sqrt{\mathcal{L}}} f \right\|_{T^{\infty}_2}.$$

The space T^{∞}_2 is the class of functions F defined on $\mathbb{R}^{n+1}_+$ for which $\mathfrak{C}(F) \in L^{\infty}(\mathbb{R}^n)$ and the norm $\|F\|_{T^{\infty}_2} = \|\mathfrak{C}(F)\|_{L^{\infty}}$, where

$$\mathfrak{C}(F)(x) = \sup_{x\in B} \left(r_B^{-n} \iint_{\widehat{B}} |F(y,t)|^2 \frac{dy\,dt}{t} \right)^{1/2}.$$

It's well known from the Carleson measure that $f \in \mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ if and only if $t\sqrt{\mathcal{L}}e^{-t\sqrt{\mathcal{L}}} f \in T^{\infty}_2$. Moreover, we say $F \in T^{\infty}_{2,C}$ if $F \in T^{\infty}_2$ and

$$\text{(i)}\quad \eta_1(F) := \lim_{a\to 0} \sup_{B:\, r_B \leq a} \left(r_B^{-n} \iint_{\widehat{B}} |F(y,t)|^2 \frac{dy\,dt}{t} \right)^{1/2} = 0,$$

$$\text{(ii)}\quad \eta_2(F) := \lim_{a\to +\infty} \sup_{B:\, r_B \geq a} \left(r_B^{-n} \iint_{\widehat{B}} |F(y,t)|^2 \frac{dy\,dt}{t} \right)^{1/2} = 0,$$

$$\text{(iii)}\quad \eta_3(F) := \lim_{a\to +\infty} \sup_{B:\, B\subseteq (B(0,a))^c} \left(r_B^{-n} \iint_{\widehat{B}} |F(y,t)|^2 \frac{dy\,dt}{t} \right)^{1/2} = 0,$$

where $\widehat{B}$ is the classical tent of B. Clearly, one may replace $\widehat{B}$ by $B\times(0,r_B)$, and by a similar argument, one may also characterize $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$ in terms of the heat semigroup of $\mathcal{L}$ rather than its Poisson counterpart. That is, the condition $t\sqrt{\mathcal{L}}e^{-t\sqrt{\mathcal{L}}}f \in T^{\infty}_{2,C}$ involved in Theorem 5.2 can be replaced by $F'(y,t) := t^2\mathcal{L}e^{-t^2\mathcal{L}}f \in T^{\infty}_{2,C}$. This observation implies that one may verify $e^{-s\mathcal{L}}f \in \mathrm{CMO}_{\mathcal{L}}$ for any fixed $s>0$ in a similar manner.

Proof of Lemma 5.1. For any fixed $s>0$, let

$$F_s(y,t) := t\sqrt{\mathcal{L}}e^{-t\sqrt{\mathcal{L}}}e^{-s\sqrt{\mathcal{L}}}f(y).$$

Step I. we claim that $F_s \in T^{\infty}_2$, that is, $\mathfrak{C}(F_s)\in L^\infty$.

To see it, for any $x\in\mathbb{R}^n$ and for any ball $B=B(x_B,r_B)$ containing x, if $s\le r_B$, then

$$\begin{aligned}
&\left(r_B^{-n}\int_0^{r_B}\int_B |F_s(y,t)|^2\frac{dy\,dt}{t}\right)^{1/2}\\
=&\left(r_B^{-n}\int_0^{r_B}\int_B t|\sqrt{\mathcal{L}}e^{-(t+s)\sqrt{\mathcal{L}}}f(y)|^2dydt\right)^{1/2}\\
\le& C\left((2r_B)^{-n}\int_0^{2r_B}\int_{2B}|\tau\sqrt{\mathcal{L}}e^{-\tau\sqrt{\mathcal{L}}}f(y)|^2\frac{dyd\tau}{\tau}\right)^{1/2}\\
\le& C\mathfrak{C}\left(t\sqrt{\mathcal{L}}e^{-t\sqrt{\mathcal{L}}}f\right)(x).
\end{aligned}\tag{5.2}$$

