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Orientation reversing finite non-abelian actions on $\mathbb{RP}^3$ which respect a Heegaard decomposition

John Kalliongis and Ryo Ohashi

Abstract. We classify, up to equivalence, the orientation-reversing finite non-abelian actions on $\mathbb{RP}^3$ which respect a genus 1 Heegaard decomposition. A description is given for each action along with its quotient type. There are seven different quotient types, and for each quotient type there is only one equivalence class. Examples are given of non-abelian actions which do not respect a genus 1 Heegaard decomposition.

Contents

1. Introduction

The symmetries of manifolds are frequently studied in low-dimensional topology, and one of the fundamental problems concerning symmetries is to classify all the group actions on a given manifold up to equivalence. The following is a list of some of the papers which deal with these questions: [BM84], [KM91], [KM03], [KO18], [KO21a], [KO22], and [Zim19]. The classification (up to conjugation) for symmetries of the orientable and non-orientable 3-dimensional handlebodies of genus one is obtained in [KM91], and in [KO18] a similar classification is obtained for I-bundles over the projective space. For a lens space $L(p,q)$ where $p > 2$, a complete classification of finite group actions which preserve a genus 1 Heegaard decomposition is obtained in [KM03] by restricting the action to an invariant Heegaard torus. If $p = 1$ or 2, then an

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action on $L(p,q)$ may contain an element which when restricted to two different invariant Heegaard tori are not equivalent (See Examples on page 28 in [KM03]). For orientation reversing actions, it is known that the 3-sphere and $\mathbb{RP}^3$ are the only 3-dimensional lens spaces which admit orientation-reversing PL maps of period $4k$ where $k \geq 1$ (see [Kim77]). If M is an elliptic 3-manifold which is not a lens space, then by [NR78] (pp. 188-190), M does not admit an orientation-reversing homeomorphism. Furthermore, no lens space other than the 3-sphere $\mathbb{S}^3$ and $\mathbb{RP}^3$ admits an orientation-reversing involution by [Kwu70]. When $q^2 \equiv -1 \pmod p$, the lens space $L(p,q)$ admits orientation reversing actions. These were classified in [KM08] when $p > 2$ and the action leaves a Heegaard torus invariant whose sides are exchanged by an orientation-reversing element. In [KO22], a classification is given when $p = 2$ for the orientation-reversing abelian actions on $\mathbb{RP}^3$. In [Zim96] and [Zim20], it was indicated that closed 3-manifolds have equivariant genus g Heegaard splittings where strong genus g-actions and maximal order groups are studied.

In this paper, continuing the study for $p = 2$, we consider the orientation-reversing non-abelian actions on the three-dimensional projective space $\mathbb{RP}^3 = L(2,1)$ which preserves a genus 1 Heegaard decomposition. We are able to classify, up to equivalence, these actions and compute their quotient spaces. There are seven different quotient types and an explicit construction is given for each *Standard Quotient Type i Non-Abelian Action* ($1 \leq i \leq 7$), representing each equivalence class. In the last section of the paper, we give examples of actions on $\mathbb{RP}^3$ which do not preserve a genus 1 Heegaard decomposition. We remark that $\mathbb{RP}^3$ is an elliptic 3-manifold with a geometric structure, and we may assume by [DL09] Theorem E, which follows from Perelman's results in [Per02], [Per03a], [Per03b], that a finite action on $\mathbb{RP}^3$ acts as a group of isometries. We work in the PL category.

We will view $\mathbb{RP}^3 = L(2,1)$ as follows: Parameterize the unit disk D^2 as $\{z \in \mathbb{C} \mid ||z|| \leq 1\}$ with boundary S^1. Let V_i be the solid torus $S^1 \times D^2$ for $i = 1, 2$ and define a homeomorphism $\alpha : \partial V_1 \to \partial V_2$ by $\alpha(u,v) = (u^{-1}v^2, v)$ for $(u,v) \in \partial V_1$. Identifying ∂V_1 to ∂V_2 via α, we obtain $\mathbb{RP}^3 = V_1 \cup_\alpha V_2$ which is the lens space $L(2,1)$. We note that $\mathbb{RP}^3$ is isomorphic to the special orthogonal group $SO(3)$, and refer to [Hal03] for details.

A G-action on a manifold M is a monomorphism $\varphi : G \to \mathrm{Homeo}_{PL}(M)$, where the set $\mathrm{Homeo}_{PL}(M)$ is the group of PL-homeomorphisms of M. Two G-actions φ and ψ are equivalent if their images are conjugate in $\mathrm{Homeo}_{PL}(M)$. When G is finite the quotient space is an orbifold which we denote by M/φ. We will always assume G is finite.

Let $\varphi : G \to \mathrm{Homeo}_{PL}(\mathbb{RP}^3)$ be an orientation-reversing non-abelian action which preserves a genus 1 Heegaard splitting $V_1 \cup_\alpha V_2$. Restricting this action to each invariant solid torus V_i determines an orbifold quotient V_i/φ whose Euler number is zero. A complete list of all the handlebody orbifolds whose Euler number is zero is given in [KM91]. For any positive integer n, the orientable orbifolds are denoted by $(A0, n)$ and $(B0, n)$, while the non-orientable ones are denoted by $(A1, n)$,..., $(A3, n)$, $(B1, n)$,..., $(B8, n)$. If X and Y are any of these non-orientable orbifolds having homeomorphic boundaries, and $\xi : \partial X \to \partial Y$ is a homeomorphism, denote by $O_\xi(X, Y)$ or $X \cup_\xi Y$ the orbifold obtained by identifying ∂X to ∂Y via ξ. The orbifolds in the main theorem are obtained by identifying the boundaries of the non-orientable orbifolds via explicitly defined homeomorphisms h_j from [KO21b]. If two groups G_1 and G_2 are isomorphic we may write $G_1 \cong G_2$. For a group H, we use the notation "$/ \sim$" to indicate that a pair of elements in the group have been identified, thus adding a new relation to the presentation and denoting its quotient group by $H/\sim$. If two orbifolds O_1 and O_2 are homeomorphic we may write $O_1 \simeq O_2$.

The main result of our paper is the following theorem, where the presentation of each of these groups may be found in Section 5.

Theorem 5.1. *Let $\varphi : G \to Homeo_{PL}(\mathbb{RP}^3)$ be an orientation-reversing finite non-abelian action which preserves a genus 1 Heegaard decomposition. Then one of the following cases is true where G is isomorphic to G_i for $1 \le i \le 7$:*

1) *$G_1 = (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_{2m})/\sim$, where n and m are even and*
$$\mathbb{RP}^3/\varphi \simeq O_{h_1}((A1, n), (B5, m)).$$
2) *$G_2 = Dih(\mathbb{Z}_n) \circ \mathbb{Z}_m/\sim$ if n and m are even, or $G_2 = \mathbb{Z}_n \circ_{-1} \mathbb{Z}_{2m}$ if n and m are both odd, and*
$$\mathbb{RP}^3/\varphi \simeq O_{h_2}((A3, n), (B4, m))$$
in both cases.
3) *$G_3 = Dih(\mathbb{Z}_n) \times \mathbb{Z}_m/\sim$, where $n > 2$, m are even and*
$$\mathbb{RP}^3/\varphi \simeq O_{h_3}((A2, n), (B3, m)).$$
4) *$G_4 = (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_{2m}) \circ \mathbb{Z}_2/\sim$, where n and m are even and*
$$\mathbb{RP}^3/\varphi \simeq O_{h_4}((B2, n), (B2, m)).$$
5) *$G_5 = Dih(\mathbb{Z}_n) \times Dih(\mathbb{Z}_m)/\sim$, where n and m are even and*
$$\mathbb{RP}^3/\varphi \simeq O_{h_5}((B6, n), (B6, m)).$$
6) *$G_6 = Dih(\mathbb{Z}_m) \circ Dih(\mathbb{Z}_{2n})/\sim$, where $m = n(mod\ 2)$ and*
$$\mathbb{RP}^3/\varphi \simeq O_{h_6}((B7, n), (B7, m)).$$
7) *$G_7 = \mathbb{Z}_n \circ Dih(\mathbb{Z}_{2m})/\sim$, where m and n are even and*
$$\mathbb{RP}^3/\varphi \simeq O_{h_7}((B1, n), (B8, m)).$$

Furthermore, in each individual case i), where $1 \le i \le 7$, φ is equivalent to φ_i, the Standard Quotient Type i Non-Abelian Action.

The arguments used in the proof of this theorem involve listing, up to homeomorphism, the orbifolds having Euler number zero Heegaard decomposition which have a finite fundamental group (These are listed in items 1-7 of the theorem.), and identifying the $\mathbb{Z}_2$-normal subgroups for which the covering corresponding to these groups is $\mathbb{RP}^3$. Next, finite groups acting on $\mathbb{RP}^3$ are explicitly constructed yielding the appropriate quotient space and $\mathbb{Z}_2$-normal subgroup.

The paper is organized as follows. Section 2 is devoted to some introductory remarks and definitions concerning orbifolds and Heegaard decomposition into orbifolds having Euler number zero. Section 3 is devoted to describing the *Standard Quotient Type i Non-Abelian Action* for $1 \leq i \leq 7$ on $\mathbb{RP}^3$ together with their quotient types. We show that for each quotient type, there is only one action up to equivalence. In Section 4, we prove our main result by using the results in Section 3. Examples are given in Section 5 of G-actions on $\mathbb{RP}^3$, both orientation preserving and reversing, which do not preserve a genus 1 Heegaard decomposition. They do, however, leave a one-sided (Heegaard surface) projective plane invariant.

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2. Orbifolds preliminaries and Heegaard decompositions

In this section, we give brief preliminary remarks about orbifolds and non-orientable orbifolds having Euler number zero Heegaard decompositions. We also list which of the orbifolds having Euler number zero Heegaard decomposition have finite fundamental groups in Theorem 2.1.

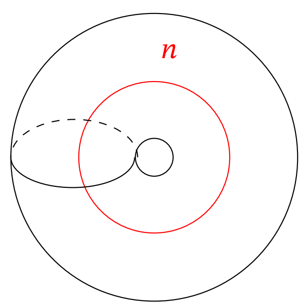
Satake introduced orbifolds in [Sat56] and [Sat57], and Thurston developed the topic more fully in [Thu78]. For more details, we refer the reader to the following references: M. Yokoyama [Yok16]; M. Boileau, S. Maillot and J. Porti [BMP03]; S. Choi [Cho00]; W. Dunbar [Dun88]; D. Cooper, C. Hodgson and S. Kerckhoff [CHK12].

A space which is locally modeled as the quotient space of $\mathbb{R}^n$ by a finite linear group is called an *orbifold*. More specifically, consider $(\widetilde{U}, G)$ where $\widetilde{U} \subseteq \mathbb{R}^n$ is homeomorphic to an open n-ball and G is a finite group of self-diffeomorphisms on $\widetilde{U}$. Let $U = \widetilde{U}/G$ be the quotient space, which is called a *local model*, and $\nu : \widetilde{U} \to U$ denote the quotient map. If $G_{\tilde{x}}$ is the stabilizer for any $\tilde{x} \in \widetilde{U}$ and $G_{\tilde{x}} \neq 1$, then $\nu(\tilde{x})$ is called an *singular point* in U; it may be labelled with the order of $G_{\tilde{x}}$. An *orbifold map* ψ between local models U and U' consists of a pair $(\widetilde{\psi}, \gamma)$, where $\widetilde{\psi} : \widetilde{U} \to \widetilde{U}'$ is a smooth map and $\gamma : G \to G'$ is a group homomorphism such that $\widetilde{\psi}(g(\tilde{x})) = \gamma(g)\widetilde{\psi}(\tilde{x})$ for all $\tilde{x} \in \widetilde{U}$, $g \in G$ and $\nu'\widetilde{\psi} = \psi\nu$. An *orbifold* is a space which consists of local models glued together by orbifold maps. The set of singular points is referred to as the *exceptional set* or the *singular locus*. An *orbifold O with boundary ∂O* is defined similarly by replacing $\mathbb{R}^n$ with the closed half space $\mathbb{R}^n_+$ to obtain local models for $x \in \partial O$. Suppose M is an n-manifold and G is a group of diffeomorphisms which acts properly discontinuously on M, which means that for any compact subset $K \subset M$ the set $\{g \in G \mid g(K) \cap K \neq \emptyset\}$ is finite. Under these assumptions the quotient space M/G is an orbifold. The orbifolds $(A0, n)$ and $(B0, n)$, defined below, are good examples of 3-dimensional orbifolds with Euler number zero. These orbifolds will cover all non-orientable orbifolds used in the genus 1 Heegaard decomposition.

An orbifold handlebody O is formed by gluing together orbifold 0-handles (3-orbifolds covered by the 3-ball B^3) and orbifold 1-handles (products with 2-orbifolds covered by the disk D^2) so that the singular points of the same type are identified. See [KM91] for more details. If the handlebody orbifold is orientable, then the underlying space is a handlebody. When there is a n-sheeted covering space $H \to O$ where H is a handlebody, then the Euler number $\chi(O) = \frac{1}{n}\chi(H)$. Interested readers can check D. Cooper, C. Hodgson and S. Kerckhoff in [CHK12] regarding the Euler number. An *Euler number* $1-g$ *Heegaard decomposition* of an orbifold O is an ordered triple (Σ, O_1, O_2) where $\Sigma \subset O$ is a closed 2-orbifold, O_i is an orbifold handlebody having Euler number $1-g$, $\Sigma = \partial O_i = O_1 \cap O_2$ and $O = O_1 \cup O_2$.

We will be concerned with Euler number zero Heegaard decompositions where the orbifolds O_i, for $i = 1, 2$, will come from the list of the non-orientable orbifolds covered by $(A0, n)$ and $(B0, n)$. See [KM91] for more details. We now describe the orbifolds $(A0, n)$ and $(B0, n)$.

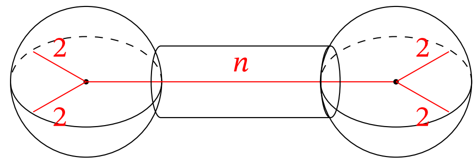
Let $V = S^1 \times D^2$ denote the solid torus, and define a $\mathbb{Z}_n$-action on V by $h(u, v) = (u, ve^{\frac{2\pi i}{n}})$. The orbifold quotient space $V/\langle h \rangle$ is denoted by $V(n)$ or $(A0, n)$. The quotient space is a solid torus containing a core consisting of singular points of order n (See Figure 1).



$(A0, n)$
Figure 1

The orbifold fundamental group is $\pi_1(V(n)) = \langle l_1, m_1 \mid [l_1, m_1] = m_1^n = 1 \rangle \cong \mathbb{Z} \times \mathbb{Z}_n$.

Next, we consider a map $\tau : V(n) \to V(n)$ defined by $\tau(u, v) = (\overline{u}, \overline{v})$, which is an involution. The orbifold $V(n)/\langle \tau \rangle$ is denoted by $(B0, n)$. Its underlying space is a 3-ball which has an exceptional set consisting of an embedded tree with five edges, one edge labeled with n and the other four edges each labeled with 2. The boundary is a Conway sphere with four cone points of order 2 (See Figure 2).



$(B0, n)$
Figure 2

We obtain a covering map $\nu : V(n) \to (B0, n) = V(n)/\langle\tau\rangle$ giving a split exact sequence

$$1 \to \pi_1(V(n)) \to \pi_1((B0, n)) \to \mathbb{Z}_2 \to 1.$$

Let $\nu_*(l_1) = l$ and $\nu_*(m_1) = m$. Since τ inverts both generators of $\pi_1(V(n))$, we obtain the following fundamental groups:

$\pi_1((B0, n)) = \langle l, m, t \mid [l, m] = m^n = t^2 = 1, tlt^{-1} = l^{-1}, tmt^{-1} = m^{-1}\rangle = \mathrm{Dih}(\mathbb{Z} \times \mathbb{Z}_n)$

and $\pi_1(\partial(B0, n)) = \langle l, m, t \mid [l, m] = t^2 = 1, tlt^{-1} = l^{-1}, tmt^{-1} = m^{-1}\rangle = \mathrm{Dih}(\mathbb{Z} \times \mathbb{Z})$.

It follows that $(A0, n)$ will double cover the non-orientable orbifolds $(A1, n)$, $(A2, n)$, $(A3, n)$, $(B3, n)$, $(B4, n)$, and $(B5, n)$; and the orbifold $(B0, n)$ will double cover the non-orientable orbifolds $(B1, n)$, $(B2, n)$, $(B6, n)$, $(B7, n)$, $(B8, n)$. Furthermore, these orbifolds are described along with their fundamental groups in [KM91], [KO21b] and [KO22].

Recall that the notation $O_\xi(X, Y)$ or $X \cup_\xi Y$ is used to denote the orbifold obtained by identifying ∂X to ∂Y via a homeomorphism $\xi : \partial X \to \partial Y$, where X, Y will come from the list of non-orientable orbifolds whose boundaries are homeomorphic, and the gluing map $\xi = h_i$ for $1 \le i \le 7$ defined in [KO21b]. For groups A and B, we adopt the notation $A \circ B$ to indicate the semidirect product $A \rtimes B$, and use $A \circ_{-1} B$ to represent the specific action $bab^{-1} = a^{-1}$ for every $a \in A$ and $b \in B$. Thus the dihedral group $\mathrm{Dih}(\mathbb{Z}_n)$ may be written by $\mathbb{Z}_n \circ_{-1} \mathbb{Z}_2$. Moreover, from [KO21b], we have the following theorem:

Theorem 2.1. *Let X and Y be any of the orbifolds*

$$(A1, n), \cdots, (A3, n), (B1, n), \cdots, (B8, n),$$

and let $\xi : \partial X \to \partial Y$ be a homeomorphism. If $\pi_1(O_\xi(X, Y))$ is finite, then $O_\xi(X, Y)$ is homeomorphic to one of the orbifolds listed in the following table with the corresponding fundamental group.

