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Signed clasp numbers of knots and four-genus bounds

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Abstract. There exist knots that have positive and negative 4-dimensional clasp numbers zero but have four-genus, and hence clasp number, arbitrarily large. Such examples were first constructed by Allison Miller, answering a question of Juhász–Zemke. Further examples are constructed here, complementing those of Miller in that they are of infinite order in the concordance group, rather than being two-torsion. More precisely, for each knot K considered here, the four-genus satisfies $\lim_{n\to\infty} g_4(nK) = \infty$. An added feature of the examples here is their simplicity; all are two-bridge knots and include the two-bridge knot $B(25,2)$, the first algebraically slice knot that was proved to be non-slice by Casson and Gordon in 1973.

1. Introduction

Let $q = p^2$, where p is an odd prime integer. We consider the two-bridge knot $B(q,2)$, abbreviated B_q, which can also be described as the k–twisted positive Whitehead double of the unknot, $D_+(U,k)$, where $k = (q-1)/4$. Figure 1 is an illustration of B_q in which the k in the box denotes k full right-handed twists. If $k = 1$ in the diagram, the resulting knot is the figure eight. We prove the following theorem, which holds in the smooth and topological locally-flat categories.

Theorem 1.1. *If $p \geq 5$ is prime and $q = p^2$, then there exists a real number $c_p > 0$ such that the four-genus satisfies $g_4(nB_q) \geq c_p n$ for all $n > 0$.*

Finding this result was motivated by a question of Juhász–Zemke [7] concerning signed 4-dimensional clasp numbers. Let $c(K)$ denote the *4-dimensional clasp number*: this is the minimum value of m for which K bounds a smooth immersed disk in B^4 with m double points. Let $c^+(K)$ denote the minimum value of m for which K bounds a smooth disk in B^4 with m positive double points, and define $c^-(K)$ similarly, minimizing negative double points. In [7] it was asked whether $c(K) - (c^+(K) + c^-(K))$ can be arbitrarily large.

Miller [10] provided the first examples answering the Juhász–Zemke question positively. It follows from Theorem 1.1 that the knots B_q are also examples.

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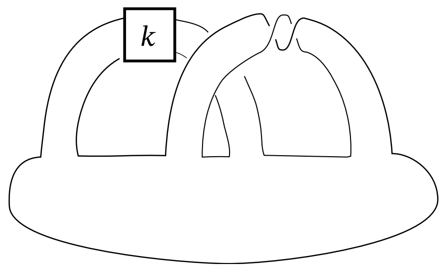

FIGURE 1. The knot B_q, where $k = (q-1)/4$ denotes full right-handed twists.

From the diagram, it is clear that B_q, and hence nB_q, can be unknotted using only positive, or only negative, crossing changes. Hence, $c^{\pm}(nB_q) = 0$. If nB_q bounds a disk in B^4 with a double points, then those double points could be resolved to form an embedded surface of genus a; it follows that $c(nB_q) \geq g_4(nB_q)$ and thus Theorem 1.1 implies $c(nB_q) \geq c_p n$.

The examples of this paper are complementary to Miller's. The examples of [10] are all amphichiral knots and thus satisfy $c^+(2K) = c^-(2K) = c(2K) = 0$; stated differently, the knots are of order two in the knot concordance group. In contrast, the fact that $c(nB_q) > 0$ for all $n > 0$ implies that B_q is of infinite order in the concordance group.

Remarks.

- The key ingredients of the proof of Theorem 1.1 come from the work of Casson and Gordon [1] and Gilmer [4]. The knot B_{25} appears in [1] as the first example of an algebraically slice knot that is not slice: that is, $g_4(B_{25}) > 0$. A theorem of Jiang [6] demonstrates the linear independence of the set $\{B_{p^2}\}_{p\geq 5}$ in concordance, thus implying that $g_4(nB_{p^2}) > 0$ for all $p \geq 5$ and all $n \geq 1$. It is a theorem from [3] that permits Jiang's result to be improved to give a linearly increasing genus bound.
- The proof of Theorem 1.1 provides a specific value of c_p that is close to, but always less than, $1/2$. For instance, we find $c_5 = 1/4$ and $c_7 = 5/14$. Finding a genus one knot K for which $c^+(K) = c^-(K) = 0$ and $g_4(nK) \geq n/2$ for all $n \geq 1$ appears to be especially challenging.
- The standard Seifert surface F_q for B_q contains simple closed curves of framing $\frac{q-1}{4}$ and -1 that are unknotted in S^3. Using these curves we can construct an unknotted essential curve α on the Seifert surface G for $B_q \# \frac{q-1}{4}B_q = \frac{q+3}{4}B_q$ of Seifert framing 0. Surgery can be performed on G along α to produce a surface in B^4 bounded by $\frac{q+3}{4}B_q$ of genus one less than the genus of G; that is, it is of genus $\frac{q-1}{4}$. Thus, $g_4(\frac{q+3}{4}B_q) \leq \frac{q-1}{4}$. It follows that $g_4(nB_q)$ is asymptotically bounded above by $(\frac{q-1}{q+3})n$,