Otherwise, $s\ge r_B$, then for any $y\in B$ and $0<t<r_B$,

$$\begin{aligned}
&|F_s(y,t)|\\
=&\left|e^{-s\sqrt{\mathcal{L}}}\left(t\sqrt{\mathcal{L}}e^{-t\sqrt{\mathcal{L}}}f\right)(y)\right|\\
=&\left|\left\{\int_{B(y,s)}+\sum_{k=1}^{\infty}\int_{B(y,2^ks)\setminus B(y,2^{k-1}s)}\right\}K_{e^{-s\sqrt{\mathcal{L}}}}(y,z)t\sqrt{\mathcal{L}}e^{-t\sqrt{\mathcal{L}}}f(z)dz\right|\\
\le& C\sum_{k=0}^{\infty}\frac{1}{2^k}\left|t\sqrt{\mathcal{L}}e^{-t\sqrt{\mathcal{L}}}f\right|_{B(y,2^ks)}\\
\le& C\sum_{k=0}^{\infty}\frac{1}{2^k}\left|t\sqrt{\mathcal{L}}e^{-t\sqrt{\mathcal{L}}}f\right|_{B(x,2^{k+1}s)},
\end{aligned}\tag{5.3}$$

then

$$\begin{aligned}
&\left(r_B^{-n}\int_0^{r_B}\int_B |F_s(y,t)|^2\frac{dy\,dt}{t}\right)^{1/2}\\
\lesssim&\sum_{k=0}^{\infty}\frac{1}{2^k}\left(\int_0^{r_B}\left|t\sqrt{\mathcal{L}}e^{-t\sqrt{\mathcal{L}}}f\right|^2_{B(x,2^{k+1}s)}\frac{dt}{t}\right)^{1/2}
\end{aligned}$$

$$\lesssim \sum_{k=0}^{\infty} \frac{1}{2^k} \left(\int_0^{r_B} \frac{1}{|B(x, 2^{k+1}s)|} \int_{B(x,2^{k+1}s)} \left| t\sqrt{\mathcal{L}} e^{-t\sqrt{\mathcal{L}}} f(z) \right|^2 \frac{dz dt}{t} \right)^{1/2}$$
$$\leq \mathfrak{C} \left(t\sqrt{\mathcal{L}} e^{-t\sqrt{\mathcal{L}}} f \right)(x). \tag{5.4}$$

Hence, for any $x \in \mathbb{R}^n$,

$$\mathfrak{C}(F_s)(x) \leq C\mathfrak{C} \left(t\sqrt{\mathcal{L}} e^{-t\sqrt{\mathcal{L}}} f \right)(x)$$

and the constant $C > 0$ is independent of x and $s > 0$.

Due to the characterization of $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ via the Carleson measure, we obtain (5.1).

Step II. We continue to verify that $\eta_i(F_s) = 0$ for $i = 1, 2, 3$, which ensures $e^{-s\sqrt{\mathcal{L}}} f \in \mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$.

For any given $\varepsilon > 0$, by $\eta_i(t\sqrt{\mathcal{L}} e^{-t\sqrt{\mathcal{L}}} f) = 0$ for $i = 1, 2, 3$, there exist two integers $\mathcal{I}_\varepsilon >> 1$ and $\mathcal{J}_\varepsilon >> 1$ such that

$$\sup_{B:\, r_B \leq 2^{-\mathcal{I}_\varepsilon}} \left(r_B^{-n} \int_0^{r_B} \int_B |t\sqrt{\mathcal{L}} e^{-t\sqrt{\mathcal{L}}} f(y)|^2 \frac{dy\, dt}{t} \right)^{1/2} < \varepsilon, \tag{5.5a}$$

$$\sup_{B:\, r_B \geq 2^{\mathcal{J}_\varepsilon}} \left(r_B^{-n} \int_0^{r_B} \int_B |t\sqrt{\mathcal{L}} e^{-t\sqrt{\mathcal{L}}} f(y)|^2 \frac{dy\, dt}{t} \right)^{1/2} < \varepsilon, \tag{5.5b}$$

$$\sup_{B:\, B \subseteq (B(0, 2^{\mathcal{J}_\varepsilon}))^c} \left(r_B^{-n} \int_0^{r_B} \int_B |t\sqrt{\mathcal{L}} e^{-t\sqrt{\mathcal{L}}} f(y)|^2 \frac{dy\, dt}{t} \right)^{1/2} < \varepsilon. \tag{5.5c}$$

Let's consider $\eta_1(F_s)$.

