New York Journal of Mathematics

New York J. Math. **31** (2025) 1324–1344.

Permutable subgroups of linear groups

M. Trombetti and **B.A.F. Wehrfritz**

ABSTRACT. A subgroup H of a group G is said to be *permutable* in G if $HK = KH$ for every $K \leq G$. This type of subgroups often plays a major role in the context of subnormality, in the study of the subgroup lattice of groups, and in many questions concerning finite and infinite groups. The aim of this paper is to study the general properties of permutable subgroups within the universe of linear groups with respect to three major topics: subnormality conditions, analogs of the Maier–Schmid theorem, and transitivity of permutability. In particular, the behaviour of Zariski closed permutable subgroups is studied in these respects and our main results show for example that the Zariski closure of a permutable subgroup of a linear group is always permutable and subnormal, and that the theorem of Maier–Schmid always hold for Zariski closed subgroups of a linear group.

CONTENTS

1. Introduction 1324
2. Subnormality of permutable subgroups 1326
3. Extensions of the Maier–Schmid theorem 1332
4. Linear PT-groups 1336
References 1343

1. Introduction

Historically, the first criteria for the subnormality of a subgroup of a group arose in the attempt to study the implications for the normal structure of a group G possessing a *permutable* subgroup, that is, a subgroup X such that $XY = YX$ for every $Y \leq G$. In fact, Ore [20] proved that a permutable subgroup is normal provided that it is also a maximal subgroup, and that a permutable subgroup of a finite group is always subnormal, so no finite simple group can have proper non-trivial permutable subgroups. Some years later, Stonehewer [25] proved that when we leave the class of finite groups, permutable subgroups (although not always subnormal) still satisfy some weaker form of subnormality, and in fact they are *ascendant* (that is, there is an ascendant series of subgroups connecting them to the whole group) — see also Example 2.16, which is

Received April 2, 2025.
2020 *Mathematics Subject Classification.* 20E15, 20F19, 20H20.
Key words and phrases. linear group; permutable subgroup; PT-group; Maier–Schmid theorem; Zariski closed subgroup.

ISSN 1076-9803/2025

a linear group with non-subnormal permutable subgroups. Note that there exist infinite simple groups with non-trivial proper ascendant subgroups, so this property alone is not sufficient to exclude the existence of proper non-trivial permutable subgroups in arbitrary simple groups. However, Stonehewer also proved that core-free permutable subgroups have in general a good behaviour. In fact, every core-free permutable subgroup X of a group G is a subdirect product of finite nilpotent groups (this implies that a perfect permutable subgroup is always normal), and every finitely generated subgroup of X^G is actually a subdirect product of finite nilpotent groups. As a consequence of the latter fact and the fact that locally (residually soluble) groups have a normal series with abelian factors, we have that no simple group can have a proper non-trivial permutable subgroup, which is a pretty nice simplicity criterion.

The previous results can be strengthened in the finite case and were inspired by a well-known theorem of Maier–Schmid [17] stating that a core-free permutable subgroup of a finite group is always contained in some term of the upper central series of the group. Again, an analog of the Maier–Schmid theorem is not true even in the universe of linear groups (see Example 2.4), but many analogs have been proved in some restricted environments. For example, it has been proved in [8] that core-free permutable subgroups of nilpotent-by-finite groups are always contained in some finite term of the upper central series. A similar statement holds in case the group is a homomorphic image of a periodic linear group (see [7], Theorem 7) or is a finitely generated linear group (see [7], Theorem 8). Other results of this type are quoted and proved in the papers [7] and [8]. We explicitly note that in any reasonably well-behaved universe of groups in which a complete analog of the Maier–Schmid holds, every permutable subgroup is subnormal.

In Section 2, we investigate subnormality criteria for permutable subgroups. In particular, we prove the following results:

- Permutable subgroups of nilpotent-by-Černikov groups are always subnormal; see Corollary 2.3).
- A subgroup of a homomorphic image of a periodic linear group is subnormal provided that it permutes with its conjugates (see Theorem 2.5). A similar result holds for finitely generated soluble-by-finite linear groups (see Theorem 2.10 and subsequent corollaries).
- A permutable subgroup of a soluble-by-periodic linear group is subnormal provided that either the characteristic of the field is positive or the unipotent radical is trivial (see Theorem 2.15).

Examples 2.4 and 2.16 show that some of the previous statements cannot be much improved.

One of the auxiliary results we prove in Section 2 states that a Černikov permutable subgroup is always subnormal. This generalizes the well-known fact that finite permutable subgroups are subnormal, and should be seen in relation to other results of this type, such as the fact that locally cyclic permutable subgroups are subnormal (see [8]). Now, linear groups can be endowed with the Zariski topology, and although this topology does not make them topological groups, it has a strong influence on their structure. So in order to deal with the topological normal structure of a linear group, understanding permutability properties of closed subgroups is relevant. This is essentially addressed in Section 3. The main result of the section (and probably of the paper) is the following one:

- The Zariski closure Y of any permutable subgroup X of a linear group G is permutable and subnormal (see Theorem 3.1). Moreover, Y^G/Y_G is always contained in a finite term of the upper central series of G/Y_G (see Theorem 3.4).

In the same section we also prove that a linear group whose subgroups are permutable must be nilpotent (see Theorem 3.8). In fact, one of the main areas of investigations concerns the existence of many permutable (or somewhat permutable) subgroups. For example, in finite groups we have already mentioned that permutability implies subnormality, but what if also subnormality implies permutability? It has been shown by Zacher [30] that soluble finite groups in which permutability is a transitive relation are precisely those soluble finite groups in which permutable subgroups coincide with the subnormal ones. This result does not hold anymore for arbitrary soluble groups (even linear ones), but we show in Section 4 that it is true in case of homomorphic images of periodic linear groups (see Theorem 4.1). Besides this result, we also prove the following relevant theorem:

- The connected component of a soluble linear group whose cyclic subgroups are permutable is abelian (see Lemma 4.3). As a consequence, in a soluble-by-finite linear group permutability is transitive if and only if subnormality coincides with permutability (see Theorem 4.4).

In the final part of Section 4, we study soluble (linear) groups whose cyclic subnormal subgroups are permutable (see Lemma 4.9 and Theorem 4.10) and homomorphic images of periodic linear groups in which cyclic subnormal subgroups permute with the Sylow subgroups (see Theorem 4.12).

Our notation is mostly standard and can be found in [27]. The m-th term of the upper central series of a group G we denote by $\zeta_m(G)$, and the *hypercentre* of G (that is, the last term of the upper central series of G) we denote by $\zeta(G)$. For recent results concerning permutable subgroups and their generalizations, we refer the reader to the monographs [2], [3] and [23].

2. Subnormality of permutable subgroups

Let G be a group. If X and Y are subgroups of G, then the condition $XY = YX$ is equivalent to the fact that XY is a subgroup of G. This circumstance is usually referred to in different ways, such as: X *permutes* with Y (or, correspondingly, Y *permutes* with X); the subgroups X and Y permute with each other; the *product* XY of X and Y is a subgroup. A subgroup of G is said to be *permutable* in G if it permutes with every subgroup of G.

As we have already mentioned in the introduction, any permutable subgroup of an arbitrary group is ascendant (see [25], Theorem A), while permutable subgroups of finitely generated groups are subnormal (see [25], Theorem B). In particular, permutable maximal subgroups of arbitrary groups are normal, and hence every permutable subgroup is normalized by the elements of prime order. The latter fact can be easily employed to prove that every permutable finite subgroup is subnormal, and our first lemma generalizes this result to Černikov subgroups.