- The invariants studied here are all knot concordance invariants. From a modern perspective, it would be interesting to prove the analog of Theorem 1.1 for a family of topologically slice knots.

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2. Casson–Gordon invariants and four-genus bounds

For a knot K, let $M_2(K)$ denote its 2–fold branched cover and let

$$\chi : H_1(M_2(K)) \to \mathbb{C}^*$$

be a character taking values in the group of units generated by $e^{2\pi i/q}$, where q is a prime power. (Such characters are naturally identified with homomorphisms $\chi : H_1(M_2(K)) \to \mathbb{Z}_q$.) In [1], Casson and Gordon defined two rational-valued invariants, $\sigma(K,\chi)$ and $\sigma_1\tau(K,\chi)$. The first is more readily computable in the case that $M_2(K)$ is a lens space; the second provides an obstruction to a knot being slice. They are related by the following result, an immediate consequence of [1, Theorem 3].

Theorem 2.1. *If $M_2(K)$ is a lens space and $\chi : H_1(M_2(K)) \to \mathbb{Z}_q$ is a nontrivial character, then*

$$\left|\sigma(K,\chi) - \sigma_1\tau(K,\chi)\right| \leq 1.$$

2.1. Computing $\sigma(B_q,\chi^r)$ for $r \neq 0 \mod p$.

We have the following result.

Theorem 2.2. *Let $q = p^2$, where p is an odd prime. Let χ denote a character that takes value $e^{2\pi i/p}$ on some generator of $H_1(M_2(B_q)) \cong \mathbb{Z}_q$. Then*

$$\{\sigma(B_q,\chi^r)\}_{0<r<p} = \{4r^2 - 2pr + 1\}_{0<r<p}.$$

Proof. This is essentially the key numeric computation of [1]. The invariant $\sigma(K,\chi)$ is defined in terms of signatures of Hermitian forms and is thus symmetric: $\sigma(K,\chi^r) = \sigma(K,\chi^{-r}) = \sigma(K,\chi^{p-r})$. This permits us to restrict attention to even values of r: $\{\sigma(B_q,\chi^{2r})\}_{0<r<p/2}$. In [1] it is shown that $\sigma(B_q,\chi^{2r}) = 4r^2 - 2pr + 1$ for $0 < r < p/2$. (The result appears on page 196, with the values m and n' there having the value p in our application.) ☐

2.2. Computing $\sigma_1\tau(B_q,\chi^0)$.

In general, there are few methods available for computing $\sigma_1\tau(K,\chi)$. However, in the case that K is of three-genus one and is algebraically slice, the invariant is determined by the Levine–Tristram signature functions of certain knots formed as simple closed curves on a genus one Seifert surface. This is a consequence of results related to companionship proved independently by Cooper [2], Gilmer [4], and Litherland [8]. The paper [5] presents a more recent exposition. We isolate the result we need. In this statement, $\sigma_K(\omega)$ denotes the Levine–Tristram signature function defined on the unit circle in $\mathbb{C}^*$.

Theorem 2.3. *Suppose that K bounds a genus one Seifert surface F and $H_1(M_2(K)) \cong \mathbb{Z}_q$ with $q = p^2$ for some prime p. Suppose that α is an essential simple closed curve on F for which the $V([\alpha],[\alpha]) = 0$, where V is the Seifert form of F. Then for $\chi : H_1(M_2(K)) \to \mathbb{Z}_p \subset \mathbb{C}^*$,*

$$\sigma_1\tau(K,\chi) = 2\sigma_\alpha(\zeta^r),$$

for some r, where $\chi(x) = \zeta \in \mathbb{C}^*$ for a generator $x \in H_1(M_2(K))$.