Orbifolds	Fundamental Group
$O_{h_1}((A1,n),(B5,m))$	$\langle a,b \mid a^2=b^2,\ a^{2m}=b^{2m}=(ba^{-1})^n=1\rangle \cong \mathbb{Z}_n\circ_{-1}\mathbb{Z}_{2m}$
$O_{h_2}((A3,n),(B4,m))$	$\langle a,b,c\vert a^n=b^2=c^{2m}=1, bab^{-1}=a^{-1}, cac^{-1}=a^{-1}, cbc^{-1}=ba\rangle \cong \mathrm{Dih}(\mathbb{Z}_n)\circ\mathbb{Z}_{2m}$
$O_{h_3}((A2,n),(B3,m))$	$\langle a,b,c\vert[a,b]=[a,c]=1, a^m=b^n=c^2=1, cbc^{-1}=b^{-1}\rangle \cong \mathrm{Dih}(\mathbb{Z}_n)\times\mathbb{Z}_m$
$O_{h_4}((B2,n),(B2,m))$	$\langle a,b \mid a^{2n}=b^2=1, ba^2b^{-1}=a^{-2},(ab)^{2m}=1\rangle \cong (\mathbb{Z}_n\circ_{-1}\mathbb{Z}_{2m})\circ\mathbb{Z}_2$
$O_{h_5}((B6,n),(B6,m))$	$\langle a,b,c,d\ \vert a^2=b^2=c^2=(bc)^n=d^2=(ad)^m=1,\{a,d\}\leftrightarrow\{b,c\}\rangle \cong \mathrm{Dih}(\mathbb{Z}_n)\times\mathrm{Dih}(\mathbb{Z}_m)$
$O_{h_6}((B7,n),(B7,m))$	$\langle a,b,c\ \vert a^2=b^{2n}=(ab^{-1}ab)^m=c^2=1, a\leftrightarrow\{b^2,c\}, b^c=b^{-1}\rangle \cong \mathrm{Dih}(\mathbb{Z}_m)\circ\mathrm{Dih}(\mathbb{Z}_{2n})$
$O_{h_7}((B1,n),(B8,m))$	$\langle a,b,c \mid a^n=b^2=c^2=1, bab^{-1}=a^{-1},[a,c]=1,(cb)^{2m}=1\rangle \cong \mathbb{Z}_n\circ\mathrm{Dih}(\mathbb{Z}_{2m})$

Notation: $x^y = yxy^{-1}$, and if x and y commute we write $x \leftrightarrow y$

3. Standard orientation reversing non-Abelian actions on $\mathbb{RP}^3$

In this section, we will define some standard orientation reversing non-abelian actions on $\mathbb{RP}^3$. In addition, we calculate the quotient spaces of these actions. These actions will be sorted by their quotient types, Quotient Type i for $1 \le i \le 7$. A standard action with Quotient Type i will be called the *Standard Quotient Type i Non-Abelian Action.* Since the later cases are similar to the previous cases, some of the details will be omitted. We will begin each quotient type with a brief description of its Euler zero Heegaard decomposition into two orbifolds which are finitely covered by a solid torus. Details may be found in [KO21b] and [KO22].

Quotient Type 1: Orbifold $(A1,n)\cup_{h_1}(B5,m)$

Orbifold $(A1,n)$:

Define maps $a,b:\mathbb{R}\times D^2\to\mathbb{R}\times D^2$ as follows: $a(t,v)=(t-\frac{1}{2},\overline{v})$ and $b(t,v)$ $=(t-\frac{1}{2},\overline{v}e^{\frac{2\pi i}{n}})$. The orbifold $(A1,n)=\mathbb{R}\times D^2/\langle a,b\rangle$ and its fundamental group is

$$\begin{aligned}\pi_1((A1,n)) &= \langle a,b \mid a^2=b^2,(ab^{-1})^n=1\rangle\\ &= \langle ba^{-1}\rangle\circ_{-1}\langle a\rangle = \mathbb{Z}_n\circ_{-1}\mathbb{Z}.\end{aligned}$$

The orbifold $(A1,n)$ is a solid Klein bottle with a simple closed curve core of singular points of order n whose boundary $\partial(A1,n)$ is a Klein bottle.

Orbifold $(B5,m)$:

Let $x,y:\mathbb{R}\times D^2\to\mathbb{R}\times D^2$ be maps defined by $x(t,v)=(-t,ve^{\frac{-\pi i}{m}})$ and $y(t,v)$ $=(-t+1,ve^{\frac{-\pi i}{m}})$. The orbifold $(B5,m)=\mathbb{R}\times D^2/\langle x,y\rangle$ and its fundamental group is

$$\pi_1((B5,m))=\langle x,y \mid x^2=y^2,\ x^{2m}=y^{2m}=1\rangle.$$

The boundary $\partial(B5,m)$ is a Klein bottle.

Orbifold $O_{h_1}((A1,n),(B5,m))$:

There is a homeomorphism $h_1:\partial(A1,n)\to\partial(B5,m)$ inducing an isomorphism which identifies a with x and b with y. This follows from [KO21b] and [KO22]. When we identify $\partial(A1,n)$ to $\partial(B5,m)$ via h_1, we obtain the orbifold

$O_{h_1}((A1,n),(B5,m))$, and its orbifold fundamental group is

$$\begin{aligned}\pi_1(O_{h_1}((A1,n),(B5,m))) &= \langle a,b \mid a^2=b^2, a^{2m}=b^{2m}=(ba^{-1})^n=1\rangle \\ &= \langle ba^{-1}\rangle\circ_{-1}\langle a\rangle = \mathbb{Z}_n\circ_{-1}\mathbb{Z}_{2m}.\end{aligned}$$

We note that both a and b are orientation reversing elements.

Proposition 3.1. *Let H be an orientable normal subgroup of*

$$\pi_1(O_{h_1}((A1,n),(B5,m))) = \mathbb{Z}_n\circ_{-1}\mathbb{Z}_{2m}$$

that is isomorphic to $\mathbb{Z}_2$, and let

$$Q = \pi_1(O_{h_1}((A1,n),(B5,m)))/H$$

be the quotient group. Let $w = ba^{-1}$, and suppose Q is not abelian. Then one of the following is true:

1) *$n>2$ and m are both even, $H=\langle w^{\frac{n}{2}}a^m\rangle$, and*

$$Q = \langle w,a \mid awa^{-1}=w^{-1}, w^n=a^{2m}=1, w^{\frac{n}{2}}a^m=1\rangle/\sim = (\mathbb{Z}_n\circ_{-1}\mathbb{Z}_{2m})/\sim .$$

2) *$n>2$ and m is even, $H=\langle a^m\rangle$, and*

$$Q = \langle w,a \mid awa^{-1}=w^{-1}, w^n=1, a^m=1\rangle = \mathbb{Z}_n\circ_{-1}\mathbb{Z}_m.$$

3) *$n>4$ is even, $H=\langle w^{\frac{n}{2}}\rangle$, and*

$$Q = \langle w,a \mid awa^{-1}=w^{-1}, w^{\frac{n}{2}}=a^{2m}=1\rangle = \mathbb{Z}_{\frac{n}{2}}\circ_{-1}\mathbb{Z}_{2m}.$$

Proof. If $w=ba^{-1}$, we may write $H=\langle w^s a^t\rangle$ where $0\le s<n$ and $o\le t<2m$. Note that $n>2$, otherwise Q is abelian.

Assume first that s and t are both non-zero. Since H is normal, $w^s a^t = a(w^s a^t)a^{-1} = w^{-s}a^t$, which implies $w^{2s}=1$ or $s=\frac{n}{2}$. Note that $1=(w^{\frac{n}{2}}a^t)^2 = a^{2t}$, for either t even or odd. This implies $t=m$, and thus $H=\langle w^{\frac{n}{2}}a^m\rangle$. Suppose m is odd. Then $w^{\frac{n}{2}}a^m = w(w^{\frac{n}{2}}a^m)w^{-1} = w^{(\frac{n}{2}+1)}a^m w^{-1}a^{-m}a^m = w^{(\frac{n}{2}+2)}a^m$. This implies $w^2=1$, and thus $n=2$. This implies $\pi_1(O_{h_1}((A1,n),(B5,m))) = \mathbb{Z}_2\circ_{-1}\mathbb{Z}_{2m}$, which is abelian and ruling out this case. This implies m is even and $Q=\langle w,a \mid awa^{-1}=w^{-1}, w^n=a^{2m}=1, w^{\frac{n}{2}}a^m=1\rangle = (\mathbb{Z}_n\circ_{-1}\mathbb{Z}_{2m})/\sim$. Since Q is not abelian, we must have $n>2$.

Suppose $s=0$ and $t\ne 0$. It follows that $H=\langle a^m\rangle$. If m is odd, then $a^m = wa^m w^{-1} = w^2 a^m$. This implies $n=2$ again giving an abelian quotient Q. Thus m must be even and $Q=\langle w,a \mid awa^{-1}=w^{-1}, w^n=1, a^m=1\rangle = \mathbb{Z}_n\circ_{-1}\mathbb{Z}_m$.

Finally, assume $s\ne 0$ and $t=0$. It follows that $s=\frac{n}{2}$ and $H=\langle a^{\frac{n}{2}}\rangle$. Furthermore, $Q=\langle w,a \mid awa^{-1}=w^{-1}, w^{\frac{n}{2}}=a^{2m}=1\rangle = \mathbb{Z}_{\frac{n}{2}}\circ_{-1}\mathbb{Z}_{2m}$. □

Standard Quotient Type 1 Non-Abelian Action

Let n and s be positive integers with n even. Recall that $w = ba^{-1}$. Consider the subgroup $N = \langle w^{\frac{n}{2}} a^{2s} \rangle$. Now $N \trianglelefteq \pi_1((A1, n))$, which is isomorphic to $\mathbb{Z}$, and $w^{\frac{n}{2}} a^{2s}(t, v) = (t - s, -v)$. There is a covering map $p : \mathbb{R} \times D^2 \to \mathbb{R} \times D^2/N = V_1$, where V_1 is a solid torus $S^1 \times D^2$, defined by $p(t, v) = (e^{\frac{2\pi i t}{s}}, ve^{\frac{\pi i t}{s}})$. The induced maps $a_1, w_1 : V_1 \to V_1$ are defined by $a_1(u, v) = (ue^{\frac{-\pi i}{s}}, u\overline{v}e^{\frac{-\pi i}{2s}})$ and $w_1(u, v) = (u, ve^{\frac{2\pi i}{n}})$. We note that $w_1^{\frac{n}{2}} a_1^{2s} = 1$. The group generated by w_1 and a_1 is denoted by G_1 and

$$\begin{aligned} G_1 &= \langle w_1, a_1 \mid a_1 w_1 a_1^{-1} = w_1^{-1}, w_1^n = a_1^{4s} = 1, w_1^{\frac{n}{2}} a_1^{2s} = 1 \rangle \\ &= (\langle w_1 \rangle \circ_{-1} \langle a_1 \rangle)/\sim = (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_{4s})/\sim. \end{aligned}$$

Note that $V_1/G_1 = (A1, n)$.

Recall that $\mathbb{RP}^3 = V_1 \cup_\alpha V_2$ where $\alpha : \partial V_1 \to \partial V_2$ by $\alpha(u, v) = (u^{-1}v^2, v)$ for $(u, v) \in \partial V_1$. We will show that w_1 and a_1 extend over V_2, thus extending G_1 over $L(2, 1)$ and that $V_2/G_1 = (B5, 2s)$.

Now $\alpha w_1 \alpha^{-1}(u, v) = \alpha w_1(u^{-1}v^2, v) = \alpha(u^{-1}v^2, ve^{\frac{2\pi i}{n}}) = (ue^{\frac{4\pi i}{n}}, ve^{\frac{2\pi i}{n}})$. We also have $\alpha a_1 \alpha^{-1}(u, v) = \alpha a_1(u^{-1}v^2, v_1) = \alpha(u^{-1}v^2 e^{\frac{-\pi i}{s}}, u^{-1}ve^{\frac{-\pi i}{2s}}) = (\overline{u}, \overline{u}ve^{\frac{-\pi i}{2s}})$. Thus, these maps extend to V_2, and so we define w_1 and a_1 on V_2 by $w_1(u, v) = (ue^{\frac{4\pi i}{n}}, ve^{\frac{2\pi i}{n}})$ and $a_1(u, v) = (\overline{u}, \overline{u}ve^{\frac{-\pi i}{2s}})$.

To compute the quotient, since $w_1^{\frac{n}{2}}(u, v) = (u, -v)$, we obtain a covering map $p_1 : V_2 \to V_2/\langle w_1 \rangle = V_2(2)$ defined by $p_1(u, v) = (u^{\frac{n}{2}}, u\overline{v}^2)$. The induced map $a_2 : V_2(2) \to V_2(2)$ is defined by $a_2(u, v) = (\overline{u}, ve^{\frac{\pi i}{s}})$. Let $p_2 : V_2(2) \to V_2(2)/\langle a_2^2 \rangle = V_2(2s)$. The induced map $a_3 : V_2(2s) \to V_2(2s)$ is defined by $a_3(u, v) = (\overline{u}, -v)$. Clearly, $V_2(2s)/\langle a_3 \rangle = (B5, 2s)$. This shows $V_2/G_1 = (B5, 2s)$. Let $m = 2s$.

Summarizing the results above, define homeomorphisms w_1 and a_1 on $\mathbb{RP}^3$ as follows:

$$w_1(u, v) = \begin{cases} (u, ve^{\frac{2\pi i}{n}}), & \text{if } (u, v) \in V_1 \\ (ue^{\frac{4\pi i}{n}}, ve^{\frac{2\pi i}{n}}), & \text{if } (u, v) \in V_2 \end{cases}$$

$$a_1(u, v) = \begin{cases} (ue^{\frac{-2\pi i}{m}}, u\overline{v}e^{\frac{-\pi i}{m}}), & \text{if } (u, v) \in V_1 \\ (\overline{u}, \overline{u}ve^{\frac{-\pi i}{m}}), & \text{if } (u, v) \in V_2. \end{cases}$$

The group generated by w_1 and a_1 is G_1. This defines an action $\varphi_1 : G_1 \to \mathrm{Homeo}_{PL}(\mathbb{RP}^3)$ such that the quotient space $\mathbb{RP}^3/\varphi_1 = (A1, n) \cup_{h_1'} (B5, 2s) = O_{h_1'}((A1, n), (B5, m))$ for some homeomorphism $h_1' : \partial(A1, n) \to \partial(B5, m)$. It follows by Lemma 21 in [KO21b], that $O_{h_1'}((A1,n),(B5,m))$ is homeomorphic to $O_{h_1}((A1,n),(B5,m))$. Composing with a fixed homeomorphism, fix a covering map $\nu_1 : \mathbb{RP}^3 \to O_{h_1}((A1,n),(B5,m))$ and note that $\varphi_1(G_1)$ is the group of covering translations. We call φ_1 a *Standard Quotient Type 1 Non-Abelian Action* on $\mathbb{RP}^3$ with quotient type $O_{h_1}((A1,n),(B5,m))$.

Proposition 3.2. *Let* $\nu : \mathbb{RP}^3 \to O_{h_1}((A1,n),(B5,m))$ *be a regular covering map such that* $\pi_1(O_{h_1}((A1,n),(B5,m)))/\nu_*(\pi_1(\mathbb{RP}^3))$ *is not abelian. Then* $\nu_*(\pi_1(\mathbb{RP}^3)) = \langle w^{\frac{n}{2}}a^m\rangle$ *and* $n > 2$ *and* m *are both even.*

Proof. Let $\nu_*(\pi_1(\mathbb{RP}^3)) = H$ be a subgroup of $\pi_1(O_{h_1}((A1,n),(B5,m)))$. We will show that cases 2) and 3) in Proposition 3.1 cannot happen. Suppose $H = \langle a^m\rangle$ where m is even. Now a^m is identified with x^m, which on the universal covering space of $(B5,m)$ is defined by $x^m(t,v) = (t,-v)$. Since $x^m \in \pi_1((B5,m))$ has a fixed point, it follows by Corollary 5.2 in [KO22] that the covering corresponding to $\langle a^m\rangle$ is not a manifold. Similarly, if $H = \langle w^{\frac{n}{2}}\rangle$, then $w^{\frac{n}{2}}$ acts on the universal covering space of $(A1,n)$ by $w^{\frac{n}{2}}(t,v) = (t,-v)$, which is not fixed-point free. Again applying Corollary 5.2 in [KO22] the covering corresponding to $\langle w^{\frac{n}{2}}\rangle$ is not a manifold. □

Theorem 3.3. *Let* $\varphi : G \to Homeo_{PL}(\mathbb{RP}^3)$ *be a finite non-abelian action such that the quotient space* $\mathbb{RP}^3/\varphi$ *is homeomorphic to* $O_{h_1}((A1,n),(B5,m))$*. Then* $n > 2$ *and* m *are both even and the following is true*:

1) G is isomorphic to $G_1 = \langle w_1, a_1 \mid a_1w_1a_1^{-1} = w_1^{-1}, w_1^n = a_1^{2m} = 1, w_1^{\frac{n}{2}}a_1^m = 1\rangle = (\langle w_1\rangle \circ_{-1} \langle a_1\rangle)/\sim\, = (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_{2m})/\sim$.

