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Carleson embeddings and pointwise multipliers between Hardy-Orlicz and Bergman-Orlicz spaces of the upper half-plane

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ABSTRACT. In this article, we give a general characterization of Carleson measures involving concave or convex growth functions. We use these characterizations to establish continuous injections and pointwise multipliers between Hardy-Orlicz spaces and Bergman-Orlicz spaces.

CONTENTS

1. Introduction.

Let $\mathbb{D}$ be the unit disc of $\mathbb{C}$. For $\alpha > -1$, and $0 < p < \infty$, the Bergman space $A^p_\alpha(\mathbb{D})$ consists of all holomorphic functions f on $\mathbb{D}$ such that

$$\|f\|^p_{p,\alpha} := \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha d\nu(z) < \infty. \tag{1.1}$$

Here, $d\nu(z)$ is the normalized area measure on $\mathbb{D}$.

When $\alpha \to -1$, the corresponding space $A^p_{-1}(\mathbb{D})$ is the Hardy space $H^p(\mathbb{D})$ that consists of all holomorphic functions f on $\mathbb{D}$ such that

$$\|f\|^p_p := \|f\|^p_{p,-1} := \sup_{0\le r<1} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty. \tag{1.2}$$

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One of the most studied questions on holomorphic function spaces and their operators is the notion of Carleson measures for these spaces. In the unit disc, this is about characterizing all positive measures μ on $\mathbb{D}$ such that for some constant $C > 0$, and for any $f \in A^p_\alpha(\mathbb{D})$, $\alpha \geq -1$,

$$\int_{\mathbb{D}} |f(z)|^q d\mu(z) \leq C\|f\|^q_{p,\alpha}. \tag{1.3}$$

This problem was first solved by L. Carleson in [3, 4] for Hardy spaces in the case $p = q$. Extension of this result for $p < q$ was obtained by P. Duren in [14]. The case with loss $p > q$ was solved by I. V. Videnskii in [37]. The corresponding results for Bergman spaces of the unit disc and the unit ball were obtained by W. Hastings and D. Luecking, J. A. Cima and W. Wogen in [8, 16, 20, 21, 22, 23]. For other contributions, we refer the reader to the following [17, 26, 36].

Our interest in this paper is about the inequality (1.3) in the case where the power functions t^q and t^p are replaced by some continuous increasing and onto functions on $[0, \infty)$, Φ_2 and Φ_1 respectively. In the unit ball of $\mathbb{C}^n$, this problem was solved in the case where $t \mapsto \frac{\Phi_2(t)}{\Phi_1(t)}$ is nondecreasing for Hardy and Bergman spaces in the following and the references therein [5, 6, 30]. The case where $t \mapsto \frac{\Phi_2(t)}{\Phi_1(t)}$ is nonincreasing was handled in [29] for the Bergman-Orlicz spaces.

In this paper, our setting is the upper half-plane $\mathbb{C}_+$ and we still consider the problem (1.3) for the growth functions Φ_1 and Φ_2. In [12], we considered this question for the case where Φ_1 and Φ_2 are convex and, $t \mapsto \frac{\Phi_2(t)}{\Phi_1(t)}$ is non-decreasing. We present here a more general result which encompasses the case where Φ_1 and Φ_2 are concave, still with $t \mapsto \frac{\Phi_2(t)}{\Phi_1(t)}$ non-decreasing. This case was missing in [12]. We note that even in the case of power functions, the study of Carleson measures for Bergman spaces of the upper half-plane with exponent in $(0, 1]$ seems to have never been considered before. In [12], the method used required boundedness of the Hardy-Littlewood maximal function on the Orlicz space but this does not hold when the associated growth function is concave. Our work still uses the boundedness of the maximal function but through a modified approach that allows us to include the case where the growth functions are concave. For the proofs, we are still combining techniques from analytic function spaces and methods from dyadic harmonic analysis.

2. Statement of main results.

We start by recalling some known notions in the literature. In this paper, a continuous and nondecreasing function Φ from $\mathbb{R}_+$ onto itself is called a growth function. Observe that if Φ is a growth function, then $\Phi(0) = 0$ and $\lim_{t\to+\infty} \Phi(t) = +\infty$. If $\Phi(t) > 0$ for all $t > 0$ then Φ is a homeomorphism of $\mathbb{R}_+$ onto $\mathbb{R}_+$.

Let $p > 0$ be a real and Φ a growth function. We say that Φ is of upper-type (resp. lower-type) $p > 0$ if there exists a constant $C_p > 0$ such that for all $t \geq 1$ (resp. $0 < t \leq 1$),

$$\Phi(st) \leq C_p t^p \Phi(s), \ \forall \, s > 0. \tag{2.1}$$

We denote by $\mathscr{U}^p$ (resp. $\mathscr{L}_p$) the set of all growth functions of upper-type $p \geq 1$ (resp. lower-type $0 < p \leq 1$) such that the function $t \mapsto \frac{\Phi(t)}{t}$ is non decreasing (resp. non-increasing) on $\mathbb{R}_+^* = \mathbb{R}_+ \backslash \{0\}$. We put $\mathscr{U} := \bigcup_{p \geq 1} \mathscr{U}^p$ (resp. $\mathscr{L} := \bigcup_{0<p\leq 1} \mathscr{L}_p$).
Any element belonging to $\mathscr{L} \cup \mathscr{U}$ is a homeomorphism of $\mathbb{R}_+$ onto $\mathbb{R}_+$.

We say that two growth functions Φ_1 and Φ_2 are equivalent, if there exists a constant $c > 0$ such that

$$c^{-1}\Phi_1(c^{-1}t) \leq \Phi_2(t) \leq c\Phi_1(ct), \ \forall \, t > 0. \tag{2.2}$$

We will assume in the sequel that any growth function belonging to $\mathscr{U}$ (resp. $\mathscr{L}$) belongs to $\mathscr{C}^1(\mathbb{R}_+)$ and is convex (resp. concave). Moreover,

$$\Phi'(t) \approx \frac{\Phi(t)}{t}, \ \forall \, t > 0,$$

(see for example [2, 11, 12, 13, 31]).

Let I be an interval of non-zero length. The Carleson square associated with I, Q_I is the subset of $\mathbb{C}_+$ defined by

$$Q_I := \{x + iy \in \mathbb{C}_+ : x \in I \text{ and } 0 < y < |I|\}. \tag{2.3}$$

Definition 2.1. Let $s > 0$ be a real, Φ a growth function and μ a positive Borel measure on $\mathbb{C}_+$. We say that μ is a $(s, \Phi)-$Carleson measure if there is a constant $C > 0$ such that for any interval I of nonzero length

$$\mu(Q_I) \leq \frac{C}{\Phi\left(\frac{1}{|I|^s}\right)}. \tag{2.4}$$

- When $\Phi(t) = t$, these measures are known as $s-$Carleson measures and if moreover, $s = 1$, we obtain the classical Carleson measures.
- When $s = 1$, we say that μ is a $\Phi-$Carleson measure. If moreover, $\Phi(t) = t$, μ is called a Carleson measure.
- When $s = 2 + \alpha$, with $\alpha > -1$, we say that μ is a $(\alpha, \Phi)-$Carleson measure.

Let $\alpha > -1$ be a real and Φ a growth function.

- The Hardy$-$Orlicz space on $\mathbb{C}_+$, $H^\Phi(\mathbb{C}_+)$ is the space of analytic functions F on $\mathbb{C}_+$ that satisfy

$$\|F\|_{H^\Phi}^{lux} := \sup_{y>0} \inf\left\{\lambda > 0 : \int_{\mathbb{R}} \Phi\left(\frac{|F(x+iy)|}{\lambda}\right) dx \leq 1\right\} < \infty.$$

- The Bergman–Orlicz space on $\mathbb{C}_+$, $A^\Phi_\alpha(\mathbb{C}_+)$ is the space of analytic functions F on $\mathbb{C}_+$ that satisfy
$$\|F\|^{lux}_{A^\Phi_\alpha} := \inf\left\{\lambda > 0 : \int_{\mathbb{C}_+} \Phi\left(\frac{|F(x+iy)|}{\lambda}\right) dV_\alpha(x+iy) \leq 1\right\} < \infty,$$
where $dV_\alpha(x+iy) := y^\alpha dxdy$.

If Φ is convex and $\Phi(t) > 0$ for all $t > 0$, then
$$\left(H^\Phi(\mathbb{C}_+), \|.\|^{lux}_{H^\Phi}\right) \quad \text{and} \quad (A^\Phi_\alpha(\mathbb{C}_+), \|.\|^{lux}_{A^\Phi_\alpha})$$
are Banach spaces (see [12, 34, 35]). The spaces $H^\Phi(\mathbb{C}_+)$ and $A^\Phi_\alpha(\mathbb{C}_+)$ generalize respectively the Hardy space $H^p(\mathbb{C}_+)$ and the Bergman space $A^p_\alpha(\mathbb{C}_+)$ for $0 < p < \infty$.

Let Φ be a growth function. We say that Φ satisfies the Δ_2–condition (or $\Phi \in \Delta_2$) if there exists a constant $K > 1$ such that
$$\Phi(2t) \leq K\Phi(t),\ \forall\, t > 0. \tag{2.5}$$
It is obvious that any growth function $\Phi \in \mathscr{L} \cup \mathscr{U}$ satisfies the Δ_2–condition.

Let Φ be a convex growth function. The complementary function of Φ is the function Ψ defined by
$$\Psi(s) = \sup_{t\geq 0}\{st - \Phi(t)\},\ \forall\, s \geq 0.$$

Let Φ be a convex growth function. We say that Φ satisfies ∇_2–condition (or $\Phi \in \nabla_2$) if Φ and its complementary function both satisfy the Δ_2–condition.

Let $\Phi \in \mathscr{C}^1(\mathbb{R}_+)$ be a growth function. The lower and the upper indices of Φ are respectively defined by
$$a_\Phi := \inf_{t>0} \frac{t\Phi'(t)}{\Phi(t)} \qquad \text{and} \qquad b_\Phi := \sup_{t>0} \frac{t\Phi'(t)}{\Phi(t)}.$$

In [12], Theorem 2.2 asserts the following.

Theorem 2.2. *Let Φ_1, Φ_2 be two growth functions in $\mathscr{U}$, and μ a positive Borel measure on $\mathbb{C}_+$. Assume that Φ_1 satisfies the ∇_2–condition and that $t \mapsto \frac{\Phi_2(t)}{\Phi_1(t)}$ is non-decreasing. Then the following assertions are equivalent.*

(i) *μ is a $\Phi_2 \circ \Phi_1^{-1}$–Carleson measure.*

(ii) *There exists a constant $C_1 > 0$ such that for all $z = x + iy \in \mathbb{C}_+$,*
$$\int_{\mathbb{C}_+} \Phi_2\left(\Phi_1^{-1}\left(\frac{1}{y}\right)\frac{y^2}{|\omega - \overline{z}|^2}\right) d\mu(\omega) \leq C_1. \tag{2.6}$$

(iii) *There exists a constant $C_2 > 0$ such that for all $0 \not\equiv F \in H^{\Phi_1}(\mathbb{C}_+)$,*
$$\int_{\mathbb{C}_+} \Phi_2\left(\frac{|F(z)|}{\|F\|^{lux}_{H^{\Phi_1}}}\right) d\mu(z) \leq C_2. \tag{2.7}$$

This theorem is quite restrictive as one can easily see. First, it is requiring both Φ_1 and Φ_2 to be in the class $\mathscr{U}$. Second, it is requiring Φ_1 to satisfy the ∇_2-condition. This last requirement is due to the need to have that the Hardy-Littlewood maximal function is bounded on $L^{\Phi_1}(\mathbb{R})$. In summary, the above theorem does not provide any characterization when Φ_1 does not satisfy the ∇_2-condition nor when any of the growth functions is in the class $\mathscr{L}$. This is the main motivation for this paper.

Our first main result is the following which extend [12, Theorem 2.2] to Hardy-Orlicz spaces defined with concave growth functions.

Theorem 2.3. *Let $\Phi_1, \Phi_2 \in \mathscr{L} \cup \mathscr{U}$ and μ a positive Borel measure on $\mathbb{C}_+$. If the function $t \mapsto \frac{\Phi_2(t)}{\Phi_1(t)}$ is non-decreasing on $\mathbb{R}_+^* = (0, \infty)$, then the following assertions are equivalent.*

(i) *μ is a $\Phi_2 \circ \Phi_1^{-1}-$Carleson measure.*

(ii) *There exist some constants $\rho \in \{1; a_{\Phi_1}\}$ and $C_1 > 0$ such that for all $z = x + iy \in \mathbb{C}_+$,*

$$\int_{\mathbb{C}_+} \Phi_2\left(\Phi_1^{-1}\left(\frac{1}{y}\right)\frac{y^{2/\rho}}{|\omega - \overline{z}|^{2/\rho}}\right) d\mu(\omega) \leq C_1. \tag{2.8}$$

(iii) *There exists a constant $C_2 > 0$ such that for all $0 \not\equiv F \in H^{\Phi_1}(\mathbb{C}_+)$,*

$$\int_{\mathbb{C}_+} \Phi_2\left(\frac{|F(z)|}{\|F\|_{H^{\Phi_1}}^{lux}}\right) d\mu(z) \leq C_2. \tag{2.9}$$

(iv) *There exists a constant $C_3 > 0$ such that for all $F \in H^{\Phi_1}(\mathbb{C}_+)$,*

$$\sup_{\lambda > 0} \Phi_2(\lambda)\mu\left(\{z \in \mathbb{C}_+ : |F(z)| > \lambda \|F\|_{H^{\Phi_1}}^{lux}\}\right) \leq C_3. \tag{2.10}$$

As a consequence, we have the following.

Corollary 2.4. *Let $\alpha > -1$ and $\Phi_1, \Phi_2 \in \mathscr{L} \cup \mathscr{U}$ such that $t \mapsto \frac{\Phi_2(t)}{\Phi_1(t)}$ is non-decreasing on $\mathbb{R}_+^*$. The Hardy-Orlicz space $H^{\Phi_1}(\mathbb{C}_+)$ embeds continuously into the Bergman-Orlicz space $A_\alpha^{\Phi_2}(\mathbb{C}_+)$ if and only if there exists a constant $C > 0$ such that for all $t > 0$,*

$$\Phi_1^{-1}(t) \leq \Phi_2^{-1}(Ct^{2+\alpha}). \tag{2.11}$$

Our second main result generalizes [12, Theorem 2.4].

Theorem 2.5. *Let $\alpha > -1$, $\Phi_1, \Phi_2 \in \mathscr{L} \cup \mathscr{U}$ and μ a positive Borel measure on $\mathbb{C}_+$. If the function $t \mapsto \frac{\Phi_2(t)}{\Phi_1(t)}$ is non-decreasing on $\mathbb{R}_+^*$, then the following assertions are equivalent.*

(i) *μ is a $(\alpha, \Phi_2 \circ \Phi_1^{-1})-$Carleson measure.*

(ii) *There exist some constants* $\rho \in \{1; a_{\Phi_1}\}$ *and* $C_1 > 0$ *such that for all* $z = x + iy \in \mathbb{C}_+$,

$$\int_{\mathbb{C}_+} \Phi_2\left(\Phi_1^{-1}\left(\frac{1}{y^{2+\alpha}}\right)\frac{y^{(4+2\alpha)/\rho}}{|\omega - \overline{z}|^{(4+2\alpha)/\rho}}\right) d\mu(\omega) \leq C_1. \tag{2.12}$$

(iii) *There exists a constant* $C_2 > 0$ *such that for all* $0 \not\equiv F \in A_\alpha^{\Phi_1}(\mathbb{C}_+)$,

$$\int_{\mathbb{C}_+} \Phi_2\left(\frac{|F(z)|}{\|F\|_{A_\alpha^{\Phi_1}}^{lux}}\right) d\mu(z) \leq C_2. \tag{2.13}$$

(iv) *There exists a constant* $C_3 > 0$ *such that for all* $F \in A_\alpha^{\Phi_1}(\mathbb{C}_+)$,

$$\sup_{\lambda > 0} \Phi_2(\lambda)\mu\left(\{z \in \mathbb{C}_+ : |F(z)| > \lambda \|F\|_{A_\alpha^{\Phi_1}}^{lux}\}\right) \leq C_3. \tag{2.14}$$

The following embedding result follows from the above.

Corollary 2.6. *Let* $\alpha, \beta > -1$ *and* $\Phi_1, \Phi_2 \in \mathscr{L} \cup \mathscr{U}$ *such that* $t \mapsto \frac{\Phi_2(t)}{\Phi_1(t)}$ *is non-decreasing on* $\mathbb{R}_+^*$. *The Bergman-Orlicz space* $A_\alpha^{\Phi_1}(\mathbb{C}_+)$ *embeds continuously into the Bergman-Orlicz space* $A_\beta^{\Phi_2}(\mathbb{C}_+)$ *if and only if there exists a constant* $C > 0$ *such that for all* $t > 0$,

$$\Phi_1^{-1}(t^{2+\alpha}) \leq \Phi_2^{-1}(Ct^{2+\beta}). \tag{2.15}$$

One of the applications of Carleson embeddings is the characterization of pointwise multipliers between different analytic function spaces. To state these applications of the previous results, we introduce some further definitions.

Let $p, q > 0$ and let Φ be a growth function. We say that Φ belongs to $\widetilde{\mathscr{U}}^q$ (resp. $\widetilde{\mathscr{L}_p}$) if the following assertions are satisfied

(a) $\Phi \in \mathscr{U}^q$ (resp. $\Phi \in \mathscr{L}_p$).

(b) there exists a constant $C_1 > 0$ such that for all $s, t > 0$,

$$\Phi(st) \leq C_1 \Phi(s)\Phi(t). \tag{2.16}$$

(c) there exists a constant $C_2 > 0$ such that for all $s, t \geq 1$

$$\Phi\left(\frac{s}{t}\right) \leq C_2 \frac{\Phi(s)}{t^q} \tag{2.17}$$

resp.

$$\Phi\left(\frac{s}{t}\right) \leq C_2 \frac{s^p}{\Phi(t)}. \tag{2.18}$$

We put $\widetilde{\mathscr{U}} := \bigcup_{q \geq 1} \widetilde{\mathscr{U}}^q$ (resp. $\widetilde{\mathscr{L}} := \bigcup_{0 < p \leq 1} \widetilde{\mathscr{L}_p}$).

Let $\omega : \mathbb{R}_+^* \longrightarrow \mathbb{R}_+^*$ be a function. An analytic function F on $\mathbb{C}_+$ is said to be in $H_\omega^\infty(\mathbb{C}_+)$ if

$$\|F\|_{H_\omega^\infty} := \sup_{z \in \mathbb{C}_+} \frac{|f(z)|}{\omega(\mathrm{Im}(z))} < \infty. \tag{2.19}$$

If ω is continuous, then $(H^\infty_\omega(\mathbb{C}_+), \|.\|_{H^\infty_\omega})$ is a Banach space.