For any ball $B' = B(x_{B'}, r_{B'})$ with $r_{B'} < 2^{-\mathcal{I}_\varepsilon - 1}$ sufficiently small, if $s \leq r_{B'}$, then combine (5.2) and (5.5a) to obtain

$$\left(r_{B'}^{-n} \int_0^{r_{B'}} \int_{B'} |F_s(y, t)|^2 \frac{dy\, dt}{t} \right)^{1/2} \leq C\varepsilon.$$

Otherwise, $r_{B'} < s$, then apply (5.3) to see

$$\begin{aligned} |F_s(y, t)| &= \left| e^{-s\sqrt{\mathcal{L}}/2} \left(t\sqrt{\mathcal{L}} e^{-(t+s/2)\sqrt{\mathcal{L}}} f \right)(y) \right| \\ &\leq C \sum_{k=0}^{\infty} \frac{1}{2^k} \left| t\sqrt{\mathcal{L}} e^{-(t+s/2)\sqrt{\mathcal{L}}} f \right|_{B(x_{B'}, 2^k s)}, \end{aligned}$$

and so

$$\left(r_{B'}^{-n} \int_0^{r_{B'}} \int_{B'} |F_s(y, t)|^2 \frac{dy\, dt}{t} \right)^{1/2} \leq C \sum_{k=0}^{\infty} \frac{1}{2^k} \left(\int_0^{r_B} \frac{1}{|B(x_{B'}, 2^{k+1}s)|} \right.$$
$$\left. \cdot \int_{B(x_{B'}, 2^{k+1}s)} \frac{t}{t + s/2} (t + s/2) \left| \sqrt{\mathcal{L}} e^{-(t+s/2)\sqrt{\mathcal{L}}} f(z) \right|^2 dz dt \right)^{1/2}$$

$$\begin{aligned}
&\leq C\sqrt{\frac{r_{B'}}{s}}\sum_{k=0}^{\infty}\frac{1}{2^k}\left(\int_0^{2^{k+1}s}\frac{1}{|B(x_{B'},2^{k+1}s)|}\right.\\
&\qquad\qquad\left.\cdot\int_{B(x_{B'},2^{k+1}s)}\left|\tau\sqrt{\mathcal{L}}e^{-\tau\sqrt{\mathcal{L}}}f(z)\right|^2\frac{dzd\tau}{\tau}\right)^{1/2}\\
&\leq C\sqrt{\frac{r_{B'}}{s}}\left\|t\sqrt{\mathcal{L}}e^{-t\sqrt{\mathcal{L}}}f\right\|_{T_2^\infty}\\
&\leq C\sqrt{\frac{r_{B'}}{s}}\|f\|_{\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)}.
\end{aligned}$$

Note that $s>0$ is fixed, thus

$$\sup_{B':\, r_{B'}\leq s\varepsilon^2}\left(r_{B'}^{-n}\int_0^{r_{B'}}\int_{B'}|F_s(y,t)|^2\frac{dy\,dt}{t}\right)^{1/2}\leq C\|f\|_{\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)}\varepsilon. \tag{5.6}$$

Consequently, $\eta_1(F_s)=0$ from these two cases.

To continue, we consider $\eta_2(F_s)$. For any ball $B'=B(x_{B'},r_{B'})$ with $r_{B'}\geq 2^{\mathcal{J}_\varepsilon}$ sufficiently large, if $s\leq r_{B'}$, then combine (5.2) and (5.5b) to obtain

$$\left(r_{B'}^{-n}\int_0^{r_{B'}}\int_{B'}|F_s(y,t)|^2\frac{dy\,dt}{t}\right)^{1/2}\leq C\varepsilon$$

as well. If $s>r_{B'}$, then $s>2^{\mathcal{J}_\varepsilon}$, and it follows from (5.3), (5.4) and (5.5b) to see

$$\left(r_{B'}^{-n}\int_0^{r_{B'}}\int_{B'}|F_s(y,t)|^2\frac{dy\,dt}{t}\right)^{1/2}\leq C\sum_{k=0}^{\infty}\frac{1}{2^k}\varepsilon\leq C\varepsilon.$$

Thus $\eta_2(F_2)=0$.