2) The action φ *is conjugate to* φ_1*, the Standard Quotient Type* 1 *Non-Abelian Action.*

Proof. $\varphi : G \to Homeo_{PL}(\mathbb{RP}^3)$ be a finite non-abelian action and $\mathbb{RP}^3/\varphi$ an orbifold homeomorphic to $O_{h_1}((A1,n),(B5,m))$. Choosing a fixed homeomorphism, we obtain a covering $\nu : \mathbb{RP}^3 \to O_{h_1}((A1,n),(B5,m))$ where $\varphi(G)$ is the group of covering translations. By Proposition 3.2, $\nu_{1*}(\pi_1(\mathbb{RP}^3)) = \langle w^{\frac{n}{2}}a^m\rangle = \nu_*(\pi_1(\mathbb{RP}^3))$ and $n > 2$ and m are both even. This implies that ν lifts to a homeomorphism k of $\mathbb{RP}^3$ so that the following diagram commutes.

$$\begin{array}{ccc} \mathbb{RP}^3 & \xrightarrow{k} & \mathbb{RP}^3 \\ \downarrow \nu & & \downarrow \nu_1 \\ O_{h_1}((A1,n),(B5,m)) & \xrightarrow{id} & O_{h_1}((A1,n),(B5,m)) \end{array}$$

For any $z \in \mathbb{RP}^3$ and any $\varphi(g)$ for $g \in G$, we have $\nu_1(k\varphi(g)k^{-1})(z) = \nu\varphi(g)k^{-1}(z) = \nu k^{-1}(z) = \nu_1(z)$. Thus $k\varphi(g)k^{-1}$ is a covering translation for

$\nu_1 : \mathbb{RP}^3 \to O_{h_1}((A1,n),(B5,m))$, showing $k\varphi(G)k^{-1} = \varphi_1(G_1)$, proving the result. □

Quotient Type 2: Orbifold $(A3,n) \cup_{h_2} (B4,m)$

Orbifold $(A3,n)$:

The universal covering space of $(A3,n)$ is $\mathbb{R}\times D^2$, and the covering transformation maps a,b,c on $\mathbb{R}\times D^2$ are defined as follows: $a(t,v) = (t, ve^{\frac{2\pi i}{n}})$, $b(t,v) = (t,\overline{v})$ and $c(t,v) = (t+\frac{1}{2}, \overline{v}e^{\frac{-\pi i}{n}})$. The orbifold $(A3,n) = \mathbb{R}\times D^2/\langle a,b,c\rangle$. A computation shows the following: $cac^{-1} = a^{-1}$, $cbc^{-1} = ba$, $bab^{-1} = a^{-1}$, $a^n = b^2 = 1$. Hence the fundamental group is

$$\begin{aligned}\pi_1((A3,n)) &= \langle a,b,c \mid a^n = b^2 = 1, bab^{-1} = a^{-1}, cac^{-1} = a^{-1}, cbc^{-1} = ba\rangle\\ &= (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_2)\circ\mathbb{Z}.\end{aligned}$$

Orbifold $(B4,m)$:

The covering translations on the universal covering space $\mathbb{R}\times D^2$ of $(B4,m)$ are defined as follows: $x(t,v) = (t+1,v)$, $y(t,v) = (t, ve^{\frac{2\pi i}{m}})$, $z(t,v) = (-t, ve^{\frac{2\pi i t}{m}})$. The orbifold $(B4,m) = \mathbb{R}\times D^2/\langle x,y,z\rangle$ and the fundamental group

$$\begin{aligned}\pi_1((B4,m)) &= \langle x,y,z \mid [x,y] = 1, y^m = z^2 = 1, zxz^{-1} = x^{-1}y, zyz^{-1} = y\rangle\\ &= (\mathbb{Z}\times\mathbb{Z}_n)\circ\mathbb{Z}_2.\end{aligned}$$

Orbifold $O_{h_2}((A3,n),(B4,m))$:

There is a homeomorphism $h_2 : \partial(A3,n) \to \partial(B4,m)$ inducing an isomorphism which makes the following identifications: $b = z$, $ba = x^{-2}yz$, $c = zx$ and $c^2 = y$. This follows from [KO21b] and [KO22]. Thus, the orbifold fundamental group $\pi_1(O_{h_2}((A3,n),(B4,m)))$ is

$$\begin{aligned}&\langle a,b,c \mid a^n = b^2 = c^{2m} = 1, bab^{-1} = a^{-1}, cac^{-1} = a^{-1}, cbc^{-1} = ba\rangle\\ &\qquad = (\langle a\rangle \circ_{-1} \langle b\rangle)\circ\langle c\rangle = \mathrm{Dih}(\mathbb{Z}_n)\circ\mathbb{Z}_{2m}.\end{aligned}$$

The elements b and c are orientation reversing elements in the fundamental group.

Proposition 3.4. *Let H be an orientable normal subgroup of the fundamental group $\pi_1(O_{h_2}((A3,n),(B4,m)))$, which is isomorphic to $\mathbb{Z}_2$, and let*

$$Q = \pi_1(O_{h_2}((A3,n),(B4,m)))/H$$

be the quotient group. If Q is not abelian, then one of the following is true:

1) *m is even, $H = \langle c^m\rangle$, and*

$$\begin{aligned}Q &= \langle a,b,c \mid a^n = b^2 = c^{2m} = 1, bab^{-1} = a^{-1}, cac^{-1} = a^{-1}, cbc^{-1} = ba, c^m = 1\rangle\\ &= (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_2)\circ\mathbb{Z}_m.\end{aligned}$$

2) *$n \geq 4$ is even, $H = \langle a^{\frac{n}{2}} \rangle$, and*

$$Q = \langle a, b, c \mid a^n = b^2 = c^{2m} = 1, bab^{-1} = a^{-1}, cac^{-1} = a^{-1}, cbc^{-1} = ba, a^{\frac{n}{2}} = 1 \rangle$$
$$= (\mathbb{Z}_{\frac{n}{2}} \circ_{-1} \mathbb{Z}_2) \circ \mathbb{Z}_{2m}.$$

3) *m and n are both even, $H = \langle a^{\frac{n}{2}} c^m \rangle$, and*

$$Q = \langle a, b, c \mid a^n = b^2 = c^{2m} = 1, bab^{-1} = a^{-1}, cac^{-1} = a^{-1}, cbc^{-1} = ba, a^{\frac{n}{2}} c^m = 1 \rangle$$
$$= (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_2) \circ \mathbb{Z}_{2m} / \sim .$$

4) *$m \geq 1$ and $n \geq 3$ are both odd, $H = \langle a^{\frac{n-1}{2}} bc^m \rangle$, and*

$$Q = \langle a, b, c \mid a^n = b^2 = 1, bab^{-1} = a^{-1}, cac^{-1} = a^{-1}, cbc^{-1} = ba, a^{\frac{n-1}{2}} bc^m = 1 \rangle$$
$$= \mathbb{Z}_n \circ_{-1} \mathbb{Z}_{2m}.$$

Proof. Recall that $cac^{-1} = a^{-1}$, $cbc^{-1} = ba$ and c^2 commutes with both a and b. As the orbifold fundamental group is a semi-direct product, we may write $H = \langle a^s b^\epsilon c^t \rangle$ where $0 \leq s < n$, $\epsilon = 0$ or 1, and $0 \leq t < 2m$.

We first assume $\epsilon = 0$, and so $H = \langle a^s c^t \rangle$. Since H is normal, $a^s c^t = ca^s c^t c^{-1} = a^{-s} c^t$, which indicates $a^{2s} = 1$. Observe that this implies $1 = (a^s c^t)^2 = c^{2t}$ whether t is either even or odd. There are three cases to consider depending on the values of s and t.

Suppose $s = 0$, and therefore $t \neq 0$ and $H = \langle c^t \rangle$. Since $c^{2t} = 1$, it follows that $t = m$ and $H = \langle c^m \rangle$. If m is odd, then $bc^m b^{-1} = ac^m$, contradicting normality. If m is even, then $H = \langle c^m \rangle$ is normal and condition 1) of the proposition satisfied.

Assume $t = 0$, and so $s \neq 0$ and $n \neq 1$. Furthermore, since $a^{2s} = 1$, it follows that $s = \frac{n}{2}$ and $H = \langle a^{\frac{n}{2}} \rangle$. Now $H = \langle a^{\frac{n}{2}} \rangle$, showing condition 2) of the proposition.

Next, we assume $s \neq 0$ and $t \neq 0$, and therefore $n \neq 1$. Since $a^{2s} = 1$ and $c^{2t} = 1$, it follows that $H = \langle a^{\frac{n}{2}} c^m \rangle$. By normality if m is odd, $a^{\frac{n}{2}} c^m = b(a^{\frac{n}{2}} c^m) b^{-1} = a^{\frac{n}{2}} ac^m$, which implies $a = 1$ giving a contradiction. Thus, m is even, and it follows that $H = \langle a^{\frac{n}{2}} c^m \rangle$ is a normal subgroup proving condition 3) in the proposition.

Finally, assume $\epsilon = 1$ and $H = \langle a^s bc^t \rangle$. An argument similar to the above shows that $H = \langle a^{\frac{n-1}{2}} bc^m \rangle$ where m and n are both odd and $Q = \mathbb{Z}_n \circ_{-1} \mathbb{Z}_{2m}$. □

Standard O.R. Quotient Type 2 Non-Abelian Action

Let n and m be positive integers. Consider the subgroup $N = \langle (bc)^n c^{m-n} \rangle$ of $\pi_1((A3, n))$. It follows that $N \trianglelefteq \pi_1((A3, n))$ if $m = n (\text{mod } 2)$. A calculation shows $(bc)^n c^{m-n}(t, v) = (t + \frac{m}{2}, -v)$. If $w = (bc)^n c^{m-n}$, then w is a fixed-point free $\mathbb{Z}$-action on $\mathbb{R} \times D^2$ with quotient space a solid torus V_1. A computation shows the following:

$$(bc)^n c^{m-n} = \begin{cases} a^{\frac{n}{2}} c^m, & \text{if } n \text{ and } m \text{ are both even.} \\ a^{\frac{n-1}{2}} bc^m, & \text{if } n \text{ and } m \text{ are both odd.} \end{cases}$$

Let $p : \mathbb{R} \times D^2 \to \mathbb{R} \times D^2/\langle w \rangle = V_1$ be a covering defined by $p(t, v) = (e^{\frac{4\pi it}{m}}, ve^{\frac{2\pi it}{m}})$. The induced maps a_1, b_1, $c_1 : V_1 \to V_1$ are defined as follows: $a_1(u, v) = (u, ve^{\frac{2\pi i}{n}})$, $b_1(u, v) = (u, u\overline{v})$, $c_1(u, v) = (ue^{\frac{2\pi i}{m}}, u\overline{v}e^{\frac{\pi i}{m}}e^{\frac{-\pi i}{n}})$. The group generated by a_1, b_1 and c_1 is denoted by G_2 and

$$\begin{aligned} G_2 = \langle a_1, b_1, c_1 | a_1^n = b_1^2 = c_1^{2m} = [a_1, c_1] = (b_1c_1)^n c_1^{m-n} = 1, b_1a_1b_1^{-1} = \\ a_1^{-1}, c_1b_1c_1^{-1} = b_1a_1\rangle \\ = ((\langle a_1\rangle \circ_{-1} \langle b_1\rangle) \circ \langle c_1\rangle / \sim = (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_2) \circ \mathbb{Z}_{2m} / \sim . \end{aligned}$$

Note that $V_1/G_2 = (A3, n)$.

We shall see that a_1, b_1 and c_1 extend over V_2, thus extending G_2 over $L(2, 1) = V_1 \cup_\alpha V_2$ and that $V_2/G_2 = (B4, m)$. Now $\alpha a_1 \alpha^{-1}(u, v) = \alpha a_1(u^{-1}v^2, v) = \alpha(u^{-1}v^2, ve^{\frac{2\pi i}{n}}) = (ue^{\frac{4\pi i}{n}}, ve^{\frac{2\pi i}{n}})$. Similar computations show $\alpha b_1 \alpha^{-1}(u, v) = (\overline{u}, \overline{u}v)$ and $\alpha c_1 \alpha^{-1}(u, v) = (\overline{u}e^{\frac{-2\pi i}{n}}, \overline{u}ve^{\frac{\pi i}{m}}e^{\frac{-\pi i}{n}})$. Note that these maps extend to V_2 and so we define a_1, b_1 and c_1 on V_2 by $a_1(u, v) = (ue^{\frac{4\pi i}{n}}, ve^{\frac{2\pi i}{n}})$, $b_1(u, v) = (\overline{u}, \overline{u}v)$ and $c_1(u, v) = (\overline{u}e^{\frac{-2\pi i}{n}}, \overline{u}ve^{\frac{\pi i}{m}}e^{\frac{-\pi i}{n}})$. A computation shows that $(b_1c_1)(u, v) = (ue^{\frac{2\pi i}{n}}, ve^{\frac{\pi i}{m}}e^{\frac{\pi i}{n}})$.

We will need to consider two cases, whether n and m are both even or both odd.

Suppose n and m are both even. In this case, the quotient space $V_2/\langle a_1\rangle$ is an orbifold whose underlying space is a solid torus having a core of singular points of order 2. We obtain a covering map $p_1 : V_2 \to V_2/\langle a_1\rangle = V_2(2)$ defined by $p_1(u, v) = (u^{\frac{n}{2}}, u\overline{v}^2)$. Let b_2 and c_2 be the induced maps on $V_2(2)$. Since $p_1(u^{\frac{2}{n}}, u^{\frac{1}{n}}\overline{v}^{\frac{1}{2}}) = (u, v)$, it follows that $(b_2c_2)(u, v) = p_1(b_1c_1)(u^{\frac{2}{n}}, u^{\frac{1}{n}}\overline{v}^{\frac{1}{2}}) = p_1(u^{\frac{2}{n}}e^{\frac{2\pi i}{n}}, u^{\frac{1}{n}}\overline{v}^{\frac{1}{2}}e^{\frac{\pi i}{m}}e^{\frac{\pi i}{n}}) = (-u, ve^{\frac{-2\pi i}{m}})$. Similarly, $b_2(u, v) = (\overline{u}, v)$. Let $p_2 : V_2(2) \to V_2(2)/\langle b_2c_2\rangle = V_2(m)$ be the covering map defined by $p_2(u, v) = (u^2, uv^{\frac{m}{2}})$. Letting $b_3 : V_2(m) \to V_2(m)$ be the induced map, we see that $b_3(u, v) = (\overline{u}, \overline{u}v)$. It follows that $V_2(m)/\langle b_3\rangle = (B4, m)$.

When n and m are both odd, the case is similar to the preceding one, and so we give a brief outline. The map a_1 is fixed-point free, and so we obtain a manifold covering $p_1 : V_2 \to V_2/\langle a_1\rangle = V_2(1)$ defined by $p_1(u, v) = (u^n, u^{\frac{n-1}{2}}v)$. If b_2 and c_2 are the induced maps on $V_2(1)$, then $(b_2c_2) : V_2(1) \to V_2(1)$ is defined by $(b_2c_2)(u, v) = (u, -ve^{\frac{\pi i}{m}})$. The map $b_2(u, v) = (\overline{u}, \overline{u}v)$. The covering map $p_2 : V_2(1) \to V_2(1)/\langle (b_2c_2)\rangle = V_2(m)$ is defined by $p_2(u, v) = (u, v^m)$. The induced map b_3 on $V_2(m)$ is defined by $b_3(u,v) = (\overline{u}, \overline{u}^m v)$. Since the homeomorphism f of $V_2(m)$ defined by $f(u,v) = (u, u^{\frac{-m+1}{2}} v)$ has the property that $f b_3 f^{-1}(u,v) = (\overline{u}, \overline{u}v)$, it follows that $V_2(m)/\langle b_3\rangle = (B4, m)$.

Summarizing the above results, we define homeomorphisms a_1, b_1 and c_1 on $\mathbb{RP}^3$ as follows:

$$a_1(u,v) = \begin{cases} (u, ve^{\frac{2\pi i}{n}}), & \text{if } (u,v) \in V_1 \\ (ue^{\frac{4\pi i}{n}}, ve^{\frac{2\pi i}{n}}), & \text{if } (u,v) \in V_2 \end{cases}$$

$$b_1(u,v) = \begin{cases} (u, u\overline{v}), & \text{if } (u,v) \in V_1 \\ (\overline{u}, \overline{u}v), & \text{if } (u,v) \in V_2 \end{cases}$$

$$c_1(u,v) = \begin{cases} (ue^{\frac{2\pi i}{m}}, u\overline{v}e^{\frac{\pi i}{m}}e^{\frac{-\pi i}{n}}), & \text{if } (u,v) \in V_1 \\ (\overline{u}e^{\frac{-2\pi i}{n}}, \overline{u}ve^{\frac{\pi i}{m}}e^{\frac{-\pi i}{n}}), & \text{if } (u,v) \in V_2\,. \end{cases}$$

Thus if $m = n (\mathrm{mod}\ 2)$, the group generated by a_1, b_1 and c_1 is G_2. This defines an action $\varphi_2 : G_2 \to \mathrm{Homeo}_{PL}(\mathbb{RP}^3)$ such that $\mathbb{RP}^3/\varphi_2 = (A3, n) \cup_{h_2'} (B4, m) = O_{h_2'}((A3, n), (B4, m))$ for some homeomorphism $h_2' : \partial(A3, n) \to \partial(B4, m)$. It follows by Lemma 23 in [KO21b] that $O_{h_2'}((A3, n), (B4, m))$ is homeomorphic to $O_{h_2}((A3, n), (B4, m))$. Composing with a fixed homeomorphism, fix a covering map $\nu_2 : \mathbb{RP}^3 \to O_{h_2}((A3, n), (B4, m))$ and note that $\varphi_2(G_2)$ is the group of covering translations. We call φ_2 a *Standard Quotient Type 2 Non-Abelian Action* on $\mathbb{RP}^3$ with quotient type orbifold $O_{h_2}((A3, n), (B4, m))$.