Let X and Y be two analytic function spaces which are metric spaces, with respective metrics d_X and d_Y. An analytic function g is said to be a multiplier from X to Y, if there exists a constant $C > 0$ such that for any $f \in X$,

$$d_Y(fg, 0) \leq C d_X(f, 0). \tag{2.20}$$

We denote by $\mathcal{M}(X, Y)$ the set of multipliers from X to Y.

The following is a characterization of pointwise multipliers from a Hardy-Orlicz space to a Bergman-Orlicz space. It is an extension of [12, Theorem 2.7].

Theorem 2.7. *Let $\Phi_1 \in \mathscr{L} \cup \mathscr{U}$ and $\Phi_2 \in \widetilde{\mathscr{L}} \cup \widetilde{\mathscr{U}}$ such that the function $t \mapsto \frac{\Phi_2(t)}{\Phi_1(t)}$ is non-decreasing on $\mathbb{R}_+^*$. Let $\alpha > -1$ and put*

$$\omega(t) = \frac{\Phi_2^{-1}\left(\frac{1}{t^{2+\alpha}}\right)}{\Phi_1^{-1}\left(\frac{1}{t}\right)}, \ \forall\, t > 0.$$

The following assertions are satisfied.

(i) *If $0 < a_{\Phi_1} \leq b_{\Phi_1} < a_{\Phi_2} \leq b_{\Phi_2} < \infty$, then*

$$\mathcal{M}(H^{\Phi_1}(\mathbb{C}_+), A^{\Phi_2}_\alpha(\mathbb{C}_+)) = H^\infty_\omega(\mathbb{C}_+).$$

(ii) *If $\omega \approx 1$, then*

$$\mathcal{M}(H^{\Phi_1}(\mathbb{C}_+), A^{\Phi_2}_\alpha(\mathbb{C}_+)) = H^\infty(\mathbb{C}_+).$$

(ii) *If ω is decreasing and $\lim_{t\to 0} \omega(t) = 0$, then*

$$\mathcal{M}(H^{\Phi_1}(\mathbb{C}_+), A^{\Phi_2}_\alpha(\mathbb{C}_+)) = \{0\}.$$

The following is a characterization of pointwise multipliers between two Bergman-Orlicz spaces. It is an extension of [12, Theorem 2.8].

Theorem 2.8. *Let $\Phi_1 \in \mathscr{L} \cup \mathscr{U}$ and $\Phi_2 \in \widetilde{\mathscr{L}} \cup \widetilde{\mathscr{U}}$ such that the function $t \mapsto \frac{\Phi_2(t)}{\Phi_1(t)}$ is non-decreasing on $\mathbb{R}_+^*$. Let $\alpha, \beta > -1$ and put*

$$\omega(t) = \frac{\Phi_2^{-1}\left(\frac{1}{t^{2+\beta}}\right)}{\Phi_1^{-1}\left(\frac{1}{t^{2+\alpha}}\right)}, \ \forall\, t > 0.$$

The following assertions are satisfied.

(i) *If $0 < a_{\Phi_1} \leq b_{\Phi_1} < a_{\Phi_2} \leq b_{\Phi_2} < \infty$ then*

$$\mathcal{M}\left(A^{\Phi_1}_\alpha(\mathbb{C}_+), A^{\Phi_2}_\beta(\mathbb{C}_+)\right) = H^\infty_\omega(\mathbb{C}_+).$$

(ii) *If $\omega \approx 1$, then*

$$\mathcal{M}\left(A^{\Phi_1}_\alpha(\mathbb{C}_+), A^{\Phi_2}_\beta(\mathbb{C}_+)\right) = H^\infty(\mathbb{C}_+).$$

(iii) *If ω is decreasing and* $\lim_{t\to 0}\omega(t) = 0$*, then*

$$\mathcal{M}\left(A_\alpha^{\Phi_1}(\mathbb{C}_+), A_\beta^{\Phi_2}(\mathbb{C}_+)\right) = \{0\}.$$

The paper is organized as follows. In Section 3, we provide some further definitions and useful results on growth functions, Hardy-Orlicz and Bergman-Orliz spaces. Indeed, there is currently no comprehensive reference in the literature for these spaces. For this reason, we establish several essential related results in our study. In Section 4, we provide characterizations of Carleson measures, including a general result that encompasses assertions (ii) of Theorem 2.3 and Theorem 2.5. Our main results are presented in Section 5.

3. Some definitions and useful properties

We present in this section some useful results needed in our presentation.

3.1. Some properties of growth functions. In this subsection, we present results on growth functions that are necessary for our study.

We recall the following useful results (see [12, 31]).

Lemma 3.1. *Let $\Phi \in \mathscr{C}^1(\mathbb{R}_+)$ a growth function. The following assertions are satisfied.*

(i) *If $\Phi \in \mathscr{L} \cup \mathscr{U}$ then $0 < a_\Phi \le b_\Phi < \infty$.*
(ii) *$\Phi \in \mathscr{U}$ if and only if $1 \le a_\Phi \le b_\Phi < \infty$. Moreover, $\Phi \in \mathscr{U} \cap \nabla_2$ if and only if $1 < a_\Phi \le b_\Phi < \infty$.*
(iii) *If $0 < a_\Phi \le b_\Phi < \infty$ then the function $t \mapsto \frac{\Phi(t)}{t^{a_\Phi}}$ is increasing on $\mathbb{R}_+^*$ while the function $t \mapsto \frac{\Phi(t)}{t^{b_\Phi}}$ is decreasing on $\mathbb{R}_+^*$.*

The following relation between growth functions in $\mathscr{U}$ and growth functions $\mathscr{L}$ will be used below (see [32, Proposition 2.1]).

Lemma 3.2. *Let Φ be a growth function and $q > 0$. If Φ is a one-to-one growth function then $\Phi \in \mathscr{U}^q$ if and only if $\Phi^{-1} \in \mathscr{L}_{1/q}$.*

The following provides equivalent characterizations of elements in ∇_2.

Lemma 3.3 (Lemma 3.1, [12])**.** *Let $\Phi \in \mathscr{U}$. The following assertions are equivalent.*

(i) $\Phi \in \nabla_2$.
(ii) *There exists a constant $C_1 > 0$ such that for all $t > 0$,*

$$\int_0^t \frac{\Phi(s)}{s^2} ds \le C_1 \frac{\Phi(t)}{t}. \tag{3.1}$$

(iii) *There exists a constant $C_2 > 1$ such that for all $t > 0$,*

$$\Phi(t) \le \frac{1}{2C_2}\Phi(C_2 t). \tag{3.2}$$

Let us make the following observation.

Lemma 3.4. *Let* $\Phi \in \mathscr{C}^1(\mathbb{R}_+)$ *be a growth function such that* $0 < a_\Phi \leq b_\Phi < \infty$. *For* $s > 0$, *consider* Φ_s *the function defined by*

$$\Phi_s(t) = \Phi\left(t^s\right), \ \forall\, t \geq 0.$$

Then $sa_\Phi \leq a_{\Phi_s} \leq b_{\Phi_s} \leq sb_\Phi$.

Proof. For $t > 0$, we have

$$(\Phi_s(t))' = st^{s-1}\Phi'\left(t^s\right) \Rightarrow \frac{t\,(\Phi_s(t))'}{\Phi_s(t)} = s \times \frac{t^s\Phi'\left(t^s\right)}{\Phi\left(t^s\right)}.$$

It follows that

$$sa_\Phi \leq \frac{t\,(\Phi_s(t))'}{\Phi_s(t)} \leq sb_\Phi, \ \forall\, t > 0.$$

□

We then deduce the following useful facts.

Corollary 3.5. *Let* $s \geq 1$ *and* $\Phi \in \mathscr{C}^1(\mathbb{R}_+)$ *be a growth function such that* $0 < a_\Phi \leq b_\Phi < \infty$. *For* $t \geq 0$, *put*

$$\Phi_s(t) = \Phi\left(t^{s/a_\Phi}\right).$$

The following assertions are satisfied.

(i) *If* $s = 1$ *then* $\Phi_s \in \mathscr{U}$.
(ii) *If* $s > 1$ *then* $\Phi_s \in \mathscr{U} \cap \nabla_2$.

Proof. Following Lemma 3.4, we have that

$$\frac{s}{a_\Phi} \times a_\Phi \leq a_{\Phi_s} \leq b_{\Phi_s} \leq b_\Phi \times \frac{s}{a_\Phi}.$$

Hence, we deduce from Lemma 3.1 that if $s = 1$ (resp. $s > 1$) then $\Phi_s \in \mathscr{U}$ since $1 \leq a_{\Phi_s} \leq b_{\Phi_s} < \infty$ (resp. $\Phi_s \in \mathscr{U} \cap \nabla_2$ since $1 < a_{\Phi_s} \leq b_{\Phi_s} < \infty$). □

It follows from Corollary 3.5 that, for any growth function of both lower and upper type, it is possible to construct an auxiliary growth function that is convex and satisfies the ∇_2-condition. The advantage of such an auxiliary function is that it allows certain results established for convex growth functions to be extended to concave growth functions.

Let us prove the following estimates of the upper and lower indices of the composition of two growth functions.

Proposition 3.6. *Let* $\Phi_1, \Phi_2 \in \mathscr{C}^1(\mathbb{R}_+)$ *be two growth functions such that* $0 < a_{\Phi_1} \leq b_{\Phi_1} < \infty$ *and* $0 < a_{\Phi_2} \leq b_{\Phi_2} < \infty$. *Then* $\Phi_1 \circ \Phi_2 \in \mathscr{C}^1(\mathbb{R}_+)$ *growth function and*

$$a_{\Phi_1} a_{\Phi_2} \leq a_{\Phi_1 \circ \Phi_2} \leq b_{\Phi_1 \circ \Phi_2} \leq b_{\Phi_1} b_{\Phi_2}.$$

Proof. For $t > 0$, we have

$$(\Phi_1 \circ \Phi_2)'(t) = \Phi_1'(\Phi_2(t))\,\Phi_2'(t) \Rightarrow \frac{t\,(\Phi_1\circ\Phi_2)'(t)}{\Phi_1\circ\Phi_2(t)} = \frac{\Phi_2(t)\Phi_1'(\Phi_2(t))}{\Phi_1(\Phi_2(t))} \times \frac{t\Phi_2'(t)}{\Phi_2(t)}.$$

It follows that

$$a_{\Phi_1} a_{\Phi_2} \leq \frac{t\,(\Phi_1\circ\Phi_2)'(t)}{\Phi_1\circ\Phi_2(t)} \leq b_{\Phi_1} b_{\Phi_2},\ \forall\, t > 0.$$

□

The following result shows that the inversion operation preserves the order between the lower and upper indices.

Proposition 3.7. *Let $\Phi \in \mathscr{C}^1(\mathbb{R}_+)$ a growth function. The following assertions are equivalent.*

(i) $0 < a_\Phi \leq b_\Phi < \infty$.
(ii) $0 < a_{\Phi^{-1}} \leq b_{\Phi^{-1}} < \infty$.

Moreover, $a_{\Phi^{-1}} = 1/b_\Phi$ *and* $b_{\Phi^{-1}} = 1/a_\Phi$.

Proof. Show that $i)$ implies $ii)$. We have

$$\left(\Phi^{-1}\right)'(t) = \frac{1}{\Phi'\left(\Phi^{-1}(t)\right)},\ \forall\, t > 0.$$

It follows that

$$\begin{aligned} 0 < a_\Phi \leq b_\Phi < \infty &\Rightarrow 0 < a_\Phi \leq \frac{t\Phi'(t)}{\Phi(t)} \leq b_\Phi < \infty,\ \forall\, t > 0 \\ &\Rightarrow 0 < a_\Phi \leq \frac{\Phi^{-1}(t)\Phi'\left(\Phi^{-1}(t)\right)}{\Phi\left(\Phi^{-1}(t)\right)} \leq b_\Phi < \infty,\ \forall\, t > 0 \\ &\Rightarrow \frac{1}{b_\Phi} \leq \frac{t}{\Phi^{-1}(t)\Phi'\left(\Phi^{-1}(t)\right)} \leq \frac{1}{a_\Phi},\ \forall\, t > 0 \\ &\Rightarrow \frac{1}{b_\Phi} \leq \frac{t\left(\Phi^{-1}\right)'(t)}{\Phi^{-1}(t)} \leq \frac{1}{a_\Phi},\ \forall\, t > 0. \end{aligned}$$

We deduce on the one hand that

$$\frac{1}{b_\Phi} \leq a_{\Phi^{-1}} \leq b_{\Phi^{-1}} \leq \frac{1}{a_\Phi}. \tag{3.3}$$

Reasoning as above, we obtain that (ii) implies (i) and we deduce on the other hand that

$$\frac{1}{b_{\Phi^{-1}}} \leq a_\Phi \leq b_\Phi \leq \frac{1}{a_{\Phi^{-1}}}. \tag{3.4}$$

From (3.3) and (3.4) we conclude that $a_{\Phi^{-1}} = 1/b_\Phi$ and $b_{\Phi^{-1}} = 1/a_\Phi$. □

Let us prove the following equivalent properties. The relevance of this result is discussed below.

Proposition 3.8. *Let $\Phi_1, \Phi_2 \in \mathscr{L} \cup \mathscr{U}$. The following assertions are equivalent.*

(i) *The function* $t \mapsto \frac{\Phi_2(t)}{\Phi_1(t)}$ *is non-decreasing on* $\mathbb{R}_+^*$.
(ii) *The function* $t \mapsto \frac{\Phi_2 \circ \Phi_1^{-1}(t)}{t}$ *is non-decreasing on* $\mathbb{R}_+^*$.
(iii) *The function* $\Phi_2 \circ \Phi_1^{-1}$ *belongs to* $\mathscr{U}^{b_{\Phi_2}/a_{\Phi_1}}$.

Proof. The equivalence between (i) and (ii) is obvious. That (iii) implies (ii) is also immediate.

Let us now show that (ii) implies (iii).

Since the functions $t \mapsto \frac{\Phi_1^{-1}(t)}{t^{1/a_{\Phi_1}}}$ and $t \mapsto \frac{\Phi_2(t)}{t^{b_{\Phi_2}}}$ are non-increasing on $\mathbb{R}_+^*$, we deduce that for all $s > 0$ and $t \geq 1$

$$\Phi_1^{-1}(st) \leq t^{1/a_{\Phi_1}} \Phi_1^{-1}(s)$$

and

$$\Phi_2\left(t^{1/a_{\Phi_1}} \Phi_1^{-1}(s)\right) \leq t^{b_{\Phi_2}/a_{\Phi_1}} \Phi_2\left(\Phi_1^{-1}(s)\right).$$

It follows that

$$\Phi_2\left(\Phi_1^{-1}(st)\right) \leq t^{b_{\Phi_2}/a_{\Phi_1}} \Phi_2\left(\Phi_1^{-1}(s)\right).$$

□

In Proposition 3.8, we show that the nondecreasing nature of the function $t \mapsto \frac{\Phi_2(t)}{\Phi_1(t)}$ is a necessary and sufficient condition for the function $\Phi_2 \circ \Phi_1^{-1}$ to be convex. This property will play an important role when extending certain results known for convex growth functions to the case of concave growth functions.

In relation with the above result, we first prove the following proposition.

Proposition 3.9. *Let* Φ *be a growth function such that* $\Phi(t) > 0$ *for all* $t > 0$. *Consider* $\widetilde{\Omega}$ *the function defined by*

$$\widetilde{\Omega}(t) = \frac{1}{\Phi\left(\frac{1}{t}\right)}, \ \forall\, t > 0 \quad \text{and} \quad \widetilde{\Omega}(0) = 0.$$

The following assertions are satisfied.

(i) $\Phi \in \mathscr{U}^q$ *(resp.* $\mathscr{L}_p$*) if and only if* $\widetilde{\Omega} \in \mathscr{U}^q$ *(resp.* $\mathscr{L}_p$*).*
(ii) $\Phi \in \mathscr{U} \cap \nabla_2$ *if and only if* $\widetilde{\Omega} \in \mathscr{U} \cap \nabla_2$.

Proof. *i*) Suppose that $\Phi \in \mathscr{U}^q$. For $0 < t_1 \leq t_2$, we have

$$\begin{aligned}
\frac{\Phi(t_1)}{t_1} \leq \frac{\Phi(t_2)}{t_2} &\Leftrightarrow \frac{\Phi(1/t_2)}{1/t_2} \leq \frac{\Phi(1/t_1)}{1/t_1} \\
&\Leftrightarrow \frac{1}{t_1}\frac{1}{\Phi(1/t_1)} \leq \frac{1}{t_2}\frac{1}{\Phi(1/t_2)} \\
&\Leftrightarrow \frac{\widetilde{\Omega}(t_1)}{t_1} \leq \frac{\widetilde{\Omega}(t_2)}{t_2}.
\end{aligned}$$

Since Φ is of upper type q then so is the function $\widetilde{\Omega}$. Indeed, for all $s > 0$ and $t \geq 1$,

$$\Phi\left(\frac{1}{s}\right) = \Phi\left(t \times \frac{1}{st}\right) \leq C_q t^q \Phi\left(\frac{1}{st}\right) \Rightarrow \frac{1}{C_q t^q \Phi\left(\frac{1}{st}\right)} \leq \frac{1}{\Phi\left(\frac{1}{s}\right)} \Rightarrow \widetilde{\Omega}(st) \leq C_q t^q \widetilde{\Omega}(s).$$

The converse is obtained in a similar way. We conclude that $\Phi \in \mathscr{U}^q$ if and only if $\widetilde{\Omega} \in \mathscr{U}^q$.
Reasoning in the same way, we also show that $\Phi \in \mathscr{L}_p$ if and only if $\widetilde{\Omega} \in \mathscr{L}_p$.
(ii) We suppose that $\Phi \in \mathscr{U} \cap \nabla_2$. For $t > 0$, we have

$$\Phi\left(\frac{1}{t}\right) \leq \frac{1}{2C}\Phi\left(\frac{C}{t}\right) \Rightarrow \frac{2C}{\Phi\left(\frac{C}{t}\right)} \leq \frac{1}{\Phi\left(\frac{1}{t}\right)} \Rightarrow 2C\widetilde{\Omega}\left(\frac{t}{C}\right) \leq \widetilde{\Omega}(t),$$

according to the Lemma 3.3. We deduce that $\widetilde{\Omega} \in \mathscr{U} \cap \nabla_2$.
The converse is obtained similarly. □

The following follows from the above, and it is useful for our characterization of Carleson measures.

Lemma 3.10. *Let $\Phi_1, \Phi_2 \in \mathscr{L} \cup \mathscr{U}$ and put*

$$\widetilde{\Omega}_3(t) = \frac{1}{\Phi_2 \circ \Phi_1^{-1}\left(\frac{1}{t}\right)}, \ \forall\, t > 0 \quad \textit{and} \quad \widetilde{\Omega}_3(0) = 0.$$

If the function $t \mapsto \frac{\Phi_2(t)}{\Phi_1(t)}$ is non-decreasing on $\mathbb{R}_+^*$ then $\widetilde{\Omega}_3 \in \mathscr{U}$.