Lemma 2.1. *Let G be a group and let X be a Černikov permutable subgroup of G. Then X is subnormal in G.*

Proof. Set $X = FD$, where D is the finite residual of X and F is a finite subgroup of G. We use induction on the order of F. If $F = \{1\}$, then the ascendant subgroup X is contained in the periodic divisible abelian radical of G (see for example [21], Lemma 4.46), and consequently X is subnormal in G (of defect at most 2). Assume now that $F \neq \{1\}$. If M is the (characteristic) subgroup generated by all the elements of prime order of G, then $F \cap M \neq \{1\}$, so XM/M is subnormal in G/M by induction. On the other hand, X is normalized by every element of prime order of G, so X is normal in XM and hence subnormal in G. □

Corollary 2.2. *Let the group $G = XN$ be the product of a permutable subgroup X and a normal subgroup N. If $G/C_G(N)$ is Černikov, then X is subnormal in G.*

Proof. Clearly, $C_X(N)$ is normal in G, and $X/C_X(N)$ is a Černikov permutable subgroup of $G/C_X(N)$, so it follows from Lemma 2.1 that X is subnormal in G. □

Corollary 2.3. *Let G be a group having a nilpotent normal subgroup N such that G/N is Černikov. If X is any permutable subgroup of G, then X is subnormal in G.*

Proof. It follows from Corollary 2.2 that X is subnormal in $X\zeta_1(N)$. By induction on the nilpotency class of N, we then have that $X\zeta_1(N)/\zeta_1(N)$ is subnormal in $G/\zeta_1(N)$, so X is subnormal in G. □

It is possible to replace Černikov groups in the above statements by periodic groups which are (divisible abelian)-by-finite (the proof being the same), but of course it is not possible to replace them by arbitrary periodic groups, as shown by the following example.

Example 2.4. *There exist periodic and torsion-free metabelian groups having a non-subnormal permutable subgroup.*

Proof. Let p be an odd prime, and let n be a positive integer with $n \geq 2$. Define a semidirect product $G_{p,n} = \langle a_{p,n} \rangle \ltimes \langle b_{p,n} \rangle$, where $a_{p,n}$ has order p^{n-1}, $b_{p,n}$ has order p^n, and $G'_{p,n} = \langle b_{p,n}^p \rangle$. Clearly, $\langle a_{p,n} \rangle$ is subnormal of defect n. Also, it is not difficult to see that $\langle a_{p,n} \rangle$ is permutable in $G_{p,n}$.

Now, let G be the direct product of the groups $G_{q,q}$, where q ranges on all odd prime numbers. Then $X = \langle a_{q,q} \,:\, q \text{ a prime} \rangle$ is a permutable subgroup of G that is not subnormal. Clearly, G is periodic and metabelian.

In order to obtain a torsion-free variation of this example, one can simply recall that free metabelian groups are torsion-free (recall that free metabelian groups are triangularizable linear groups of degree 2 and characteristic 0 — see [27], Lemma 2.10 and Theorem 2.11), so by the first half of the proof it must contain a non-subnormal permutable subgroup. A more concrete and easy example is the following one. For each odd prime p and positive integer $n \geq 2$, we consider the semidirect product $W_{p,n} = \langle c_{n,p} \rangle \ltimes B_{n,p}$, where $c_{n,p}$ is an element of infinite order, $B_{n,p}$ is isomorphic to the direct product of p^{n-1} copies of $\mathbb{Z}$, and $c_{n,p}$ acts on $B_{n,p}$ in such a way that $c_{n,p}^{p^{n-1}} \in \zeta_1(W_{p,n})$ and $W_{p,n}/\langle c_{n,p}^{p^{n-1}} \rangle \simeq \mathbb{Z} \wr \mathbb{Z}_{p^{n-1}}$. Now,

$$W_{p,n}/\langle c_{n,p}^{p^{n-1}}, B_{p,n}^{p^n} \rangle \simeq \mathbb{Z}_{p^n} \wr \mathbb{Z}_{p^{n-1}}$$

contains a copy $H_{p,n}/\langle c_{n,p}^{p^{n-1}}, B_{p,n}^{p^n} \rangle$ of $G_{p,n}$ (this is the group defined in the first half of the proof). Thus, if H is the direct product of the groups $H_{q,q}$, where q ranges on all odd prime numbers, then one easily sees that H is a torsion-free metabelian group having a non-subnormal permutable subgroup. □

Now, we turn our attention to linear groups. In this case, we aim to extend the above-mentioned result of [7] showing that a permutable subgroup of a periodic linear group is always subnormal. A well-known theorem of Szép [26] states that if X is any *conjugate-permutable* subgroup of a group G (that is, if $XX^g = X^gX$ for all $g \in G$), then X is subnormal in G provided G is finite (see also [16], Theorem 7.2.1). In the infinite case, conjugacy-permutable subgroups are more difficult to deal with than permutable subgroups, and in fact they have only been studied by Kurdachenko et al. in [15] and Koppe [14]. In particular, it is proved in [15] that conjugacy-permutable subgroups X of a group G are subnormal provided that G is either soluble-by-finite reduced minimax or Černikov, while they are just ascendant when G is soluble-by-finite and minimax. Moreover, it is proved in [14] that conjugacy-permutable subgroups X of arbitrary groups are at least serial (see also [16], Theorem 7.2.7) — here, *serial* means that there is a (possibly not ascendant nor descendant) series of subgroups connecting X to G. Thus, it immediately follows from [11], Theorem 2.14, that if G is periodic linear, then X is ascendant in G. Our next theorem shows that in this case X is even subnormal, thus generalizing one of the results of [15].

In order to prove this result, we first recall that if G is a linear group, then $u(G)$ denotes the *unipotent radical* of G (that is, the largest unipotent radical normal subgroup of G), while G^0 denotes the connected component of G containing the identity. Note that G^0 centralizes every element with finitely many conjugates.

Theorem 2.5. *Let G be a homomorphic image of a periodic linear group and let X be a subgroup of G such that $XX^g = X^gX$ for all $g \in G$. Then X is subnormal in G.*

Proof. We may assume that G itself is linear. Now, it follows from a combination of [11], Theorem 2.14, and [14] that X is ascendant in G. Let S be the soluble radical of G, and set $U = u(G)$. By [11], Corollary 2.5, X is subnormal in XU, while [11], Lemma 3.4, yields that XS/S is subnormal in G/S. Since XS^0 is trivially subnormal in XS, it is enough to prove that XU is subnormal in XS^0U. Thus, moving to the quotient G/U, we may suppose $G = XA$, where A is a diagonalizable normal subgroup of G and hence by [27], 2.6, satisfies the minimal condition on p-subgroups for all primes p. Now, $X/C_X(A)$ is a finite (see [27], Lemma 1.12) ascendant subgroup of $G/C_X(A)$, and hence the normal closure $Y/C_X(A)$ of $X/C_X(A)$ in $G/C_X(A)$ is Černikov. Finally, an application of [16], Theorem 7.3.10, yields that X is subnormal in Y and so also in G. Therefore X is subnormal in G. □

Corollary 2.6. *Let G be a homomorphic image of a periodic linear group and let X be a permutable subgroup of G. Then X is subnormal in G.*

Remark 2.7. Corollary 2.6 also follows from [7, Theorem 7 and its proof] and [11, Theorem 4.3].

Corollary 2.8. *Let G be a linear group and let X be a periodic permutable subgroup of G. Then X is subnormal in G.*

Proof. Since X is ascendant, so X is contained in the periodic radical T of G. Now, it follows from Corollary 2.6 that X is subnormal in T, and hence even in G. □

In the conjugacy-permutability context, the answer to the following question seems to be unknown.

Question 2.9. *Let G be a group and X a subgroup of G such that $XX^g = X^gX$ for all $g \in G$. Is it true that X is at least ascendant?*

Note that any counterexample to the above question would also result in a linear counterexample by using free groups. In the context of linear groups, it would be interesting to know the answer in the soluble case. We now prove that for finitely generated soluble-by-finite linear groups the answer is positive. First, we need the following result which extends the aforementioned results of [15].

Theorem 2.10. *Let G be a soluble-by-finite group. If X is a polycyclic-by-finite (resp., soluble minimax) conjugacy-permutable subgroup of G, then X is ascendant.*

Proof. Let N be an abelian normal subgroup of G. In order to complete the proof, it is enough to show that X is ascendant in XN. Thus, without any loss of generality, we may assume $G = XN$, $X \cap N = \{1\} \neq N$ and that $C_N(X) = \{1\}$. If X is normal in G, then we are done. Assume not. Then there is $g \in G$ such that $X^g \neq X$.