Proposition 3.5. *Let $\nu : \mathbb{RP}^3 \to O_{h_2}((A3, n), (B4, m))$ be a regular covering map such that $\pi_1(O_{h_2}((A3, n), (B4, m)))/\nu_*(\pi_1(\mathbb{RP}^3))$ is not abelian and suppose $m = n (mod\ 2)$. Then $\nu_*(\pi_1(\mathbb{RP}^3)) = \langle (b_1c_1)^n c_1^{m-n}\rangle$.*

Proof. If $\nu : \mathbb{RP}^3 \to \mathbb{RP}^3/\varphi = O_{h_2}((A3, n), (B4, m))$ is the covering map, then $\nu_*(\pi_1(\mathbb{RP}^3)) = H$ is a $\mathbb{Z}_2$ normal subgroup of $\pi_1(O_{h_2}((A3, n), (B4, m)))$. We will show that cases 1) and 2) of Proposition 3.4 can be excluded. Suppose $H = \langle c^m \rangle$ where m is even. The homeomorphism h_2 identifies c^2 with y, and thus c^m is identified with $y^{\frac{m}{2}}$. Since y has a fixed point as a map in the universal covering space of $(B4, m)$, it follows by Corollary 5.2 in [KO22] that the covering corresponding to $\langle c^m\rangle$ is not a manifold. If $H = \langle a^{\frac{n}{2}}\rangle$, then a has a fixed point as a map in the universal covering of $(A3, n)$, which is again excluded by Corollary 5.2 in [KO22]. Thus we have shown that $\nu_*(\pi_1(\mathbb{RP}^3)) = \langle a^{\frac{n}{2}} c^m\rangle$ or $\langle a^{\frac{n-1}{2}} bc^m\rangle$. Recall that $(b_1c_1)^n c_1^{m-n} = a^{\frac{n}{2}} c^m$ if n and m are both even, or $a^{\frac{n-1}{2}} bc^m$ if n and m are both odd. □

Theorem 3.6. *Let $\varphi : G \to Homeo_{PL}(\mathbb{RP}^3)$ be a finite non-abelian action such that the quotient space $\mathbb{RP}^3/\varphi$ is homeomorphic to $O_{h_2}((A3,n),(B4,m))$. Then $m = n(\text{mod } 2)$ and the following is true:*

1) G is isomorphic to $G_2 = \langle a_1, b_1, c_1 \mid a_1^n = b_1^2 = c_1^{2m} = 1, [a_1, c_1] = 1, b_1 a_1 b_1^{-1} = a_1^{-1}, c_1 b_1 c_1^{-1} = b_1 a_1, (b_1 c_1)^n c_1^{m-n} = 1\rangle$.

2) $G_2 = (\langle a_1\rangle \circ_{-1} \langle b_1\rangle)\circ\langle c_1\rangle/\sim = Dih(\mathbb{Z}_n)\circ\mathbb{Z}_{2m}/\sim$, if n and m are both even.

3) $G_2 = \langle a_1\rangle\circ_{-1}\langle c_1\rangle = \mathbb{Z}_n\circ_{-1}\mathbb{Z}_{2m}$, if n and m are both odd.

4) The action φ is conjugate to φ_2, the Standard Quotient Type 2 Non-Abelian Action.

Proof. The proof is similar to that of Theorem 3.3 and uses Proposition 3.5. □

Quotient Type 3: Orbifold $(A2,n)\cup_{h_3}(B3,m)$

Orbifold $(A2,n)$: Define the maps a, b, $c : \mathbb{R}\times D^2 \to \mathbb{R}\times D^2$ as follows: $a(t,v) = (t+1,v)$, $b(t,v) = (t, ve^{\frac{2\pi i}{n}})$, $c(t,v) = (t,\overline{v})$. The orbifold $(A2,n) = \mathbb{R}\times D^2/\langle a,b,c\rangle$, and its fundamental group of $(A2,n)$ is

$$\begin{aligned}\pi_1((A2,n)) &= \langle a,b,c \mid b^n = c^2 = [a,b] = [a,c] = 1, cbc^{-1} = b^{-1}\rangle\\ &= (\langle b\rangle\circ_{-1}\langle c\rangle)\times\langle a\rangle = \mathrm{Dih}(\mathbb{Z}_n)\times\mathbb{Z}.\end{aligned}$$

Orbifold $(B3,m)$: On $\mathbb{R}\times D^2$, define maps x, y and z by $x(t,v) = (t+1,v)$, $y(t,v) = (t, ve^{\frac{2\pi i}{m}})$ and $z(t,v) = (-t,v)$. The orbifold $(B3,m) = \mathbb{R}\times D^2/\langle x,y,z\rangle$, with the corresponding fundamental group is

$$\begin{aligned}\pi_1((B3,m)) &= \langle x,y,z \mid y^m = z^2 = [x,y] = [y,z] = 1, zxz^{-1} = x^{-1}\rangle\\ &= (\langle x\rangle\circ_{-1}\langle z\rangle)\times\langle y\rangle = \mathrm{Dih}(\mathbb{Z})\times\mathbb{Z}_m.\end{aligned}$$

Orbifold $O_{h_3}((A2,n),(B3,m))$:

By [KO21b] and [KO22], there is a homeomorphism $h_3 : \partial(A2,n)\to\partial(B3,m)$ inducing an isomorphism which makes the following identifications: $a = y$, $b = x$ and $c = z$. Hence the orbifold fundamental group is

$$\begin{aligned}\pi_1(O_{h_3}((A2,n),(B3,m))) &= \langle a,b,c \mid a^m = b^n = c^2 = [a,b] = [a,c] = 1,\\ &\qquad cbc^{-1} = b^{-1}\rangle\\ &= (\langle b\rangle\circ_{-1}\langle c\rangle)\times\langle a\rangle = \mathrm{Dih}(\mathbb{Z}_n)\times\mathbb{Z}_m.\end{aligned}$$

The element c is an orientation reversing element.

Proposition 3.7. *Let H be an orientable normal subgroup of the fundamental group $\pi_1(O_{h_3}((A2,n),(B3,m)))$, which is isomorphic to $\mathbb{Z}_2$, and let*

$$Q = \pi_1(O_{h_3}((A2,n),(B3,m)))/H$$

be the quotient group. If Q is not abelian, then one of the following is true:

1) m is even, $n > 2$, $H = \langle a^{\frac{m}{2}}\rangle$ and $Q = \langle a,b,c \mid a^m = b^n = c^2 = [a,b] = [a,c] = 1, cbc^{-1} = b^{-1}, a^{\frac{m}{2}} = 1\rangle = (\mathbb{Z}_n\circ_{-1}\mathbb{Z}_2)\times\mathbb{Z}_{\frac{m}{2}}$.

2) $n > 4$ is even, $H = \langle b^{\frac{n}{2}} \rangle$ and $Q = \langle a, b, c \mid a^m = b^n = c^2 = [a, b] = [a, c] = 1, cbc^{-1} = b^{-1}, b^{\frac{n}{2}} = 1 \rangle = (\mathbb{Z}_{\frac{n}{2}} \circ_{-1} \mathbb{Z}_2) \times \mathbb{Z}_m$.

3) m and $n > 2$ are both even, $H = \langle b^{\frac{n}{2}} a^{\frac{m}{2}} \rangle$ and $Q = \langle a, b, c \mid a^m = b^n = c^2 = [a, b] = [a, c] = 1, cbc^{-1} = b^{-1},\ b^{\frac{n}{2}} a^{\frac{m}{2}} = 1 \rangle = (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_2) \times \mathbb{Z}_m / \sim$.

Proof. We may write $H = \langle b^s c^\epsilon a^t \rangle$ where $0 \le s < n$, $\epsilon = 0$ or 1, and $0 \le t < m$. If $\epsilon = 1$, then since H is normal, $b^s c a^t = b(b^s c a^t) b^{-1} = b^{s+2} c a^t$. This implies $b^2 = 1$ or $n = 2$, and thus $\pi_1(O_{h_3}((A2, n), (B3, m)))$ and Q are abelian. Since we are excluding the case where Q abelian, we must have $H = \langle b^s a^t \rangle$.

Suppose $s = 0$ and thus $H = \langle a^t \rangle$. Since $(a^t)^2 = 1$, it follows that $t = \frac{m}{2}$ and $H = \langle a^{\frac{m}{2}} \rangle$. Since Q is abelian if $n = 2$, statement 1) in the proposition follows. If $t = 0$, a similar argument shows $H = \langle b^{\frac{n}{2}} \rangle$, and since Q is not abelian $n > 4$. Assume $s \neq 0$ and $t \neq 0$, and thus $H = \langle b^s a^t \rangle$. Now $1 = (b^s a^t)^2 = b^{2s} a^{2t}$, which implies $s = \frac{n}{2}$ and $t = \frac{m}{2}$. Hence $H = \langle b^{\frac{n}{2}} a^{\frac{m}{2}} \rangle$. The condition $n > 2$ guarantees Q not being abelian. □

Standard O.R. Quotient Type 3 Non-Abelian Action

Let n and s be positive integers with n even. Consider the normal subgroup $\langle b^{\frac{n}{2}} a^s \rangle$ of $\pi_1((A2, n))$. We see that $b^{\frac{n}{2}} a^s(t, v) = (t + s, -v)$. There is a covering map $p : \mathbb{R} \times D^2 \to \mathbb{R} \times D^2 / \langle b^{\frac{n}{2}} a^s \rangle = V_1$ defined by $p(t, v) = (e^{\frac{2\pi i t}{s}}, v e^{\frac{\pi i t}{s}})$. The induced maps $a_1, b_1, c_1 : V_1 \to V_1$ are defined as follows: $a_1(u, v) = (u e^{\frac{2\pi i}{s}}, v e^{\frac{\pi i}{s}})$, $b_1(u, v) = (u, v e^{\frac{2\pi i}{n}})$, $c_1(u, v) = (u, u\overline{v})$. The group generated by a_1, b_1 and c_1 is denoted by G_3 and

$$
\begin{aligned}
G_3 = \langle a_1, b_1, c_1 \mid [a_1, b_1] = [a_1, c_1] = a_1^{2s} = b_1^n = c_1^2 = 1, c_1 b_1 c_1^{-1} = b_1^{-1},\ b_1^{\frac{n}{2}} a_1^s = 1 \rangle \\
= (\langle b_1 \rangle \circ_{-1} \langle c_1 \rangle) \times \langle a_1 \rangle / \sim = (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_2) \times \mathbb{Z}_{2s} / \sim.
\end{aligned}
$$

Note that $V_1 / G_3 = (A2, n)$.

We shall see that a_1, b_1 and c_1 extend over V_2, thus extending G_3 over $L(2, 1) = V_1 \cup_\alpha V_2$ and that $G_3 = (B3, 2s)$.

First,

$$
\begin{aligned}
\alpha a_1 \alpha^{-1}(u, v) &= \alpha a_1(u^{-1} v^2, v) = \alpha(u^{-1} v^2 e^{\frac{2\pi i}{s}}, v e^{\frac{\pi i}{s}}) \\
&= ((u^{-1} v^2 e^{\frac{2\pi i}{s}})^{-1} (v e^{\frac{\pi i}{s}})^2, v e^{\frac{\pi i}{s}}) = (u, v e^{\frac{\pi i}{s}}).
\end{aligned}
$$

Similar computations show that $\alpha b_1 \alpha^{-1}(u, v) = (u e^{\frac{4\pi i}{n}}, v e^{\frac{2\pi i}{n}})$ and $\alpha c_1 \alpha^{-1}(u, v) = (\overline{u}, \overline{u} v)$. Since these maps extend to V_2, we may define a_1, b_1, $c_1 : V_2 \to V_2$ as follows: $a_1(u, v) = (u, v e^{\frac{\pi i}{s}})$, $b_1(u, v) = (u e^{\frac{4\pi i}{n}}, v e^{\frac{2\pi i}{n}})$ and $c_1(u, v) = (\overline{u}, \overline{u} v)$.

Next, we obtain a covering map $p_1 : V_2 \to V_2/\langle b_1\rangle = V_2(2)$ defined by $p_1(u,v) = (u^{\frac{n}{2}}, u\overline{v}^2)$. If a_2 and c_2 are the induced maps on $V_2(2)$, then $a_2(u,v) = (u, ve^{\frac{-2\pi i}{s}})$ and $c_2(u,v) = (\overline{u}, v)$. Further, we obtain a covering map $p_2 : V_2(2) \to V_2(2)/\langle a_2\rangle = V_2(2s)$ defined by $p_2(u,v) = (u, v^s)$. It follows that the induce map c_3 on $V_2(2s)$ is defined by $c_3(u,v) = (\overline{u}, v)$, and thus $V_2(2s)/\langle c_3\rangle = (B3, 2s)$. Let $m = 2s$.

Summarizing the above results, we define homeomorphisms a_1, b_1 and c_1 on $\mathbb{RP}^3 = V_1 \cup_\alpha V_2$ as follows:

$$a_1(u,v) = \begin{cases} (ue^{\frac{4\pi i}{m}}, ve^{\frac{2\pi i}{m}}), & \text{if } (u,v) \in V_1 \\ (u, ve^{\frac{2\pi i}{m}}), & \text{if } (u,v) \in V_2 \end{cases}$$

$$b_1(u,v) = \begin{cases} (u, ve^{\frac{2\pi i}{n}}), & \text{if } (u,v) \in V_1 \\ (ue^{\frac{4\pi i}{n}}, ve^{\frac{2\pi i}{n}}), & \text{if } (u,v) \in V_2 \end{cases}$$

$$c_1(u,v) = \begin{cases} (u, u\overline{v}), & \text{if } (u,v) \in V_1 \\ (\overline{u}, \overline{u}v), & \text{if } (u,v) \in V_2. \end{cases}$$

The group generated by a_1, b_1 and c_1 is G_3. This defines an action $\varphi_3 : G_3 \to \text{Homeo}_{PL}(\mathbb{RP}^3)$ such that $\mathbb{RP}^3/\varphi_3 = (A2,n) \cup_{h_3'} (B3, 2s) = O_{h_3'}((A2,n),(B3,m))$ for some homeomorphism $h_3' : \partial(A2,n) \to \partial(B3,m)$. It follows by Lemma 22 in [KO21b] that $O_{h_3'}((A2,n),(B3,m))$ is homeomorphic to $O_{h_3}((A2,n),(B3,m))$. Composing with a fixed homeomorphism, fix a covering map $\nu_3 : \mathbb{RP}^3 \to O_{h_3}((A2,n),(B3,m))$ and note that $\varphi_3(G_3)$ is the group of covering translations. We call φ_3 a *Standard Quotient Type 3 Non-Abelian Action* on $\mathbb{RP}^3$ with quotient type $O_{h_3}((A2,n),(B3,m))$.

Proposition 3.8. *Let $\nu : \mathbb{RP}^3 \to O_{h_3}((A2,n),(B3,m))$ be a regular covering map such that $\pi_1(O_{h_3}((A2,n),(B3,m)))/\nu_*(\pi_1(\mathbb{RP}^3))$ is not abelian. Then $\nu_*(\pi_1(\mathbb{RP}^3)) = \langle b^{\frac{n}{2}} a^{\frac{m}{2}}\rangle$.*

Proof. If $\nu : \mathbb{RP}^3 \to \mathbb{RP}^3/\varphi = O_{h_3}((A2,n),(B3,m))$ be the covering map, then $\nu_*(\pi_1(\mathbb{RP}^3)) = H$ is a $\mathbb{Z}_2$ normal subgroup of $\pi_1(O_{h_3}((A2,n),(B3m)))$. We will show that the first two cases in Proposition 3.7 are excluded. If $H = \langle a^{\frac{m}{2}}\rangle$, recall that a is identified with y, which has a fixed point as a map in the universal covering space of $(B3,m)$. By Corollary 5.2 in [KO22], the covering corresponding to $a^{\frac{m}{2}}$ is not a manifold. Similarly, if $H = \langle b^{\frac{n}{2}}\rangle$, then b has a fixed point and thus excluded. This leaves case 3) in Proposition 3.7 where $H = \langle b^{\frac{n}{2}} a^{\frac{m}{2}}\rangle = \nu_*(\pi_1(\mathbb{RP}^3))$. □

Theorem 3.9. *Let $\varphi : G \to Homeo_{PL}(\mathbb{RP}^3)$ be a finite non-abelian action such that the quotient space $\mathbb{RP}^3/\varphi = O_{h_3}((A2,n),(B3,m))$. Then m and $n > 2$ are both even and the following is true:*

1) G is isomorphic to $G_3 = \langle a_1, b_1, c_1 \mid a_1^m = b_1^n = c_1^2 = [a_1,b_1] = [a_1,c_1] = 1, c_1b_1c_1^{-1} = b_1^{-1}, b_1^{\frac{n}{2}}a_1^{\frac{m}{2}} = 1\rangle = (\langle b_1\rangle \circ_{-1}\langle c_1\rangle) \times \langle a_1\rangle / \sim = (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_2) \times \mathbb{Z}_m / \sim$.