Proof. The proof follows from Proposition 3.8 and Proposition 3.9. □

We have the following useful property of the classes $\widetilde{\mathscr{L}}$ and $\widetilde{\mathscr{U}}$ defined in the previous section. It is needed for the proof of our results on the pointwise multipliers between the analytic function spaces considered in this paper.

Lemma 3.11. *Let $\Phi \in \widetilde{\mathscr{L}} \cup \widetilde{\mathscr{U}}$. There exists a constant $C > 0$ such that*

$$\Phi\left(\frac{s}{t}\right) \leq C\frac{\Phi(s)}{\Phi(t)}, \ \forall\, s, t > 0. \tag{3.5}$$

Proof. The inequality (3.5) is true for $\Phi \in \widetilde{\mathscr{U}}$ (see [13, Lemma 4.3]).
For $0 < p \leq 1$, suppose that $\Phi \in \widetilde{\mathscr{L}_p}$. For $s, t > 0$, we have

$$\Phi\left(\frac{s}{t}\right) \leq C_1 \Phi(s) \Phi\left(\frac{1}{t}\right),$$

since the inequality (2.16) is satisfied.
If $0 < t < 1$ then we have

$$\Phi(t) = \Phi\left(\frac{1}{\frac{1}{t}}\right) \leq C_2 \frac{1^p}{\Phi\left(\frac{1}{t}\right)},$$

thanks to (2.18). It follows that

$$\Phi\left(\frac{1}{t}\right) \le C_2 \frac{1}{\Phi(t)}. \tag{3.6}$$

If $t \ge 1$, then we have

$$\Phi\left(\frac{1}{t}\right) = \Phi\left(\frac{1}{t} \times 1\right) \le C_p \left(\frac{1}{t}\right)^p \Phi(1),$$

since Φ is of lower type p. It follows that

$$\Phi\left(\frac{1}{t}\right) \le \frac{C_2}{\Phi(1)} \frac{1}{\Phi(t)}, \tag{3.7}$$

since from (2.18), we have also

$$\Phi(t) = \Phi\left(\frac{t}{1}\right) \le C_2 \frac{t^p}{\Phi(1)}.$$

From (3.6) and (3.7), we deduce that

$$\Phi\left(\frac{1}{t}\right) \lesssim \frac{1}{\Phi(t)}.$$

Therefore,

$$\Phi\left(\frac{s}{t}\right) \lesssim \frac{\Phi(s)}{\Phi(t)}.$$

□

In this part, we have mainly constructed and verified the properties of two auxiliary functions: the function Φ_s from Corollary 3.5 and the function $\widetilde{\Omega}_3$ from Lemma 3.10. These results deserve some comments before moving forward. In Corollary 3.5, starting from a growth function $\Phi \in \mathscr{C}^1(\mathbb{R}_+)$ such that $0 < a_\Phi \le b_\Phi < \infty$, we succeed in constructing Φ_s, an auxiliary convex growth function satisfying the ∇_2-condition, which enables us to extend several known results for convex growth functions (see Subsections 3.2 and 3.3) to concave growth functions. In Lemma 3.10, we construct another auxiliary function $\widetilde{\Omega}_3$, which is convex whenever the function $t \mapsto \frac{\Phi_2(t)}{\Phi_1(t)}$ is non-decreasing on $\mathbb{R}_+^*$. This will be useful for the characterization of Carleson measures in Section 4. Finally, in Lemma 3.11, the inequality (3.5) will play a crucial role in the study of pointwise multipliers in Section 5.

3.2. Some properties of Orlicz spaces. Let $(X, \sum, \mu)$ be a measure space and Φ a growth function. The Orlicz space on X, $L^\Phi(X, d\mu)$ is the set of all equivalent classes (in the usual sense) of measurable functions $f : X \longrightarrow \mathbb{C}$ which satisfy

$$||f||^{lux}_{L^\Phi_\mu} := \inf\left\{\lambda > 0 : \int_X \Phi\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) \le 1\right\} < \infty.$$

If Φ is convex then $(L^\Phi(X, d\mu), ||.||^{lux}_{L^\Phi_\mu})$ is a Banach space (see [7, 19, 28]). The space L^Φ generalizes the Lebesgue space L^p for $0 < p < \infty$.

Let Φ be a growth function. Let $f \in L^{\Phi}(X, d\mu)$ and put

$$\|f\|_{L^{\Phi}_{\mu}} := \int_X \Phi(|f(x)|)\, d\mu(x).$$

If $\Phi \in \mathscr{C}^1(\mathbb{R}_+)$ is a growth function such that $0 < a_{\Phi} \leq b_{\Phi} < \infty$, then we have the following inequalities

$$\|f\|_{L^{\Phi}_{\mu}} \lesssim \max\left\{\left(\|f\|^{lux}_{L^{\Phi}_{\mu}}\right)^{a_{\Phi}}; \left(\|f\|^{lux}_{L^{\Phi}_{\mu}}\right)^{b_{\Phi}}\right\}$$

and

$$\|f\|^{lux}_{L^{\Phi}_{\mu}} \lesssim \max\left\{\left(\|f\|_{L^{\Phi}_{\mu}}\right)^{1/a_{\Phi}}; \left(\|f\|_{L^{\Phi}_{\mu}}\right)^{1/b_{\Phi}}\right\}.$$

We will simply denote $L^{\Phi}(\mathbb{R}) = L^{\Phi}(\mathbb{R}, dx)$, where dx is the Lebesgue measure on $\mathbb{R}$.

Let Φ be a convex growth function. We have the following inclusion

$$L^{\Phi}(\mathbb{R}) \subset L^1\left(\mathbb{R}, \frac{dt}{1+t^2}\right).$$

Let $\alpha > -1$ and E be a measurable set of $\mathbb{C}_+$. We denote

$$|E|_{\alpha} := \int_E dV_{\alpha}(x+iy).$$

Let I be an interval and Q_I its associated Carleson square. It is easy to see that

$$|Q_I|_{\alpha} = \frac{1}{1+\alpha}|I|^{2+\alpha}. \tag{3.8}$$

Fix $\beta \in \{0; 1/3\}$. An interval $\beta-$dyadic is any interval I of $\mathbb{R}$ of the form

$$2^{-j}([0,1) + k + (-1)^j\beta),$$

where $k, j \in \mathbb{Z}$. We denote by $\mathcal{D}^{\beta}_j$ the set of $\beta-$dyadic intervals I such that $|I| = 2^{-j}$. Put $\mathcal{D}^{\beta} := \bigcup_j \mathcal{D}^{\beta}_j$.

We have the following properties (see for example [9, 33]):

- for all $I, J \in \mathcal{D}^{\beta}$, we have $I \cap J \in \{\emptyset; I; J\}$,
- for each fixed $j \in \mathbb{Z}$, if $I \in \mathcal{D}^{\beta}_j$ then there exists a unique $J \in \mathcal{D}^{\beta}_{j-1}$ such that $I \subset J$,
- for each fixed $j \in \mathbb{Z}$, if $I \in \mathcal{D}^{\beta}_j$ then there exists $I_1, I_2 \in \mathcal{D}^{\beta}_{j+1}$ such that $I = I_1 \cup I_2$ and $I_1 \cap I_2 = \emptyset$.

We refer to [18, 27] for the following.

Lemma 3.12. *Let I be an interval. There exist $\beta \in \{0, 1/3\}$ and $J \in \mathcal{D}^{\beta}$ such that $I \subset J$ and $|J| \leq 6|I|$.*

Let $\alpha > -1$ and f a measurable function on $\mathbb{R}$ (resp. $\mathbb{C}_+$). The Hardy-Littlewood maximal function and its upper half-plane analogue for a function of f are respectively defined by

$$\mathcal{M}_{HL}(f)(x) := \sup_{I\subset\mathbb{R}} \frac{\chi_I(x)}{|I|}\int_I |f(t)|dt,\ \forall\, x \in \mathbb{R},$$

and

$$\mathcal{M}_{V_\alpha}(f)(z) := \sup_{I\subset\mathbb{R}} \frac{\chi_{Q_I}(z)}{|Q_I|_\alpha}\int_{Q_I} |f(\omega)|dV_\alpha(\omega),\ \forall\, z \in \mathbb{C}_+,$$

where the supremum is taken over all intervals of $\mathbb{R}$. Similarly, for $\beta \in \{0; 1/3\}$, we define their dyadic versions $\mathcal{M}_{HL}^{\mathcal{D}^\beta}(f)$ and $\mathcal{M}_{V_\alpha}^{\mathcal{D}^\beta}(f)$ as above but with the supremum taken this time on the intervals in the dyadic grid $\mathcal{D}^\beta$. We have

$$\mathcal{M}_{HL}(f) \le 6 \sum_{\beta\in\{0;1/3\}} \mathcal{M}_{HL}^{\mathcal{D}^\beta}(f) \tag{3.9}$$

and

$$\mathcal{M}_{V_\alpha}(f) \le 6^{2+\alpha} \sum_{\beta\in\{0;1/3\}} \mathcal{M}_{V_\alpha}^{\mathcal{D}^\beta}(f). \tag{3.10}$$

We have the following control of the level sets of the above maximal functions. It is needed in the characterization of Carleson measures and to prove boundedness of the above maximal functions.

Proposition 3.13. *Let $\beta \in \{0; 1/3\}$, $\alpha > -1$, $0 < \gamma < \infty$ and Φ a growth function. Put*

$$\Phi_\gamma(t) := \Phi(t^\gamma),\ \forall\, t \ge 0.$$

If Φ_γ is convex then the following assertions are satisfied

(i) *for all $0 \not\equiv f \in L^\Phi(\mathbb{R})$ and for $\lambda > 0$,*

$$\left|\left\{x \in \mathbb{R} : \left(\mathcal{M}_{HL}^{\mathcal{D}^\beta}\left(\left(\frac{|f|}{\|f\|_{L^\Phi}^{lux}}\right)^{1/\gamma}\right)(x)\right)^\gamma > \lambda\right\}\right| \le \frac{1}{\Phi(\lambda)}.$$

(ii) *for all $0 \not\equiv f \in L^\Phi(\mathbb{C}_+, dV_\alpha)$ and for $\lambda > 0$*

$$\left|\left\{z \in \mathbb{C}_+ : \left(\mathcal{M}_{V_\alpha}^{\mathcal{D}^\beta}\left(\left(\frac{|f|}{\|f\|_{L_\alpha^\Phi}^{lux}}\right)^{1/\gamma}\right)(z)\right)^\gamma > \lambda\right\}\right|_\alpha \le \frac{1}{\Phi(\lambda)}.$$

Proof. *i*) Let $0 \not\equiv f \in L^\Phi(\mathbb{R})$ and put

$$g := \frac{|f|^{1/\gamma}}{\left(\|f\|_{L^\Phi}^{lux}\right)^{1/\gamma}}.$$

We have

$$\int_{\mathbb{R}} \Phi_\gamma(|g(x)|)dx = \int_{\mathbb{R}} \Phi_\gamma\left(\left(\frac{|f(x)|}{\|f\|_{L^\Phi}^{lux}}\right)^{1/\gamma}\right)dx = \int_{\mathbb{R}} \Phi\left(\frac{|f(x)|}{\|f\|_{L^\Phi}^{lux}}\right)dx \leq 1.$$

We deduce that $g \in L^{\Phi_\gamma}(\mathbb{R})$ and $\|g\|_{L^{\Phi_\gamma}}^{lux} \leq 1$.
For $\lambda > 0$, we can therefore find $\{I_j\}_{j\in\mathbb{N}}$ a family of pairwise disjoint $\beta-$dyadic intervals such that

$$\left\{x \in \mathbb{R} : \mathcal{M}_{HL}^{\mathcal{D}^\beta}(g)(x) > \lambda^{1/\gamma}\right\} = \bigcup_{j\in\mathbb{N}} I_j,$$

and

$$\lambda^{1/\gamma} < \frac{1}{|I_j|}\int_{I_j} |g(y)|dy, \ \forall\, j \in \mathbb{N}.$$

For $j \in \mathbb{N}$, we have

$$\Phi(\lambda) = \Phi_\gamma\left(\lambda^{1/\gamma}\right) \leq \Phi_\gamma\left(\frac{1}{|I_j|}\int_{I_j} |g(y)|dy\right) \leq \frac{1}{|I_j|}\int_{I_j} \Phi_\gamma(|g(y)|)dy,$$

thanks to Jensen's inequality. We deduce that

$$|I_j| \leq \frac{1}{\Phi(\lambda)}\int_{I_j} \Phi_\gamma(|g(y)|)dy, \ \forall\, j \in \mathbb{N}.$$

It follows that

$$\begin{aligned}\left|\left\{x \in \mathbb{R} : \mathcal{M}_{HL}^{\mathcal{D}^\beta}(g)(x) > \lambda^{1/\gamma}\right\}\right| &= \sum_j |I_j| \\ &\leq \sum_j \frac{1}{\Phi(\lambda)}\int_{I_j} \Phi_\gamma(|g(y)|)dy \\ &= \frac{1}{\Phi(\lambda)}\int_{\bigcup_j I_j} \Phi_\gamma(|g(y)|)dy \leq \frac{1}{\Phi(\lambda)}.\end{aligned}$$

In the same way, we prove the inequality of the point (ii). □

Since it is possible to construct from any growth function, a convex growth function that is both of lower and upper type (see Corollary 3.5), the inequalities established in Proposition 3.13 also hold for concave growth functions.

We then have the following boundedness of the above maximal functions between different Orlicz spaces.

Theorem 3.14. *Let $\alpha > -1$ and $\Phi_1, \Phi_2 \in \mathscr{U}$. The following assertions are equivalent.*

(i) *There exists a constant $C_1 > 0$ such that for all $t > 0$,*

$$\int_0^t \frac{\Phi_2(s)}{s^2}ds \leq C_1\frac{\Phi_1(t)}{t}. \tag{3.11}$$

(ii) *There exists a constant* $C_2 > 0$ *such that for all* $f \in L^{\Phi_1}(\mathbb{R})$,

$$\|\mathcal{M}_{HL}(f)\|^{lux}_{L^{\Phi_2}} \le C_2 \|f\|^{lux}_{L^{\Phi_1}}. \tag{3.12}$$

(iii) *There exists a constant* $C_3 > 0$ *such that for all* $f \in L^{\Phi_1}(\mathbb{C}_+, dV_\alpha)$,

$$\|\mathcal{M}_{V_\alpha}(f)\|^{lux}_{L^{\Phi_2}_{V_\alpha}} \le C_3 \|f\|^{lux}_{L^{\Phi_1}_{V_\alpha}}. \tag{3.13}$$

Proof. $i) \Leftrightarrow ii)$ This equivalence follows from the [10, Lemma 3.15].
$(i) \Rightarrow (iii)$ The proof of this implication is identical to that of the [12, Proposition 3.12].

Let us show that (iii) implies (i). Assume that inequality (3.11) is not satisfied. We can find a sequence of positive reals $(t_k)_{k\ge1}$ such that

$$\int_0^{t_k} \frac{\Phi_2(s)}{s^2} ds \ge \frac{2^k \Phi_1(2^k t_k)}{t_k}, \ \forall\, k \ge 1. \tag{3.14}$$

For $k \ge 1$, put

$$f_k := 2^k t_k \chi_{Q_{I_k}},$$

where Q_{I_k} is the Carleson square associated with the interval I_k given as follows:

$$I_k := \left\{ x \in \mathbb{R} \,:\, \sum_{j=0}^{k-1} \left(\frac{\alpha+1}{2^j \Phi_1(2^j t_j)} \right)^{\frac{1}{\alpha+2}} \le x < \sum_{j=0}^{k} \left(\frac{\alpha+1}{2^j \Phi_1(2^j t_j)} \right)^{\frac{1}{\alpha+2}} \right\}$$

From (3.8), we have

$$|Q_{I_k}|_\alpha = \frac{1}{1+\alpha} |I_k|^{2+\alpha} = \frac{1}{2^k \Phi_1(2^k t_k)}.$$

It follows that $f_k \in L^{\Phi_1}(\mathbb{C}_+, dV_\alpha)$. Indeed

$$\int_{\mathbb{C}_+} \Phi_1(|f_k(z)|) dV_\alpha(z) = \int_{Q_{I_k}} \Phi_1(2^k t_k) dV_\alpha(z) = \Phi_1(2^k t_k)|Q_{I_k}|_\alpha = \frac{1}{2^k} < \infty.$$

Following Lemma 3.12, there exists a dyadic interval $J_k \in \mathcal{D}^\beta$ such that $I_k \subset J_k$ and $|J_k| \le 6|I_k|$. Let $z \in Q_{I_k}$. We have

$$|f_k(z)| = \frac{1}{|Q_{I_k}|_\alpha} \int_{Q_{I_k}} 2^k t_k \chi_{Q_{I_k}}(\omega) dV_\alpha(\omega) \le 6^{2+\alpha} \frac{\chi_{Q_{J_k}}(z)}{|Q_{J_k}|_\alpha} \int_{Q_{J_k}} |f_k(\omega)| dV_\alpha(\omega),$$

where Q_{J_k} is the Carleson square associated with J_k. We deduce that

$$|f_k(z)| \le 6^{2+\alpha} \mathcal{M}^{\mathcal{D}^\beta}_{V_\alpha}(f_k)(z), \ \forall\, z \in \mathbb{C}_+.$$

It follows that for $\lambda > 0$,

$$\frac{1}{\lambda} \int_{\{z \in \mathbb{C}_+ \,:\, |f_k(z)| > \lambda\}} |f_k(z)| dV_\alpha(z) \le 2^{2+\alpha} \left| \left\{ z \in \mathbb{C}_+ \,:\, \mathcal{M}^{\mathcal{D}^\beta}_{V_\alpha}(6^{2+\alpha} f_k)(z) > \lambda \right\} \right|_\alpha. \tag{3.15}$$

Put

$$f(z)=\sum_{k=1}^{\infty}6^{2+\alpha}f_k(z),\ \forall\, z\in\cup_{k\geq 1}Q_{I_k}\quad\text{and}\quad f(z)=0,\ \forall\, z\in\mathbb{C}_+\setminus\cup_{k\geq 1}Q_{I_k}.$$

Since the I_k are pairwise disjoint, the same is true for the Q_{I_k}. So we have

$$\begin{aligned}\int_{\mathbb{C}_+}\Phi_1(|f(z)|)dV_\alpha(z) &\lesssim \sum_{k=1}^{\infty}\int_{\mathbb{C}_+}\Phi_1(|f_k(z)|)dV_\alpha(z)\\ &= \sum_{k=1}^{\infty}\int_{Q_{I_k}}\Phi_1(2^k t_k)\chi_{Q_{I_k}}(z)dV_\alpha(z)\\ &= \sum_{k=1}^{\infty}\frac{1}{2^k}\\ &< \infty.\end{aligned}$$

We deduce that $f\in L^{\Phi_1}(\mathbb{C}_+,dV_\alpha)$.
Since the inequalities (3.14) and (3.15) are satisfied, we have

$$\begin{aligned}\int_{\mathbb{C}_+}\Phi_2\left(\mathcal{M}_{V_\alpha}(f)(z)\right)dV_\alpha(z) &\gtrsim \int_0^{\infty}\Phi_2'(\lambda)\left|\left\{z\in\mathbb{C}_+\,:\,\mathcal{M}_{V_\alpha}^{\mathcal{D}^\beta}(6^{2+\alpha}f_k)(z)>\lambda\right\}\right|_\alpha d\lambda\\ &\gtrsim \int_0^{\infty}\Phi_2'(\lambda)\left(\frac{1}{\lambda}\int_{\{\omega\in\mathbb{C}_+\,:\,|f_k(\omega)|>\lambda\}}|f_k(z)|dV_\alpha(z)\right)d\lambda\\ &\gtrsim \int_{\mathbb{C}_+}|f_k(z)|\left(\int_0^{|f_k(z)|}\frac{\Phi_2(\lambda)}{\lambda^2}d\lambda\right)dV_\alpha(z)\\ &\gtrsim 2^k t_k|Q_{I_k}|_\alpha\left(\int_0^{2^k t_k}\frac{\Phi_2(\lambda)}{\lambda^2}d\lambda\right)\\ &\gtrsim 2^k.\end{aligned}$$

We also deduce that $\mathcal{M}_{V_\alpha}(f)\notin L^{\Phi_2}(\mathbb{C}_+,dV_\alpha)$. □

We can derive the following (see also [15]).