The Levine–Tristram signature function satisfies $\sigma_K(1) = 0$ for all K. Thus we have the following corollary when applied to χ^0, which is trivial.

Corollary 2.4. *Suppose that K bounds a genus one Seifert surface F, $H_1(M_2(K)) \cong \mathbb{Z}_q$, and $V(\alpha,\alpha) = 0$ for a simple closed curve α representing a nontrivial homology class, $[\alpha] \in H_1(F)$. Then $\sigma_1\tau(K,\chi^0) = 0$ for all χ.*

2.3. Bounds on $\sigma_1\tau(B_q,\chi^r)$.

Theorem 2.5. *Assume $q = p^2$ where $p \geq 5$ is an odd prime.*

- *There exists a generator χ of the group of order p characters on $H_1(M_2(B_q))$ such that $\sigma_1\tau(B_q,\chi) \leq \frac{9-p^2}{4}$.*
- *$\sigma_1\tau(B_q,\chi^r) \leq 0$ for all r.*

Proof. We consider the function $f(r) = 4r^2 - 2pr + 1$ that appears in Theorem 2.2 as a real quadratic in the variable r. Its minimum occurs at $p/4$. The closest integer point to $p/4$ is either $(p-1)/4$ or $(p+1)/4$ depending on whether $p \equiv 1 \mod 4$ or $p \equiv 3 \mod 4$. In both cases the value at this point is $(5-p^2)/4 < -1$. Since $\sigma(B_q,\chi^r)$ and $\sigma_1\tau(B_q,\chi^r)$ differ by at most one, we have the first statement.

For integers r with $1 \leq r \leq p/2$, the maximum value of the quadratic $f(r)$ must be at an endpoint, either $r = 1$ or $r = (p-1)/2$. We compute $f(1) = (5-2p)$ and $f((p-1)/2) = 2-p$. The larger of the two is $2-p < 1$. Even upon adding 1, this is negative. Thus, if $\sigma_1\tau(B_q,\chi^r)$ were to be positive for some r, it would have to be at $r = 0$, where the value was shown to be 0 in Corollary 2.4. □

3. The genus bound

The proof of Theorem 1.1 depends on the following special case of a theorem of Gilmer [3, Theorem 1] that relates values of $\sigma_1\tau(K,\chi)$ to $g_4(K)$.

Theorem 3.1. *Let K be a knot for which $H_1(M_2(K)) \cong (\mathbb{Z}_q)^n$, where q is a prime power. If $g_4(K) \leq n/2$ and the classical signature of K satisfies $\sigma(K) = 0$, then there is a subgroup $\mathcal{M} \subset (\mathbb{Z}_q)^n \subset H_1(M_2(K))$ of order at least $q^{(n-2g_4(K))/2}$ such that for all $\chi \in \mathcal{M}$,*

$$\left|\sigma_1\tau(K,\chi)\right| \leq 4g_4(K).$$

We will refer to the subgroup $\mathcal{M}$ as a *metabolizer*. In the statement of Gilmer's theorem in [3] there is an additional term $\mu(K, \chi)$, but prior to the statement of that theorem he points out that $\mu(K, \chi) = 0$ in the case of characters χ of prime power order.

3.1. Proof of Theorem 1.1. The continuing assumption is that $q = p^2$ where $p \geq 5$ is a prime. Here is a restatement of the theorem with the value of c_p specified.

Theorem 1.1. *For every odd prime $p \geq 5$, let $c_p = \frac{1}{2} - \frac{8}{p^2+7}$. Then for $q = p^2$, $g_4(nB_q) \geq c_p n$.*

Proof. We will first assume that n is such that $g_4(nB_q) < n/2$ and find a value of $c_p < 1/2$ for which $g_4(nB_q) \geq c_p n$ for all such n. Then, in any cases that $g_4(nB_q) \geq n/2$ we will certainly also have that $g_4(nB_q) \geq c_p n$. We abbreviate $g_4(nB_q) = g$.