2) The action φ is conjugate to φ_3, the Standard O.R. Quotient Type 3-Action.

Proof. The proof is similar to that of Theorem 3.3 and uses Proposition 3.8. □

Quotient Type 4: Orbifold $(B2,n) \cup_{h_4} (B2,m)$

Orbifold $(B2,n)$: Define maps $a, b : \mathbb{R}\times D^2 \to \mathbb{R}\times D^2$ by $a(t,v) = (-t + \frac{1}{2}, ve^{\frac{\pi i}{n}})$ and $b(t,v) = (-t, \overline{v})$. The orbifold $(B2,n) = \mathbb{R}\times D^2/\langle a,b\rangle$. The orbifold fundamental group of $(B2,n)$ is

$$\begin{aligned}\pi_1((B2,n)) &= \langle a,b \mid a^{2n} = b^2 = 1, ba^2b^{-1} = a^{-2}\rangle \\ &= (\langle a^2\rangle \circ_{-1} \langle ab\rangle)\circ\langle b\rangle = (\mathbb{Z}_n \circ_{-1} \mathbb{Z})\circ\mathbb{Z}_2.\end{aligned}$$

Note that $b(ab)b^{-1} = a^{-2}(ab)^{-1}$.

Orbifold $(B2,m)$: Define maps $x, y : \mathbb{R}\times D^2 \to \mathbb{R}\times D^2$ by $x(t,v) = (-t + \frac{1}{2}, ve^{\frac{\pi i}{m}})$ and $y(t,v) = (-t, \overline{v})$. The orbifold $(B2,m) = \mathbb{R}\times D^2/\langle x,y\rangle$. The orbifold fundamental group of $(B2,m)$ is

$$\begin{aligned}\pi_1((B2,m)) &= \langle x,y \mid x^{2m} = y^2 = 1, yx^2y^{-1} = x^{-2}\rangle \\ &= (\langle x^2\rangle \circ_{-1} \langle xy\rangle)\circ\langle y\rangle = (\mathbb{Z}_m \circ_{-1} \mathbb{Z})\circ\mathbb{Z}_2.\end{aligned}$$

Orbifold $O_{h_4}((B2,n),(B2,m))$:

There is by [KO22] and [Kim77] a homeomorphism $h_4 : \partial(B2,n) \to \partial(B2,m)$ inducing an isomorphism which makes the following identifications: $a = yx^{-1}$, $b = xyx$. Hence the orbifold fundamental group is

$$\begin{aligned}\pi_1(O_{h_4}((B2,n),(B2,m))) &= \langle a,b \mid a^{2n} = b^2 = 1, ba^2b^{-1} = a^{-2}, (ab)^{2m} = 1\rangle \\ &= (\langle a^2\rangle \circ_{-1} \langle ab\rangle)\circ\langle b\rangle = (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_{2m})\circ\mathbb{Z}_2.\end{aligned}$$

Proposition 3.10. *Let H be an orientable normal subgroup of the fundamental group $\pi_1(O_{h_4}((B2,n),(B2,m)))$, which is isomorphic to $\mathbb{Z}_2$, and let*

$$Q = \pi_1(O_{h_4}((B2,n),(B2,m)))/H$$

be the quotient group. If Q is not abelian, then one of the following is true:

1) m and n are even, $H = \langle a^n(ab)^m\rangle$ and $Q = \langle a,b \mid a^{2n} = b^2 = 1, ba^2b^{-1} = a^{-2}, (ab)^{2m} = 1, a^n(ab)^m = 1\rangle = (\langle a^2\rangle \circ_{-1} \langle ab\rangle)\circ\langle b\rangle / \sim = (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_{2m})\circ\mathbb{Z}_2 / \sim$.

2) m is even, $H = \langle (ab)^m\rangle$ and $Q = \langle a,b \mid a^{2n} = b^2 = 1, ba^2b^{-1} = a^{-2}, (ab)^m = 1\rangle = (\langle a^2\rangle \circ_{-1} \langle ab\rangle)\circ\langle b\rangle = (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_m)\circ\mathbb{Z}_2$.

3) $n \geq 4$ is even, $m \neq 1$, $H = \langle a^n\rangle$ and $Q = \langle a,b \mid a^n = b^2 = 1, ba^2b^{-1} = a^{-2}, (ab)^{2m} = 1\rangle = (\langle a^2\rangle \circ_{-1} \langle ab\rangle)\circ\langle b\rangle = (\mathbb{Z}_{\frac{n}{2}} \circ_{-1} \mathbb{Z}_{2m})\circ\mathbb{Z}_2$.

Proof. Write $H = \langle a^{2s}(ab)^t b^\epsilon\rangle$ where $0 \le s < n$, $0 \le t < 2m$ and $\epsilon = 0$ or 1. Since H is an orientable subgroup, a is orientation reversing but b is orientation preserving, it follows that t must be even.

First, we consider the case when $\epsilon = 0$, and thus $H = \langle a^{2s}(ab)^t\rangle$. Suppose first that $s \ne 0$ and $t \ne 0$. We will also assume $a^{2xs} \ne 1$, since this case will be handled later. Since t is even, and therefore $(ab)^t$ commutes with a^2, we have 1 $= (a^{2s}(ab)^t)^2 = a^{4s}(ab)^{2t}$. This implies $2s = n$ and $t = m$, hence $H = \langle a^n(ab)^m\rangle$. Note that m and n are both even. A computation shows that $b(ab)^2b^{-1} = (ab)^{-2}$ and $a(ab)^2a^{-1} = (ab)^{-2}$. This implies that H is a normal subgroup. Thus $Q = \pi_1(O_{h_4}((B2,n),(B2,m)))/H = \langle a, b \mid a^{2n} = b^2 = 1, ba^2b^{-1} = a^{-2}, (ab)^{2m} = 1, a^n(ab)^m = 1\rangle$ proving 1). Clearly, if $s = 0$ and $t \ne 0$, then $H = \langle (ab)^m\rangle$ and $Q = \langle a, b, \mid a^{2n} = b^2 = 1, ba^2b^{-1} = a^{-2}, (ab)^m = 1\rangle = (\langle a^2\rangle \circ_{-1} \langle ab\rangle)\circ\langle b\rangle = (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_2)\circ\mathbb{Z}_2$. Further, if $s \ne 0$ and $t = 0$, then $H = \langle a^n\rangle$ and $Q = \langle a, b \mid a^n = b^2 = 1, ba^2b^{-1} = a^{-2}, (ab)^{2m} = 1\rangle = (\langle a^2\rangle\circ_{-1}\langle ab\rangle)\circ\langle b\rangle = (\mathbb{Z}_{\frac{n}{2}} \circ_{-1} \mathbb{Z}_{2m})\circ\mathbb{Z}_2$.

Next, suppose $\epsilon = 1$, and thus $H = \langle a^{2s}(ab)^t b\rangle$. We will show that, in this case, assuming H is a normal subgroup leads to a contradiction. Recall that t is even.

Assume first that $s \ne 0$, $t \ne 0$, $a^{2s} \ne 1$ and $(ab)^t \ne 1$. It follows that $(a^{2s}(ab)^t b)^2 = 1$. Consider $a^2(a^{2s}(ab)^t b)a^{-2} = a^{2s+4}(ab)^t b$, which by normality must equal $a^{2s}(ab)^t b$. This implies $n = 2$ and $s = 1$ so that $H = \langle a^2(ab)^t b\rangle$. Conjugating this element by (ab) we see $(ab)(a^2(ab)^t b)(ab)^{-1} = (ab)^{t+2}b$, which must equal $a^2(ab)^t b$. This implies $a^2 = 1$ and $(ab)^2 = 1$, contradicting $a^{2s} \ne 1$ and $(ab)^t \ne 1$.

If we suppose $s = 0$, $t \ne 0$ and $(ab)^t \ne 1$, then $H = \langle (ab)^t b\rangle$. We compute $(ab)(ab)^t b(ab)^{-1} = a^2(ab)^{t+2}b$, and setting this equal to $(ab)^t b$ we see that $(ab)^2 = 1$. This shows $t = m = 1$, contradicting t being even.

Assume now that $s \ne 0$, $t = 0$ and $a^{2s} \ne 1$. Write $H = \langle a^{2s}b\rangle$. Then $a^2(a^{2s}b)a^{-2} = a^{2s+4}b$, which must equal $a^{2s}b$. This implies $a^4 = 1$, $n = 2$ and $s = 1$. Thus $H = \langle a^2 b\rangle$. Conjugating by (ab) we have $(ab)(a^2b)(ab)^{-1} = (ab)^2 b$, which again must equal a^2b. This implies $a^2 = 1$, contradicting $a^{2s} \ne 1$.

Finally, suppose $s = 0$ and $t = 0$, and so $H = \langle b\rangle$. Computing $(ab)b(ab)^{-1}$ $= a^2(ab)^2 b$ and setting it equal to b, we see that $a^2 = (ab)^2 = 1$ resulting $n = m = 1$. Consequently, $\pi_1(O_{h_4}((B2,1),(B2,1))) = \mathbb{Z}_2 \times \mathbb{Z}_2$, which is abelian and ruling out this case. □

Standard O.R. Quotient Type 4 Non-Abelian Action

Let n and s be positive integers with n even. Consider the normal subgroup $N = \langle a^n(ab)^{2s}\rangle$ of $\pi_1((B2,n))$, which is isomorphic to $\mathbb{Z}$. A computation shows that $a^n(ab)^{2s}(t,v) = (t+s, -v)$ for $(u,v) \in \mathbb{R}\times D^2$. Let $p : \mathbb{R}\times D^2 \to \mathbb{R}\times D^2/\langle a^n(ab)^{2s}\rangle = V_1$ be the covering map where V_1 is a solid torus, defined by $p(t,v) = (e^{\frac{2\pi i t}{s}}, ve^{\frac{\pi i t}{s}})$. The induced maps a_1 and b_1 on V_1 are defined by $a_1(u,v)$ $= (\overline{u}e^{\frac{\pi i}{s}}, \overline{u}ve^{\frac{\pi i}{n}}e^{\frac{\pi i}{2s}})$ and $b_1(u,v) = (\overline{u},\overline{v})$. The group generated by a_1 and b_1 is denoted by G_4 and

$$\begin{aligned} G_4 &= \langle a_1, b_1 \mid a_1^{2n} = b_1^2 = (a_1b_1)^{4s} = 1, b_1a_1^2b_1^{-1} = a_1^{-2}, a_1^n(a_1b_1)^{2s} = 1\rangle \\ &= (\langle a_1^2\rangle \circ_{-1} \langle a_1b_1\rangle)\circ\langle b_1\rangle/\sim = (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_{4s})\circ \mathbb{Z}_2/\sim. \end{aligned}$$

Note that $V_1/G_4 = (B2, n)$.

We shall see that a_1, b_1 extend over V_2, thus extending G_4 over $L(2,1) = V_1 \cup_\alpha V_2$ and that $V_2/G_4 = (B2, 2s)$.

First, $\alpha a_1 \alpha^{-1}(u,v) = \alpha a_1(u^{-1}v^2, v) = \alpha(u\overline{v}^2 e^{\frac{\pi i}{s}}, u\overline{v}^2 v e^{\frac{\pi i}{n}} e^{\frac{\pi i}{2s}}) = \alpha(u\overline{v}^2 e^{\frac{\pi i}{s}}, u\overline{v} e^{\frac{\pi i}{n}} e^{\frac{\pi i}{2s}}) = ((u\overline{v}^2 e^{\frac{\pi i}{s}})^{-1}(u\overline{v} e^{\frac{\pi i}{n}} e^{\frac{\pi i}{2s}})^2, u\overline{v} e^{\frac{\pi i}{n}} e^{\frac{\pi i}{2s}}) = (ue^{\frac{2\pi i}{n}}, u\overline{v} e^{\frac{\pi i}{n}} e^{\frac{\pi i}{2s}})$. A similar computation shows $\alpha a_1 \alpha^{-1}(u,v) = (\overline{u}, \overline{v})$. Since these maps extend to V_2, we may define $a_1, b_1 : V_2 \to V_2$ by $a_1(u,v) = (ue^{\frac{2\pi i}{n}}, u\overline{v} e^{\frac{\pi i}{n}} e^{\frac{\pi i}{2s}})$ and $b_2(u,v) = (\overline{u}, \overline{v})$. We check that $a^2(u,v) = (ue^{\frac{4\pi i}{n}}, ve^{\frac{2\pi i}{n}})$. We obtain a covering map $p_1 : V_2 \to V_2/\langle a_1^2\rangle = V_2(2)$ defined by $p_1(u,v) = (u^{\frac{n}{2}}, u\overline{v}^2)$. The induced maps a_2 and b_2 on $V_2(2)$ are defined by $a_2(u,v) = (-u, \overline{v}e^{\frac{-\pi i}{s}})$ and $b_2(u,v) = (\overline{u}, \overline{v})$. Since $(a_2b_2)^2(u,v) = (u, ve^{\frac{-2\pi i}{2s}})$, there is a covering map $p_2 : V_2(2) \to V_2(2)/\langle (a_2b_2)^2\rangle = V_2(2s)$ defined by $p_2(u,v) = (u, v^s)$. The induced maps a_3 and b_3 on $V_2(2s)$ are defined by $a_3(u,v) = (-u, -\overline{v})$ and $b_3(u,v) = (\overline{u}, \overline{v})$. Since $a_3b_3(u,v) = (-\overline{u}, -v)$, it follows that $V_2(2s)/\langle a_3b_3\rangle = (B5, 2s)$. Moreover, if b_4 is the induced map on $(B5, 2s)$, then $(B5, 2s)/\langle b_4\rangle = (B2, 2s)$. Let $m = 2s$.

Summarizing the above results, we define homeomorphisms a_1, b_1 on $\mathbb{RP}^3 = V_1 \cup_\alpha V_2$ as follows:

$$a_1(u,v) = \begin{cases} (\overline{u}e^{\frac{2\pi i}{m}}, \overline{u}ve^{\frac{\pi i}{n}} e^{\frac{\pi i}{m}}), & \text{if } (u,v) \in V_1 \\ (ue^{\frac{2\pi i}{n}}, u\overline{v}e^{\frac{\pi i}{n}} e^{\frac{\pi i}{m}}), & \text{if } (u,v) \in V_2 \end{cases}$$

$$b_1(u,v) = \begin{cases} (\overline{u}, \overline{v}), & \text{if } (u,v) \in V_1 \\ (\overline{u}, \overline{v}), & \text{if } (u,v) \in V_2. \end{cases}$$

The group generated by a_1 and b_1 is G_4. This defines an action $\varphi_4 : G_4 \to \text{Homeo}_{PL}(\mathbb{RP}^3)$ such that the quotient space $\mathbb{RP}^3/\varphi_4 = (B2, n) \cup_{h_4'} (B2, 2s) = O_{h_4'}((B2, n), (B2, m))$ for some homeomorphism $h_4' : \partial(B2, n) \to \partial(B2, m)$. It follows by Lemma 25 in [KO21b] that $O_{h_4'}((B2, n), (B2, m))$ is homeomorphic to $O_{h_4}((B2, n), (B2, m))$. Composing with a fixed homeomorphism, fix a covering map $\nu_4 : \mathbb{RP}^3 \to O_{h_4}((B2, n), (B2, m))$ and note that $\varphi_4(G_4)$ is the group of covering translations. We call φ_4 a *Standard Quotient Type 4 Non-Abelian Action* on $\mathbb{RP}^3$ with quotient type orbifold $O_{h_4}((B2, n), (B2, m))$.

Proposition 3.11. *Let $\nu : \mathbb{RP}^3 \to O_{h_4}((B2, n), (B2, m))$ be a regular covering map such that $\pi_1(O_{h_4}((B2, n), (B2, m)))/\nu_*(\pi_1(\mathbb{RP}^3))$ is not abelian. Then $\nu_*(\pi_1(\mathbb{RP}^3)) = \langle a^n(ab)^m\rangle$ where n and m are both even.*

Proof. If $\nu : \mathbb{RP}^3 \to \mathbb{RP}^3/\varphi = O_{h_4}((B2, n), (B2, m))$ is the covering map, then $\nu_*(\pi_1(\mathbb{RP}^3)) = H$ is a $\mathbb{Z}_2$ normal subgroup of $\pi_1(O_{h_4}((A2, n), (B3m)))$. We will show that the second and third cases in Proposition 3.10 are excluded. If $H = \langle (ab)^m \rangle$, recall that a is identified with yx^{-1} and b is identified with xyx. Thus $(ab)^m$ is identified with x^m where m is even. Since this has a fixed point in the universal covering space of $(B2, m)$, by Corollary 5.2 in [KO22], the covering corresponding to $(ab)^m$ is not a manifold. Since a^2 has a fixed point in the universal covering space of $(B2, n)$ and n is even, again applying Corollary 5.2 in [KO22] that the covering corresponding to a^n is not a manifold. Thus case 1) of Proposition 3.10 applies where $H = \langle a^n(ab)^m \rangle = \nu_*(\pi_1(\mathbb{RP}^3))$. □

Theorem 3.12. *Let $\varphi : G \to Homeo_{PL}(\mathbb{RP}^3)$ be a finite non-abelian action such that the quotient space $\mathbb{RP}^3/\varphi = O_{h_4}((B2, n), (B2, m))$. Then m and n are both even and the following is true:*

1) G is isomorphic to $G_4 = \langle a_1, b_1 | \ a_1^{2n} = b_1^2 = (a_1b_1)^{2m} = 1, b_1a_1^2b_1^{-1} = a_1^{-2}, a_1^n(a_1b_1)^m = 1\rangle = (\langle a_1^2\rangle \circ_{-1} \langle a_1b_1\rangle)\circ\langle b_1\rangle/\sim = (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_{2m})\circ\mathbb{Z}_2/\sim$.