Corollary 3.15. *Let $\alpha>-1$ and $\Phi\in\mathscr{U}$. The following assertions are equivalent.*

(i) $\Phi\in\nabla_2$.
(ii) $\mathcal{M}_{HL}\,:\,L^\Phi(\mathbb{R})\longrightarrow L^\Phi(\mathbb{R})$ *is bounded.*
(iii) $\mathcal{M}_{V_\alpha}\,:\,L^\Phi(\mathbb{C}_+,dV_\alpha)\longrightarrow L^\Phi(\mathbb{C}_+,dV_\alpha)$ *is bounded.*

The above results on the maximal functions $\mathcal{M}_{HL}$ and $\mathcal{M}_{V_\alpha}$ and the established results will be used in the proof of our main results in the last section.

3.3. Some properties of Hardy-Orlicz and Bergman-Orlicz spaces on $\mathbb{C}_+$. Several properties of Hardy-Orlicz and Bergman-Orlicz spaces are needed in our presentation. We give and prove them in this section.

Let Φ be a growth function and $F \in H^{\Phi}(\mathbb{C}_+)$. Put

$$\|F\|_{H^\Phi} := \sup_{y>0} \int_{\mathbb{R}} \Phi\left(|F(x+iy)|\right) dx.$$

Let $\Phi \in \mathscr{C}^1(\mathbb{R}_+)$ a growth function such that $0 < a_\Phi \le b_\Phi < \infty$. We have the following inequalities

$$\|F\|_{H^\Phi} \lesssim \max\left\{\left(\|F\|_{H^\Phi}^{lux}\right)^{a_\Phi}; \left(\|F\|_{H^\Phi}^{lux}\right)^{b_\Phi}\right\}$$

and

$$\|F\|_{H^\Phi}^{lux} \lesssim \max\left\{\left(\|F\|_{H^\Phi}\right)^{1/a_\Phi}; \left(\|F\|_{H^\Phi}\right)^{1/b_\Phi}\right\}.$$

Let Ω be an open set of $\mathbb{C}$ and $F : \Omega \longrightarrow]-\infty, +\infty]$ a function. We say that F is subharmonic if the following assertions are satisfied:

(i) F is upper semicontinuous on Ω

$$F(z_0) \ge \varlimsup_{z\to z_0} F(z),\ \forall\, z_0 \in \Omega,$$

(ii) for all $z_0 \in \Omega$, there exists $r(z_0) > 0$ such that $\mathcal{D}(z_0, r(z_0)) = \{z \in \Omega : |z - z_0| < r(z_0)\}$ is contained in Ω and such that for all $r < r(z_0)$

$$F(z_0) \le \frac{1}{\pi r^2} \int\int_{|x+iy-z_0|<r} F(x+iy)dxdy. \tag{3.16}$$

We have the following pointwise estimate of elements of Hardy-Orlicz spaces.

Proposition 3.16. *Let Φ be a growth function such that $\Phi(t) > 0$ for all $t > 0$. If Φ is convex or belongs to $\mathscr{L}$ then for $F \in H^{\Phi}(\mathbb{C}_+)$, we have*

$$|F(x+iy)| \le \Phi^{-1}\left(\frac{2}{\pi y}\right) \|F\|_{H^\Phi}^{lux},\ \forall\, x+iy \in \mathbb{C}_+. \tag{3.17}$$

Proof. For $t \ge 0$, put

$$\Phi_\rho(t) = \Phi\left(t^{1/\rho}\right),$$

where $\rho = 1$ if Φ is convex and $\rho = a_\Phi$ if $\Phi \in \mathscr{L}$. By construction, Φ_ρ is a convex growth function. Let $0 \not\equiv F \in H^{\Phi}(\mathbb{C}_+)$, and $z_0 = x_0 + iy_0 \in \mathbb{C}_+$ and $r = \frac{y_0}{2}$. Since $|F|^\rho$ is subharmonic on $\mathbb{C}_+$, we have

$$|F(z_0)|^\rho \le \frac{1}{\pi r^2} \int\int_{\overline{\mathcal{D}(z_0,r)}} |F(u+iv)|^\rho dudv,$$

where $\mathcal{D}(z_0, r)$ is the disk centered at z_0 and of radius r. By Jensen's inequality, it follows that

$$\begin{aligned}\Phi\left(\frac{|F(z_0)|}{\|F\|^{lux}_{H^\Phi}}\right) &\le \Phi_\rho\left(\frac{1}{\pi r^2}\int\int_{\mathcal{D}(z_0,r)}\left(\frac{|F(u+iv)|}{\|F\|^{lux}_{H^\Phi}}\right)^\rho dudv\right)\\ &\le \frac{1}{\pi r^2}\int\int_{\mathcal{D}(z_0,r)}\Phi\left(\frac{|F(u+iv)|}{\|F\|^{lux}_{H^\Phi}}\right)dudv\\ &\le \frac{1}{\pi r^2}\int_0^{2r}\int_{\mathbb{R}}\Phi\left(\frac{|F(u+iv)|}{\|F\|^{lux}_{H^\Phi}}\right)dudv \le \frac{1}{\pi r^2}\int_0^{2r} dv.\end{aligned}$$

We deduce that

$$\Phi\left(\frac{|F(z_0)|}{\|F\|^{lux}_{H^\Phi}}\right) \le \frac{2}{\pi r},\ \forall\, r < y_0.$$

□

The following will be used in Proposition 3.18 below, which extends to Hardy-Orlicz spaces, some known and useful characterizations of Hardy spaces.

Lemma 3.17. *Let Φ be a growth function such that $\Phi(t) > 0$ for all $t > 0$. If Φ is convex or belongs to $\mathscr{L}$, then for $F \in H^\Phi(\mathbb{C}_+)$ and for $\beta > 0$, we have*

$$\Phi(|F(z+i\beta)|) \le \frac{1}{\pi}\int_{\mathbb{R}}\frac{y}{(x-t)^2+y^2}\Phi(|F(t+i\beta)|)dt,\ \forall\, z = x+iy \in \mathbb{C}_+. \quad (3.18)$$

Proof. For $t \ge 0$, put

$$\Phi_\rho(t) = \Phi\left(t^{1/\rho}\right),$$

where $\rho = 1$ if Φ is convex and $\rho = a_\Phi$ if $\Phi \in \mathscr{L}$.
Let $0 \not\equiv F \in H^\Phi(\mathbb{C}_+)$ and $\beta > 0$. For $z \in \mathbb{C}_+$, put

$$U_\beta(z) = |F(z+i\beta)|^\rho.$$

By construction, U_β is continuous on $\overline{\mathbb{C}_+} := \mathbb{C}_+ \cup \mathbb{R}$ and subharmonic on $\mathbb{C}_+$. For $z = x+iy \in \overline{\mathbb{C}_+}$, we have

$$\begin{aligned}|U_\beta(z)| &= |F(x+i(y+\beta))|^\rho\\ &\le \left(\Phi^{-1}\left(\frac{2}{\pi(y+\beta)}\right)\|F\|^{lux}_{H^\Phi}\right)^\rho\\ &\le \left(\Phi^{-1}\left(\frac{2}{\pi\beta}\right)\|F\|^{lux}_{H^\Phi}\right)^\rho,\end{aligned}$$

according to Proposition 3.16. We deduce that U_β is bounded on $\overline{\mathbb{C}_+}$. It follows that

$$|F(z+i\beta)|^\rho \le \frac{1}{\pi}\int_{\mathbb{R}}\frac{y}{(x-t)^2+y^2}|F(t+i\beta)|^\rho dt,\ \forall\, z = x+iy \in \mathbb{C}_+,$$

thanks to [24, Corollary 10.15]. Since Φ_ρ is convex, by Jensen's inequality we deduce that

$$\Phi(|F(z+i\beta)|) \leq \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-t)^2+y^2}\Phi(|F(t+i\beta)|)dt,\ \forall\, z=x+iy\in\mathbb{C}_+.$$

□

We have the following equivalent definition of Hardy-Orlicz spaces.

Proposition 3.18. *Let Φ be a growth function such that $\Phi(t)>0$ for all $t>0$ and F an analytic function on $\mathbb{C}_+$. If Φ is convex or belongs to $\mathscr{L}$, then the following assertions are equivalent.*

(i) $F\in H^\Phi(\mathbb{C}_+)$.

(ii) *The function $y\mapsto \|F(.+iy)\|_{L^\Phi}^{lux}$ is non-increasing on $\mathbb{R}_+^*$ and*

$$\lim_{y\to 0}\|F(.+iy)\|_{L^\Phi}^{lux}<\infty.$$

Moreover,

$$\|F\|_{H^\Phi}^{lux}=\lim_{y\to 0}\|F(.+iy)\|_{L^\Phi}^{lux}.$$

Proof. The implication $(ii)\Rightarrow(i)$ is immediate.
Let us now show that (i) implies (ii). Suppose that $F\not\equiv 0$ is non-identically zero because there is nothing to show when $F\equiv 0$. Let $0<y_1<y_2$. According to Lemma 3.17 and Fubbini's theorem, we have

$$\begin{aligned}
L &= \int_{\mathbb{R}}\Phi\left(\frac{|F(x+iy_2)|}{\|F(.+iy_1)\|_{L^\Phi}^{lux}}\right)dx\\
&= \int_{\mathbb{R}}\Phi\left(\frac{|F(x+i(y_2-y_1)+iy_1)|}{\|F(.+iy_1)\|_{L^\Phi}^{lux}}\right)dx\\
&\leq \int_{\mathbb{R}}\frac{1}{\pi}\int_{\mathbb{R}}\frac{(y_2-y_1)}{(x-t)^2+(y_2-y_1)^2}\Phi\left(\frac{|F(t+iy_1)|}{\|F(.+iy_1)\|_{L^\Phi}^{lux}}\right)dtdx\\
&= \int_{\mathbb{R}}\Phi\left(\frac{|F(t+iy_1)|}{\|F(.+iy_1)\|_{L^\Phi}^{lux}}\right)\left(\frac{1}{\pi}\int_{\mathbb{R}}\frac{(y_2-y_1)}{(x-t)^2+(y_2-y_1)^2}dx\right)dt\\
&= \int_{\mathbb{R}}\Phi\left(\frac{|F(t+iy_1)|}{\|F(.+iy_1)\|_{L^\Phi}^{lux}}\right)dt\leq 1.
\end{aligned}$$

We deduce that $\|F(.+iy_2)\|_{L^\Phi}^{lux}\leq\|F(.+iy_1)\|_{L^\Phi}^{lux}$. Therefore,

$$\sup_{y>0}\|F(.+iy)\|_{L^\Phi}^{lux}=\lim_{y\to 0}\|F(.+iy)\|_{L^\Phi}^{lux}.$$

□

Let Φ be a growth function. The Hardy space on $\mathbb{D}$, $H^{\Phi}(\mathbb{D})$ is the set of analytic function G on $\mathbb{D}$ which satisfy

$$\|G\|_{H^{\Phi}(\mathbb{D})}^{lux} := \sup_{0\leq r<1} \inf\left\{\lambda > 0 : \frac{1}{2\pi}\int_0^{2\pi} \Phi\left(\frac{|G(re^{i\theta})|}{\lambda}\right) d\theta \leq 1\right\} < \infty.$$

Let Φ be a growth function. If Φ is convex or belongs to $\mathscr{L}$, then for some $\rho \in \{1; a_{\Phi}\}$,

$$H^{\Phi}(\mathbb{D}) \subseteq H^{\rho}(\mathbb{D}). \tag{3.19}$$

The proof of the following result is identical to that of [10, Theorem 3.11]. Therefore, the proof will be omitted.

Theorem 3.19. *Let Φ be a growth function such that $\Phi(t) > 0$ for all $t > 0$. If Φ is convex or belongs to $\mathscr{L}$, then for $F \in H^{\Phi}(\mathbb{C}_+)$, the function G defined by*

$$G(\omega) = F\left(i\frac{1-\omega}{1+\omega}\right), \ \forall\, \omega \in \mathbb{D},$$

is in $H^{\Phi}(\mathbb{D})$. Moreover,

$$\|G\|_{H^{\Phi}(\mathbb{D})}^{lux} \leq \|F\|_{H^{\Phi}(\mathbb{C}_+)}^{lux}.$$

Denote by B the function beta defined by

$$B(m,n) = \int_0^{\infty} \frac{u^{m-1}}{(1+u)^{m+n}} du, \ \forall\, m,n > 0.$$

The following two results can be found for example, in [1].

Lemma 3.20. *Let $y > 0$ and $\alpha \in \mathbb{R}$. The integral*

$$J_{\alpha}(y) = \int_{\mathbb{R}} \frac{dx}{|x+iy|^{\alpha}},$$

converges if and only if $\alpha > 1$. In this case,

$$J_{\alpha}(y) = B\left(\frac{1}{2}, \frac{\alpha-1}{2}\right) y^{1-\alpha}.$$

Lemma 3.21. *Let $\alpha, \beta \in \mathbb{R}$ and $t > 0$. The integral*

$$I(t) = \int_0^{\infty} \frac{y^{\alpha}}{(t+y)^{\beta}} dy, \tag{3.20}$$

converges if and only if $\alpha > -1$ and $\beta > \alpha + 1$. In this case,

$$I(t) = B(1+\alpha, \beta-\alpha-1)t^{-\beta+\alpha+1}. \tag{3.21}$$

The Nevanlinna's class on $\mathbb{C}_+$, $\mathscr{N}(\mathbb{C}_+)$ is the set of holomorphic functions F on $\mathbb{C}_+$ such that

$$\sup_{y>0} \int_{\mathbb{R}} \log\left(1 + |F(x+iy)|\right) dx < \infty.$$

For $0 \not\equiv F \in \mathcal{N}(\mathbb{C}_+)$, there exists a unique function f measurable on $\mathbb{R}$ such that $\log|f| \in L^1\left(\mathbb{R}, \frac{dt}{1+t^2}\right)$ and

$$\lim_{y\to 0} F(x+iy) = f(x),$$

for almost all $x \in \mathbb{R}$, (see [25]).

We have the following embedding relations between the Nevanlinna class and Hardy-Orlicz spaces.

Proposition 3.22. *Let $\Phi \in \mathscr{C}^1(\mathbb{R}_+)$ be a growth function such that $0 < a_\Phi \leq b_\Phi < \infty$. The following assertions are satisfied.*

(i) *If $0 < a_\Phi \leq b_\Phi \leq 1$, then $H^\Phi(\mathbb{C}_+) \subset \mathcal{N}(\mathbb{C}_+)$.*

(ii) *If $1 < a_\Phi \leq b_\Phi < \infty$, then $H^\Phi(\mathbb{C}_+) \not\subset \mathcal{N}(\mathbb{C}_+)$.*

Proof. (i) For $0 \not\equiv F \in H^\Phi(\mathbb{C}_+)$, put

$$F_1 = F\chi_{0<\{|F|\leq 1\}} \quad \text{and} \quad F_2 = F\chi_{\{|F|\geq 1\}}.$$

For $z \in \mathbb{C}_+$, we have

$$\log(1+|F_1(z)|) \leq |F_1(z)| \leq |F_1(z)|^{b_\Phi} \leq \frac{1}{\Phi(1)} \times \Phi(|F_1(z)|)$$

and

$$\log(1+|F_2(z)|) = \frac{1}{a_\Phi}\log(1+|F_2(z)|)^{a_\Phi} \leq \frac{2^{a_\Phi}}{a_\Phi}|F_2(z)|^{a_\Phi} \leq \frac{2^{a_\Phi}}{a_\Phi}\frac{1}{\Phi(1)} \times \Phi(|F_2(z)|),$$

since the function $t \mapsto \frac{\Phi(t)}{t^{a_\Phi}}$ (resp. $t \mapsto \frac{\Phi(t)}{t^{b_\Phi}}$) is non-decreasing (resp. non-increasing) on $\mathbb{R}_+^*$. Using the sub-additivity of the logarithmic function on $(1,\infty)$, we deduce that

$$\log(1+|F(z)|) \lesssim \log(1+|F_1(z)|+|F_2(z)|) \lesssim (\Phi(|F_1(z)|)+\Phi(|F_2(z)|)).$$

It follows that $F \in \mathcal{N}(\mathbb{C}_+)$. Indeed, for $y > 0$, we have

$$\begin{aligned}\int_{\mathbb{R}} \log(1+|F(x+iy)|)dx &\lesssim \int_{\mathbb{R}} \Phi(|F_1(x+iy)|)dx + \int_{\mathbb{R}} \Phi(|F_2(x+iy)|)dx \\ &\lesssim \sup_{y>0}\int_{\mathbb{R}} \Phi(|F(x+iy)|)dx < \infty.\end{aligned}$$

(ii) Let $\alpha \in \mathbb{R}$ such that $1/a_\Phi < \alpha < 1$. For $z \in \mathbb{C}_+$, put

$$F_\alpha(z) = \frac{1}{(z+i)^\alpha}.$$

By construction, F_α is an analytic function on $\mathbb{C}_+$ and

$$|F_\alpha(z)| = \frac{1}{|x+i(1+y)|^\alpha} < 1, \ \forall\, z = x+iy \in \mathbb{C}_+.$$

We deduce that

$$\log(1+|F_\alpha(z)|) \geq \frac{1}{2}\frac{1}{|x+i(1+y)|^\alpha}, \ \forall\, z = x+iy \in \mathbb{C}_+$$

and

$$\Phi\left(|F_\alpha(z)|\right) \leq \Phi(1)\frac{1}{|x+i(1+y)|^{\alpha a_\Phi}},\ \forall\, z = x+iy \in \mathbb{C}_+,$$

since $|F_\alpha| < 1$ and the function $t \mapsto \frac{\Phi(t)}{t^{a_\Phi}}$ is non-decreasing on $\mathbb{R}_+^*$. It follows that $F_\alpha \in H^\Phi(\mathbb{C}_+)$ and $F_\alpha \notin \mathscr{N}(\mathbb{C}_+)$. Indeed, for $y > 0$, we have

$$\int_{\mathbb{R}} \Phi\left(|F_\alpha(x+iy)|\right) dx \lesssim B\left(\frac{1}{2}, \frac{\alpha a_\Phi - 1}{2}\right)(1+y)^{1-\alpha a_\Phi} \leq B\left(\frac{1}{2}, \frac{\alpha a_\Phi - 1}{2}\right) < +\infty$$

and

$$\int_{\mathbb{R}} \log\left(1 + |F_\alpha(x+iy)|\right) dx \geq \frac{1}{2}\int_{\mathbb{R}} \frac{dx}{|x+i(1+y)|^\alpha} = +\infty,$$

according to Lemma 3.20. □

The above embedding of the Hardy-Orlicz spaces $H^\Phi(\mathbb{C}_+)$ into the Nevanlinna's class will allow us to apply properties of elements in the Nevanlinna's class to functions in $H^\Phi(\mathbb{C}_+)$.