The knot B_q is a two-bridge knot having two-fold branched cyclic cover the lens space $L(q, 2)$; this is used in [1] and described in detail in [11]. In particular, the first homology of the cover is $\mathbb{Z}_q$. We have the $H_1(M_2(nB_q)) \cong (\mathbb{Z}_q)^n$. The metabolizer $\mathcal{M}$ given by Theorem 3.1 has order at least $p^{(n-2g)}$. Since each element in $\mathcal{M}$ has order at most p^2, an independent set of generators of $\mathcal{M}$ must have at least $p^{(n-2g)/2}$ elements. Since the value is an integer, we can take the ceiling and let $d = \lceil \frac{n-2g}{2} \rceil$.

To simplify the following discussion, we will explicitly identify the set of $\mathbb{Z}_q$-valued character on $H_1(B_q) \cong \mathbb{Z}_q$ with $\mathbb{Z}_q$, as follows. Let $\theta \in H_1(B_q)$ be a generator. Then we identify χ with $\chi(\theta)$.

Represent a set of generators of $\mathcal{M}$ as a set of vectors in $(\mathbb{Z}_q)^n$. Together these can be used to form the rows of a matrix with at least d rows. Row operations and column interchanges can convert this into a matrix for which the top left $d \times d$ block is an upper triangular matrix with nonzero diagonal entries and with the further property that rows corresponding to diagonal entries that are divisible by p have all their entries divisible by p. If the leading entry of a row is not divisible by p, then a multiple of that row by some invertible element in $\mathbb{Z}_q$ equals 1. If the leading entry is divisible by p, then some multiple of that row by an invertible element in $\mathbb{Z}_q$ has leading entry p. Thus, for $i \leq d$, row i can be assumed to be of the form

$$r_i = (0, \ldots, 0, a_i, a_i^{i+1}, a_i^{i+2}, \ldots, a_i^d, \alpha_i^1, \ldots, \alpha_i^{n-d}),$$

where either $a_i = 1$ or $a_i = p$ and all entries to the right of a_i are divisible by p.

If we form elements v_i in $(\mathbb{Z}_q)^n$ by multiplying the r_i that begin with 1 by p and leave the other r_i unchanged, we form a set of d elements

$$v_i = (0, \ldots, 0, p, b_i^{i+1}, b_i^{i+2}, \ldots, b_i^d, \beta_i^1, \ldots, \beta_i^{n-d}),$$

where all b_i^j and all β_i^j are divisible by p. There are $i - 1$ leading 0 entries in v_i.

One can form a linear combination of these elements to construct an element $v \in \mathcal{M}$ of the form

$$v = (p, p, \dots, p, \gamma_1, \dots, \gamma_{n-d}),$$

where the first d entries are p and the γ_i are divisible by p.

Theorem 2.5 asserts the existence of an element χ in the group of order p characters on $H_1(M_2(B_q))$ with specified properties. The element $p \in \mathbb{Z}_q \cong H_1(M_2(B_q))$ is a generator, so the character χ corresponds to $kp \in H_1(M_2(B_q))$ for some k. Multiplying v by k we have

$$kv = (kp, kp, \dots, kp, \gamma'_1, \dots, \gamma'_{n-d}) \in \mathcal{M},$$

for some set of γ'_j all divisible by p. If we express v in terms of characters, we have found that an element

$$w = (\chi, \chi, \dots, \chi, \chi_1, \dots, \chi_{n-d}) \in \mathcal{M},$$

where the first d entries are the specified χ. The vector w corresponds to a character $\overline{\chi} : H_1(M_2(nB_q)) \to \mathbb{Z}_p$.

Recall that the character χ from Theorem 2.5 satisfies $\sigma_1\tau(B_q, \chi) \leq (9 - p^2)/4$. Applying the fact that $\sigma_1\tau(B_q, \chi) \leq 0$ for all χ, along with the additivity of $\sigma_1\tau$ (discussed in the subsection below), after taking absolute values we have

$$d\Big(\frac{p^2-9}{4}\Big) = \frac{(n-2g)(p^2-9)}{8} \leq \left|\sigma_1\tau(nB_q, \overline{\chi})\right| \leq 4g,$$

where the second inequality comes from Theorem 3.1.