By Theorem 4.4.2 (resp., Theorems 4.6.11 and 4.6.13) of [2], we have that X^gX is minimax, so the previsouly mentioned theorem of [15] shows that X is ascendant in X^gX. But

$$X^gX = X^gX \cap XN = X(X^gX \cap N),$$

so $C_N(X) \neq \{1\}$, a contradiction. □

Corollary 2.11. *Let G be a nilpotent-by-polycyclic-by-finite group. If X is any conjugacy-permutable subgroup of G, then X is ascendant in G.*

Proof. Let N be a nilpotent normal subgroup of G such that G/N is polycyclic-by-finite, and let $Z = \zeta_1(N)$. Then the polycyclic-by-finite group $X/C_X(Z)$ is ascendant in $XZ/C_X(Z)$ by Theorem 2.10, and hence X is ascendant in XZ. Now, a simple induction argument shows that X is ascendant in XN. Finally, since G/N is polycyclic-by-finite, XN is subnormal in G, and consequently X is ascendant in G. □

Corollary 2.12. *Let G be a finitely generated soluble-by-finite linear group. If X is any conjugacy-permutable subgroup of G, then X is ascendant in G.*

Question 2.13. *Let G be a finitely generated soluble-by-finite linear group and X a subgroup of G such that $XX^g = X^gX$ for all $g \in G$. Is it true that X is subnormal?*

Our next aim is to give some sufficient conditions for a permutable subgroup of a soluble-by-periodic linear group to be subnormal.

Lemma 2.14. *Let G be a group, and let N be a nilpotent normal subgroup of finite exponent of G. If H is a permutable subgroup of G, then H is subnormal in HN. In particular, if G is a soluble-by-finite linear group of positive characteristic, then H is subnormal in G.*

Proof. Let

$$\{1\} = Z_0 \leq Z_1 \leq \ldots \leq Z_n = N$$

be a (finite) central series of N with factors of prime exponent. If C is any cyclic subgroup of Z_1, then either H is a maximal subgroup of HC, or $C \leq H$; in either case, H is normal in HC. The arbitrariness of C shows that H is normal in HZ_1. Similarly, we prove that HZ_i is normal in HZ_{i+1} for every i, and hence that H is subnormal in HN.

Finally, suppose G is a soluble-by-finite linear group of positive characteristic p and degree n. Then H is subnormal in $H \cdot u(G)$ (of defect at most $n-1$), while $H \cdot u(G)$ is subnormal in G by Corollary 2.3 because $G/u(G)$ is abelian-by-finite. □

Theorem 2.15. *Let $\mathcal{K}$ be a field, and let H be a soluble-by-periodic, permutable subgroup of the subgroup G of* $\mathrm{GL}(n, \mathcal{K})$*. If either* $\mathrm{char}(\mathcal{K}) \neq 0$ *or* $u(G) = \{1\}$*, then H is subnormal in G.*

Proof. Clearly we may assume $\mathcal{K}$ is algebraically closed. Since locally soluble subgroups of linear groups are soluble, so the soluble radical S of G is soluble and contains the soluble radical of the ascendant subgroup H. Similarly, G/S is linear, and so the periodic radical T/S of G/S contains HS/S. Replacing G by T, it is therefore possible to assume that G is soluble-by-periodic.

By Lemma 2.14, H is subnormal in HU, where $U = u(S^0)$ (this being trivial whenever $U = \{1\}$). Moreover, G/U is isomorphic to a linear group over $\mathcal{K}$ in which the image of S^0 is an abelian d-group and hence diagonalizable. It follows from Lemma 1.12 of [27] that $G/C_G(S^0/U)$ is finite, so HU/U is subnormal in HS^0/U by Corollary 2.2. Finally, HS^0/S^0 is a subnormal subgroup of the linear group G/S^0 by Corollary 2.6. □

Our next example shows that Theorem 2.15 cannot really be improved. Recall that $\mathrm{Tr}(n, \mathcal{K})$ denotes the set of all triangular matrices of degree n over the field $\mathcal{K}$.

Example 2.16. *There is a reduced, torsion-free, metabelian, minimax group G with Hirsch number 2 such that G has a subgroup H that is permutable but not subnormal in G. Note also that H/H_G is infinite cyclic, and that G is isomorphic to a triangular linear group of degree 2 over the rationals. Further, G is not hypercentral.*

Proof. Let p and q be primes with either p odd and dividing $q-1$, or $p = 2$ and 4 dividing $q-1$. Let A denote the additive group of the ring $J = \mathbb{Z}[1/p, 1/q]$, and consider the automorphism

$$x : a \in A \mapsto qa \in A.$$

Clearly, x has infinite order. Let G denote the split extension $\langle x \rangle \ltimes A$ of A by $\langle x \rangle$. Then G is a reduced, torsion-free, metabelian, minimax group with Hirsch number 2. Moreover, G is isomorphic to a subgroup of $\mathrm{Tr}(2, J)$. Set $A_q = \mathbb{Z}[1/q] \leq A$, so A_q is a normal subgroup of G.

We claim that $H = \langle x \rangle A_q$ is permutable but not subnormal in G. Let φ be the natural epimorphism of G onto $K = G/A_q$. Then $K = \langle y \rangle C$, where $C = A^{\varphi}$ is an additive Prüfer p-group and $y = x^{\varphi}$ acts on C by $c^y = qc$ for $c \in C$. If $c \in C$ has order $p^r > q^s$, then $c^{y^s} = q^s c \neq 0$, so the action of y is faithful. It follows from Theorem 2.4.11 of [23] that $\langle y \rangle$ is permutable but not subnormal in G, and the claim is proved. Finally, G is not hypercentral because it contains no non-trivial cyclic normal subgroup. □

3. Extensions of the Maier–Schmid theorem

In this section we deal with Zariski closures of permutable subgroups in linear groups, and we prove our main results. We start by showing that the Zariski closure of a permutable subgroup not only inherits the permutability but has large pieces which are normal in the whole group.

Theorem 3.1. *Let X be a permutable subgroup of the linear group G. Then the Zariski closure Y of X in G is permutable and subnormal in G. Moreover, $Y^0 \trianglelefteq G$ and Y^G/Y_G is nilpotent.*

Proof. Let $g \in G$. If g has infinite order and $X \cap \langle g \rangle = \{1\}$, then g normalizes X by [23], Lemma 6.2.3. In this case, set $X_g = X$. If there is some positive power g^r (possibly $= 1$) of g lying in X, then $|\langle g \rangle X : X| \leq r$ and g normalizes some normal subgroup X_g of X of finite index ($\leq r!$ for example). The Zariski closure Y_g of X_g in G is also normalized by g and $Y = XY_g$, $|Y : Y_g|$ is finite, $Y^0 = (Y_g)^0$. Thus, Y^0 is normalized by g, and Y^0 is normal in G. Also, XY^0 is closed in G and hence $Y = XY^0 = XY_G$, because $Y^0 \leq Y_G \leq Y$.

If L is any subgroup of G, then $XL = LX$ by permutability and $Y^0L = LY^0$ by normality. Since $Y = XY^0$, so Y is permutable in G. Now, Y/Y_G is a finite permutable subgroup of G/Y_G, and hence Lemma 2.1 yields that Y is subnormal in G. By [25] or [16], Theorem 7.1.10, the group Y/Y_G is nilpotent. But G/Y_G is linear, so the Fitting subgroup of G/Y_G is nilpotent and contains Y/Y_G, implying that Y^G/Y_G is nilpotent and completing the proof. □

Now, in order to prove the extension of the Maier–Schmid theorem for closed permutable subgroups of linear groups, we need a couple of auxiliary results.