Let f be a measurable function on $\mathbb{R}$. The Poisson integral U_f of f is the function defined by

$$U_f(x+iy) := \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-t)^2+y^2} f(t)dt,\ \forall\, x+iy \in \mathbb{C}_+,$$

when it makes sense.
If $f \in L^1\left(\mathbb{R}, \frac{dt}{1+t^2}\right)$ then U_f is a harmonic function on $\mathbb{C}_+$ and

$$\lim_{y\to 0} U_f(x+iy) = f(x),$$

for almost all $x \in \mathbb{R}$ (see [24]).

The following result gives a representation of functions in Hardy-Orlicz spaces in terms of the Poisson integral.

Lemma 3.23 (Lemma 4.1, [10]). *Let Φ be a convex growth function such that $\Phi(t) > 0$ for all $t > 0$ and $0 \not\equiv F$ an analytic function on $\mathbb{C}_+$. The following assertions are equivalent.*

(i) $F \in H^\Phi(\mathbb{C}_+)$.

(ii) *There exists a unique function $f \in L^\Phi(\mathbb{R})$ such that* $\log|f| \in L^1\left(\mathbb{R}, \frac{dt}{1+t^2}\right)$ *and*

$$F(x+iy) = U_f(x+iy),\ \forall\, x+iy \in \mathbb{C}_+.$$

Moreover,

$$\|F\|_{H^\Phi}^{lux} = \lim_{y\to 0}\|F(.+iy)\|_{L^\Phi}^{lux} = \|f\|_{L^\Phi}^{lux}.$$

We next show the existence of a radial limit for functions in a Hardy-Orlicz space.

Theorem 3.24. *Let Φ be a growth function such that $\Phi(t) > 0$ for all $t > 0$. If Φ is convex or belongs to $\mathscr{L}$, then for $0 \not\equiv F \in H^{\Phi}(\mathbb{C}_+)$, there exists a unique function $f \in L^{\Phi}(\mathbb{R})$ such that $\log|f| \in L^1\left(\mathbb{R}, \frac{dt}{1+t^2}\right)$,*

$$f(x) = \lim_{y\to 0} F(x+iy),$$

for almost all $x \in \mathbb{R}$, $f(t) \neq 0$ for almost all $t \in \mathbb{R}$,

$$\log|F(x+iy)| \leq \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-t)^2+y^2}\log|f(t)|dt,\ \forall\, x+iy \in \mathbb{C}_+$$

and

$$\|F\|_{H^{\Phi}}^{lux} = \lim_{y\to 0}\|F(.+iy)\|_{L^{\Phi}}^{lux} = \|f\|_{L^{\Phi}}^{lux}. \tag{3.22}$$

Proof. Let $0 \not\equiv F \in H^{\Phi}(\mathbb{C}_+)$. There exists a unique measurable function f on $\mathbb{R}$ such that $\log|f| \in L^1\left(\mathbb{R}, \frac{dt}{1+t^2}\right)$ and

$$\lim_{y\to 0} F(x+iy) = f(x),$$

for almost all $x \in \mathbb{R}$, according to point (i) of Proposition 3.22 and Lemma 3.23. Suppose that there exists A a measurable subset of $\mathbb{R}$ with Lebesgue measure $|A| > 0$, and

$$f(x) = 0,\ \forall\, x \in A.$$

We have

$$+\infty = \int_A |\log|f(t)||\frac{dt}{1+t^2} \leq \int_{\mathbb{R}} |\log|f(t)||\frac{dt}{1+t^2}.$$

We deduce that $\log|f| \notin L^1\left(\mathbb{R}, \frac{dt}{1+t^2}\right)$. Which is absurd. Hence, $f(t) \neq 0$, for almost all $t \in \mathbb{R}$. For $\omega \in \mathbb{D}$, put

$$G(\omega) = F\left(i\frac{1-\omega}{1+\omega}\right).$$

Since $G \in H^{\Phi}(\mathbb{D}) \subset H^p(\mathbb{D})$, with $p > 0$, there exists a unique function $g \in L^{\Phi}(\mathbb{T})$ such that $\log|g| \in L^1(\mathbb{T})$ and

$$\lim_{r\to 1} G(re^{i\theta}) = g(e^{i\theta}),$$

for almost all $\theta \in \mathbb{R}$ and

$$\log|G(re^{i\theta})| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(u-\theta)+r^2}\log|g(e^{iu})|du,\ \forall\, re^{i\theta} \in \mathbb{D}.$$

Moreover,

$$\log|g(e^{i\theta})| = \lim_{r\to 1}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(u-\theta)+r^2}\log|g(e^{iu})|du\right), \tag{3.23}$$

for almost all $\theta \in \mathbb{R}$.
Consider φ, the map defined by

$$\varphi(\omega) = i\frac{1-\omega}{1+\omega},\ \forall\, \omega \in \mathbb{D} \cup \mathbb{T}\backslash\{-1\},$$

where $\mathbb{T}$ is the complex unit circle. Note that the restriction of φ to $\mathbb{D}$ (resp. $\mathbb{T}\backslash\{-1\}$) is an analytic function on $\mathbb{D}$ with values in $\mathbb{C}_+$ (resp. a homeomorphism from $\mathbb{T}\backslash\{-1\}$ onto $\mathbb{R}$).
For $z = x + iy \in \mathbb{C}_+$ and $\omega = re^{iu} \in \mathbb{D}$ such that $z = i\frac{1-\omega}{1+\omega}$, using

$$y = \frac{1-r^2}{1+r^2+2r\cos u}$$

and (3.23), we deduce that

$$|f(x)| = |g\circ\varphi^{-1}(x)|,$$

for almost all $x \in \mathbb{R}$. Therefore,

$$\log|F(x+iy)| \le \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-t)^2+y^2}\log|f(t)|dt,\ \forall\, x+iy \in \mathbb{C}_+. \tag{3.24}$$

Indeed,

$$\begin{aligned}\log|F(x+iy)| &= \log|G(re^{iu})| \\ &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(u-\theta)+r^2}\log|g(e^{i\theta})|d\theta \\ &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-t)^2+y^2}\log|g\circ\varphi^{-1}(t)|dt \\ &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-t)^2+y^2}\log|f(t)|dt.\end{aligned}$$

Let us prove (3.22). By Fatou's lemma, we have

$$\begin{aligned}\int_{\mathbb{R}} \Phi\left(\frac{|f(x)|}{\|F\|_{H^\Phi}^{lux}}\right)dx &\le \liminf_{y\to 0}\int_{\mathbb{R}} \Phi\left(\frac{|F(x+iy)|}{\|F\|_{H^\Phi}^{lux}}\right)dx \\ &\le \sup_{y>0}\int_{\mathbb{R}} \Phi\left(\frac{|F(x+iy)|}{\|F\|_{H^\Phi}^{lux}}\right)dx \le 1.\end{aligned}$$

We deduce that $f \in L^\Phi(\mathbb{R})$ and

$$\|f\|_{L^\Phi}^{lux} \le \|F\|_{H^\Phi}^{lux}. \tag{3.25}$$

Put

$$\Phi_\rho(t) = \Phi\left(t^{1/\rho}\right),\ \forall\, t \ge 0,$$

where $\rho = 1$ if Φ is convex and $\rho = a_\Phi$ if $\Phi \in \mathscr{L}$.
Using Jensen's inequality and (3.24), we deduce that

$$|F(x+iy)|^\rho \le \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-t)^2+y^2}|f(t)|^\rho dt,\ \forall\, x+iy \in \mathbb{C}_+.$$

Fix $y > 0$. We have

$$\begin{aligned}\int_{\mathbb{R}} \Phi\left(\frac{|F(x+iy)|}{\|f\|_{L^\Phi}^{lux}}\right)dx &\le \int_{\mathbb{R}} \Phi_\rho\left(\frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-t)^2+y^2}\left(\frac{|f(t)|}{\|f\|_{L^\Phi}^{lux}}\right)^\rho dt\right)dx\\ &\le \int_{\mathbb{R}} \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-t)^2+y^2}\Phi_\rho\left(\left(\frac{|f(t)|}{\|f\|_{L^\Phi}^{lux}}\right)^\rho\right)dtdx\\ &= \int_{\mathbb{R}} \Phi\left(\frac{|f(t)|}{\|f\|_{L^\Phi}^{lux}}\right)\left(\frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-t)^2+y^2}dx\right)dt\\ &= \int_{\mathbb{R}} \Phi\left(\frac{|f(t)|}{\|f\|_{L^\Phi}^{lux}}\right)dt \le 1.\end{aligned}$$

We deduce that

$$\|F\|_{H^\Phi}^{lux} \le \|f\|_{L^\Phi}^{lux}. \tag{3.26}$$

From (3.25) and (3.26) and also from Proposition 3.18, it follows that

$$\|F\|_{H^\Phi}^{lux} = \lim_{y\to 0}\|F(.+iy)\|_{L^\Phi}^{lux} = \|f\|_{L^\Phi}^{lux}.$$

□

We have the following pointwise estimate of functions in a Bergman-Orlicz space. It will be used in the characterization of pointwise multipliers from Hardy-Orlicz or Bergman-Orlicz spaces to Bergman-Orlicz spaces.

Lemma 3.25. *Let $\alpha > -1$ and Φ a one-to-one growth function. If Φ is convex or belongs to $\mathscr{L}$, then there exists a constant $C := C_{\alpha,\Phi} > 1$ such that for $F \in A_\alpha^\Phi(\mathbb{C}_+)$,*

$$|F(x+iy)| \le C\Phi^{-1}\left(\frac{1}{y^{2+\alpha}}\right)\|F\|_{A_\alpha^\Phi}^{lux},\ \forall\, x+iy \in \mathbb{C}_+. \tag{3.27}$$

Proof. For $t \ge 0$, put

$$\Phi_\rho(t) = \Phi\left(t^{1/\rho}\right),$$

where $\rho = 1$ if Φ is convex and $\rho = a_\Phi$ if $\Phi \in \mathscr{L}$.

Let $0 \not\equiv F \in A_\alpha^\Phi(\mathbb{C}_+)$. Fix $z_0 = x_0 + iy_0 \in \mathbb{C}_+$ and put $r = \frac{y_0}{2}$. Since $|F|^\rho$ is subharmonic on $\mathbb{C}_+$, we have

$$|F(z_0)|^\rho \le \frac{1}{\pi r^2}\int\int_{\mathcal{D}(z_0,r)} |F(u+iv)|^\rho dudv.$$

For $u + iv \in \overline{\mathcal{D}(z_0,r)}$, we have

$$r \le v \le 3r \Rightarrow 0 < \frac{1}{v^\alpha} \le 2^\alpha \times \frac{1}{y_0^\alpha},\ \text{if } \alpha \ge 0$$

and

$$0 < \frac{1}{v^\alpha} \le \left(\frac{2}{3}\right)^\alpha \times \frac{1}{y_0^\alpha},\ \text{if } -1 < \alpha < 0.$$

We deduce that

$$0 < \frac{1}{v^\alpha} \leq C_\alpha \frac{1}{y_0^\alpha},\ \forall\, u+iv \in \overline{\mathcal{D}(z_0,r)}, \tag{3.28}$$

where $C_\alpha := \max\left\{2^\alpha; (2/3)^\alpha\right\}$. By Jensen's inequality, we have

$$\begin{aligned}\Phi\left(\left(\frac{\pi}{4C_\alpha}\right)^{1/\rho} \times \frac{|F(z_0)|}{\|F\|_{A_\alpha^\Phi}^{lux}}\right) &\leq \frac{\pi}{4C_\alpha}\Phi_\rho\left(\frac{1}{\pi r^2}\int\int_{\overline{\mathcal{D}(z_0,r)}}\left(\frac{|F(u+iv)|}{\|F\|_{A_\alpha^\Phi}^{lux}}\right)^\rho dudv\right)\\ &\leq \frac{\pi}{4C_\alpha} \times \frac{4}{\pi y_0^2} \times \frac{C_\alpha}{y_0^\alpha}\int\int_{\overline{\mathcal{D}(z_0,r)}} \Phi\left(\frac{|F(u+iv)|}{\|F\|_{A_\alpha^\Phi}^{lux}}\right) v^\alpha dudv\\ &\leq \frac{1}{y_0^{2+\alpha}}\int_{\mathbb{C}_+}\Phi\left(\frac{|F(u+iv)|}{\|F\|_{A_\alpha^\Phi}^{lux}}\right)dV_\alpha(u+iv) \leq \frac{1}{y_0^{2+\alpha}}.\end{aligned}$$

We deduce that

$$|F(z_0)| \leq \left(\frac{4C_\alpha}{\pi}\right)^{1/\rho}\Phi^{-1}\left(\frac{1}{y_0^{2+\alpha}}\right)\|F\|_{A_\alpha^\Phi}^{lux}.$$

□

We will need the following result to find equivalent definitions of Carleson measures.

Proposition 3.26. *Let $\alpha > -1$. There exist $C := C_\alpha > 0$ and $\beta \in \{0, 1/3\}$ such that for any analytic function F on $\mathbb{C}_+$ and for all $0 < \gamma < \infty$,*

$$|F(z)|^\gamma \leq C\mathcal{M}_{V_\alpha}^{\mathcal{D}^\beta}(|F|^\gamma)(z),\ \forall\, z \in \mathbb{C}_+. \tag{3.29}$$

Proof. Let $0 < \gamma < \infty$ and $0 \not\equiv F$ be an analytic function on $\mathbb{C}_+$. Fix $z_0 = x_0 + iy_0 \in \mathbb{C}_+$ and $r = \dfrac{y_0}{2}$. From (3.28), we have

$$0 < \frac{1}{v^\alpha} \leq \max\left\{2^\alpha; (2/3)^\alpha\right\}\frac{1}{y_0^\alpha},\ \forall\, u+iv \in \overline{\mathcal{D}(z_0,r)}.$$

Let I be an interval centered at x_0 and of length $|I| = 2y_0$. Consider Q_I the Carleson square associated with I. According to Lemma 3.12, there exist $\beta \in \{0, 1/3\}$ and $J \in \mathcal{D}^\beta$ such that $I \subset J$ and $|J| \leq 6|I|$. From Relation (3.8) we have

$$|Q_J|_\alpha = \frac{1}{1+\alpha}|J|^{2+\alpha} \leq \frac{6^{2+\alpha}}{1+\alpha}|I|^{2+\alpha} = \frac{12^{2+\alpha}}{1+\alpha}y_0^{2+\alpha}.$$

Since $|F|^{\gamma}$ is subharmonic on $\mathbb{C}_+$ and $\overline{\mathcal{D}(z_0,r)}$ is contained in Q_I we have

$$\begin{aligned}|F(z_0)|^{\gamma} &\leq \frac{1}{\pi r^2}\int\int_{\overline{\mathcal{D}(z_0,r)}}|F(u+iv)|^{\gamma}dudv\\ &\leq \frac{4}{\pi y_0^2}\times\frac{\max\left\{2^{\alpha};(2/3)^{\alpha}\right\}}{y_0^{\alpha}}\int\int_{\overline{\mathcal{D}(z_0,r)}}|F(u+iv)|^{\gamma}v^{\alpha}dudv\\ &\leq C_{\alpha}\frac{\chi_{Q_J}(z_0)}{|Q_J|_{\alpha}}\int\int_{Q_J}|F(u+iv)|^{\gamma}v^{\alpha}dudv \leq C_{\alpha}\mathcal{M}_{V_{\alpha}}^{\mathcal{D}^{\beta}}\left(|F|^{\gamma}\right)(z_0),\end{aligned}$$

where $C_{\alpha} := \frac{4}{\pi}\times\frac{12^{2+\alpha}}{1+\alpha}\times\max\left\{2^{\alpha};(2/3)^{\alpha}\right\}$. □

We give below some test functions that will be used in the proof of Theorem 2.5.

Proposition 3.27. *Let $\alpha > -1$ and Φ a one-to-one growth function. If Φ is convex or belongs to $\mathscr{L}$ then there exists some constants $\rho \in \{1; a_{\Phi}\}$ and*

$$C_{\alpha} := B(1+\alpha, 2+\alpha)\,B\left(\frac{1}{2},\frac{3+2\alpha}{2}\right), \tag{3.30}$$

such that for all $z = x+iy \in \mathbb{C}_+$ the functions F_z and G_z defined respectively by

$$F_z(\omega) = \Phi^{-1}\left(\frac{1}{\pi y}\right)\frac{y^{2/\rho}}{(\omega-\overline{z})^{2/\rho}},\ \forall\,\omega\in\mathbb{C}_+ \tag{3.31}$$

and

$$G_z(\omega) = \Phi^{-1}\left(\frac{1}{C_{\alpha}y^{2+\alpha}}\right)\frac{y^{(4+2\alpha)/\rho}}{(\omega-\overline{z})^{(4+2\alpha)/\rho}},\ \forall\,\omega\in\mathbb{C}_+, \tag{3.32}$$

are analytic functions belonging respectively to $H^{\Phi}(\mathbb{C}_+)$ and $A_{\alpha}^{\Phi}(\mathbb{C}_+)$. Moreover, $\|F_z\|_{H^{\Phi}}^{lux} \leq 1$ and $\|G_z\|_{A_{\alpha}^{\Phi}}^{lux} \leq 1$.