It remains to consider $\eta_3(F_s)$. For any ball $B'=B(x_{B'},r_{B'})$ which is far away from the origin, it suffices to assume that $r_{B'}<2^{\mathcal{J}_\varepsilon}$ due to the argument of $\eta_2(F_s)=0$. Furthermore, assume that $B'\subseteq(B(0,2^{\mathcal{J}_\varepsilon+1}))^c$, then $2B'\subseteq(B(0,2^{\mathcal{J}_\varepsilon}))^c$. This, combined with (5.2) and (5.5c), implies that

$$\left(r_{B'}^{-n}\int_0^{r_{B'}}\int_{B'}|F_s(y,t)|^2\frac{dy\,dt}{t}\right)^{1/2}\leq C\varepsilon\ \text{ if }\ s\leq r_{B'}.$$

Otherwise, if $s>r_{B'}$, then

$$\left(r_{B'}^{-n}\int_0^{r_{B'}}\int_{B'}|F_s(y,t)|^2\frac{dy\,dt}{t}\right)^{1/2}\leq C\sum_{k=0}^{\infty}\frac{1}{2^k}I_k^{1/2},$$

where

$$I_k:=\int_0^{2^{k+1}s}\frac{1}{|B(x_{B'},2^{k+1}s)|}\int_{B(x_{B'},2^{k+1}s)}\left|\tau\sqrt{\mathcal{L}}e^{-\tau\sqrt{\mathcal{L}}}f(z)\right|^2\frac{dzd\tau}{\tau}.$$

Using (5.5b) again, it suffices to consider the case $r_{B'}<s<2^{\mathcal{J}_\varepsilon}$.

Let $N_\varepsilon \in \mathbb{N}_+$ such that $\sum_{k=N_\varepsilon+1}^{\infty} 2^{-k} < \varepsilon$, then whenever

$$B' \subseteq (B(0, 2^{\mathcal{J}_\varepsilon+N_\varepsilon+1}))^c,$$

it's clear that $B(x_{B'}, 2^{k+1}s) \subseteq (B(0, 2^{\mathcal{J}_\varepsilon}))^c$ for $0 \le k \le N_\varepsilon$. Hence we use (5.5c) to see

$$\begin{aligned}\left(r_{B'}^{-n} \int_0^{r_{B'}} \int_{B'} |F_s(y,t)|^2 \frac{dy\,dt}{t}\right)^{1/2} &\le C \sum_{k=0}^{N_\varepsilon} \frac{1}{2^k}\varepsilon + \sum_{k=N_\varepsilon+1}^{\infty} \frac{1}{2^k} \|f\|_{\mathrm{BMO}_\mathcal{L}} \\ &\le C(\|f\|_{\mathrm{BMO}_\mathcal{L}} + 1)\varepsilon.\end{aligned}$$

Hence $\eta_3(F_s) = 0$. We complete the proof of Lemma 5.1. □

Remark 5.3. *Note that the constant C in (5.2) and (5.3) is independent of $s > 0$, hence*

$$\begin{aligned}\sup_{s>0} \left\| e^{-s\sqrt{\mathcal{L}}} f \right\|_{\mathrm{BMO}_\mathcal{L}} &\approx \sup_{s>0} \left\| t\sqrt{\mathcal{L}} e^{-t\sqrt{\mathcal{L}}} (e^{-s\sqrt{\mathcal{L}}} f) \right\|_{T_2^\infty} \\ &\le C \left\| t\sqrt{\mathcal{L}} e^{-t\sqrt{\mathcal{L}}} f \right\|_{T_2^\infty} \approx \|f\|_{\mathrm{BMO}_\mathcal{L}}.\end{aligned}$$

Remark 5.4. *It's natural to continue to study the behavior of the maximal operator $\mathcal{P}^*$ defined by*

$$\mathcal{P}^* f(x) = \sup_{s>0} \left| e^{-s\sqrt{\mathcal{L}}} f(x) \right|,$$

on $\mathrm{CMO}_\mathcal{L}(\mathbb{R}^n)$. Recall that it has been shown in [9] that $\mathcal{P}^*$ is bounded on $\mathrm{BMO}_\mathcal{L}(\mathbb{R}^n)$. On one hand, this result cannot be used to deduce our (5.1). On the other hand, the condition $r_{B'} \le s\varepsilon^2$ in (5.6) indicates that estimating the limit behaviour $\eta_1(\mathcal{P}^* f)$ is non-trivial.