2) The action φ is conjugate to φ_4, the Standard O.R. Quotient Type 4-Action.

Proof. We apply Proposition 3.11, and the proof is similar to that of Theorem 3.3. □

Quotient Type 5: Orbifold $(B6, n) \cup_{h_5} (B6, m)$

Orbifold $(B6, n)$: Define maps $a, b, c, d : \mathbb{R} \times D^2 \to \mathbb{R} \times D^2$ by $a(t, v) = (-t, v)$, $b(t, v) = (t, \overline{v})$, $c(t, v) = (t, \overline{v}e^{\frac{-2\pi i}{n}})$ and $d(t, v) = (-t-1, v)$. We obtain the orbifold $(B6, n) = \mathbb{R} \times D^2/\langle a, b, c, d\rangle$. The orbifold fundamental group of $(B6, n)$ is

$$\pi_1((B6, n)) = \langle a, b, c, d \mid a^2 = b^2 = c^2 = (bc)^n = d^2 = 1, [a, b] = [a, c] = [b, d] = [c, d] = 1\rangle$$
$$= (\langle bc\rangle \circ_{-1} \langle c\rangle) \times (\langle ad\rangle \circ_{-1} \langle d\rangle) = (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_2) \times (\mathbb{Z} \circ_{-1} \mathbb{Z}_2).$$

Orbifold $(B6, m)$:

Define maps $x, y, z, w : \mathbb{R} \times D^2 \to \mathbb{R} \times D^2$ by $x(t, v) = (-t, v)$, $y(t, v) = (t, \overline{v})$, $z(t, v) = (t, \overline{v}e^{\frac{-2\pi i}{m}})$ and $w(t, v) = (-t - 1, v)$. We obtain the orbifold $(B6, m) = \mathbb{R} \times D^2/\langle x, y, z, w\rangle$. The orbifold fundamental group of $(B6, m)$ is

$$\pi_1((B6, m)) = \langle x, y, z, w | x^2 = y^2 = z^2 = (yz)^m = w^2 = 1$$
$$[x, y] = [x, z] = [y, w] = [z, w] = 1\rangle$$
$$= (\langle yz\rangle \circ_{-1} \langle z\rangle) \times (\langle xw\rangle \circ_{-1} \langle w\rangle) = (\mathbb{Z}_m \circ_{-1} \mathbb{Z}_2) \times (\mathbb{Z} \circ_{-1} \mathbb{Z}_2).$$

Orbifold $O_{h_5}((B6, n), (B6, m))$:

By [KO21b] and [KO22], there is a homeomorphism $h_5 : \partial(B6, n) \to \partial(B6, m)$ inducing an isomorphism which makes the following identifications: $a = y$, $b = xwx$, $c = x$ and $d = z$. Hence the orbifold fundamental group is

$$\pi_1(O_{h_5}((B6, n), (B6, m))) = \langle a, b, c, d \mid a^2 = b^2 = c^2 = (bc)^n = d^2 = 1,$$
$$[a, b] = [a, c] = [b, d] = [c, d] = 1, (ad)^m = 1\rangle$$
$$= (\langle bc\rangle \circ_{-1} \langle c\rangle) \times (\langle ad\rangle \circ_{-1} \langle a\rangle)$$
$$= (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_2) \times (\mathbb{Z}_m \circ_{-1} \mathbb{Z}_2).$$

Proposition 3.13. *Let H be an orientable normal subgroup of the fundamental group $\pi_1(O_{h_5}((B6,n),(B6,m)))$, which is isomorphic to $\mathbb{Z}_2$, and let*

$$Q = \pi_1(O_{h_5}((B6,n),(B6,m)))/H$$

be the quotient group. If Q is not abelian, then one of the following is true:

1) m is even, $H = \langle (ad)^{\frac{m}{2}} \rangle$ and $Q = \langle a,b,c,d \mid a^2 = b^2 = c^2 = (bc)^n = d^2 = 1, [a,b] = [a,c] = [b,d] = [c,d] = 1, (ad)^{\frac{m}{2}} = 1\rangle = (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_2) \times (\mathbb{Z}_{\frac{m}{2}} \circ_{-1} \mathbb{Z}_2)$.

2) n is even, $H = \langle (bc)^{\frac{n}{2}} \rangle$ and $Q = \langle a,b,c,d \mid a^2 = b^2 = c^2 = (bc)^{\frac{n}{2}} = d^2 = 1, [a,b] = [a,c] = [b,d] = [c,d] = 1, (ad)^m = 1\rangle = (\mathbb{Z}_{\frac{n}{2}} \circ_{-1} \mathbb{Z}_2) \times (\mathbb{Z}_m \circ_{-1} \mathbb{Z}_2)$.

3) n and m are both even, $H = \langle (bc)^{\frac{n}{2}}(ad)^{\frac{m}{2}} \rangle$ and $Q = \langle a,b,c,d \mid a^2 = b^2 = c^2 = (bc)^n = d^2 = 1, [a,b] = [a,c] = [b,d] = [c,d] = 1, (ad)^m = 1, (bc)^{\frac{n}{2}}(ad)^{\frac{m}{2}} = 1\rangle = (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_2) \times (\mathbb{Z}_m \circ_{-1} \mathbb{Z}_2)/\sim$.

Proof. Write $H = \langle (bc)^s c^{\epsilon_1} (ad)^t a^{\epsilon_2} \rangle$ where $0 \le s < n$, $0 \le t < m$ and $\epsilon_i = 0$ or 1. Since a, b, c and d are all orientation reversing and H is an orientation preserving subgroup, it follows that either $\epsilon_1 = \epsilon_2 = 0$, or $\epsilon_1 = \epsilon_2 = 1$.

Assume first that $\epsilon_1 = \epsilon_2 = 0$ and thus $H = \langle (bc)^s (ad)^t \rangle$. If $s = 0$ and $t \neq 0$, then it follows that $H = \langle (ad)^{\frac{m}{2}} \rangle$ and $Q = (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_2) \times (\mathbb{Z}_{\frac{m}{2}} \circ_{-1} \mathbb{Z}_2)$ giving case 1). If $s \neq 0$ and $t = 0$, then $H = \langle (bc)^{\frac{n}{2}} \rangle$ and $Q = (\mathbb{Z}_{\frac{n}{2}} \circ_{-1} \mathbb{Z}_2) \times (\mathbb{Z}_m \circ_{-1} \mathbb{Z}_2)$ proving case 2). Finally, if $s \neq 0$ and $t \neq 0$, then $H = \langle (bc)^{\frac{n}{2}} (ad)^{\frac{m}{2}} \rangle$ and $Q = (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_2) \times (\mathbb{Z}_m \circ_{-1} \mathbb{Z}_2)/\sim$ proving case 3).

Suppose $\epsilon_1 = \epsilon_2 = 1$ and therefore $H = \langle (bc)^s c (ad)^t a \rangle$. We will show that this case does not occur.

Suppose $s = 0$ and $t \neq 0$, and so $H = \langle c(ad)^t a \rangle$. A computation shows $(ad)[c(ad)^t a](ad)^{-1} = c(ad)^{t+2} a$ and $(bc)[c(ad)^t a](bc)^{-1} = (bc)^2 [c(ad)^t a]$. Since N is a normal subgroup, it follows that $(ad)^2 = 1$ implying $m = 2$, and $(bc)^2 = 1$ implying $n = 2$. Hence would be an abelian group contradicting the assumption of a non abelian quotient Q.

Assume now that $s \neq 0$ and $t = 0$ and hence $H = \langle (bc)^s ca \rangle$. We see that $(bc)[(bc)^s ca](bc)^{-1} = (bc)^{s+2} ca$ and $(ad)[(bc)^s ca](ad)^{-1} = (bc)^s c (ad)^2 a$. This implies $n = m = 2$, again as above excluding this case.

Finally assume $s = t = 0$ and $H = \langle ca \rangle$. Conjugating ca by (bc) and (ad) will show as above that $n = m = 2$, excluding this case also. □

Standard O.R. Quotient Type 5 Non-Abelian Action

Let n and s be a positive integers with n even. Let $N = \langle (bc)^{n/2} (ad)^s \rangle$. Then, $N \trianglelefteq \pi_1((B6,n))$ isomorphic to $\mathbb{Z}$. Computing we have $(bc)^{\frac{n}{2}} (ad)^s (t,v) = (t + s, -v)$. Let $p : \mathbb{R} \times D^2 \to \mathbb{R} \times D^2 / \langle (bc)^{\frac{n}{2}} (ad)^s \rangle = V_1$ be the covering map where V_1 is a solid torus, defined by $p(t,v) = (e^{\frac{2\pi i t}{s}}, v e^{\frac{\pi i t}{s}})$. The induced maps a_1, b_1, c_1 and d_1 on V_1 are defined as follows: $a_1(u,v) = (\overline{u}, \overline{u} v)$, $b_1(u,v) = (u, u\overline{v})$,

$c_1(u,v) = (u, u\overline{v}e^{\frac{-2\pi i}{n}})$ and $d_1(u,v) = (\overline{u}e^{\frac{-2\pi i}{s}}, \overline{u}ve^{\frac{-\pi i}{s}})$. The group generated by a_1, b_1, c_1 and d_1 is denoted by G_5 and

$$\begin{aligned} G_5 = \langle a_1, b_1, c_1, d_1 \mid a_1^2 = b_2^2 = c_1^2 = d_1^2 = (b_1c_1)^n = (a_1d_1)^m = 1, \\ [a_1, b_1] = [a_1, c_1] = [b_1, d_1] = [c_1, d_1] = 1, (b_1c_1)^{\frac{n}{2}}(a_1d_1)^{\frac{m}{2}} = 1\rangle \\ = (\langle b_1c_1\rangle \circ_{-1} \langle c_1\rangle) \times (\langle a_1d_1\rangle \circ_{-1} \langle a_1\rangle)/\sim = (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_2) \times (\mathbb{Z}_m \circ_{-1} \mathbb{Z}_2)/\sim. \end{aligned}$$

It follows that $V_1/G_5 = (B6, n)$. We now extend G_5 over V_2, thus extending G_5 over $L(2,1) = V_1 \cup_\alpha V_2$. Since $\alpha : V_1 \to V_2$ is defined by $\alpha(u,v) = (\overline{u}v^2, v)$, a computation shows the following: $\alpha a_1 \alpha^{-1}(u,v) = (u, u\overline{v})$, $\alpha b_1 \alpha^{-1}(u,v) = (\overline{u}, \overline{u}v)$, $\alpha c_1 \alpha^{-1}(u,v) = (\overline{u}e^{\frac{-4\pi i}{n}}, \overline{u}ve^{\frac{-2\pi i}{n}})$ and $\alpha d_1 \alpha^{-1}(u,v) = (u, u\overline{v}e^{\frac{-\pi i}{s}})$.

Since $b_1c_1(u,v) = (ue^{\frac{4\pi i}{n}}, ve^{\frac{2\pi i}{n}})$, we obtain a covering map $p_1 : V_2 \to V_2/\langle b_1c_1\rangle = V_2(2)$ defined by $p_1(u,v) = (u^{\frac{n}{2}}, u\overline{v}^2)$. The induced maps a_2, b_2 and d_2 on $V_2(2)$ are defined by $a_2(u,v) = (u, \overline{v})$, $b_2(u,v) = (\overline{u}, v)$ and $d_2(u,v) = (u, \overline{v}e^{\frac{2\pi i}{s}})$. A computation shows that $a_2d_2(u,v) = (u, ve^{\frac{-2\pi i}{s}})$. We obtain a covering map $p_2 : V_2(2) \to V_2(2)/\langle a_2d_2\rangle = V_2(2s)$. The induced map a_3 on $V_2(2s)$ is defined by $a_3(u,v) = (u, \overline{v})$. It follows that $V_2(2s)/\langle a_3\rangle = (B6, 2s)$. Let $m = 2s$.

Summarizing the above results, we define homeomorphisms a_1, b_1, c_1 and d_1 on $\mathbb{RP}^3 = V_1 \cup_\alpha V_2$ as follows:

$$a_1(u,v) = \begin{cases} (\overline{u}, \overline{u}v), & \text{if } (u,v) \in V_1 \\ (u, u\overline{v}), & \text{if } (u,v) \in V_2 \end{cases}$$

$$b_1(u,v) = \begin{cases} (u, u\overline{v}), & \text{if } (u,v) \in V_1 \\ (\overline{u}, \overline{u}v), & \text{if } (u,v) \in V_2 \end{cases}$$

$$c_1(u,v) = \begin{cases} (u, u\overline{v}e^{\frac{-2\pi i}{n}}), & \text{if } (u,v) \in V_1 \\ (\overline{u}e^{\frac{-4\pi i}{n}}, \overline{u}ve^{\frac{-2\pi i}{n}}), & \text{if } (u,v) \in V_2 \end{cases}$$

$$d_1(u,v) = \begin{cases} (\overline{u}e^{\frac{-4\pi i}{m}}, \overline{u}ve^{\frac{-2\pi i}{m}}), & \text{if } (u,v) \in V_1 \\ (u, u\overline{v}e^{\frac{-2\pi i}{m}}), & \text{if } (u,v) \in V_2. \end{cases}$$

The group generated by a_1, b_1, c_1 and d_1 is G_5. This defines an action $\varphi_5 : G_5 \to \mathrm{Homeo}_{PL}(\mathbb{RP}^3)$ such that

$$\mathbb{RP}^3/\varphi_5 = (B6, n) \cup_{h_5'} (B6, m) = O_{h_5'}((B6, n), (B6, m))$$

for some homeomorphism $h_5' : \partial(B6, n) \to \partial(B6, m)$. It follows by Lemma 26 in [KO21b] that $O_{h_5'}((B6, n), (B6, m))$ is homeomorphic to $O_{h_5}((B6, n), (B6, m))$. Composing with a fixed homeomorphism, fix a covering map $\nu_5 : \mathbb{RP}^3 \to O_{h_5}((B6, n), (B6, m))$ and note that $\varphi_5(G_5)$ is the group of covering translations. We call φ_5 a *Standard Quotient Type 5 Non-Abelian Action* on $\mathbb{RP}^3$ with quotient type $O_{h_5}((B6, n), (B6, m))$.

Proposition 3.14. *Let* $\nu : \mathbb{RP}^3 \to O_{h_5}((B6,n),(B6,m))$ *be a regular covering map such that* $\pi_1(O_{h_5}((B6,n),(B6,m)))/\nu_*(\pi_1(\mathbb{RP}^3))$ *is not abelian. Then* $\nu_*(\pi_1(\mathbb{RP}^3)) = \langle (bc)^{\frac{n}{2}}(ad)^{\frac{m}{2}} \rangle$ *where* n *and* m *are both even.*

Proof. If $\nu : \mathbb{RP}^3 \to \mathbb{RP}^3/\varphi = O_{h_5}((B6,n),(B6,m))$ is the covering map, then $\nu_*(\pi_1(\mathbb{RP}^3)) = H$ is a $\mathbb{Z}_2$ normal subgroup of $\pi_1(O_{h_5}((B6,n),(B6,m)))$. We will show that the first and second cases in Proposition 3.13 are excluded. If $H = \langle (ad)^{\frac{m}{2}} \rangle$, then note that (ad) is identified with (yz), which has a fixed point in the universal covering space of $(B6,m)$. By Corollary 5.2 in [KO22], the covering corresponding to $(ad)^{\frac{m}{2}}$ is not a manifold. When $H = \langle (bc)^{\frac{n}{2}} \rangle$, we see that (bc) has a fixed point in the universal covering space of $(B6,n)$ excluding this case also. Thus case 3) of Proposition 3.13 applies where $H = \langle (bc)^{\frac{n}{2}}(ad)^{\frac{m}{2}} \rangle = \nu_*(\pi_1(\mathbb{RP}^3))$. □

Theorem 3.15. *Let* $\varphi : G \to Homeo_{PL}(\mathbb{RP}^3)$ *be a finite non-abelian action such that the quotient space* $\mathbb{RP}^3/\varphi = O_{h_5}((B6,n),(B6,m))$*. Then* m *and* n *are both even and the following is true:*

1) G is isomorphic to $G_5 = \langle a_1, b_1, c_1, d_1 \mid a_1^2 = b_2^2 = c_1^2 = d_1^2 = (b_1c_1)^n = (a_1d_1)^m = 1, [a_1,b_1] = [a_1,c_1] = [b_1,d_1] = [c_1,d_1] = 1, (b_1c_1)^{\frac{n}{2}}(a_1d_1)^{\frac{m}{2}} = 1 \rangle$
$= (\langle b_1c_1 \rangle \circ_{-1} \langle c_1 \rangle) \times (\langle a_1d_1 \rangle \circ_{-1} \langle a_1 \rangle)/\sim = (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_2) \times (\mathbb{Z}_m \circ_{-1} \mathbb{Z}_2)/\sim.$

2) The action φ *is conjugate to* φ_5*, the Standard O.R. Quotient Type 5-Action.*

Proof. The proof is similar to that of Theorem 3.3 together with Proposition 3.14 is used. □

Quotient Type 6: Orbifold $(B7,n) \cup_{h_6} (B7,m)$

Orbifold $(B7,n)$: Define maps a, b, c on $\mathbb{R} \times D^2$ as follows: $a(t,v) = (-t-1, v)$, $b(t,v) = (-t, ve^{\frac{\pi i}{n}})$ and $c(t,v) = (t, \overline{v})$. The orbifold $(B7,n) = \mathbb{R} \times D^2/\langle a,b,c \rangle$ and its fundamental group is

$$\begin{aligned}\pi_1((B7,n)) &= \langle a,b,c \mid a^2 = b^{2n} = c^2 = [a,b^2] = [a,c] = 1, cbc^{-1} = b^{-1} \rangle \\ &= (\langle ab^{-1}ab \rangle \circ_{-1} \langle a \rangle) \circ (\langle b \rangle \circ_{-1} \langle c \rangle) = (\mathbb{Z} \circ_{-1} \mathbb{Z}_2) \circ (\mathbb{Z}_{2n} \circ_{-1} \mathbb{Z}_2).\end{aligned}$$

Note also that $bab^{-1} = b^{-1}ab$. For convenience if we let $d = ab^{-1}ab$, then $bdb^{-1} = d^{-1}$, $bab^{-1} = ad$ and $cdc^{-1} = d$. A computation shows that if n and s are both even, then $d^{\frac{s}{2}}b^n = b^{n-s}(ab)^s$. If n and s are both odd, then $d^{\frac{s-1}{2}}ab^n = b^{n-s}(ab)^s$.