Proof. Fix $z = x+iy \in \mathbb{C}_+$. By construction F_z ad G_z are analytic functions which does not vanish on $\mathbb{C}_+$. For $\omega = u+iv \in \mathbb{C}_+$, we have

$$\frac{y^2}{|(u-x)+i(y+v)|^2} \leq 1.$$

Put $\rho = 1$ if Φ is convex and $\rho = a_{\Phi}$ if $\Phi \in \mathscr{L}$, and

$$C_{\alpha} := B(1+\alpha, 2+\alpha)\,B\left(\frac{1}{2},\frac{3+2\alpha}{2}\right).$$

Since the function $t \mapsto \frac{\Phi(t)}{t^{\rho}}$ is non-decreasing on $\mathbb{R}_+^*$, we deduce that

$$\int_{\mathbb{R}}\Phi\left(|F_z(u+iv)|\right)du \lesssim \frac{y}{\pi}\int_{\mathbb{R}}\frac{1}{|(u-x)+i(y+v)|^2}du$$

and

$$\int_{\mathbb{C}_+}\Phi(|G_z(\omega)|)dV_{\alpha}(\omega) \lesssim \frac{y^{2+\alpha}}{C_{\alpha}}\int_0^{\infty}\left(\int_{\mathbb{R}}\frac{du}{|(u-x)+i(v+y)|^{4+2\alpha}}\right)v^{\alpha}dv.$$

Following Lemma 3.20, we have

$$\int_{\mathbb{R}} \frac{1}{|(u-x)+i(y+v)|^2} du = B\left(\frac{1}{2}, \frac{1}{2}\right) \frac{1}{y+v}$$

and

$$\int_{\mathbb{R}} \frac{du}{|(u-x)+i(v+y)|^{4+2\alpha}} = B\left(\frac{1}{2}, \frac{3+2\alpha}{2}\right) \frac{1}{(v+y)^{3+2\alpha}}.$$

We deduce that

$$\int_{\mathbb{R}} \Phi\left(|F_z(u+iv)|\right) du \lesssim 1, \ \forall\, v > 0$$

and

$$\int_{\mathbb{C}_+} \Phi(|G_z(\omega)|) dV_\alpha(\omega) \lesssim 1,$$

since

$$\int_0^\infty \frac{v^\alpha}{(y+v)^{3+2\alpha}} dv = B(1+\alpha, 2+\alpha) \frac{1}{y^{2+\alpha}},$$

thanks to Lemma 3.21. Therefore, $F_z \in H^\Phi(\mathbb{C}_+)$ with $\|F_z\|_{H^\Phi}^{lux} \leq 1$ and $G_z \in A_\alpha^\Phi(\mathbb{C}_+)$ with $\|G_z\|_{A_\alpha^\Phi}^{lux} \leq 1$. □

4. Some characterizations of Carleson measures.

In this section, we give among others, a general characterization of a (s, Φ)-Carleson measure. We start with the following elementary result whose proof is left to the interested reader.

Proposition 4.1. *Let $s > 0$, $\alpha > -1$ and Φ_1, Φ_2 be two one-to-one growth functions. The following assertions are equivalent.*

(i) *V_α is a $(s, \Phi_2 \circ \Phi_1^{-1})-$Carleson measure.*

(ii) *There exists a constant $C > 0$ such that for all $t > 0$*

$$\Phi_1^{-1}(t^s) \leq \Phi_2^{-1}(Ct^{2+\alpha}). \tag{4.1}$$

We have the following example of a $(s, \Phi)-$Carleson measure.

Proposition 4.2. *Let $s \geq 1$ and $\Phi \in \mathscr{U}$. Put*

$$d\mu(x+iy) = \frac{dxdy}{y^2 \Phi\left(\frac{1}{y^s}\right)}, \ \forall\, x+iy \in \mathbb{C}_+.$$

If $\Phi \in \nabla_2$, then μ is a $(s, \Phi)-$Carleson measure. In particular, the converse is true for $s = 1$.

Proof. Put

$$\widetilde{\Omega}(t) = \frac{1}{\Phi\left(\frac{1}{t}\right)}, \ \forall\, t > 0 \quad \text{and} \quad \widetilde{\Omega}(0) = 0.$$

According to Proposition 3.9, $\Omega \in \mathscr{U} \cap \nabla_2$.
Let I be an interval of nonzero length and Q_I the Carleson square associated with I. We have

$$\mu(Q_I) = \int_0^{|I|}\int_I \frac{\widetilde{\Omega}(y^s)}{y^2}dxdy = |I|\int_0^{|I|}\frac{\widetilde{\Omega}(y^s)}{y^{2s}}y^{s-1}y^{s-1}dy$$
$$\leq s^{-1}|I|^s\int_0^{|I|^s}\frac{\widetilde{\Omega}(y)}{y^2}dy \leq s^{-1}|I|^s C\frac{\widetilde{\Omega}(|I|^s)}{|I|^s} = \frac{C/s}{\Phi\left(\frac{1}{|I|^s}\right)},$$

thanks to Lemma 3.3. In particular, for $s = 1$, we have

$$\mu(Q_I) \lesssim \widetilde{\Omega}(|I|) \Leftrightarrow \int_0^{|I|}\frac{\widetilde{\Omega}(y)}{y^2}dy \lesssim \frac{\widetilde{\Omega}(|I|)}{|I|}.$$

Hence by Lemma 3.3 and Proposition 3.9, $\Phi \in \mathscr{U} \cap \nabla_2$. □

We have the following equivalent characterizations of Carleson measures. They will be used in the proof our main results.

Lemma 4.3. *Let $\alpha > -1$, $\Phi \in \mathscr{U}$ and μ be a positive Borel measure on $\mathbb{C}_+$. Put*

$$\widetilde{\Omega}(t) = \frac{1}{\Phi\left(\frac{1}{t}\right)},\ \forall\, t > 0 \quad \textit{and} \quad \widetilde{\Omega}(0) = 0.$$

The following assertions are satisfied

(i) *μ is a $\Phi-$Carleson measure if and only if there exists a constant $C_1 > 0$ such that for all $f \in L^1\left(\mathbb{R}, \frac{dt}{1+t^2}\right)$ and any $\lambda > 0$,*

$$\mu\left(\{z \in \mathbb{C}_+ : |U_f(z)| > \lambda\}\right) \leq C_1\widetilde{\Omega}\left(|\{x \in \mathbb{R} : \mathcal{M}_{HL}(f)(x) > \lambda\}|\right), \qquad (4.2)$$

where U_f is the Poisson integral of f.

(ii) *μ is a $(\alpha, \Phi)-$Carleson measure if and only if there exists a constant $C_2 > 0$ such that for $f \in L^\Phi(\mathbb{C}_+, dV_\alpha)$ and $\lambda > 0$,*

$$\mu\left(\left\{z \in \mathbb{C}_+ : \mathcal{M}_{V_\alpha}^{\mathcal{D}^\beta}(f)(z) > \lambda\right\}\right) \leq C_2\widetilde{\Omega}\left(\left|\left\{z \in \mathbb{C}_+ : \mathcal{M}_{V_\alpha}^{\mathcal{D}^\beta}(f)(z) > \lambda\right\}\right|_\alpha\right). \quad (4.3)$$

Proof. (i) That μ is a $\Phi-$Carleson measure implies that (4.2) holds, has already been proved in [12, Lemma 4.2].
Let us suppose that (4.2) is satisfied and show that μ is a $\Phi-$Carleson measure. Let I be an interval of $\mathbb{R}$ of non-zero length and Q_I the Carleson square associated with I. Put

$$\lambda = \frac{1}{2}\Phi^{-1}\left(\frac{1}{|I|}\right)$$

and

$$f = 2\lambda\chi_I.$$

By construction, $f \in L^{\Phi}(\mathbb{R})$ and $\|f\|_{L^{\Phi}}^{lux} \leq 1$. Indeed

$$\int_{\mathbb{R}} \Phi(|f(x)|)dx = \int_I \Phi\left(\Phi^{-1}\left(\frac{1}{|I|}\right)\right)dx = 1.$$

Let $x_0 + iy_0 \in Q_I$. We have

$$\lambda < f(x_0) = \liminf_{y\to 0} U_f(x_0+iy) \leq U_f(x_0+iy_0),$$

where U_f is the Poisson integral of f. We deduce that

$$Q_I \subset \left\{z \in \mathbb{C}_+ : |U_f(z)| > \lambda\right\}.$$

Since inequality (4.2) is satisfied, we have

$$\begin{aligned}\mu(Q_I) &\lesssim \mu\left(\left\{z \in \mathbb{C}_+ : |U_f(z)| > \lambda\right\}\right)\\ &\lesssim \widetilde{\Omega}\left(\left|\{x \in \mathbb{R} : \mathcal{M}_{HL}(f)(x) > \lambda\}\right|\right)\\ &\lesssim \widetilde{\Omega}\left(\frac{1}{\Phi(\lambda)}\right) \lesssim \widetilde{\Omega}\left(|I|\right).\end{aligned}$$

(ii) Again, that μ is a $(\alpha, \Phi)-$Carleson measure implies that (4.3) holds was proved in [12, Lemma 4.3]. Let us prove the converse. Let I be an interval of nonzero length and Q_I the Carleson square associated with I. Put

$$\lambda = \frac{1}{2}\Phi^{-1}\left(\frac{1+\alpha}{|I|^{2+\alpha}}\right)$$

and

$$f = 2\lambda \chi_{Q_I}.$$

By construction $f \in L^{\Phi}(\mathbb{C}_+, dV_\alpha)$ and $\|f\|_{L^{\Phi}_{\alpha}}^{lux} \leq 1$. Indeed,

$$\int_{\mathbb{C}_+} \Phi(|f(z)|)dV_\alpha(z) \leq \int_{Q_I} \Phi\left(\Phi^{-1}\left(\frac{1+\alpha}{|I|^{2+\alpha}}\right)\right)dV_\alpha(z) = 1.$$

By Lemma 3.12, there are $\beta \in \{0, 1/3\}$ and $J \in \mathcal{D}^{\beta}$ such that $I \subset J$ and $|J| \leq 6|I|$. Consider Q_J the Carleson square associated with J. Let $z \in Q_I$. We have

$$\lambda < \frac{\chi_{Q_I}(z)}{|Q_I|_\alpha}\int_{Q_I} f(\omega)dV_\alpha(\omega) \lesssim \frac{\chi_{Q_J}(z)}{|Q_J|_\alpha}\int_{Q_J} f(\omega)dV_\alpha(\omega) \lesssim \mathcal{M}_{V_\alpha}^{\mathcal{D}^{\beta}} f(z).$$

We deduce that

$$Q_I \subset \left\{z \in \mathbb{C}_+ : \mathcal{M}_{V_\alpha}^{\mathcal{D}^{\beta}} f(z) > \lambda\right\}.$$

Since the inequality (4.3) is satisfied and by Chebychev's inequality, we have

$$\begin{aligned}\mu(Q_I) &\lesssim \mu\left(\left\{z\in\mathbb{C}_+ : \mathcal{M}_{V_\alpha}^{\mathcal{D}^\beta} f(z) > \lambda\right\}\right)\\ &\lesssim \widetilde{\Omega}\left(\left|\left\{z\in\mathbb{C}_+ : \mathcal{M}_{V_\alpha}^{\mathcal{D}^\beta} f(z) > \lambda\right\}\right|_\alpha\right)\\ &\lesssim \widetilde{\Omega}\left(\frac{1}{\Phi\left(\Phi^{-1}\left(\frac{1}{|I|^{2+\alpha}}\right)\right)}\right) \lesssim \widetilde{\Omega}\left(|I|^{2+\alpha}\right).\end{aligned}$$

□

The following is a generalization of [12, Theorem 4.1].

Theorem 4.4. *Let $s>0$ be a real, Φ_1,Φ_2 two one-to-one growth functions and μ a positive Borel measure on $\mathbb{C}_+$. If $\Phi_2\in\mathscr{L}\cup\mathscr{U}$ and Φ_1 is convex or belongs $\mathscr{L}$, then the following assertions are equivalent.*

(i) *μ is a $(s,\Phi_2\circ\Phi_1^{-1})$–Carleson measure.*

(ii) *There exist some constants $\rho\in\{1;a_{\Phi_1}\}$ and $C:=C_{s,\Phi_1,\Phi_2}>0$ such that for all $z=x+iy\in\mathbb{C}_+$*

$$\int_{\mathbb{C}_+}\Phi_2\left(\Phi_1^{-1}\left(\frac{1}{y^s}\right)\frac{y^{2s/\rho}}{|\omega-\overline{z}|^{2s/\rho}}\right)d\mu(\omega)\leq C. \tag{4.4}$$

Proof. Let us show that (ii) implies (i). We assume that the inequality (4.4) holds.

Let I be an interval of nonzero length and Q_I its Carleson square.

Fix $z_0=x_0+iy_0\in\mathbb{C}_+$, and assume that x_0 is the center of I and $|I|=2y_0$.

Let $\omega=u+iv\in Q_I$. We have

$$|\omega-\overline{z_0}|^2=|(u-x_0)+i(v+y_0)|^2\leq y_0^2+(3y_0)^2=10y_0^2.$$

It follows that

$$1\leq 10^{s/\rho}\frac{y_0^{2s/\rho}}{|\omega-\overline{z_0}|^{2s/\rho}}.$$

Since Φ_1^{-1} is increasing and $t\mapsto\frac{\Phi_2(t)}{t^{b_{\Phi_2}}}$ is non-increasing on $\mathbb{R}_+^*$, we have

$$\begin{aligned}\Phi_2\circ\Phi_1^{-1}\left(\frac{1}{|I|^s}\right)&\leq\Phi_2\left(\Phi_1^{-1}\left(\frac{1}{y_0^s}\right)\frac{10^{s/\rho}y_0^{2s/\rho}}{|\omega-\overline{z_0}|^{2s/\rho}}\right)\\ &\leq 10^{sb_{\Phi_2}/\rho}\Phi_2\left(\Phi_1^{-1}\left(\frac{1}{y_0^s}\right)\frac{y_0^{2s/\rho}}{|\omega-\overline{z_0}|^{2s/\rho}}\right).\end{aligned}$$

Thus

$$\Phi_2\circ\Phi_1^{-1}\left(\frac{1}{|I|^s}\right)\leq 10^{sb_{\Phi_2}/\rho}\Phi_2\left(\Phi_1^{-1}\left(\frac{1}{y_0^s}\right)\frac{y_0^{2s/\rho}}{|\omega-\overline{z_0}|^{2s/\rho}}\right),\ \forall\,\omega\in Q_I.$$

Since the inequality (4.4) is satisfied, we obtain

$$\begin{aligned}\Phi_2\circ\Phi_1^{-1}\left(\frac{1}{|I|^s}\right)\mu(Q_I) &= \int_{Q_I}\Phi_2\circ\Phi_1^{-1}\left(\frac{1}{|I|^s}\right)d\mu(\omega)\\ &\leq 10^{sb_{\Phi_2}/\rho}\int_{\mathbb{C}_+}\Phi_2\left(\Phi_1^{-1}\left(\frac{1}{y_0^s}\right)\frac{y_0^{2s/\rho}}{|\omega-\overline{z_0}|^{2s/\rho}}\right)d\mu(\omega)\\ &\leq 10^{sb_{\Phi_2}/\rho}C_2.\end{aligned}$$

We deduce that

$$\mu(Q_I)\leq\frac{10^{sb_{\Phi_2}/\rho}C_2}{\Phi_2\circ\Phi_1^{-1}\left(\frac{1}{|I|^s}\right)}.$$

For the converse, we assume that the inequality (2.4) holds.
Put

$$\rho=\begin{cases}1 & \text{if } \Phi_1 \text{ is convex}\\ a_{\Phi_1} & \text{if } \Phi_1\in\mathscr{L}\end{cases}$$

Fix $z_0=x_0+iy_0\in\mathbb{C}_+$ and let $j\in\mathbb{N}$. Consider I_j the centered interval x_0 with $|I_j|=2^{j+1}y_0$ and Q_{I_j} its Carleson square. Put

$$E_j:=Q_{I_j}\backslash Q_{I_{j-1}},\ \forall\, j\geq 1 \text{ and } E_0=Q_{I_0}.$$

Fix $j\in\mathbb{N}$ and let $\omega=u+iv\in\mathbb{C}_+$.
If $\omega\in E_0$, then we have

$$|\omega-\overline{z_0}|^2=|(u-x_0)+i(v+y_0)|^2\geq(v+y_0)^2\geq y_0^2\geq 2^{-2}y_0^2.$$

If $\omega\in E_j$ with $j\geq 1$ then we have

$$|\omega-\overline{z_0}|^2=|(u-x_0)+i(v+y_0)|^2\geq(u-x_0)^2\geq 2^{2(j-1)}y_0^2.$$

We deduce that

$$\frac{y_0^{2s/\rho}}{|\omega-\overline{z_0}|^{2s/\rho}}\leq\frac{1}{2^{2(j-1)s/\rho}},\ \forall\,\omega\in E_j,\ \forall\, j\geq 0.$$

Fix $j\in\mathbb{N}$ and let $\omega\in E_j$. Since the functions $t\mapsto\frac{\Phi_1^{-1}(t)}{t^{1/\rho}}$ and $t\mapsto\frac{\Phi_2(t)}{t^{b_{\Phi_2}}}$ are non-increasing on $\mathbb{R}_+^*$ and $t\mapsto\frac{\Phi_2(t)}{t^{a_{\Phi_2}}}$ is non-decreasing on $\mathbb{R}_+^*$, we have

$$\begin{aligned}\Phi_2\left(\Phi_1^{-1}\left(\frac{1}{y_0^s}\right)\frac{y_0^{2s/\rho}}{|\omega-\overline{z_0}|^{2s/\rho}}\right)&\leq\Phi_2\left(\Phi_1^{-1}\left(\frac{1}{y_0^s}\right)\frac{1}{2^{2(j-1)s/\rho}}\right)\\ &=\Phi_2\left(\Phi_1^{-1}\left(\frac{1}{y_0^s}\right)\frac{1}{2^{(j+1)s/\rho}}\times\frac{1}{2^{js/\rho}}\times\frac{1}{2^{-3s/\rho}}\right)\\ &\leq\frac{1}{2^{-3sb_{\Phi_2}/\rho}}\times\frac{1}{2^{jsa_{\Phi_2}/\rho}}\times\Phi_2\circ\Phi_1^{-1}\left(\frac{1}{|I_j|^s}\right).\end{aligned}$$

We deduce that

$$\Phi_2\left(\Phi_1^{-1}\left(\frac{1}{y_0^s}\right)\frac{y_0^{2s/\rho}}{|\omega-\overline{z_0}|^{2s/\rho}}\right)\leq\frac{1}{2^{-3sb_{\Phi_2}/\rho}}\times\frac{1}{2^{jsa_{\Phi_2}/\rho}}\times\Phi_2\circ\Phi_1^{-1}\left(\frac{1}{|I_j|^s}\right),\ \forall\,\omega\in E_j.$$

Since the inequality (2.4) holds, it follows that

$$\begin{aligned}
L &:= \int_{E_j}\Phi_2\left(\Phi_1^{-1}\left(\frac{1}{y_0^s}\right)\frac{y_0^{2s/\rho}}{|\omega-\overline{z_0}|^{2s/\rho}}\right)d\mu(\omega)\\
&\leq \int_{E_j}\frac{1}{2^{-3sb_{\Phi_2}/\rho}}\times\frac{1}{2^{jsa_{\Phi_2}/\rho}}\times\Phi_2\circ\Phi_1^{-1}\left(\frac{1}{|I_j|^s}\right)d\mu(\omega)\\
&\leq \frac{1}{2^{-3sb_{\Phi_2}/\rho}}\times\frac{1}{2^{jsa_{\Phi_2}/\rho}}\times\Phi_2\circ\Phi_1^{-1}\left(\frac{1}{|I_j|^s}\right)\mu(Q_{I_j})\\
&\leq \frac{1}{2^{-3sb_{\Phi_2}/\rho}}\times\frac{1}{2^{jsa_{\Phi_2}/\rho}}\times C_1.
\end{aligned}$$

We deduce that

$$\int_{E_j}\Phi_2\left(\Phi_1^{-1}\left(\frac{1}{y_0^s}\right)\frac{y_0^{2s/\rho}}{|\omega-\overline{z_0}|^{2s/\rho}}\right)d\mu(\omega)\leq\frac{C_1}{2^{-3sb_{\Phi_2}/\rho}}\times\frac{1}{2^{jsa_{\Phi_2}/\rho}},\ \forall\,j\geq 0.$$

By construction, the E_j are pairwise disjoint and form a partition of $\mathbb{C}_+$. So we have

$$\begin{aligned}
L &:= \int_{\mathbb{C}_+}\Phi_2\left(\Phi_1^{-1}\left(\frac{1}{y_0^s}\right)\frac{y_0^{2s/\rho}}{|\omega-\overline{z_0}|^{2s/\rho}}\right)d\mu(\omega)\\
&= \sum_{j=0}^{\infty}\int_{E_j}\Phi_2\left(\Phi_1^{-1}\left(\frac{1}{y_0^s}\right)\frac{y_0^{2s/\rho}}{|\omega-\overline{z_0}|^{2s/\rho}}\right)d\mu(\omega)\\
&\leq \frac{C_1}{2^{-3sb_{\Phi_2}/\rho}}\times\sum_{j=0}^{\infty}\frac{1}{2^{jsa_{\Phi_2}/\rho}}<\infty.
\end{aligned}$$

□

5. Proofs of main results.

This section is devoted to the proofs of our main results. Let us start with the proof of Theorem 2.3.