Based on Lemma 5.1, we continue to finish the remaining argument of Theorem 1.3.

Proof of Theorem 1.3. For any $f \in \mathrm{CMO}_\mathcal{L}(\mathbb{R}^n)$, note that $\mathrm{CMO}_\mathcal{L}(\mathbb{R}^n)$ is the closure of $C_c^\infty(\mathbb{R}^n)$ in $\mathrm{BMO}_\mathcal{L}(\mathbb{R}^n)$, hence there exists a sequence $\{f_k\}_k$ in $C_c^\infty(\mathbb{R}^n)$ such that

$$\lim_{k\to\infty} \|f_k - f\|_{\mathrm{BMO}_\mathcal{L}(\mathbb{R}^n)} = 0,$$

which, combined with the (uniform) boundedness of $e^{-t\sqrt{\mathcal{L}}}$ on $\mathrm{BMO}_\mathcal{L}(\mathbb{R}^n)$ for $t > 0$, deduces that for any $k \in \mathbb{N}_+$,

$$\begin{aligned}&\left\| e^{-t\sqrt{\mathcal{L}}} f - f \right\|_{\mathrm{BMO}_\mathcal{L}} \\ \le& \left\| e^{-t\sqrt{\mathcal{L}}} (f - f_k) \right\|_{\mathrm{BMO}_\mathcal{L}} + \|f_k - f\|_{\mathrm{BMO}_\mathcal{L}} + \left\| e^{-t\sqrt{\mathcal{L}}} f_k - f_k \right\|_{\mathrm{BMO}_\mathcal{L}} \\ \le& C \|f_k - f\|_{\mathrm{BMO}_\mathcal{L}} + \left\| e^{-t\sqrt{\mathcal{L}}} f_k - f_k \right\|_{\mathrm{BMO}_\mathcal{L}},\end{aligned}$$

where the positive constant C is independent of t.

Hence, to prove (1.7), it suffices to verify it for $f \in C_c^\infty(\mathbb{R}^n)$.

We start by showing that for any $f \in C_c^\infty(\mathbb{R}^n)$, $\lim_{t\to 0} e^{-t\sqrt{\mathcal{L}}} f(x) = f(x)$ uniformly for all $x \in \mathbb{R}^n$, which is crucial for our purpose.

Note that by the Kato–Trotter formula, there exists constants C, $c > 0$ such that for all $x, y \in \mathbb{R}^n$ and $t > 0$,

$$\begin{aligned} 0 &\le K_{e^{t\Delta}}(x,y) - K_{e^{-t\mathcal{L}}}(x,y) \\ &\le C\left(\frac{\sqrt{t}}{\sqrt{t}+\rho(x)}\right)^{2-\frac{n}{q_1}} \frac{1}{t^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{ct}\right), \end{aligned} \tag{5.7}$$

where $q_1 > q \ge n/2$ is the index in the proof of Theorem 1.2; see [4, Proposition 7.13]. Let

$$\delta = \min\left\{2 - \frac{n}{q_1}, \frac{1}{2}\right\},$$

then combining the Bochner subordination formula

$$e^{-t\sqrt{\mathcal{L}}} = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{t}{\sqrt{s}} \exp\left(-\frac{t^2}{4s}\right) e^{-s\mathcal{L}} \frac{ds}{s},$$