Solving for g we find

$$g \geq \Big(\frac{p^2-9}{2p^2+14}\Big)n = \Big(\frac{1}{2} - \frac{8}{p^2+7}\Big)n.$$

□

3.2. Additivity. The form of additivity of Casson–Gordon invariants that we used above states that given pairs (K_1, χ_1) and (K_2, χ_2), where $\chi_i : H_1(M_2(K_i)) \to \mathbb{Z}_p$, one has

$$\sigma_1\tau(K_1 \# K_2, \chi_1 \oplus \chi_2) = \sigma_1\tau(K_1, \chi_1) + \sigma_1\tau(K_2, \chi_2).$$

This is a consequence of results that were proved independently by Gilmer [4, Proposition (3.2)] and Litherland [8, Theorem 2]. Both sources describe a more general form of additivity than what we are using. Litherland considered satellite knots of arbitrary winding number; a connected sum of knots is a winding number one satellite. Gilmer's presentation is restricted to connected sums, so we explain the connection between his result and additivity as we use it

A general form of the Casson–Gordon invariant is denoted $\tau(K, \chi)$ and takes values in $W(\mathbb{C}(t), J) \otimes \mathbb{Q}$. In the notation of [3]: $K \subset S^3$ is a knot; χ is a character on $H_1(M_2(K))$ taking values in $\mathbb{Q}/\mathbb{Z}$; $\mathbb{C}(t)$ is the field of fractions of the polynomial ring $\mathbb{C}[t]$; J is the involution of $\mathbb{C}(t)$ induced by $t \to t^{-1}$; and $W(\mathbb{C}(t), J)$ is the Witt group of J–hermitian inner products on finite dimensional $\mathbb{C}(t)$–vector spaces. The Witt group is an abelian group, and thus tensoring with $\mathbb{Q}$ is a well defined operation yielding a $\mathbb{Q}$–vector space. In [3], Proposition 3.2 states the additivity of τ.

An element of $W(\mathbb{C}(t), J)$ can be represented by a hermitian matrix $A(t)$ with entries in $\mathbb{C}(t)$. For all but a finite set of $\omega \in S^1 \subset \mathbb{C}$, the matrix $A(\omega)$ is a well-defined complex hermitian matrix and has a signature $\sigma(A(\omega))$. The limit $\lim_{\omega\to 1}(\sigma(A(\omega))$ is well-defined. This limit defines a map $W(\mathbb{C}(t), J) \to \mathbb{Z}$, which extends to the tensor product to give a well-defined function $\sigma_1 : W(\mathbb{C}(t), J) \otimes \mathbb{Q} \to \mathbb{Q}$. The Casson–Gordon invariant $\sigma_1\tau$ is the composition of τ and σ_1.

The additivity of matrix signatures under connected sums implies the additivity of σ_1. This, along with the additivity of τ proved by Gilmer yields the additivity of $\sigma_1\tau$ that we use above, except for one technical point. Our characters take value in $\mathbb{Z}_p$ and in Gilmer's paper they take value in $\mathbb{Q}/\mathbb{Z}$. There is an natural inclusion $\mathbb{Z}_p \subset \mathbb{Q}/\mathbb{Z}$, and this completes the connection between Gilmer's result and additivity as we use it.

4. Observations and questions

(1) *The stable clasp number.* A function $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is called *subadditive* if $f(a+b) \leq f(a) + f(b)$ for all a and b. For any such function, $\lim_{n\to\infty} f(n)/n$ exists. In [9] this is used to define the stable four-genus of a knot K: $g_s(K) = \lim_{n\to\infty} g_4(nK)/n$. In the exact same way, one can define the stable clasp number of a knot K to be $c_s(K) = \lim_{n\to\infty} c(nK)/n$. For Miller's examples [10], $c_s(K) = 0$. We have

$$\frac{q-9}{2q-14} \leq c_s(B_q) \leq \frac{q-1}{q+3}.$$

The right inequality follows from the third remarks at the end of the introduction. Here are two problems. Determine $c_s(B_q)$ exactly. Find any knot K for which $c_s(K) \notin \mathbb{Q}$.

(2) Find topologically slice knots K_n for which $c(K_n) - (c^+(K_n) + c^-(K_n))$ goes to infinity as n increases. Can such example be found for which $c^+(K_n) = 0 = c^-(K_n)$ for all n?

(3) The examples in this paper and those in [10] depended on estimates of the four-genus. Are there examples of knots K for which $c^+(K) = 0 = c^-(K)$ and $c(K) > g_4(K)$?

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