Proof of Theorem 2.3. The equivalence $(i)\Leftrightarrow(ii)$ is given by Theorem 4.4. The implication $(iii)\Rightarrow(iv)$ is obvious. Let us prove that $(i)\Rightarrow(iii)$ and $(iv)\Rightarrow(i)$ which is enough to conclude.

$(i) \Rightarrow (iii)$: Let $0 \not\equiv F \in H^{\Phi_1}(\mathbb{C}_+)$. Following Theorem 3.24, there exists a unique function $f \in L^\Phi(\mathbb{R})$ such that $\log|f| \in L^1\left(\mathbb{R}, \frac{dt}{1+t^2}\right)$ and

$$\log|F(x+iy)| \leq \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-t)^2+y^2}\log|f(t)|dt,\ \forall\, x+iy \in \mathbb{C}_+ \tag{5.1}$$

and $\|F\|_{H^\Phi}^{lux} = \|f\|_{L^\Phi}^{lux}$. Using Jensen's inequality in (5.1), we deduce that

$$|F(x+iy)| \lesssim \left(\mathcal{M}_{HL}(|f|^{a_{\Phi_1}/2})(x)\right)^{2/a_{\Phi_1}},\ \forall\, x+iy \in \mathbb{C}_+.$$

Fix $\lambda > 0$, and put

$$E_\lambda := \left\{x \in \mathbb{R} : \left(\mathcal{M}_{HL}\left(\frac{|f|}{\|f\|_{L^\Phi}^{lux}}\right)^{a_{\Phi_1}/2}(x)\right)^{2/a_{\Phi_1}} > \lambda\right\}.$$

From (3.9), we deduce that

$$|E_\lambda| \lesssim \sum_{\beta\in\{0;1/3\}}\left|\left\{x \in \mathbb{R} : \left(\mathcal{M}_{HL}^{\mathcal{D}^\beta}\left(\frac{|f|}{\|f\|_{L^\Phi}^{lux}}\right)^{a_{\Phi_1}/2}(x)\right)^{2/a_{\Phi_1}} > \frac{\lambda}{12}\right\}\right|.$$

Put

$$\Phi_a(t) = \Phi_1\left(t^{2/a_{\Phi_1}}\right),\ \forall\, t \geq 0.$$

From Proposition 3.5, we deduce that $\Phi_a \in \mathscr{U} \cap \nabla_2$. It follows from Proposition 3.13 that

$$\left|\left\{x \in \mathbb{R} : \left(\mathcal{M}_{HL}^{\mathcal{D}^\beta}\left(\frac{|f|}{\|f\|_{L^\Phi}^{lux}}\right)^{a_{\Phi_1}/2}(x)\right)^{2/a_{\Phi_1}} > \frac{\lambda}{12}\right\}\right| \lesssim \frac{1}{\Phi_1(\lambda)},\ \forall\, \beta \in \{0; 1/3\}.$$

We deduce that

$$|E_\lambda| \lesssim \frac{1}{\Phi_1(\lambda)}.$$

Put

$$\widetilde{\Omega}_3(t) = \frac{1}{\Phi_2\circ\Phi_1^{-1}\left(\frac{1}{t}\right)},\ \forall\, t > 0 \quad \text{and} \quad \widetilde{\Omega}_3(0) = 0.$$

From Lemma 3.10, we deduce that $\widetilde{\Omega}_3 \in \mathscr{U}$. Since μ is an $\Phi_2\circ\Phi_1^{-1}$–Carleson measure and $t \mapsto \frac{\widetilde{\Omega}_3(t)}{t}$ is non-decreasing on $\mathbb{R}_+^*$, by Lemma 4.3, we have

$$\begin{aligned}\mu\left(\left\{z \in \mathbb{C}_+ : |F(z)| > \lambda\|f\|_{L^{\Phi_1}}^{lux}\right\}\right) &\lesssim \mu\left(\left\{z \in \mathbb{C}_+ : |U_f(z)| > \lambda\|f\|_{L^{\Phi_1}}^{lux}\right\}\right)\\ &\lesssim \widetilde{\Omega}_3\left(|E_\lambda|\right)\\ &\lesssim \Phi_1(\lambda)\widetilde{\Omega}_3\left(\frac{1}{\Phi_1(\lambda)}\right)|E_\lambda|.\end{aligned}$$

As

$$\Phi_1(\lambda)\widetilde{\Omega}_3\left(\frac{1}{\Phi_1(\lambda)}\right) = \Phi_1(\lambda)\frac{1}{\Phi_2(\lambda)} = \frac{\Phi_1(\lambda)}{\lambda} \times \frac{\lambda}{\Phi_2(\lambda)} \approx \frac{\Phi_1'(\lambda)}{\Phi_2'(\lambda)}.$$

We deduce that

$$\mu\left(\left\{z \in \mathbb{C}_+ : |F(z)| > \lambda \|f\|_{L^{\Phi_1}}^{lux}\right\}\right) \lesssim \frac{\Phi_1'(\lambda)}{\Phi_2'(\lambda)} |E_\lambda|, \ \forall \lambda > 0.$$

We have

$$\begin{aligned}\int_{\mathbb{C}_+} \Phi_2\left(\frac{|F(z)|}{\|F\|_{H^{\Phi_1}}^{lux}}\right) d\mu(z) &= \int_0^\infty \Phi_2'(\lambda)\mu\left(\left\{z \in \mathbb{C}_+ : |F(z)| > \lambda\|f\|_{L^{\Phi_1}}^{lux}\right\}\right) d\lambda \\ &\lesssim \int_0^\infty \Phi_2'(\lambda)\left(\frac{\Phi_1'(\lambda)}{\Phi_2'(\lambda)} \times |E_\lambda|\right) d\lambda \\ &= \int_0^\infty \Phi_1'(\lambda) \times |E_\lambda| d\lambda \\ &= \int_{\mathbb{R}} \Phi_a\left(\mathcal{M}_{HL}^{\mathcal{D}^\beta}\left(\frac{|f|}{\|f\|_{L^\Phi}^{lux}}\right)^{a_{\Phi_1}/2}(x)\right) dx \\ &\lesssim \int_{\mathbb{R}} \Phi_a\left(\left(\frac{|f(x)|}{\|f\|_{L^{\Phi_1}}^{lux}}\right)^{a_{\Phi_1}/2}\right) dx \\ &= \int_{\mathbb{R}} \Phi_1\left(\frac{|f(x)|}{\|f\|_{L^{\Phi_1}}^{lux}}\right) dx \\ &\lesssim 1.\end{aligned}$$

$(iv) \Rightarrow (i)$: Let I be an interval of nonzero length and Q_I its Carleson square. Fix $z_0 = x_0 + iy_0 \in \mathbb{C}_+$ and we assume that x_0 is the center of I and $|I| = 2y_0$. Put

$$F_{z_0}(\omega) = \Phi_1^{-1}\left(\frac{1}{\pi y_0}\right)\frac{y_0^{2/\rho}}{(\omega - \overline{z_0})^{2/\rho}}, \ \forall \omega \in \mathbb{C}_+,$$

where $\rho = 1$ if $\Phi \in \mathscr{U}$ and $\rho = a_\Phi$ if $\Phi \in \mathscr{L}$. From Proposition 3.27, we deduce that $F_{z_0} \in H^{\Phi_1}(\mathbb{C}_+)$ and $\|F_{z_0}\|_{H^{\Phi_1}}^{lux} \leq 1$.
Let $\omega = u + iv \in Q_I$. We have

$$|\omega - \overline{z_0}|^2 = |(u - x_0) + i(v + y_0)|^2 \leq y_0^2 + (2y_0 + y_0)^2 = 10y_0^2 \Rightarrow \frac{1}{10} \leq \frac{y_0^2}{|\omega - \overline{z_0}|^2}.$$

Since the function $t \mapsto \frac{\Phi_1^{-1}(t)}{t^{1/\rho}}$ is non-increasing on $\mathbb{R}_+^*$, we have

$$\Phi_1^{-1}\left(\frac{1}{|I|}\right) < \Phi_1^{-1}\left(\frac{1}{y_0}\right) \leq \pi^{1/\rho}\Phi_1^{-1}\left(\frac{1}{\pi y_0}\right).$$

We deduce that

$$\Phi_1^{-1}\left(\frac{1}{|I|}\right) < \left(\frac{\pi}{10}\right)^{1/\rho} \Phi_1^{-1}\left(\frac{1}{\pi y_0}\right) \frac{y_0^{2/\rho}}{|\omega - \overline{z_0}|^{2/\rho}} \leq \left(\frac{\pi}{10}\right)^{1/\rho} \frac{|F_{z_0}(\omega)|}{\|F_{z_0}\|_{H^{\Phi_1}}^{lux}}.$$

Taking

$$\lambda := \left(\frac{10}{\pi}\right)^{1/\rho} \Phi_1^{-1}\left(\frac{1}{|I|}\right),$$

it follows that

$$|F_{z_0}(\omega)| > \lambda \|F_{z_0}\|_{H^{\Phi_1}}^{lux}, \ \forall\, \omega \in Q_I.$$

Therefore

$$Q_I \subset \left\{ z \in \mathbb{C}_+ : |F_{z_0}(z)| > \lambda \|F_{z_0}\|_{H^{\Phi_1}}^{lux} \right\}.$$

Since inequality (2.10) is satisfied, we have

$$\mu(Q_I) \leq \mu\left(\left\{ z \in \mathbb{C}_+ : |F_{z_0}(z)| > \lambda \|F_{z_0}\|_{H^{\Phi_1}}^{lux} \right\}\right) \leq \frac{C_1}{\Phi_2(\lambda)}.$$

As

$$\Phi_2 \circ \Phi_1^{-1}\left(\frac{1}{|I|}\right) = \Phi_2\left(\left(\frac{\pi}{10}\right)^{1/\rho} \lambda\right) \leq C_2 \Phi_2(\lambda).$$

We deduce that

$$\mu(Q_I) \leq \frac{C_3}{\Phi_2 \circ \Phi_1^{-1}\left(\frac{1}{|I|}\right)}.$$

□

Proof of Corollary 2.4. The proof of Corollary 2.4 follows from Theorem 2.3 and Proposition 4.1 for ($s = 1$). □

Proof of Theorem 2.5. The equivalence $(i) \Leftrightarrow (ii)$ is given by Theorem 4.4. The implication $(iii) \Rightarrow (iv)$ is obvious. To conclude, it is enough to prove that $(i) \Rightarrow (iii)$ and $(iv) \Rightarrow (i)$.

$(i) \Rightarrow (iii)$: Let $0 \not\equiv F \in A_\alpha^{\Phi_1}(\mathbb{C}_+)$. By Proposition 3.26, there exists $\beta \in \{0, 1/3\}$ such that

$$|G(z)| \lesssim \left(\mathcal{M}_{V_\alpha}^{\mathcal{D}^\beta}\left(|G|^{a_{\Phi_1}/2}\right)(z)\right)^{2/a_{\Phi_1}}, \ \forall\, z \in \mathbb{C}_+,$$

where $G := \frac{|F(z)|}{\|F\|_{A_\alpha^{\Phi_1}}^{lux}}$. Put

$$\widetilde{\Omega}_3(t) = \frac{1}{\Phi_2 \circ \Phi_1^{-1}\left(\frac{1}{t}\right)}, \ \forall\, t > 0 \quad \text{and} \quad \widetilde{\Omega}_3(0) = 0.$$

From Lemma 3.10, we deduce that $\widetilde{\Omega}_3 \in \mathscr{U}$. Since $t \mapsto \frac{\widetilde{\Omega}_3(t)}{t}$ is non-decreasing on $\mathbb{R}_+^*$, it follows from Proposition 3.13 that for $\lambda > 0$, we have

$$|E_\lambda|_\alpha \leq \frac{1}{\Phi_1(\lambda)} \Rightarrow \widetilde{\Omega}_3\left(|E_\lambda|_\alpha\right) \leq \Phi_1(\lambda)\widetilde{\Omega}_3\left(\frac{1}{\Phi_1(\lambda)}\right)|E_\lambda|_\alpha \lesssim \frac{\Phi_1'(\lambda)}{\Phi_2'(\lambda)}|E_\lambda|_\alpha,$$

where

$$E_\lambda := \left\{z \in \mathbb{C}_+ : \left(\mathcal{M}_{V_\alpha}^{\mathcal{D}^\beta}\left(|G|^{a_{\Phi_1}/2}\right)(z)\right)^{2/a_{\Phi_1}} > \lambda\right\}.$$

Since μ is an $(\alpha, \Phi_2 \circ \Phi_1^{-1})$−Carleson measure, by Lemma 4.3, we deduce that

$$\mu(E_\lambda) \lesssim \widetilde{\Omega}_3\left(|E_\lambda|_\alpha\right) \lesssim \frac{\Phi_1'(\lambda)}{\Phi_2'(\lambda)}|E_\lambda|_\alpha, \ \forall \lambda > 0.$$

Put

$$\Phi_a(t) = \Phi_1\left(t^{2/a_{\Phi_1}}\right), \ \forall t \geq 0.$$

From Proposition 3.5, we deduce that $\Phi_a \in \mathscr{U} \cap \nabla_2$. We have

$$\begin{aligned}
\int_{\mathbb{C}_+} \Phi_2\left(\frac{|F(z)|}{\|F\|_{A_\alpha^{\Phi_1}}^{lux}}\right) d\mu(z) &\lesssim \int_{\mathbb{C}_+} \Phi_2\left(\left(\mathcal{M}_{V_\alpha}^{\mathcal{D}^\beta}\left(|G|^{a_{\Phi_1}/2}\right)(z)\right)^{2/a_{\Phi_1}}\right) d\mu(z)\\
&= \int_0^\infty \Phi_2'(\lambda)\mu(E_\lambda) d\lambda\\
&\lesssim \int_0^\infty \Phi_2'(\lambda)\left(\frac{\Phi_1'(\lambda)}{\Phi_2'(\lambda)}|E_\lambda|_\alpha\right) d\lambda\\
&= \int_{\mathbb{C}_+} \Phi_a\left(\mathcal{M}_{V_\alpha}^{\mathcal{D}^\beta}\left(|G|^{a_{\Phi_1}/2}\right)(z)\right) dV_\alpha(z)\\
&\lesssim \int_{\mathbb{C}_+} \Phi_a\left(|G|^{a_{\Phi_1}/2}\right) dV_\alpha(z) \lesssim 1.
\end{aligned}$$

$(iv) \Rightarrow (i)$: Let I be an interval of nonzero length and Q_I its Carleson square. Fix $z_0 = x_0 + iy_0 \in \mathbb{C}_+$ and we assume that x_0 is the center of I and $|I| = 2y_0$. Put

$$G_{z_0}(\omega) = \Phi_1^{-1}\left(\frac{1}{C_\alpha y_0^{2+\alpha}}\right)\frac{y_0^{(4+2\alpha)/\rho}}{(\omega - \overline{z_0})^{(4+2\alpha)/\rho}}, \ \forall \omega \in \mathbb{C}_+,$$

where $\rho = 1$ if $\Phi \in \mathscr{U}$ and $\rho = a_\Phi$ if $\Phi \in \mathscr{L}$, and C_α is the constant in (3.30). From Proposition 3.27, we deduce that $G_{z_0} \in A_\alpha^{\Phi_1}(\mathbb{C}_+)$ and $\|G_{z_0}\|_{A_\alpha^{\Phi_1}}^{lux} \leq 1$.

For $\omega = u + iv \in Q_I$, we have

$$|\omega - \overline{z_0}|^2 = |(u - x_0) + i(v + y_0)|^2 \leq y_0^2 + (2y_0 + y_0)^2 = 10y_0^2 \Rightarrow \frac{1}{10} \leq \frac{y_0^2}{|\omega - \overline{z_0}|^2}.$$

Since the function $t \mapsto \frac{\Phi_1^{-1}(t)}{t^{1/\rho}}$ is non-increasing on $\mathbb{R}_+^*$, we have

$$\Phi_1^{-1}\left(\frac{1}{|I|^{2+\alpha}}\right) < \Phi_1^{-1}\left(\frac{1}{y_0^{2+\alpha}}\right) \leq (C_\alpha)^{1/\rho}\Phi_1^{-1}\left(\frac{1}{C_\alpha y_0^{2+\alpha}}\right).$$

We deduce that

$$\Phi_1^{-1}\left(\frac{1}{|I|^{2+\alpha}}\right) < \left(\frac{C_\alpha}{10}\right)^{1/\rho}\Phi_1^{-1}\left(\frac{1}{C_\alpha y_0^{2+\alpha}}\right)\frac{y_0^{(4+2\alpha)/\rho}}{|\omega-\overline{z_0}|^{(4+2\alpha)/\rho}} \leq \left(\frac{C_\alpha}{10}\right)^{1/\rho}\frac{|G_{z_0}(\omega)|}{\|G_{z_0}\|_{A_\alpha^{\Phi_1}}^{lux}}.$$

Taking

$$\lambda := \left(\frac{10}{C_\alpha}\right)^{1/\rho}\Phi_1^{-1}\left(\frac{1}{|I|^{2+\alpha}}\right),$$

it follows that

$$|G_{z_0}(\omega)| > \lambda\|G_{z_0}\|_{A_\alpha^{\Phi_1}}^{lux},\ \forall\,\omega \in Q_I.$$

Therefore

$$Q_I \subset \left\{z \in \mathbb{C}_+ : |G_{z_0}(z)| > \lambda\|G_{z_0}\|_{A_\alpha^{\Phi_1}}^{lux}\right\}.$$

Since inequality (2.14) is satisfied, we have

$$\mu(Q_I) \leq \mu\left(\left\{z \in \mathbb{C}_+ : |G_{z_0}(z)| > \lambda\|G_{z_0}\|_{A_\alpha^{\Phi_1}}^{lux}\right\}\right) \leq \frac{C_1}{\Phi_2(\lambda)}.$$

As

$$\Phi_2\circ\Phi_1^{-1}\left(\frac{1}{|I|^{2+\alpha}}\right) = \Phi_2\left(\left(\frac{C_\alpha}{10}\right)^{1/\rho}\lambda\right) \leq C_2\Phi_2(\lambda).$$

We deduce that

$$\mu(Q_I) \leq \frac{C_3}{\Phi_2\circ\Phi_1^{-1}\left(\frac{1}{|I|^{2+\alpha}}\right)}.$$

□

Proof of Corollary 2.6. The proof of Corollary 2.6 follows from Theorem 2.5 and Proposition 4.1 for ($s = 2+\alpha$). □

The following result follows from Lemma 3.25 and Proposition 3.27. Therefore, the proof will not be written.