Lemma 3.2. *Let H be a finite subnormal subgroup of a group G. If H^G is Černikov, then H^G is finite.*

Proof. Let A be the finite residual of H^G. Now, $H \cap A$ lies in some finite characteristic subgroup of A, so we may assume $H \cap A = \{1\}$. Since H is subnormal, so $[H, A] = \{1\}$ (see for example [21], Lemma 3.13). Then $[H^G, A] = \{1\}$, so H^G is central-by-finite, and hence the commutator subgroup K of H^G is finite by Schur's theorem. If m is the order of H, then H is contained in the finite G-invariant subgroup $\{x \in H^G : x^m \in K\}$. Therefore H^G is finite. □

Lemma 3.3. *Let $\mathcal{K}$ be a field. Let U be a unipotent normal subgroup of the subgroup G of $\mathrm{GL}(n, \mathcal{K})$. Suppose there exists an integer k with $W = [U,_k G]$ finite. Then there exists a function $\varepsilon = \varepsilon(n, |W|)$ of n and $|W|$ only such that $[U,_\varepsilon G] \leq W$.*

Proof. Since W is finite, so W is closed and definable by polynomials in the n^2 variables $X_{i,j}$ of total degree at most $|W|$. Then G/W is isomorphic to some linear group over $\mathcal{K}$ of degree m, where m is boundable in terms of n and $|W|$, cf. the proof of [27], Theorem 6.4. Then $[U,_{m-1} G] \leq W$ by the main result of [29] and the lemma follows. □

Theorem 3.4. *Let $\mathcal{K}$ be a field of characteristic $q \geq 0$, G a subgroup of $\mathrm{GL}(n, \mathcal{K})$, and X a closed permutable subgroup of G. Then X^G/X_G has finite exponent and is contained in $\zeta_k(G/X_G)$ for some non-negative integer k. Further, if either $q = 0$, or $q > 0$ and X/X^0 contains no element of order q, then X^G/X_G is finite, so in particular $[X^G, G^0] \leq X_G$.*

Proof. Replacing G by G/X^0, we may assume that X is finite (X^0 is normal in G by Theorem 3.1). Let $\pi = \pi(X)$. A further application of Theorem 3.1 yields that X^G/X_G is a nilpotent π-group. Thus, without loss of generality, we may assume that $X_G = \{1\}$. Now, write $X^G = U \times D$, where U is the unipotent radical of X^G and $\pi(D) \cap \pi(U) = \emptyset$. Since π is finite, then D is Černikov, so Lemma 3.2 shows that the normal closure of XU/U in G/U is finite, and hence $D \simeq X^G/U$ is finite; in particular, X^G has finite exponent because either $U = \{1\}$, or $q > 0$ and U is a nilpotent q-group of finite exponent. Thus, passing to $G/\overline{U}$ (where $\overline{U}$ is the Zariski closure of U in G) and G/D reduces us two cases: a) X^G is finite; b) X^G is unipotent. Further, if $q = 0$ or X has no element of order q, then Case b) does not arise and X^G is finite.

In both cases we use Theorem 4.2 of [27], stating that for every finitely generated subgroup H of G and for every finite subset S of H, there is finite field $\mathcal{K}_{H,S}$ (of characteristic q if the latter is positive) and a homomorphism $\varphi_{H,S}$ of H into $\mathrm{GL}(n, \mathcal{K}_{H,S})$ such that $|S^{\varphi_{H,S}}| = |S|$ and $h^{\varphi_{H,S}}$ is unipotent whenever $h \in H$ is unipotent.

Case a). Let k be the order of X^G, and let F be a finitely generated subgroup of G such that $X \leq F$ and $X^F = X^G$. For each $g \in F \setminus X$, put $S_g = X \cup \{g\}$. An application of the Maier–Schmid Theorem (see [23], Theorem 5.2.3) to $F^{\varphi_{X,S_g}}$ yields that $[X^G,_k F]$ is contained in $X \cdot \mathrm{Ker}(\varphi_{X,S_g})$. Moreover, since $|S_g| = |S_g^{\varphi_{X,S_g}}|$, it follows that g does not belong to $X \cdot \mathrm{Ker}(\varphi_{X,S_g})$. Therefore

$$[X^G,_k F] \leq \bigcap_{g \in F \setminus X} X \cdot \mathrm{Ker}(\varphi_{X,S_g}) = X.$$

The arbitrariness of F yields that $[X^G,_k G] \leq X$, and hence $[X^G,_k G] = \{1\}$ because $[X^G,_k G] \leq X_G = \{1\}$. It follows that $X^G \leq \zeta_k(G)$ completing the proof for Case a).

Case b). Let F be a finitely generated subgroup of G such that $X \leq F$. For each $g \in F \setminus X$, put $S_g = X \cup \{g\}$. As in the previous paragraph, $g \notin X \cdot \mathrm{Ker}(\varphi_{X,S_g})$, and $[X^G,_\ell F] \leq X \cdot \mathrm{Ker}(\varphi_{X,S_g})$ for some ℓ possibly depending on g and F. However, in our case X^G is unipotent, so also the image of X^F by φ_{X,S_g} is unipotent, and hence by Lemma 3.3 we can choose ℓ dependent on $|X|$ and n but independent of g and F. Again,

$$[X^G,_\ell F] \leq \bigcap_{g \in F \setminus X} X \cdot \mathrm{Ker}(\varphi_{X,S_g}) = X,$$

and so also $[X^G,_\ell G] = \{1\}$. Therefore $X^G \leq \zeta_\ell(G)$ and the proof is complete. $\square$

Corollary 3.5. *Let G be a connected linear group over a field of characteristic 0. Then the Zariski closure of any permutable subgroup is normal. In particular, all closed permutable subgroups of G are normal.*

Proof. Let X be any permutable subgroup of G and let Y be the Zariski closure of X in G. Then Y is permutable in G by Theorem 3.1 and so $[Y^G, G] = [Y^G, G^0] \leq Y_G$ by Theorem 3.4. Therefore Y is normal in G. $\square$

An analog of the Maier–Schmid theorem for reduced soluble(-by-finite) minimax groups — these groups are linear by Theorem 1.1 and Corollary 1.3 of [28] — has been obtained in [24]. In this context, core-free permutable subgroups have been proved to (be finite and to) lie in the hypercentre. We also refer the reader to [6] for the proof of the same results because the reviewer of [24] for MathRev points out that the proof of Lemma 1 should be amended, while example (i) in Section 5 is not working (our Example 2.16 can be used in place of the wrong example in Section 5 of [24]). Note that the hypercentre of a reduced soluble-by-finite minimax group coincides with a finite term of the upper central series.

Theorem 3.6. *A soluble-by-finite reduced minimax group G has finite central height.*

Proof. It follows from Theorem 1.1 and Corollary 1.3 of [28] that G embeds into $\mathrm{GL}(n, R)$ for some integer n and finitely generated subring R of the rationals $\mathbb{Q}$. Then G has finite central height by Corollary 8.9 of [27]. $\square$

It is apparently unknown if an analogue of the aforementioned Maier–Schmid result holds for a finitely generated soluble-by-finite minimax group G. Of course, in this case every core-free permutable subgroup H of G must be finite, and we may even assume that H^G is finite and abelian. Thus, a positive answer to this question is achieved in case H is closed in the profinite topology of G.

In this context, we recall a couple of interesting results due to Brewster and Lennox [5]: a closed (in the profinite topology) permutable subgroup H of a soluble minimax group is subnormal and H^G/H_G is finite.

Finally, we wish to note that a dual of Lemma 2.1 does not hold in the context of soluble-by-finite reduced minimax groups. The following example, which illustrates this fact, is due to the referee, to whom we are sincerely grateful.

Example 3.7. *There exist soluble-by-finite reduced minimax groups having polycyclic-by-finite permutable subgroups that are not subnormal.*

Proof. Let p be a prime such that 2 is a square modulo p (for instance, $p = 7$). By Hensel's lemma, 2 is the square of a p-adic integer $t = t_0 + t_1 p + t_2 p^2 + \dots$, where $0 \le t_i < p$. In particular, $t^2 \equiv_{p^i} 2$ for every positive integer i.

Let H be the abelian group generated by an element x and a sequence of elements $y_0, y_1, \dots$ with relations $y_{i+1}^p = y_i x^{t_i}$. Then H is torsion-free of rank 2, the subgroup $K = \langle x, y_0 \rangle$ is free abelian of rank 2, and H/K is a Prüfer p-group.

Let α be the automorphism of H mapping x to xy_0, y_0 to $x^2 y_0$ and more generally mapping y_{i+1} to the unique element y_{i+1}^α of H such that

$$\left(y_{i+1}^\alpha\right)^{p^{i+1}} = y_0^\alpha (x^\alpha)^{t_0 + t_1 p + \dots + t_i p^i}.$$

Let's just prove that such an assignment is possible for $i = 1$ and $i = 2$, the general case being dealt in a similar way. First, we need to solve the equation

$$y_1^{pn_1} x^{pm_1} = y_0^{n_1} x^{n_1 t_0 + pm_1} = x^{2+t_0} y_0^{1+t_0}$$

for integers n_1 and m_1. Thus,

$$n_1 = 1 + t_0 \quad \text{and} \quad n_1 t_0 + pm_1 = 2 + t_0.$$

By replacing the former equation into the latter, we obtain $t_0^2 + pm_1 = 2$, which we know is solvable by the choice of t. This defines y_1^α.