Orbifold $(B7,m)$: Define maps x, y, z on $\mathbb{R} \times D^2$ as follows: $x(t,v) = (-t-1, v)$, $y(t,v) = (-t, ve^{\frac{\pi i}{m}})$ and $z(t,v) = (t, \overline{v})$. The orbifold $(B7,m) = \mathbb{R} \times D^2/\langle x,y,z \rangle$ having fundamental group

$$\begin{aligned}\pi_1((B7,m)) &= \langle x,y,z \mid x^2 = y^{2m} = z^2 = [x,y^2] = [x,z] = 1, zyz^{-1} = y^{-1} \rangle \\ &= (\langle xy^{-1}xy \rangle \circ_{-1} \langle x \rangle) \circ (\langle y \rangle \circ_{-1} \langle z \rangle) = (\mathbb{Z} \circ_{-1} \mathbb{Z}_2) \circ (\mathbb{Z}_{2m} \circ_{-1} \mathbb{Z}_2).\end{aligned}$$

Orbifold $O_{h_6}((B7,n),(B7,m))$:

By [KO21b] and [KO22], there is a homeomorphism $h_6 : \partial(B7,n) \to \partial(B7,m)$ inducing an isomorphism which makes the following identifications: $a = z$, $b = zyx$, $c = yxy^{-1}$. Thus the fundamental group of $O_{h_6}((B7,n),(B7,m))$ is

$$\langle a,b,c \mid a^2 = b^{2n} = (ab^{-1}ab)^m = c^2 = [a,b^2] = [a,c] = 1, cbc^{-1} = b^{-1}\rangle$$
$$= (\langle ab^{-1}ab\rangle \circ_{-1} \langle a\rangle) \circ (\langle b\rangle \circ_{-1} \langle c\rangle) = (\mathbb{Z}_m \circ_{-1} \mathbb{Z}_2) \circ (\mathbb{Z}_{2n} \circ_{-1} \mathbb{Z}_2).$$

Proposition 3.16. *Let H be an orientable normal subgroup of the fundamental group $\pi_1(O_{h_6}((B7,n),(B7,m)))$, which is isomorphic to $\mathbb{Z}_2$, and let*

$$Q = \pi_1(O_{h_6}((B7,n),(B7,m)))/H$$

be the quotient group. If Q is not abelian, then one of the following is true:

1) m and n are both even, $H = \langle d^{\frac{m}{2}} b^n\rangle = \langle b^{n-m}(ab)^m\rangle$ and $Q = \langle a,b,c \mid a^2 = b^{2n} = c^2 = (ab^{-1}ab)^m = [a,b^2] = [a,c] = 1, cbc^{-1} = b^{-1}, b^{n-m}(ab)^m = 1\rangle = (\mathbb{Z}_m \circ_{-1} \mathbb{Z}_2) \circ (\mathbb{Z}_{2n} \circ_{-1} \mathbb{Z}_2)/\sim$.

2) n is even, $H = \langle b^n\rangle$ and $Q = \langle a,b,c \mid a^2 = b^{2n} = c^2 = (ab^{-1}ab)^m = [a,b^2] = [a,c] = 1, cbc^{-1} = b^{-1}, b^n = 1\rangle = (\mathbb{Z}_m \circ_{-1} \mathbb{Z}_2) \circ (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_2)$.

3) m is even, $H = \langle d^{\frac{m}{2}}\rangle$ and $Q = \langle a,b,c \mid a^2 = b^{2n} = c^2 = (ab^{-1}ab)^m = [a,b^2] = [a,c] = 1, cbc^{-1} = b^{-1}, d^{\frac{m}{2}} = 1\rangle = (\mathbb{Z}_{\frac{m}{2}} \circ_{-1} \mathbb{Z}_2) \circ (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_2)$.

4) m and n are both odd, $H = \langle b^{n-m}(ab)^m\rangle$ and $Q = \langle a,b,c \mid a^2 = b^{2n} = c^2 = (ab^{-1}ab)^m = [a,b^2] = [a,c] = 1, cbc^{-1} = b^{-1}, b^{n-m}(ab)^m = 1\rangle = (\mathbb{Z}_m \circ_{-1} \mathbb{Z}_2) \circ (\mathbb{Z}_{2n} \circ_{-1} \mathbb{Z}_2)/\sim$.

Proof. The group $H = \langle d^s a^{\epsilon_1} b^t c^{\epsilon_2}\rangle$ is isomorphic to $\mathbb{Z}_2$, where $d = ab^{-1}ab$, $0 \le s < m$, $0 \le t < 2n$ and $\epsilon_i = 0$ or 1. Since a, b and c are orientation reversing elements, it follows that d is orientation preserving. Being that H is an orientation preserving subgroup, we have the following cases to consider: Case I) t is even, and either $\epsilon_1 = \epsilon_2 = 0$ or $\epsilon_1 = \epsilon_2 = 1$; Case II) t is odd, and either $\epsilon_1 = 1$ and $\epsilon_2 = 0$ or $\epsilon_1 = 0$ and $\epsilon_2 = 1$.

Case I: t is even.

We consider first the situation when $\epsilon_1 = \epsilon_2 = 0$, and thus $H = \langle d^s b^t\rangle$. Assume $s \neq 0$ and $t \neq 0$. Since t is even, it follows that b^t commutes with d, and thus $1 = (d^s b^t)^2 = d^{2s} b^{2t}$. This implies $s = \frac{m}{2}$, $t = n$ and $H = \langle d^{\frac{m}{2}} b^n\rangle$. One can verify that H is indeed a normal subgroup showing 1) in the proposition. If $s = 0$ and $t \neq 0$, it follows that $H = \langle b^n\rangle$ where n is even. Furthermore, H is normal giving 2) in the statement of the proposition. Suppose now that $s \neq 0$ and $t = 0$. In this case, m is even and $H = \langle d^{\frac{m}{2}}\rangle$ showing 3).

Assume $\epsilon_1 = \epsilon_2 = 1$ and hence $H = \langle d^s ab^t c\rangle$. If $s = 0$ and $H = \langle ab^t c\rangle$, then we always have $1 = (ab^t c)^2$ giving no new information. By normality, $ab^t c = b(ab^t c)b^{-1} = adb^{t+2}c$, which implies $m = 1$, $b^2 = 1$ and $n = 1$. Since t is even, we must have $t = 0$. Thus $n = m = 1$, $\pi_1(O_{h_6}((B7,1),(B7,1))) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, implying Q abelian, thus we are ruling out this case. We now suppose $t = 0$

and $H = \langle d^s ac\rangle$. It always follows that $1 = (d^s ac)^2$. Normality implies $d^s ac = b(d^s ac)b^{-1} = d^{-s-1}ab^2c = d^{-s-1}b^2ac$. Therefore, $b^2 = 1$, $n = 1$ and $s = \frac{m-1}{2}$. Conjugating by a yields $d^{\frac{m-1}{2}}ac = a(d^{\frac{m-1}{2}}ac)a^{-1} = d^{\frac{1-m}{2}}ac$, which implies $d = 1$ and $m = 1$. Again, we obtain an abelian quotient ruling out this case. Finally, we suppose $s \neq 0$ and $t \neq 0$. In this case, it always follows that $1 = (d^s ab^t c)^2$ so that we do not obtain any new information. By normality, we must have $d^s ab^t c = b(d^s ab^t c)b^{-1} = d^{-s-1}ab^{t+2}c$, which implies $s = \frac{m-1}{2}$, $b^2 = 1$ and $n = 1$. Since $t < 2n$ is even, $t = 0$ giving a contradiction.

<u>Case II</u>: t is odd.

We suppose $\epsilon_1 = 1$ and $\epsilon_2 = 0$, and thus $H = \langle d^s ab^t\rangle$. If $s = 0$ and thus $H = \langle ab^t\rangle$, then $1 = (ab^t)^2 = db^{2t}$. This implies $d = 1$, $m = 1$ and $t = n$. Thus $H = \langle ab^n\rangle$ and $Q = \mathbb{Z}_{2n} \circ_{-1} \mathbb{Z}_2$. Note that $ab^n = b^{n-1}(ab)^1$ since b raised to an even power commutes with a, thus proving 4) for $m = 1$. Suppose $s \neq 0$ and $H = \langle d^s ab^t\rangle$. Since $H \cong \mathbb{Z}_2$, $1 = (d^s ab^t)^2 = d^{2s+1}b^{2t}$, so $s = \frac{m-1}{2}$ implying m is odd, and $t = n$ which is also odd. Thus $H = \langle d^{\frac{m-1}{2}}ab^n\rangle$, and one can check that this is always a normal subgroup. Recall that $d^{\frac{m-1}{2}}ab^n = b^{n-m}(ab)^m$, giving case 5).

Assume now that $\epsilon_1 = 0$ and $\epsilon_2 = 1$, and thus $H = \langle d^s b^t c\rangle$. If $s = 0$ and $H = \langle b^t c\rangle$, then it always follows that $1 = (b^t c)^2$. By normality, $b^t c = a(b^t c)a^{-1} = db^t c$. Thus $d = 1$, $m = 1$ and the orbifold fundamental group is $\mathbb{Z}_2 \times (\mathbb{Z}_{2n} \circ_{-1} \mathbb{Z}_2)$. Again, by normality, $b^t c = c(b^t c)c^{-1} = b^{-t}c$, implying $t = n$. Furthermore, $b^n c = b(b^n c)b^{-1} = b^{n+2}c$. This implies $b^2 = 1$ and $n = 1$. Thus $n = m = 1$ so that $\pi_1(O_{h_6}((B7,1),(B7,1))) = \mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_2)$, which is abelian and thus we exclude this case.

We now suppose $s \neq 0$. A computation shows that $(d^s b^t c)^2 = 1$ is always true. Suppose $m \neq 1$. By normality, $d^s b^t c = a(d^s b^t c)a^{-1} = d^{-s+1}b^t c$, and thus $s = \frac{m+1}{2} \neq 0$ and $H = \langle d^{\frac{m+1}{2}} b^t c\rangle$. Again, by normality, we have $d^{\frac{m+1}{2}} b^t c = b(d^{\frac{m+1}{2}} b^t c)b^{-1} = d^{\frac{-m-1}{2}} b^{t+2}c$, which implies $d^{m+1} = 1$. Hence $d = 1$ and $m = 1$, contradicting the fact that $m \neq 1$. Hence $m = 1$ and $H = \langle b^t c\rangle$. Using normality, we have $b^t c = b(b^t c)b^{-1} = b^{t+2}c$, or $b^2 = 1$. Thus $n = 1$, $\pi_1(O_{h_6}((B7,1),(B7,1))) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ which is abelian and thus we may exclude this case also. □

<u>**Standard O.R. Quotient Type 6 Non-Abelian Action**</u>

Let n and m be positive integers such that $n = m (\mathrm{mod}\, 2)$. Let $N = \langle b^{n-m}(ab)^m\rangle$ be a subgroup of $\pi_1((B7,n))$. Note that $N \trianglelefteq \pi_1((B7,n))$ isomorphic to $\mathbb{Z}$. A calculation shows that $b^{n-m}(ab)^m(t,v) = (t-m, -v)$. We obtain a covering map $p : \mathbb{R} \times D^2 \to \mathbb{R} \times D^2/\langle b^{n-m}(ab)^m\rangle = V_1$ defined by $p(t,v) = (e^{\frac{2\pi i t}{m}}, ve^{\frac{\pi i t}{m}})$. The induced maps a_1, b_1, c_1 on V_1 are defined as follows: $a_1(u,v) = (\overline{u}e^{\frac{-2\pi i}{m}}, \overline{u}ve^{\frac{-\pi i}{m}})$, $b_1(u,v) = (\overline{u}, \overline{u}ve^{\frac{\pi i}{n}})$, $c_1(u,v) = (u, u\overline{v})$. Thus, $d_1(u,v) = (ue^{\frac{-4\pi i}{m}}, ve^{\frac{-2\pi i}{m}})$ where $d_1 = a_1 b_1^{-1} a_1 b_1$. The group generated by a_1, b_1 and c_1 is denoted by G_6 and

$G_6=\langle a_1,b_1,c_1\,|\,a_1^2=b_1^{2n}=c_1^2=[a_1,b_1^2]=[a_1,c_1]=1, c_1b_1c_1^{-1}=b_1^{-1}, b_1^{n-m}(a_1b_1)^m=1\rangle$
$=((\langle d_1\rangle\circ_{-1}\langle a_1\rangle)\circ(\langle b_1\rangle\circ_{-1}\langle c_1\rangle))/\sim\,=(\mathbb{Z}_m\circ_{-1}\mathbb{Z}_2)\circ(\mathbb{Z}_{2n}\circ_{-1}\mathbb{Z}_2)/\sim.$

It follows that $V_1/G_6=(B7,n)$.

Next, we will extend G_6 over V_2, and thus over $L(2,1)=V_1\cup_\alpha V_2$. A computation shows $\alpha a_1\alpha^{-1}(u,v)=\alpha a_1(u^{-1}v^2,v)=\alpha(u\overline{v}^2e^{\frac{-2\pi i}{m}},u\overline{v}e^{\frac{-\pi i}{m}})=(u,u\overline{v}e^{\frac{-\pi i}{m}})$. Similar computations show $\alpha b_1\alpha^{-1}(u,v)=(ue^{\frac{2\pi i}{n}},u\overline{v}e^{\frac{\pi i}{n}})$, and $\alpha c_1\alpha^{-1}(u,v)=(\overline{u},\overline{u}v)$. These maps extend over V_2, thus the maps on V_2 are defined as follows: $a_1(u,v)=(u,u\overline{v}e^{\frac{-\pi i}{m}})$, $b_1(u,v)=(ue^{\frac{2\pi i}{n}},u\overline{v}e^{\frac{\pi i}{n}})$, and $c_1(u,v)=(\overline{u},\overline{u}v)$. A straight forward calculation shows $b_1^2(u,v)=(ue^{\frac{4\pi i}{n}},ve^{\frac{2\pi i}{n}})$. In contrast to all previous types, there are two cases to consider depending on whether m and n are both even or odd.

First, we suppose both m and n are even. Then, $b_1^n(u,v)=(u,-v)$ giving us a fixed point. We obtain a covering map $p_1\,:\,V_2\to V_2/\langle b_1^2\rangle=V_2(2)$ defined by $p_1(u,v)=(u^{\frac{n}{2}},u\overline{v}^2)$. The induced maps on $V_2(2)$ are defined by $a_2(u,v)=(u,\overline{v}e^{\frac{2\pi i}{m}})$, $b_2(u,v)=(-u,\overline{v})$ and $c_2(u,v)=(\overline{u},v)$. Since $b_2a_2(u,v)=(-u,ve^{\frac{-2\pi i}{m}})$, we obtain a covering map $p_2\,:\,V_2(2)\to V_2(2)/\langle b_2a_2\rangle=V_2(m)$ defined by $p_2(u,v)=(u^2,uv^{\frac{m}{2}})$. The induced maps b_3 and c_3 on $V_2(m)$ are defined by $b_3(u,v)=(u,-\overline{v})$ and $c_3(u,v)=(\overline{u},\overline{u}v)$. Now, $c_3b_3(u,v)=(\overline{u},-\overline{v})$ so that $V_2(m)/\langle c_3b_3\rangle=(B0,m)$. We can see that the fixed point set of the map c_3b_3 is $\mathrm{Fix}(c_3b_3)=\{(1,ri),(-1,ri)\,|\,-1\le r\le 1\}$. Further observe that $c_3(1,ri)=(1,ri)$ and $c_3(-1,ri)=(-1,-ri)$. This implies that if c_4 is the induced map on $(B0,m)$, then $(B0,m)/\langle c_4\rangle=(B7,m)=V_2/\langle a_1,b_1,c_1\rangle$.