we have that whenever $t < \rho(x)$,

$$\begin{aligned} 0 \le& K_{e^{-t\sqrt{-\Delta}}}(x,y) - K_{e^{-t\sqrt{\mathcal{L}}}}(x,y) \\ =& \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{t}{\sqrt{s}} \exp\left(-\frac{t^2}{4s}\right) \left[K_{e^{s\Delta}}(x,y) - K_{e^{-s\mathcal{L}}}(x,y)\right] \frac{ds}{s} \\ \le& C \int_0^\infty \frac{t}{\sqrt{s}} \left(\frac{\sqrt{s}}{\sqrt{s}+\rho(x)}\right)^{\delta} \frac{1}{\sqrt{s^n}} \exp\left(-\frac{t^2}{4s} - \frac{|x-y|^2}{cs}\right) \frac{ds}{s} \\ \le& C\left(\frac{t}{\rho(x)}\right)^{\delta} \int_0^{t^2+|x-y|^2} \left(\frac{t}{\sqrt{s}}\right)^{1-\delta} \frac{1}{s^{\frac{n}{2}}} \frac{s^{\frac{n+1}{2}}}{(t^2+|x-y|^2)^{\frac{n+1}{2}}} \frac{ds}{s} \\ &+ C\left(\frac{t}{\rho(x)}\right)^{\delta} \int_{t^2+|x-y|^2}^\infty \left(\frac{t}{\sqrt{s}}\right)^{1-\delta} \frac{1}{s^{\frac{n}{2}}} \frac{ds}{s} \\ \le& C\left(\frac{t}{\rho(x)}\right)^{\delta} \frac{t^{1-\delta}}{(t^2+|x-y|^2)^{\frac{n+1-\delta}{2}}}. \end{aligned} \tag{5.8}$$

For $f \in C_c^\infty(\mathbb{R}^n)$, it's clear that $e^{-t\sqrt{-\Delta}} f(x) \to f(x)$ uniformly for $x \in \mathbb{R}^n$ as $t \to 0$. That is, for any $\varepsilon > 0$, there exists some $t_0 > 0$ such that for any $t \le t_0$,

$$\left| e^{-t\sqrt{-\Delta}} f(x) - f(x) \right| < \varepsilon, \quad \forall\, x \in \mathbb{R}^n.$$

Meanwhile, suppose supp $f \subseteq B(0, M_1)$ for some $M_1 > 0$. Then there exists $M_2 > M_1$ such that for any $t \le t_0$,

$$\left|e^{-t\sqrt{\mathcal{L}}} f(x)\right| < \varepsilon \quad \text{for } |x| \ge M_2.$$

Let

$$\rho_{\min} = \inf_{x \in B(0, M_2)} \rho(x),$$

then $\rho_{\min} > 0$ by Lemma 2.2. Without loss of generality, assume that t_0 satisfies

$$\left(\frac{\sqrt{t_0}}{\rho_{\min}}\right)^{\delta} < \varepsilon.$$

Then for any $t \le t_0$ and $x \in \mathbb{R}^n$, one may combine (5.8) and $f \in C_c^\infty(\mathbb{R}^n)$ to see

- if $|x| < M_2$,

$$\begin{aligned} &\left|e^{-t\sqrt{\mathcal{L}}} f(x) - f(x)\right| \\ \le &\left|e^{-t\sqrt{\mathcal{L}}} f(x) - e^{-t\sqrt{-\Delta}} f(x)\right| + \left|e^{-t\sqrt{-\Delta}} f(x) - f(x)\right| \\ \le &CMf(x) \cdot \varepsilon + \varepsilon \\ \lesssim &\varepsilon. \end{aligned}$$

- if $|x| \ge M_2$,

$$\left|e^{-t\sqrt{\mathcal{L}}} f(x) - f(x)\right| = \left|e^{-t\sqrt{\mathcal{L}}} f(x)\right| \le \varepsilon$$

Therefore,

$$\left|e^{-t\sqrt{\mathcal{L}}} f(x) - f(x)\right| \lesssim \varepsilon \quad \text{for } t \le t_0 \text{ and } x \in \mathbb{R}^n. \tag{5.9}$$

Hence $e^{-t\sqrt{\mathcal{L}}} f \to f$ uniformly for all $x \in \mathbb{R}^n$ as $t \to 0$.

It remains to prove (1.7) for $f \in C_c^\infty(\mathbb{R}^n)$.