Next, we move to y_2^α. In this case, we need to solve the equation

$$y_2^{p^2 n_2} x^{p^2 m_2} = y_0^{n_2} x^{n_2 t_0 + n_2 t_1 p + p^2 m_2} = x^{2+t_0+t_1 p} y_0^{1+t_0+t_1 p},$$

which gives $n_2 = 1 + t_0 + t_1 p$ and $n_2 t_0 + n_2 t_1 p + p^2 m_2 = 2 + t_0 + t_1 p$. Replacing the former into the latter, we obtain $p^2 m = 2 - (t_0^2 + 2 t_0 t_1 p + t_1^2 p^2)$, which is again true by the choice of t.

Since $\langle x, y_0 \rangle = \langle xy_0, x^2 y_0 \rangle$, $K^\alpha = K$. Also, we can easily check that α acts on H/K as a non-zero p-adic integer equivalent to 1 modulo p. Let $G = \langle \alpha \rangle \ltimes H$ be the natural semidirect product. Then $X = \langle \alpha \rangle K$ is a permutable subgroup of G (because X/K is a permutable subgroup of G/K) which is not subnormal (because X/K is not subnormal in G/K). □

In the remainder of this section we deal briefly with quasihamiltonian groups. Recall that a group G is *quasihamiltonian* if all its subgroups are permutable. The structure of quasihamiltonian groups was completely described by Iwasawa (see [23], Chapter 2 for details). It follows from Iwasawa's description that any quasihamiltonian group is hypercentral and even nilpotent if its torsion subgroup has finite exponent. Although quasihamiltonian groups need not be nilpotent in general, this is the case within the universe of linear groups.

Theorem 3.8. *Let G be a linear quasihamiltonian group. Then G is nilpotent and abelian-by-finite.*

Proof. Since G is quasihamiltonian, it is hypercentral. Then by Theorems 8.6 ii) and 8.14 of [27], there is a positive integer c with $G/\zeta_c(G)$ periodic. Further

$\zeta_c(G)$ is closed in G, so $G/\zeta_c(G)$ is linear and each of its subgroups is subnormal by Corollary 2.6. But then $G/\zeta_c(G)$ is nilpotent by [11], Corollary 3.6, and hence G is nilpotent.

In order to show that G is abelian-by-finite, we may assume that G is connected, so G centralizes every finite subgroup with finitely many conjugates. If G is not periodic and not abelian, then the torsion subgroup T of G is central and G/T is locally cyclic (see [23], Theorem 2.4.11 and Lemma 2.4.8), so G is abelian and we have a contradiction. If G is periodic and non-abelian, then G has a central subgroup Z such that G/Z is locally cyclic (see [23], Theorems 2.4.13 and 2.4.14), so we again obtain a contradiction. □

4. Linear *PT*-groups

Let G be a group. It is proved by Zacher in [30] that if G is finite, then G is a *PT-group* (that is, the property of being a permutable subgroup is transitive in G) if and only if the permutable subgroups of G are precisely the subnormal ones. The structure of arbitrary soluble PT-groups has been described by Menegazzo in [18] and [19]. In particular, as remarked in Section 3 of [19], the result of Zacher is still true if the group is soluble, and either periodic or torsion-free, but is false when it is neither. In this section, we extend Zacher's theorem to periodic linear groups and to soluble linear groups.

Theorem 4.1. *Let G be a homomorphic image of a periodic linear group. Then permutability is a transitive relation in G if and only if it coincides with subnormality.*

Proof. Suppose that permutability is a transitive relation in G, so in particular each subnormal subgroup of G is permutable. On the other hand, all permutable subgroups of G are subnormal by Corollary 2.6, and so permutable subgroups and subnormal subgroups of G coincide.

Assume conversely that the subnormal subgroups of G are precisely the permutable ones. If H is a permutable subgroup of K, and K is a permutable subgroup of G, then Corollary 2.6 yields that H is subnormal in K, and that K is subnormal G. Thus, H is subnormal in G, and hence H is even permutable in G by the hypothesis. □

Remark 4.2. In the following, we are mostly dealing with groups in which the subnormal cyclic subgroups are permutable, that is, with groups G in which all subgroups of the *Baer radical* $B(G)$ (i.e. the subgroup generated by all subnormal abelian subgroups) are permutable in G — call $\mathfrak{B}$ the previous property. In particular, if $G \in \mathfrak{B}$, then $B(G)$ is quasihamiltonian. The reason for which we do not frequently make use of the Baer radical is that in the linear case and in the image of periodic linear case it coincides with the Fitting subgroup.

However, in order to better explain the relevance of the Baer radical, we are going to make a couple of remarks in the arbitrary case. First, it is easy to see (some of the arguments can be found in the next results) that every $g \in G \in \mathfrak{B}$

with $B(G) \cap \langle g \rangle = \{1\}$ normalizes all subgroups of $B(G)$, and also that $B(G)$ is Dedekind if $G/B(G)$ is not periodic. Also, it follows from Iwasawa's results that if $G \in \mathfrak{B}$, then $B(G)$ is actually the Fitting subgroup of G. Moreover, the property that all subgroups of the Fitting subgroup be permutable is weaker than $\mathfrak{B}$, *Baer groups* (groups coinciding with their Baer radical) with trivial Fitting subgroups providing obvious examples.

Lemma 4.3. *Let G be a soluble linear group whose subnormal cyclic subgroups are permutable. Then G^0 is abelian.*

Proof. Clearly, we may assume that $G = G^0$ is connected. By Theorem 3.8, we only need show that G is nilpotent.

Let U be the unipotent radical of G. Suppose first that U is torsion-free, let C be any cyclic subgroup of U, and $x \in G$. Then C is permutable in G, so if $\langle x \rangle \cap C = \{1\}$, then C is normalized by x (see also [23], Lemma 6.2.3). If $\langle x \rangle \cap C \neq \{1\}$, then $\langle x, C \rangle$ is central-by-finite, so $\langle x, C \rangle'$ is finite by Schur's theorem. But $\langle x, C \rangle' \leq U$ and so x centralizes C. In any case, G centralizes C, because G is connected. The arbitrariness of C yields that G is nilpotent.

Suppose now U is periodic. If G/U is periodic, then $\pi(U) \cap \pi(G/U) = \emptyset$. Now, every cyclic subgroup of $\zeta_1(U)$ is permutable in G and so by Dedekind Modular Law it is normalized by any $\pi(G/U)$-element of G. Thus, every element of $\zeta_1(U)$ is normalized by G, so has finitely many conjugates in G and it is consequently centralized by G. It follows that $\zeta_1(U) \leq \zeta_1(G)$. Repeating this argument yields that $U \leq \zeta_n(G)$ for some positive integer n. In particular, G is nilpotent.

Assume finally that G/U is not periodic. Since G/U is abelian, so G is generated by the elements that are of infinite order modulo U, and hence an argument similar to the one in the second paragraph shows that every cyclic subgroup of U is normal in G. On the other hand, G is connected, so U is centralized by G. Again, G is nilpotent. □

Theorem 4.4. *Let G be a soluble-by-finite linear group. The following conditions are equivalent:*

(1) *Every subnormal subgroup of G is permutable.*
(2) *The subnormal subgroups of G coincide with the permutable ones.*
(3) *G is a PT-group.*

Proof. Let S denote the soluble radical of G, so here G/S is finite. Suppose (1) holds for G. Then (1) holds for S and S^0 is abelian by Lemma 4.3. Hence every permutable subgroup of G is subnormal in G by Corollary 2.3. Thus (2) holds for G. Since subnormality is transitive, if (2) holds for G, then G is a PT-group. Finally, normal subgroups are always permutable, so (3) implies (1). □

Remark 4.5. In the statement of Theorem 4.4, we can replace the hypothesis that G is soluble-by-finite by the hypothesis that G is soluble-by-Černikov because every soluble-by-Černikov group is obviously soluble-by-finite, but not by the hypothesis that G is soluble-by-periodic (see Example 2.16).

Corollary 4.6. *Let G be a soluble linear PT-group. Then every subnormal subgroup of G is a PT-group. Moreover, G^0 is abelian.*

The structure of a non-periodic soluble linear PT-group G can be easily deduced from Lemma 1.2 and Theorem 2.3 of [19], Theorem 2.2 of [27], and the fact that the connected component G^0 centralizes every finite normal subgroup of G. Thus, for example, if the set of all periodic elements of G is a subgroup, then every Prüfer subgroup is central in G.