Lemma 5.1. *Let $\alpha,\beta > -1$, $\Phi_1,\Phi_2 \in \mathscr{L}\cup\mathscr{U}$. There are constants $C_1 := C_{\alpha,\Phi_1,\Phi_2} > 0$ and $C := C_{\alpha,\beta,\Phi_1,\Phi_2} > 0$ such that for all $F \in \mathcal{M}\left(H^{\Phi_1}(\mathbb{C}_+), A_\alpha^{\Phi_2}(\mathbb{C}_+)\right)$ and $G \in \mathcal{M}\left(A_\alpha^{\Phi_1}(\mathbb{C}_+), A_\beta^{\Phi_2}(\mathbb{C}_+)\right)$,*

$$|F(x+iy)| \leq C_1\frac{\Phi_2^{-1}\left(\frac{1}{y^{2+\alpha}}\right)}{\Phi_1^{-1}\left(\frac{1}{y}\right)},\ \forall\,x+iy \in \mathbb{C}_+ \tag{5.2}$$

and

$$|G(x+iy)| \leq C_2 \frac{\Phi_2^{-1}\left(\frac{1}{y^{2+\beta}}\right)}{\Phi_1^{-1}\left(\frac{1}{y^{2+\alpha}}\right)},\ \forall\, x+iy \in \mathbb{C}_+. \tag{5.3}$$

Proof of Theorem 2.7. The inclusion of $\mathcal{M}(H^{\Phi_1}(\mathbb{C}_+), A_\alpha^{\Phi_2}(\mathbb{C}_+))$ in $H_\omega^\infty(\mathbb{C}_+)$ follows from Lemma 5.1.
Conversely, fix $0 \not\equiv G \in H_\omega^\infty(\mathbb{C}_+)$ and let $z = x+iy \in \mathbb{C}_+$. Since $\Phi_2 \in \widetilde{\mathscr{L}} \cup \widetilde{\mathscr{U}}$, by Lemma 3.11, we have

$$\Phi_2(\omega(y)) = \Phi_2\left(\frac{\Phi_2^{-1}\left(\frac{1}{y^{2+\alpha}}\right)}{\Phi_1^{-1}\left(\frac{1}{y}\right)}\right) \lesssim \frac{\Phi_2\left(\Phi_2^{-1}\left(\frac{1}{y^{2+\alpha}}\right)\right)}{\Phi_2\left(\Phi_1^{-1}\left(\frac{1}{y}\right)\right)} = \frac{1}{y^{2+\alpha}\Phi_2\circ\Phi_1^{-1}\left(\frac{1}{y}\right)}.$$

We deduce that

$$\Phi_2\left(\frac{|G(x+iy)|}{\|G\|_{H_\omega^\infty}}\right) \lesssim \Phi_2(\omega(y)) \lesssim \frac{1}{y^{2+\alpha}\Phi_2\circ\Phi_1^{-1}\left(\frac{1}{y}\right)},\ \forall\, x+iy \in \mathbb{C}_+.$$

Put

$$d\mu(x+iy) = \frac{dxdy}{y^2\Phi_2\circ\Phi_1^{-1}(\frac{1}{y})},\ \forall\, x+iy \in \mathbb{C}_+.$$

Since $\Phi_2\circ\Phi_1^{-1} \in \nabla_2$, from Proposition 4.2, we deduce that μ is a measure $\Phi_2\circ\Phi_1^{-1}-$Carleson.
Let $0 \not\equiv F \in H^{\Phi_1}(\mathbb{C}_+)$. By Theorem 2.3, we have

$$\begin{aligned} L &:= \int_{\mathbb{C}_+} \Phi_2\left(\frac{|G(x+iy)F(x+iy)|}{\|G\|_{H_\omega^\infty}\|F\|_{H^{\Phi_1}}^{lux}}\right) dV_\alpha(x+iy) \\ &\lesssim \int_{\mathbb{C}_+} \Phi_2\left(\frac{|G(x+iy)|}{\|G\|_{H_\omega^\infty}}\right)\Phi_2\left(\frac{|F(x+iy)|}{\|F\|_{H^{\Phi_1}}^{lux}}\right) y^\alpha dxdy \\ &\lesssim \int_{\mathbb{C}_+} \Phi_2\left(\frac{|F(x+iy)|}{\|F\|_{H^{\Phi_1}}^{lux}}\right) d\mu(x+iy) \\ &\lesssim 1. \end{aligned}$$

We deduce that $G \in \mathcal{M}(H^{\Phi_1}(\mathbb{C}_+), A_\alpha^{\Phi_2}(\mathbb{C}_+))$. □

Proof of Theorem 2.8. The inclusion of $\mathcal{M}(A_\alpha^{\Phi_1}(\mathbb{C}_+), A_\beta^{\Phi_2}(\mathbb{C}_+))$ in $H_\omega^\infty(\mathbb{C}_+)$ follows from Lemma 5.1. Conversely, fix $0 \not\equiv G \in H_\omega^\infty(\mathbb{C}_+)$ and let $z = x+iy \in$

$\mathbb{C}_+$. Since $\Phi_2 \in \widetilde{\mathscr{L}} \cup \widetilde{\mathscr{U}}$, by Lemma 3.11, we have

$$\Phi_2(\omega(y)) = \Phi_2\left(\frac{\Phi_2^{-1}\left(\frac{1}{y^{2+\beta}}\right)}{\Phi_1^{-1}\left(\frac{1}{y^{2+\alpha}}\right)}\right) \lesssim \frac{\Phi_2\left(\Phi_2^{-1}\left(\frac{1}{y^{2+\beta}}\right)\right)}{\Phi_2\left(\Phi_1^{-1}\left(\frac{1}{y^{2+\alpha}}\right)\right)} = \frac{1}{y^{2+\beta}\Phi_2\circ\Phi_1^{-1}\left(\frac{1}{y^{2+\alpha}}\right)}.$$

We deduce that

$$\Phi_2\left(\frac{|G(x+iy)|}{\|G\|_{H^\infty_\omega}}\right) \lesssim \Phi_2(\omega(y)) \lesssim \frac{1}{y^{2+\beta}\Phi_2\circ\Phi_1^{-1}\left(\frac{1}{y^{2+\alpha}}\right)}, \ \forall\, x+iy \in \mathbb{C}_+.$$

Put

$$d\mu(x+iy) = \frac{dxdy}{y^2\Phi_2\circ\Phi_1^{-1}(\frac{1}{y^{2+\alpha}})}, \ \forall\, x+iy \in \mathbb{C}_+.$$

By Proposition 4.2, μ is a $(\alpha, \Phi_2\circ\Phi_1^{-1})$−Carleson measure. By Theorem 2.5, we have

$$\begin{aligned} L := &\int_{\mathbb{C}_+} \Phi_2\left(\frac{|G(x+iy)F(x+iy)|}{\|G\|_{H^\infty_\omega}\|F\|^{lux}_{A^{\Phi_1}_\alpha}}\right) dV_\beta(x+iy) \\ \lesssim &\int_{\mathbb{C}_+} \Phi_2\left(\frac{|G(x+iy)|}{\|G\|_{H^\infty_\omega}}\right)\Phi_2\left(\frac{|F(x+iy)|}{\|F\|^{lux}_{A^{\Phi_1}_\alpha}}\right) y^\beta dxdy \\ \lesssim &\int_{\mathbb{C}_+} \Phi_2\left(\frac{|F(x+iy)|}{\|F\|^{lux}_{A^{\Phi_1}_\alpha}}\right) d\mu(x+iy) \\ \lesssim &\ 1. \end{aligned}$$

We deduce that $G \in \mathcal{M}(A^{\Phi_1}_\alpha(\mathbb{C}_+), A^{\Phi_2}_\beta(\mathbb{C}_+))$. □

Statements and Declarations

The authors declare no competing interests.

References

[1] Bansah, Justice S.; Sehba, Benoit F. Boundeness of a family of Hilbert-Type operators and its Bergman-Type analogue, *Illinois Journal of Mathematics.* **59** (2015), no. 4, 949–977. MR3628296 , Zbl 1370.47029, doi: 10.1215/ijm/1488186016. 1292

[2] Bonami, Aline; Sehba, Benoit F. Hankel operators between Hardy-Orlicz spaces and products of holomorphic function, *Rev.U. Mat.Arg. Vol.* **50** (2009), no. 2, 183–196. MR2656535, Zbl 1218.47043. 1273

[3] Carleson, Lennart A. E. An interpolation problem for bounded analytic functions, *Amer. J. Math.* **80** (1958), 921–930. 1272
MR0117349, Zbl 0085.06504, doi: 10.2307/2372840.

[4] CARLESON, LENNART A. E. An interpolation by bounded analytic functions and corona problem, *Ann. of Math.(2)* **76** (1962), 547–559. 1272
MR0141789, Zbl 0112.29702, doi: 10.2307/1970375.

[5] CHARPENTIER, STÉPHANE. Composition operators on weighted Bergman–Orlicz spaces on the ball. *Compl Anal Oper Theory.* **7** (2013), no. 1, 43–68. 1272
MR3010788, Zbl 1273.47048, doi: 10.1007/s11785-011-0140-7.

[6] CHARPENTIER, STÉPHANE; SEHBA BENOIT F. Carleson measure theorems for large Hardy–Orlicz and Bergman Orlicz spaces. *J Funct Spaces Appl.* (2012) 1–21. 1272
MR2898468 , Zbl 1248.46022, doi: 10.1155/2012/792763.

[7] CHEN, SHU TAO. Geometry of Orlicz Spaces. *Dissertationes Math. (Rozprawy Mat.)* **356** (1996). MR1410390, Zbl 0873.46018. 1283

[8] CIMA, JOSEPH A.; WOGEN, WARREN R. A Carleson measure theorem for the Bergman space on the unit ball of $\mathbb{C}^n$. *J Oper Theor.* **7** (1982), no. 1, 157–165. MR0650200, Zbl 0499.42011. 1272

[9] CRUZ-URIBE, DAVID V.; MARTELL, JOSÉ M.; PÉREZ, CARLOS M. Weights, extrapolation and the theory of Rubio de Francia, *Oper. Theory Adv. Appl.*, **215** (2011), xiv+280. MR2797562, Zbl 1234.46003. 1284

[10] DJE, TANOH J. M.; FEUTO, J. Representation theorems in Hardy-Orlicz spaces on the upper complex half-plane. *In: Feuto, J., Kpata, B.A. (eds) Harmonic Analysis and Partial Differential Equations. HAAW 2023. Applied and Numerical Harmonic Analysis. Birkhäuser, Cham.* MR4103881 , Zbl 1450.42013, doi: 10.1007/978-3-031-66375-8-2. 1287, 1292, 1294

[11] DJE, TANOH J. M.; FEUTO, JUSTIN; SEHBA, BENOIT F. A generalization of the Carleson lemma and Weighted norm inequaliies for the maximal functions in the Orlicz setting. *J. Math. Anal. Appl.* **491** (2020), no. 1, 24 pp. MR4103881 , Zbl 1450.42013, doi: 10.1016/j.jmaa.2020.124248. 1273

[12] DJE, TANOH J. M.; SEHBA, BENOIT F. Carleson embeddings for Hardy-Orlicz and Bergman-Orlicz spaces of the upper half-plane. *Funct. Approx. Comment. Math.* **64** (2021), no. 2, 163–201. MR4278751 , Zbl 1478.30013, doi: 10.7169/facm/1877. 1272, 1273, 1274, 1275, 1277, 1278, 1287, 1301, 1302, 1303

[13] DJE, TANOH J. M.; SEHBA, BENOIT F. Two-weight inequalities for multilinear maximal functions in Orlicz spaces. *Banach J. Math. Anal.* **17** (2023), no. 2, 47 pp. 1273, 1282
MR4559429, Zbl 1509.42033, doi: 10.1007/s43037-023-00256-3.

[14] DUREN, PETER L. Theory of H^p Spaces, *Pure Appl. Math.*, **38** (1970). Academic Press. MR0268655, Zbl 0215.20203. 1272

[15] GALLARDO, GÓMEZ D. Orlicz spaces for which the Hardy-Littlewood maximal operator is bounded, *Publ. Mat.* **32** (1988), no. 2, 261–266. 1288
MR0975900, Zbl 0685.46016.

[16] HASTINGS, WILLIAM W. A Carleson measure theorem for Bergman spaces. *Proc Amer Math Soc.* **52** (1975), 237–241. MR0374886, Zbl 0296.31009, doi: 10.2307/2040137. 1272

[17] HÖRMANDER, LARS V. L^p-estimates for (pluri)subharmonic functions, *Math. Scand.* **20** (1967), 65–78. MR0234002. 1272

[18] HYTÖNEN, TUOMAS P.; PÉREZ, CARLOS M. Sharp weighted bounds involving A_∞, *Anal. PDE* **6** (2013), no. 4, 777-818. MR3092729, Zbl 1283.42032, doi: 10.2140/apde.2013.6.777. 1284

[19] KOKILASHVILI, VAKHTANG M.; KRBEC, MIROSLAV. Weighted Inequalities in Lorentz and Orlicz spaces, *World Scientific publishing. C.O.Pte.Ltd* (1991). ISBN: 978-981-4360-30-2. MR1156767, Zbl 0751.46021. 1283

[20] LUECKING, DANIEL H. Embedding theorems for spaces of analytic functions via Khintchine's inequality. *Michigan Math J.* **40** (1993), no. 2, 333–358. MR1226835, Zbl 0801.46019, doi: 10.1307/mmj/1029004756. 1272

[21] LUECKING, DANIEL H. A technique for characterizing Carleson measures on Bergman spaces. *Proc. Amer. Math. Soc.* **87** (1983), 656–660. MR0687635, Zbl 0521.32005, doi: 10.1090/S0002-9939-1983-0687635-6. 1272

[22] LUECKING, DANIEL H. Forward and reverse Carleson inequalities for functions in Bergman spaces and their derivatives. *Amer. J. Math.* **107** (1985), no. 1, 85–111. MR0778090, Zbl 0584.46042, doi: 10.2307/2374458. 1272

[23] LUECKING, DANIEL H. Multipliers of Bergman spaces into Lebesgue spaces. *Proc. Edingburgh Math. Soc.* **29** (1986), no. 2, 125–131. MR0829188, Zbl 0587.30048, doi: 10.1017/S001309150001748X. 1272

[24] MASHREGHI, JAVAD. Representation theorems in Hardy spaces, *Cambridge University Press, New York*, (2009). Online ISBN 9780511814525. MR2500010 , Zbl 1169.22001, doi: 10.1017/CBO9780511814525. 1291, 1294

[25] MOCHIZUKI, NOZOMU. Nevanlinna and Smirnov classes on the upper half plane, *Hokkaido Mathematical Journal* **20** (1991) 609-620. MR1134994, Zbl 0760.30016, doi: 10.14492/hokmj/1381413994 . 1293

[26] POWER, STEPHEN C. Hörmander's Carleson theorem for the ball, *Glasgow Math. J.*, **26** (1985), no. 1, 13–17. MR0776671, Zbl 0576.32007. 1272

[27] POTT, SANDRA; REGUERA, RODRIGUEZ M. C. Sharp Békollé estimates for the Bergman Projection.*Journal of Functional Analysis* **265** (2013), 3233-3244. MR3110501 , Zbl 1295.46020, doi: 10.1016/j.jfa.2013.08.018. 1284

[28] RAO, MALEMPATI M.; REN, ZHONG D. Theory of Orlicz Spaces, *Marcel Dekker,INC*, **270** (1991). MR1113700 , Zbl 0724.46032. 1283

[29] SEHBA, BENOIT F. Carleson embeddings with loss for Bergman-Orlicz spaces of the unit ball. *Complex Var. Elliptic Equ.* **66** (2021), no. 11, 1922–1936. MR4334044 , Zbl 1478.32024, doi: 10.1080/17476933.2020.1793967. 1272

[30] SEHBA, BENOIT F. Φ-Carleson measures and multipliers between Bergman–Orlicz spaces of the unit ball of $\mathbb{C}^n$. *J Aust Math Soc.* **104** (2018), no. 1, 63–79. MR3745415 , Zbl 1380.32006. 1272

[31] SEHBA, BENOIT F.; TCHOUNDJA, EDGAR L. Duality for large Bergman-Orlicz spaces Boundedness of Hankel operators, *Complex var. Elliptic Equ.*, **62** (2017). MR3691307 , Zbl 06803001, doi: 10.1080/17476933.2016.1278435. 1273, 1278

[32] SEHBA, BENOIT F.; TCHOUNDJA, EDGAR L. Hankel operators on holomorphic Hardy-Orlicz spaces, *Integr. Equ. Oper. Theory.* **73** (2012), 331–349. MR2945208 , Zbl 1275.47062, doi: 10.1007/s00020-012-1974-8. 1278

[33] SHAKARCHI, RSAMI; STEIN, ELIAS M. Real Analysis III: Measure theory, Integration and Hilbert Spaces. *Princeton University press, Princeton, New Jersey*, 2005. xx+402 pp. MR2129625, Zbl 1081.28001, doi: 10.2307/j.ctvd58v18. 1284

[34] SZAJKOWSKI, JAN. Modular spaces of analytic functions in the half-plane. I. *Funct. Approx. Comment. Math.* **13** (1982), 39–53. MR0817322 , Zbl 0517.46021. 1274

[35] SZAJKOWSKI, JAN. Modular spaces of analytic functions in the half-plane. II. *Funct. Approx. Comment. Math.* **13** (1982),55–76. MR0817322 , Zbl 0517.46022. 1274

[36] UEKI, SEI-ICHIRO. Weighted composition operators between weighted Bergman spaces in the unit ball of $\mathbb{C}^n$, *Nihonkai Math. J.*, **16** (2005), no. 1, 31-48. MR2167712 , Zbl 1097.47026. 1272

[37] VIDENSKII, IL'YA V. An analogue of Carleson measures (Russian), *Dokl. Akad. Nauk.* SSSR298(1988), no.5, 1042–1047; translation in Soviet Math. Dokl. **37** (1988), no. 1, 186-190.. MR0939673, Zbl 0687.30026. 1272

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