For any ball $B = B(x_B, r_B)$ and for any $t < t_0$,

- if $r_B < \rho(x_B)$, then by (5.9),

$$\begin{aligned} &\frac{1}{|B|} \int_B \left|e^{-t\sqrt{\mathcal{L}}} f(x) - f(x) - \left(e^{-t\sqrt{\mathcal{L}}} f - f\right)_B\right|^2 dx \\ \lesssim &\sup_{x \in B} \left|e^{-t\sqrt{\mathcal{L}}} f(x) - f(x)\right| \\ \lesssim &\varepsilon. \end{aligned}$$

- if $r_B \ge \rho(x_B)$, similarly, by (5.9) again, we also have

$$\frac{1}{|B|} \int_B \left|e^{-t\sqrt{\mathcal{L}}} f(x) - f(x)\right|^2 dx \lesssim \varepsilon.$$

Therefore, we complete the proof of Theorem 1.3. □

Remark 5.5. *Similarly, due to* (5.7) *and the fact that for any* $f \in C_c^\infty(\mathbb{R}^n)$*,* $e^{t\Delta}f(x) \to f(x)$ *uniformly for all* $x \in \mathbb{R}^n$ *as* $t \to 0$*, the approximation to the identity arising from the heat semigroup associated to* $\mathcal{L}$ *also matches* $\mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$*. That is, for any* $f \in \mathrm{CMO}_{\mathcal{L}}(\mathbb{R}^n)$*, we have*

$$\lim_{t\to 0} e^{-t\mathcal{L}} f = f \quad \text{in } \mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n).$$

In particular, if $f \in C_c^\infty(\mathbb{R}^n)$*, then we also have* $\lim_{t\to 0} e^{-t\mathcal{L}} f(x) = f(x)$ *uniformly for all* $x \in \mathbb{R}^n$*.*

Remark 5.6. *Recall that*

$$\lim_{t\to 0} e^{-t\sqrt{\mathcal{L}}} f = f \quad \text{in } L^p(\mathbb{R}^n)$$

for $1 \le p < \infty$*; see* [3]*. Our Lemma* 5.1 *and Theorem* 1.3 *can be regarded as certain endpoint results. All of these will be useful in further applications such as function spaces, density arguments, partial differential equations and so on.*

References

[1] Bennett, Colin; DeVore, Ronald A.; Sharpley, Robert. Weak-L^∞ and BMO. *Ann. of Math. (2)* **113** (1981), no. 3, 601–611. MR0621018, Zbl 0465.42015. 1347, 1348, 1351

[2] Biroli, Marco; Mosco, Umberto. A Saint-Venant type principle for Dirichlet forms on discontinuous media. *Ann. Mat. Pura Appl. (4)* **169** (1995), 125–181. MR1378473, Zbl 0851.31008. 1371

[3] Bui, Huy-Qui; Duong, Xuan Thinh; Yan, Lixin. Calderón reproducing formulas and new Besov spaces associated with operators. *Adv. Math.* **229** (2012), no. 4, 2449–2502. MR2880229, Zbl 1241.46020. 1379

[4] Bui, The Anh; Duong, Xuan Thinh; Ly, Fu Ken. Maximal function characterizations for new local Hardy-type spaces on spaces of homogeneous type. *Trans. Amer. Math. Soc.* **370** (2018), no. 10, 7229–7292. MR3841848, Zbl 1404.42043. 1377

[5] Chang, Der-Chen. The dual of Hardy spaces on a bounded domain in $\mathbb{R}^n$. *Forum Math.* **6** (1994), no. 1, 65–81. MR1253178, Zbl 0803.42014. 1369

[6] Dafni, Galia. Local VMO and weak convergence in h^1. *Canad. Math. Bull.* **45** (2002), no. 1, 46–59. MR1884133, Zbl 1004.42020. 1348, 1372

[7] Deng, Donggao; Duong, Xuan Thinh; Song, Liang; Tan, Chaoqiang; Yan, Lixin. Functions of vanishing mean oscillation associated with operators and applications. *Michigan Math. J.* **56** (2008), no. 3, 529–550. MR2488724, Zbl 1158.32002 1346, 1350