It follows from a theorem of Zacher (see [30] and [4], Theorem 1) that a subgroup of a finite soluble PT-group is itself a PT-group. This is not true of soluble linear groups in general.

Example 4.7. *There exist periodic and non-periodic soluble (affine) algebraic linear PT-groups having a finite subgroup that is not a PT-group.*

Proof. Let $\mathcal{K}$ be any algebraically closed field of characteristic not 2, and put

$$D = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \;:\; x \in \mathcal{K}^{\times} \right\}.$$

Consider the closed subgroup $G = \langle \sigma \rangle \ltimes D$ of $\mathrm{GL}(2, \mathcal{K})$, where

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Clearly, D is divisible abelian, while σ has order 2 and inverts D. Since $\langle \sigma \rangle^G = G$, the only permutable subgroups of G are G and the subgroups of D. Hence G is a PT-group. However, G has a subgroup X that is isomorphic with the dihedral group of order 8, and the latter is not a PT-group. $\square$

Remark 4.8. Further results in this context can be found in [9], where, among other things, it is shown that a (homomorphic image of a) periodic linear group is a soluble PT-group if and only if each subnormal subgroup of a Sylow subgroup is permutable in the corresponding Sylow normalizer — also a characterization of soluble PT-groups that are homomorphic images of a periodic linear group is given. Note that these results use Corollary 2.6 and Theorem 4.1.

It is not possible to weaken conditions (1) and (2) in the statement of Theorem 4.4 by just requiring that the subnormal cyclic/nilpotent subgroups are permutable (or that the cyclic/nilpotent subnormal subgroups coincide with the cyclic/nilpotent permutable subgroups), the infinite dihedral group being a counterexample (a finite example is the dihedral group of order 24); note also that the permutable subgroups of the infinite dihedral group are subnormal (see for example Corollary 2.3). Despite these examples, finite groups whose cyclic subnormal subgroups are permutable have been studied by Robinson in [22]. In the remainder of the section, we deal with certain types of infinite groups whose cyclic subnormal subgroups satisfy certain permutability conditions, such as the condition of permuting with the Sylow subgroups. In this respect, our results extend results in [1] and [22].

Lemma 4.9. *Let G be a torsion-free soluble group with all its cyclic subnormal subgroups permutable in G. Then G is abelian.*

Proof. Let F be the Fitting subgroup of G, $u \in G \setminus \{1\}$ and $v \in F \setminus \{1\}$. Then $U = \langle u \rangle$ is infinite cyclic, and $V = \langle v \rangle$ is permutable in G because every finitely generated subgroup of F is subnormal. If $U \cap V = \{1\}$, then U normalizes V by [16], Lemma 7.1.7. If $U \cap V \neq \{1\}$, then UV is central-by-finite and so even finite-by-abelian by Schur's theorem; but G is torsion-free, and hence UV is abelian. In both cases V is normalized by U. The arbitrariness of V shows that either u centralizes F or inverts F (see [23], Theorems 1.5.7 and 1.5.8). Now, G is soluble, so $C_G(F) \leq F$, and hence either G is abelian, or $G = \langle w \rangle F$, where w inverts F and $|G : F| = 2$. But $w^2 \in C_G(F) \leq F$ is clearly not inverted by w. Thus, G is abelian. □

Theorem 4.10. *Let G be a soluble group such that the subset T of all its elements of finite order is a proper subgroup of G. Assume further that G/B is periodic, where B is the Baer radical of G, and that all cyclic subnormal subgroups of G are permutable in G.*

1. *If G/T is not torsion-free abelian of rank 1, then G is abelian.*
2. *If G/T is torsion-free abelian of rank 1, then T is abelian with all its subgroups normal in G. Further, if $G/C_G(T)$ is finite, then $G/\zeta_1(G)$ is periodic.*

In particular, if T has finite exponent, then $G/C_G(T)$ is finite.

Proof. First, note that B is actually the Fitting subgroup of G (see Remark 4.2). Clearly, every finitely generated subgroup of B is subnormal in G and so is permutable in G. Also, B is not contained in T, otherwise G is periodic. If $u \in T$ and $v \in B \setminus T$, then $\langle u, v \rangle = \langle u \rangle \langle v \rangle$ and $\langle u \rangle = \langle u, v \rangle \cap T$ is normalized by v. Thus, $\langle u \rangle$ is normalized by $\langle B \setminus T \rangle = B$, so $\langle u, B \rangle$ is a Baer group. Since G is soluble, if $T \not\leq B$, then we may choose u so that $\langle u, B \rangle / B \neq \{1\}$ is subnormal in G/B, and so we obtain the contradiction $u \in \langle u, B \rangle \leq B$. Therefore $T \leq B$. Since B is quasihamiltonian and not periodic, so T is abelian. Moreover, either B is abelian, or B/T is torsion-free abelian of rank 1. In the latter case, $G/C_G(B/T)$ has index at most 2, so G/T acts trivially or as the inversion on B/T; but G/T is torsion-free and G/B is periodic, so B/T is central in G/T and consequently G/T is abelian by Schur's theorem.

Let x be a non-trivial, non-periodic element of B, and let y be any non-periodic element of G that does not normalize $\langle x \rangle$. Since $\langle x \rangle$ is subnormal and so permutable in G, $Z = \langle y \rangle \cap \langle x \rangle \neq \{1\}$. Clearly, Z is central of finite index in $\langle y \rangle \langle x \rangle$. Consequently, $[y, x] \in T$. In particular, every subgroup of B/T is normal in G/T. If B/T is not central in G/T, then G/T inverts B/T, which impossible being G/B periodic. Thus, $B/T \leq \zeta_1(G/T)$ and so $G'T/T$ is periodic by Schur's theorem. It follows that G/T is abelian.

Suppose B/T is not torsion-free abelian of rank 1, so B is abelian. Let $v \in B \setminus T$. Then there is $u \in G$ of infinite order such that $\langle u \rangle T \cap \langle v \rangle T = T$. Now, $\langle v \rangle$ is subnormal in G, so $\langle v \rangle$ is permutable, and hence $\langle v \rangle$ is normalized by u

(see [16], Lemma 7.1.7). If wT belongs to the isolator of $\langle v\rangle T/T$ in G/T, then $\langle wu\rangle T \cap \langle v\rangle T = T$, so wu, u and hence w all normalize $\langle v\rangle$. Since $G = \langle G \setminus T\rangle$, so $\langle v\rangle$ is normal in G. Consequently, every subgroup of B is normal in G.

Finally, B is Dedekind and non-periodic, so B is abelian. Since the only non-trivial power automorphism of B is the inversion, and G/B is periodic, it follows that no non-periodic element of G can invert B (otherwise they would act as the inversion on themselves), and hence $C_G(B) = G$. But G is soluble and hence $B = C_G(B) = G$ is abelian.

What we have proved shows that G/T is always abelian and that if G/T does not have rank 1, then G is abelian.

Assume finally that G/T is torsion-free abelian of rank 1 with $G/C_G(T)$ finite. Since G/T has rank 1 and $T \leq C_G(T)$, so there exists $g \in G \setminus T$ such that $G = \langle g\rangle C_G(T)$. Moreover, $C_G(T)$ is abelian because it is locally cyclic over its centre. Let s be a positive integer such that $g^s \in C_G(T)$. Then $g^s \in \zeta_1(G)$. Finally, the groups T and $G/\langle g^s\rangle T$ and hence $G/\langle g^s\rangle$ are all periodic. The statement is proved. □

Corollary 4.11. *Let G be a soluble linear group such that the subset T of all its elements of finite order is a proper subgroup of G. If all subnormal cyclic subgroups of G are permutable, then G satisfies* (1) *and* (2) *of* Theorem 4.10.

Proof. This is a consequence of Lemma 4.3 and Theorem 4.10. □

Theorem 4.12. *Let G be a soluble homomorphic image of a periodic linear group such that every cyclic subnormal subgroup of G permutes with every Sylow subgroup of G. Let F, H and R be respectively the Fitting subgroup, the Hirsch–Plotkin radical and the locally nilpotent residual of G. Then*:

(1) $R \leq F \leq C_G(R) \leq H$.
(2) *R is abelian and the p'-elements of G induce power automorphisms on the Sylow p-subgroup F_p of F for every prime p.*
(3) *G is hypercyclic, so G is nilpotent-by-finite,* (*locally nilpotent*)*-by-*(*finite abelian*)*.*
(4) *$R \leq O_{2'}(G) = O^2(G)$, so $2 \notin \pi(R)$.*

Assume further that all cyclic subnormal subgroups of G are permutable in G. Then:

(5) *$\pi(R) \cap \pi(H/R) = \emptyset$, so $C_G(R) = H$.*
(6) *G' is contained in a metabelian subgroup of H of finite index. In particular, $G''' = \{1\}$.*
(7) *G induces power automorphisms on R.*

Proof. Let p be a prime, and let N be any nilpotent normal p-subgroup of G. If Q is any Sylow q-subgroup of G for some prime $q \neq p$, then $\langle a\rangle Q$ is a subgroup for every $a \in N$. It follows that $\langle a\rangle$ is normalized by Q. Hence the p'-elements induce power automorphisms in N.

If N is non-abelian, then all p'-power automorphisms of N are trivial, so $G/C_G(N)$ is a p-group. If N is abelian, then the power automorphisms are in the centre of $\mathrm{Aut}(N)$. Therefore $G/C_G(N)$ is locally nilpotent.

Now, F has finite index in H, and the above yields $R \leq C_G(F) = \zeta_1(F)$. Thus, R is abelian and $R \leq F \leq C_G(R)$. But $C_G(R)$ is locally nilpotent, and so $C_G(R) \leq H$. This proves (1) and (2).

The fact that G is hypercyclic follows at once from (2), noting that the Sylow p-subgroups of G are hypercentral for every prime p. Then (3) essentially follows from Theorem 11.21 of [27] (see also [13], Theorem 2.12). Point (4) follows from the fact that $G/O_{2'}(G)$ is a 2-group (recall that G is hypercyclic).

Now, suppose all cyclic subnormal subgroups of G are permutable in G. Since the Sylow q-subgroups of G are either nilpotent or Černikov for every prime q, we see easily that, for any odd prime q, the Sylow q-subgroup of H is nilpotent and is contained in a finite term of the upper central series of any Sylow q-subgroup of G containing it. In particular, $O_{2'}(H) = O_{2'}(F)$.

Let $p \in \pi(R) \cap \pi(H/R)$, and consider a p'-element x of G. In particular, x induces power automorphisms on the Sylow p-subgroup H_p of H by (2). If H_p is non-abelian, then $[H_p, x] = \{1\}$. If H_p is abelian, then the fact that p is odd and that x acts trivially on H_pR/R shows that x acts trivially on H_p. However, this implies that $G/O_{p'}(G)$ is nilpotent, a contradiction. This proves (5).

By (5), H is a direct product of R and a Hall subgroup K normal in G. Since R and G/H are abelian, so G/K is metabelian. Now, H has finite index in G, so by the permutability of the subnormal cyclic subgroups of G, there is a normal subgroup M of G with G/M a finite elementary abelian 2-group such that every Prüfer subgroup of G/R is central in M, so $(H \cap M)/R$ is nilpotent. Thus $H \cap M$ is nilpotent, quasihamiltonian and hence metabelian (see [23], Theorem 2.4.22). Therefore $H \cap M$ is metabelian. Since $G' \leq H \cap M$, so (6) is proved.

If $X \leq Y$ are finite subgroups of G with (locally) nilpotent residuals R_X and R_Y, then clearly $R_X \leq R_Y \leq R$, $R_1 = \bigcup_X R_X$ is a normal subgroup of G modulo which G is locally nilpotent and hence $R_1 = R$.

Let $x \in R$ and $g \in G$. We have only to prove that g normalizes $\langle x \rangle$. Set $X = \langle x, g \rangle$. There exists a finite subgroup Y of G such that $X \leq Y$ and $x \in R_Y$. By [12], Satz VI.7.15, if C is a Carter subgroup of Y, then $Y = CR_Y$ with $C \cap R_Y = \{1\}$. Then $g = cy$ for some $c \in C$ and $y \in R_Y$. But $\langle x \rangle$ is subnormal and hence permutable, so $\langle c, x \rangle = \langle c \rangle \langle x \rangle$ and $\langle x \rangle = R_Y \cap \langle c \rangle \langle x \rangle$ is normal in $\langle c, x \rangle$. Thus, c normalizes $\langle x \rangle$, y centralizes $\langle x \rangle$ (recall R is abelian) and so $g = cy$ normalizes $\langle x \rangle$. The proof is complete. □

Remark 4.13. The situation of Theorem 4.12 just for finite soluble groups is discussed in [22]. Also, the consideration of any finite simple group shows that we cannot obtain an analogue of the above result in the soluble-by-finite context.

Of course, some of the conditions in Theorem 4.12 are even sufficient, as shown for example by the following result.

Theorem 4.14. *Let G be a homomorphic image of a periodic linear group having a normal subgroup R satisfying:*

(i) *R is abelian and G/R is residually (locally nilpotent).*
(ii) *All elements of G induce power automorphisms in R.*
(iii) *$\pi(R) \cap \pi(H/R) = \emptyset$, where H is the Hirsch–Plotkin radical of G.*
(iv) *Each subnormal cyclic subgroup of H/R is permutable in G/R.*

Then all subnormal cyclic subgroups of G are permutable in G.

Proof. Let $\langle g\rangle$ be a subnormal p-subgroup for some prime p. It suffices to prove that $\langle g\rangle$ permutes with an arbitrary cyclic q-subgroup $\langle x\rangle$ of G, where q is any prime. If $p \in \pi(R)$, then $g \in R$ by (iii), so $\langle g\rangle$ is normal in G by (ii).

Assume $p \notin \pi(R)$. If $q \neq p$, then $[x, g] \in R$. But $[x, g]$ is a p-element, so $[x, g] = 1$ since $p \notin \pi(R)$. Finally, let $q = p$. Then x and g both belong to some Sylow p-subgroup P. By condition (iv), we have $\langle x\rangle\langle g\rangle R = \langle g\rangle\langle x\rangle R$. Intersecting both sides with P and noting that $p \notin \pi(R)$, we obtain $\langle x\rangle\langle g\rangle = \langle g\rangle\langle x\rangle$. The statement is proved. □

The following example shows that Theorem 4.12 cannot be much improved.

Example 4.15. *There exists a metabelian periodic linear group G satisfying the following properties.*

(1) *The subnormal subgroups of G permute with the Sylow subgroups of G.*
(2) *$\pi(R) \cap \pi(G/R) \cap \big(\mathbb{P} \setminus \{2\}\big) \neq \emptyset$, where R denotes the locally nilpotent residual of G.*

Moreover, G can be chosen of any characteristic.

Proof. Let $\mathcal{K}$ be an algebraically closed field with $\mathrm{char}(\mathcal{K}) \neq 3$. Set

$$R = \{\mathrm{diag}(a, b, a^{-1}b^{-1}) \ : \ a, b \text{ 3-elements of } \mathcal{K}^*\},$$

so $R \leq D(3, \mathcal{K}) \cap \mathrm{SL}(3, \mathcal{F})$ is a divisible abelian 3-group of rank 2. Consider two automorphisms x and y of R, where x is induced by conjugation of R by the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and y is the inversion. Let G be the split extension of R by $\langle x, y\rangle \leq \mathrm{R}$. Then $[x, y] = 1$, G/R is cyclic of order 6, $P = \langle x\rangle R$ is the unique Sylow 3-subgroup P of G, and the Sylow 2-subgroups of G are conjugate to $\langle y\rangle$. Clearly, $[R, y] = R^2 = R = [R, x]$. Also, G is Černikov, so it is periodic linear. More precisely, by construction P is a subgroup of $\mathrm{GL}(3, \mathcal{K})$, so G can be embedded into $\mathrm{GL}(6, \mathcal{K})$. Furthermore, the locally nilpotent residual of G is R and $\pi(R) \cap \pi(G/R) = \{3\}$. This proves (2).

We need to check that every subnormal subgroup of G permutes with every Sylow subgroup of G. Let X be a subnormal subgroup of G. If $x' \in (P \cap X) \setminus R$, then $R = [R, x] = [R, x^{-1}] = [R, x']$. But $[G,_d X] \leq X$ for some $d \geq 1$, so $R \leq X$ and X is normal in G. Suppose X contains an involution. Without loss of generality, we may assume $y \in X$. Then again $R \leq X$ and X is normal in G. Finally, suppose $X \leq R$. Trivially $XP = PX$. If y' is any conjugate of y, then $y' \in yR$, so y' acts as the inversion on R, and hence $\langle y' \rangle X = X \langle y' \rangle$. This proves (1).