[8] Duong, Xuan Thinh; Yan, Lixin; Zhang, Chao. On characterization of Poisson integrals of Schrödinger operators with BMO traces. *J. Funct. Anal.* **266** (2014), no. 4, 2053–2085. MR3150151, Zbl 1292.35099. 1370

[9] Dziubański, Jacek; Garrigós, Gustavo; Martínez, Teresa; Torrea, José Luis; Zienkiewicz, Jacek. BMO spaces related to Schrödinger operators with potentials satisfying a reverse Hölder inequality. *Math. Z.* **249** (2005), 329–356. MR2115447, Zbl 1136.35018. 1346, 1348, 1351, 1376

[10] Dziubański, Jacek; Zienkiewicz, Jacek. Hardy space H^1 associated to Schrödinger operator with potential satisfying reverse Hölder inequality. *Rev. Mat. Iberoam.* **15** (1999), no. 2, 279–296. MR1715409, Zbl 0959.47028. 1368

[11] Gibara, Ryan; Kline, Josh. Fractional maximal functions and mean oscillation on bounded doubling metric measure spaces. *J. Funct. Anal.* **285** (2023), no. 10, Paper No. 110126, 31 pp. MR4631383, Zbl 1529.42021. 1347

[12] GILBARG, DAVID; TRUDINGER, NEIL S. Elliptic partial differential equations of second order. Reprint of the 1998 edition. Classics in Mathematics. *Springer-Verlag, Berlin,* 2001. MR1814364, Zbl 1042.35002. 1371

[13] JIANG, RENJIN; LI, BO. On the Dirichlet problem for the Schrödinger equation with boundary value in BMO space. *Sci. China Math.* **65** (2022), no. 7, 1431–1468. MR4444237, Zbl 1497.30020. 1364

[14] KY, LUONG DANG. On weak*-convergence in $H^1_L(\mathbb{R}^d)$. *Potential Anal.* **39** (2013), no. 4, 355–368. MR3116053, Zbl 1284.42072. 1346, 1350

[15] LIN, QINGZE; LIU, HONGSHEN; SONG, LIANG. The behavior of Hardy–Littlewood maximal operator and Littlewood–Paley operators on CMO space. *Tokyo J. Math.* **48** (2025), no. 1, 193–216. MR4949391, Zbl 8087155. 1347, 1358

[16] RUDIN, WALTER. Real and Complex Analysis. Third edition. *McGraw-Hill Book Co., New York,* 1987. MR0924157, Zbl 0925.00005. 1368

[17] SHAABANI, SHAHABODDIN. Maximal operators on BMO and slices. *Canad. Math. Bull.* **67** (2024), no. 1, 94–107. MR4706803, Zbl 1532.42024. 1347, 1348, 1351

[18] SHEN ZHONGWEI. L^p estimates for Schrödinger operators with certain potentials. *Ann. Inst. Fourier (Grenoble)* **45** (1995), 513–546. MR1343560, Zbl 0818.35021. 1346, 1349, 1351, 1358, 1361, 1362, 1364, 1365, 1367

[19] SONG, LIANG; WU LIANGCHUAN. The CMO-Dirichlet boundary value problem for the Schrödinger equation in the upper half-space and characterizations of CMO. *J. Geom. Anal.* **32** (2022), no. 4, Paper No. 130, 37 pp. MR4376641, Zbl 1485.42044. 1345, 1346, 1347, 1348, 1349, 1350, 1361, 1370, 1371, 1372

[20] STEIN, ELIAS M. Singular integrals and differentiability properties of functions. *Princeton Univ. Press, Princeton N.J.,* 1970. MR0290095, Zbl 0207.13501. 1369, 1370

[21] STEIN, ELIAS M. Harmonic analysis: Real variable methods, orthogonality and oscillatory integrals. *Princeton Univ. Press, Princeton, NJ,* 1993. MR1232192, Zbl 0821.42001. 1370

[22] WU, LIANGCHUAN; YAN, LIXIN. Heat kernels, upper bounds and Hardy spaces associated to the generalized Schrödinger operators. *J. Funct. Anal.* **270** (2016), no. 10, 3709–3749. MR3478871, Zbl 1356.42016. 1348, 1358

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