This proves the statement for all characteristic $\neq 3$. But if we replace 3 by 5 in the above argument, we obtain an example of characteristic 3, thus completing the proof of the statement. □

Acknowledgments

The preliminary research for this paper was done jointly with our colleague Francesco de Giovanni (07.10.1955 – 03.02.2024). He died before the final version was completed. The first author is supported by GNSAGA (INdAM) and is a member of the no profit association "AGTA — Advances in Group Theory and Applications" (www.advgrouptheory.com). Funded by the European Union - Next Generation EU, Missione 4 Componente 1 CUP B53D23009410006, PRIN 2022- 2022PSTWLB - Group Theory and Applications. We thank the referee for their valuable comments and suggestions, which have helped improve the paper.

References

[1] Agrawal, Ram K. Finite groups whose subnormal subgroups permute with all Sylow subgroups. *Proc. Amer. Math. Soc.* **47** (1975), 77–83. MR0364444, Zbl 0299.20014. 1338

[2] Amberg, Bernhard; Franciosi, Silvana; de Giovanni, Francesco Products of Groups. *Claredon, Oxford*, 1992. MR1211633, Zbl 0774.20001. 1326, 1330

[3] Ballester-Bolinches, Adolfo; Esteban-Romero, Ramon; Asaad, Mohamed Products of Finite Groups. *de Gruyter, Berlin*, 2010. MR2762634, Zbl 1206.20019. 1326

[4] Beidleman, James C.; Brewster, Ben; Robinson, Derek J.S. Criteria for permutability to be transitive in finite groups. *J. Algebra* **222** (1999), 400–412. MR1733679, Zbl 0948.20015. 1338

[5] Brewster, D.C.; Lennox, John C. Subnormality and quasinormality in soluble minimax groups. *J. Algebra* **48** (1977), 368–381. MR0453870, Zbl 0406.20025. 1334

[6] Celentani, Maria Rosaria; Leone, Antonella On quasinormal subgroups of infinite groups. *Ricerche Mat.* **47** (1998), 205–211. MR1760330, Zbl 0928.20024. 1334

[7] Celentani, Maria Rosaria; Leone, Antonella; Robinson, Derek J.S. The Maier–Schmid problem for infinite groups. *J. Algebra* **301** (2006), 294–307. MR2230333, Zbl 1121.20029. 1325, 1328, 1329

[8] Cutolo, Giovanni; Leone, Antonella Permutable subgroups and the Maier–Schmid theorem for nilpotent-by-finite groups. *J. Algebra* **546** (2020), 723–733. MR4037560, Zbl 1445.20025. 1325, 1326

[9] Ferrara, Maria; Trombetti, Marco Periodic linear groups in which permutability is a transitive relation. *Ann. Mat. Pura Appl. (4)* **203** (2024), 361–383. MR4685733, Zbl 1530.20073. 1338

[10] Fuchs, László Infinite Abelian Groups, Vol. 1. *Academic Press, New York & London*, 1970. MR0255673, Zbl 0209.05503.

[11] de Giovanni, Francesco; Trombetti, Marco; Wehrfritz, Bertram A.F. Subnormality in linear groups. *J. Pure Appl. Algebra* **227** (2023), 107185, 16pp. MR4457897, Zbl 1520.20060. 1328, 1329, 1336

[12] Huppert, Bertram Endliche Gruppen I. *Springer, Berlin*, 1967. MR0224703, Zbl 0412.20002. 1341

[13] IORIO, LUIGI; TROMBETTI, MARCO On some products of groups *Proc. Edinburgh Math. Soc.* **68** (2025), 869–897. MR4943973. 1341

[14] KOPPE, HANS Über Untergruppen, die mit ihren Konjugierten vertauschbar sind. *Math. Z.* **152** (1976), 99. MR0424945, Zbl 0326.20032. 1328, 1329

[15] KURDACHENKO, LEONID A.; MUÑOZ-ESCOLANO, JOSÉ M.; TURBAY, N.A. Conjugate-permutable subgroups of infinite groups. in: Proceedings of "Encuentro en la Teoríde Grupos y sus Aplicaciones" (Zaragoza, 2011). *Biblioteca de La Revista Matemática Iberoamericana*, Madrid (2011), 169–184. MR3635585, Zbl 1265.20034. 1328, 1329, 1330

[16] LENNOX, JOHN C.; STONEHEWER, STEWART E. Subnormal Subgroups of Groups. *Clarendon Press, Oxford*, 1989. MR3635585, Zbl 0606.20001. 1328, 1329, 1332, 1339, 1340

[17] MAIER, RUDOLF; SCHMID, PETER The embedding of quasinormal subgroups in finite groups. *Math. Z.* **131** (1973), 269–272. MR0364443, Zbl 0259.20017. 1325

[18] MENEGAZZO, FEDERICO Gruppi nei quali la relazione di quasi-normalità è transitiva. *Rend. Sem. Mat. Univ. Padova* **40** (1968), 347–361. MR0228584, Zbl 0174.04902. 1336

[19] MENEGAZZO, FEDERICO Gruppi nei quali la relazione di quasi-normalità è transitiva. II. *Rend. Sem. Mat. Univ. Padova* **42** (1969), 389–399. MR0257225, Zbl 0275.20057. 1336, 1338

[20] ORE, ØYSTEIN On the application of structure theory to groups. *Bull. Amer. Math. Soc.* **44** (1938), 801–806. MR1563880, Zbl 0020.34705. 1324

[21] ROBINSON, DEREK J.S. Finiteness Conditions and Generalized Soluble Groups, Part 1. *Springer, Berlin*, 1972. MR0332989, Zbl 0243.20032. 1327, 1332

[22] ROBINSON, DEREK J.S. Finite groups whose cyclic subnormal subgroups are permutable. *Algebra Colloq.* **12** (2003), 171–180. MR2127288, Zbl 1081.20030. 1338, 1341

[23] SCHMIDT, ROLAND Subgroup Lattices of Groups. *de Gruyter, Berlin*, 1994. MR1292462, Zbl 0843.20003. 1326, 1331, 1332, 1333, 1335, 1336, 1337, 1339, 1341

[24] SENEVIRATNE, H.H.G. On permutable subgroups of soluble minimax groups. *Arch. Math.* **44** (1985), 481–484. MR0797438, Zbl 0572.20021. 1334

[25] STONEHEWER, STEWART E. Permutable subgroups of infinite groups. *Math. Z.* **125** (1972), 1–16. MR0294510, Zbl 0219.20021. 1324, 1327, 1332

[26] SZÉP, JENŐ Bemerkung zu einem Satz von Ø. Ore. *Publ. Math. Debrecen* **3** (1953), 81–82. MR0061105, Zbl 0056.25602. 1328

[27] WEHRFRITZ, BERTRAM A.F. Infinite Linear Groups. *Springer, Berlin*, 1973. MR0837425, Zbl 0261.20038. 1326, 1328, 1329, 1331, 1332, 1333, 1334, 1335, 1338, 1341

[28] WEHRFRITZ, BERTRAM A.F. On the holomorphs of soluble groups of finite rank. *J. Pure Appl. Alg.* **4** (1974), 55–69. MR0347985, Zbl 0286.20031. 1334

[29] WEHRFRITZ, BERTRAM A.F. Hypercentral unipotent subgroups of linear groups. *Bull. London Math. Soc.* **10** (1978), 310–313. MR0519916, Zbl 0398.200354. 1332

[30] ZACHER, GIOVANNI I gruppi risolubili finiti in cui i sottogruppi di composizione coincidono con i sottogruppi quasi-normali *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (*8*) **37** (1964), 150–154. MR0174633, Zbl 0136.28302. 1326, 1336, 1338

(M. Trombetti) DIPARTIMENTO DI MATEMATICA E APPLICAZIONI, UNIVERSITÀ DI NAPOLI FEDERICO II, NAPOLI, ITALY
marco.trombetti@unina.it

(B.A.F. Wehrfritz) SCHOOL OF MATHEMATICAL SCIENCES, QUEEN MARY UNIVERSITY OF LONDON, LONDON, ENGLAND
b.a.f.wehrfritz@qmul.ac.uk

This paper is available via http://nyjm.albany.edu/j/2025/31-49.html.