The second case is when both m and n are odd. As a result, b_1^2 is fixed-point-free and we obtain a covering map $p_1\,:\,V_2\to V_2/\langle b_1^2\rangle=V_2(1)$ defined by $p_1(u,v)=(u^n,u^{\frac{n-1}{2}}v)$. The induced maps on $V_2(1)$ are defined by $a_2(u,v)=(u,u\overline{v}e^{\frac{-\pi i}{m}})$, $b_2(u,v)=(-u,-u\overline{v})$ and $c_2(u,v)=(\overline{u},\overline{u}v)$. The map $b_2a_2(u,v)=(u,-ve^{\frac{\pi i}{m}})$. We obtain a covering map $p_2\,:\,V_2(1)\to V_2(1)/\langle b_2a_2\rangle=V_2(m)$ defined by $p_2(u.v)=(u,v^m)$. The induced maps b_3 and c_3 are defined on $V_2(m)$ as follows: $b_3(u,v)=(u,-u^m\overline{v})$ and $c_3(u,v)=(\overline{u},\overline{u}^m v)$, hence $b_3c_3(u,v)=(\overline{u},-\overline{v})$. A similar argument as the one above shows that $V_2/\langle a_1,b_1,c_1\rangle=(B7,m)$.

Summarizing the above results, we define homeomorphisms a_1, b_1, c_1 on $\mathbb{RP}^3=V_1\cup_\alpha V_2$ as follows:

$$a_1(u,v)=\begin{cases}(\overline{u}e^{\frac{-2\pi i}{m}},\overline{u}ve^{\frac{-\pi i}{m}}), & \text{if } (u,v)\in V_1\\ (u,u\overline{v}e^{\frac{-\pi i}{m}}), & \text{if } (u,v)\in V_2\end{cases}$$

$$b_1(u,v) = \begin{cases} (\bar{u}, \bar{u}ve^{\frac{\pi i}{n}}), & \text{if } (u,v) \in V_1 \\ (ue^{\frac{2\pi i}{n}}, u\bar{v}e^{\frac{\pi i}{n}}), & \text{if } (u,v) \in V_2 \end{cases}$$

$$c_1(u,v) = \begin{cases} (u, u\bar{v}), & \text{if } (u,v) \in V_1 \\ (\bar{u}, \bar{u}v), & \text{if } (u,v) \in V_2. \end{cases}$$

Let $d_1 = a_1 b_1^{-1} a_1 b_1$. The group generated by a_1, b_1, c_1 is G_6. This defines an action $\varphi_6 : G_6 \to \mathrm{Homeo}_{PL}(\mathbb{RP}^3)$ such that $\mathbb{RP}^3/\varphi_6 = (B7,n) \cup_{h_6'} (B7,m) = O_{h_6'}((B7,n),(B7,m))$ for some homeomorphism $h_6' : \partial(B7,n) \to \partial(B7,m)$. It follows by Lemma 27 in [KO21b] that $O_{h_6'}((B7,n),(B7,m))$ is homeomorphic to $O_{h_6}((B7,n),(B7,m))$. Composing with a fixed homeomorphism, fix a covering map $\nu_6 : \mathbb{RP}^3 \to O_{h_6}((B7,n),(B7,m))$ and note that $\varphi_6(G_6)$ is the group of covering translations. We call φ_6 a *Standard Quotient Type 6 Non-Abelian Action* on $\mathbb{RP}^3$ with quotient type orbifold $O_{h_6}((B7,n),(B7,m))$.

Proposition 3.17. *Let $\nu : \mathbb{RP}^3 \to O_{h_6}((B7,n),(B7,m))$ be a regular covering map such that $\pi_1(O_{h_6}((B7,n),(B7,m)))/\nu_*(\pi_1(\mathbb{RP}^3))$ is not abelian. Then $\nu_*(\pi_1(\mathbb{RP}^3)) = \langle b^{n-m}(ab)^m\rangle$ where $m = n(mod\ 2)$.*

Proof. If $\nu : \mathbb{RP}^3 \to \mathbb{RP}^3/\varphi = O_{h_6}((B7,n),(B7,m))$ is the covering map, then $\nu_*(\pi_1(\mathbb{RP}^3)) = H$ is a $\mathbb{Z}_2$ normal subgroup of $\pi_1(O_{h_6}((B7,n),(B7,m)))$. We will show that cases 2) and 3) in by Proposition 3.16 are excluded. If $H = \langle b^n\rangle$ where n is even, then $b^n(t,v) = (t,-v)$, which has a fixed point. By Corollary 5.2 in [KO22], the covering corresponding to b^n is not a manifold. Suppose $H = \langle d^{\frac{m}{2}}\rangle$. Note that $d = ab^{-1}ab$ is identified with $z(zyx)^{-1}z(zyx) = zx^{-1}y^{-1}z^{-1}yx = x^{-1}zy^{-1}zyx = x^{-1}y^2x = y^2$. Since $y^2(t,v) = (t, ve^{\frac{2\pi i}{m}})$, it follows that $d^{\frac{m}{2}}(t,v) = (t,-v)$, which has a fixed point. Thus we also exclude this case by use of Corollary 5.2 in [KO22]. Thus cases 1) and 4) of Proposition 3.16 apply where $H = \langle b^{n-m}(ab)^m\rangle = \nu_*(\pi_1(\mathbb{RP}^3))$ and $m = n(\mathrm{mod}\ 2)$. □

Theorem 3.18. *Let $\varphi : G \to Homeo_{PL}(\mathbb{RP}^3)$ be a finite non-abelian action such that the quotient space $\mathbb{RP}^3/\varphi = O_{h_6}((B7,n),(B7,m))$. Then, $m = n(mod\ 2)$ and the following is true:*

1) $G_6 = \langle a_1, b_1, c_1 \mid a_1^2 = b_1^{2n} = c_1^2 = [a_1, b_1^2] = [a_1, c_1] = b_1^{n-m}(a_1b_1)^m = 1,$
$c_1b_1c_1^{-1} = b_1^{-1}\rangle$
$= (\langle d_1\rangle \circ_{-1} \langle a_1\rangle) \circ (\langle b_1\rangle \circ_{-1} \langle c_1\rangle)/\sim = (\mathbb{Z}_m \circ_{-1} \mathbb{Z}_2) \circ (\mathbb{Z}_{2n} \circ_{-1} \mathbb{Z}_2)/\sim$, where $d_1 = a_1b_1^{-1}a_1b_1$.

2) The action φ is conjugate to φ_6, the Standard O.R. Quotient Type 6-Action.

Proof. The proof is similar to that of Theorem 3.3 and uses Proposition 3.17. □

Quotient Type 7: Orbifold $(B1,n)\cup_{h_7}(B8,m)$

Orbifold $(B1,n)$: Define maps a, b, c on $\mathbb{R}\times D^2$ as follows: $a(t,v)=(t,ve^{\frac{2\pi i}{n}})$, $b(t,v)=(-t,\overline{v})$, and $c(t,v)=(\frac{1}{2}-t,v)$. The orbifold $(B1,n)=\mathbb{R}\times D^2/\langle a,b,c\rangle$ and its fundamental group is

$$\begin{aligned}\pi_1((B1,n)) &= \langle a,b,c \mid a^n=b^2=c^2=[a,c]=1, bab^{-1}=a^{-1}\rangle\\ &= \langle a\rangle\circ(\langle b\rangle * \langle c\rangle)=\mathbb{Z}_n\circ(\mathbb{Z}_2*\mathbb{Z}_2).\end{aligned}$$

Orbifold $(B8,m)$: Define maps x, y and z on $\mathbb{R}\times D^2$ as follows: $x(t,v)=(-t,,ve^{\frac{\pi i}{m}})$, $y(t,v)=(t,\overline{v})$ and $z(t,v)=(1-t,ve^{\frac{\pi i}{m}})$. We remark that the three maps are orientation reversing. The orbifold $(B8,m)=\mathbb{R}\times D^2/\langle x,y,z\rangle$ and the corresponding fundamental group is

$$\begin{aligned}\pi_1((B8,m)) &= \langle x,y,z \mid x^{2m}=y^2=z^{2m}=1, yxy^{-1}=x^{-1}, yzy^{-1}=z^{-1},\\ &\qquad x^2=z^2\rangle\\ &= \langle xz^{-1}\rangle\circ(\langle x\rangle\circ_{-1}\langle y\rangle)=\mathbb{Z}\circ(\mathbb{Z}_{2m}\circ_{-1}\mathbb{Z}_2).\end{aligned}$$

Orbifold $O_{h_7}((B1,n),(B8,m))$:

Again by [KO21b] and [KO22], there is a homeomorphism $h_7:\partial(B1,n)\to\partial(B8,m)$ inducing an isomorphism which makes the following identifications: $a=(xz^{-1})^{-1}=zx^{-1}$, $b=yx$ and $c=y$. As a result, the fundamental group of this quotient orbifold becomes

$$\begin{aligned}\pi_1(O_{h_7}((B1,n),(B8,m))) &= \langle a,b,c \mid a^n=b^2=c^2=[a,c]=1, bab^{-1}=a^{-1},\\ &\qquad (cb)^{2m}=1\rangle\\ &= \langle a\rangle\circ(\langle b\rangle * \langle c\rangle/\langle (cb)^{2m}\rangle)=\langle a\rangle\circ(\langle cb\rangle\circ_{-1}\langle c\rangle/\langle (cb)^{2m}\rangle)\\ &= \mathbb{Z}_n\circ\mathrm{Dih}(\mathbb{Z}_{2m}),\end{aligned}$$

where c is an orientation reversing element.

Proposition 3.19. *Let H be an orientable normal subgroup of the fundamental group $\pi_1(O_{h_7}((B1,n),(B8,m)))$, which is isomorphic to $\mathbb{Z}_2$, and let*

$$Q=\pi_1(O_{h_7}((B1,n),(B8,m)))/H$$

be the quotient group. If Q is not abelian, then one of the following is true:

1) Either $m>1$ and n is even or $m=1$ and $n\geq 6$ is even, $H=\langle a^{\frac{n}{2}}\rangle$ and $Q=\langle a,b,c \mid a^n=b^2=c^2=[a,c]=1, bab^{-1}=a^{-1}, (cb)^{2m}=a^{\frac{n}{2}}=1\rangle=\mathbb{Z}_{\frac{n}{2}}\circ Dih(\mathbb{Z}_{2m})$.

2) m is even and either $n>2$ or $m\geq 4$, $H=\langle (cb)^m\rangle$ and $Q=\langle a,b,c \mid a^n=b^2=c^2=[a,c]=1, bab^{-1}=a^{-1}, (cb)^m=1\rangle=\mathbb{Z}_n\circ Dih(\mathbb{Z}_m)$.

3) m and n are both even, $H=\langle a^{\frac{n}{2}}(cb)^m\rangle$ and $Q=\langle a,b,c \mid a^n=b^2=c^2=[a,c]=1, bab^{-1}=a^{-1}, (cb)^{2m}=a^{\frac{n}{2}}(cb)^m=1\rangle=\mathbb{Z}_n\circ Dih(\mathbb{Z}_{2m})/\sim$.

Proof. The group $H=\langle a^s(cb)^tc^\epsilon\rangle$ is isomorphic to $\mathbb{Z}_2$, where $0\leq s<n$, $0\leq t<2m$ and $\epsilon=0$ or 1. Since c reverses the orientation, either t is even and $\epsilon=0$, or t is odd and $\epsilon=1$.

Case I: t is even and $\epsilon = 0$.

If $t = 0$ and $s \neq 0$, then $H = \langle a^s \rangle$ where $a^{2s} = 1$, giving us $s = \frac{n}{2}$. Since $bab^{-1} = a^{-1}$ and $[a, c] = 1$, H is normal and Q is isomorphic to $\mathbb{Z}_{\frac{n}{2}} \circ \mathrm{Dih}(\mathbb{Z}_{2m})$. The condition in 1) is needed for Q to be a non-abelian group.

On the other hand, if $t \neq 0$ and $s = 0$, then $H = \langle (cb)^t \rangle$. As $(cb)^{2t} = 1$, we have $t = m$. Observe $[a, (cb)^2] = 1$ and $(cb)^2$ is inverted by means of conjugation by b and c respectively. Since $t = m$ is even, it follows that H is normal. Furthermore, Q is isomorphic to $\mathbb{Z}_n \circ \mathrm{Dih}(\mathbb{Z}_m)$ and the condition in 2) is needed for Q to be a non-abelian group.

Now, suppose $s \neq 0$ and $t \neq 0$, hence $H = \langle a^s (cb)^t \rangle$. It is easy to check $a^s (cb)^t a^s (cb)^t = s^{2s} (cb)^{2t}$, as t is an even number, which must equal 1. It follows that $s = \frac{n}{2}$ and $t = m$. Moreover, $[a, (cb)^2] = [b, a^{\frac{n}{2}} (cb)^m] = [c, a^{\frac{n}{2}} (cb)^m] = 1$ showing H is normal and giving 3).

Case II: t is odd and $\epsilon = 1$.

In this case, we have $H = \langle (cb)^t c \rangle$ or $H = \langle a^s (cb)^t c \rangle$ depending on $s = 0$ or $s \neq 0$ respectively. However, regardless to the value of s, the normality condition of H forces us to conclude $n = 2$ and $m = 1$ (See [KO22], Proposition 5.16 Case II of proof). Thus, $\pi_1(O_{h_7}((B1, 2), (B8, 1))) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ resulting in Case II being excluded. □

Standard O.R. Quotient Type 7 Non-Abelian Action

Let n and s be positive integers such that $n = 0 (\mathrm{mod}\ 2)$. Let $N = \langle a^{\frac{n}{2}} (cb)^{2s} \rangle$ be a subgroup of $\pi_1((B1, n))$. We see $N \trianglelefteq \pi_1((B1, n))$ isomorphic to $\mathbb{Z}$. Simple calculations show that $a^{\frac{n}{2}} (cb)^{2s} (t, v) = (t + s, -v)$, and we obtain a covering map $p : \mathbb{R} \times D^2 \to \mathbb{R} \times D^2 / \langle a^{\frac{n}{2}} (cb)^{2s} \rangle = V_1$ defined by $p(t, v) = (e^{\frac{2\pi i t}{s}}, v e^{\frac{\pi i t}{s}})$. The induced maps a_1, b_1, c_1 on V_1 are defined as follows: $a_1(u, v) = (u, v e^{\frac{2\pi i}{n}})$, $b_1(u, v) = (\overline{u}, \overline{v})$, $c_1(u, v) = (\overline{u} e^{\frac{\pi i}{s}}, \overline{u} v e^{\frac{\pi i}{2s}})$. We use G_7 to denote the group generated by a_1, b_1 and c_1, we obtain

$$
\begin{aligned}
G_7 &= \langle a_1, b_1, c_1 \mid a_1^n = b_1^2 = c_1^2 = [a_1, c_1] = 1, b_1 a_1 b_1^{-1} = a_1^{-1}, a_1^{\frac{n}{2}} (c_1 b_1)^{2s} = 1 \rangle \\
&= \langle a_1 \rangle \circ (\langle b_1 \rangle * \langle c_1 \rangle) = \mathbb{Z}_n \circ (\mathbb{Z}_2 * \mathbb{Z}_2) / \sim .
\end{aligned}
$$

It follows that $V_1 / G_7 = (B1, n)$.

The next step is to extend G_7 over V_2 and thus over $L(2, 1) = V_1 \cup_\alpha V_2$. Recall that the attaching map $\alpha : \partial V_1 \to \partial V_2$ is defined by $\alpha(u, v) = (u^{-1} v^2, v)$, hence $\alpha a_1 \alpha^{-1}(u, v) = \alpha a_1 (u^{-1} v^2, v) = \alpha(u^{-1} v^2, v e^{\frac{2\pi i}{n}}) = (u e^{\frac{4\pi i}{n}}, v e^{\frac{2\pi i}{n}})$. Likewise, $\alpha b_1 \alpha^{-1}(u, v) = (\overline{u}, \overline{v})$ and $\alpha c_1 \alpha^{-1}(u, v) = (u, u \overline{v} e^{\frac{\pi i}{2s}})$. These maps extend to V_2 and we relabel them a_1, b_1 and c_1 respectively. Since $a_1^{\frac{n}{2}}(u, v) = (u, -v)$ has fixed points, our covering map $p_1 : V_2 \to V_2 / \langle a_1 \rangle = V_2(2)$ becomes $p_1(u, v)$

$= (u^{\frac{n}{2}}, u\overline{v}^2)$. The induced maps on $V_2(2)$ are defined by $b_2(u,v) = (\overline{u}, \overline{v})$ and $c_2(u,v) = (u, \overline{v}e^{\frac{-\pi i}{s}})$. A composition of the two induced maps is $c_2b_2(u,v) = (\overline{u}, ve^{\frac{-\pi i}{s}})$, hence $(c_2b_2)^2(u,v) = (u, ve^{\frac{-2\pi i}{s}})$. Thus, we obtain a further covering map $p_2 : V_2(2) \to V_2(2)/\langle (c_2b_2)^2\rangle = V_2(2s)$ defined by $p_2(u,v) = (u, v^s)$. The induced maps b_3, c_3 on $V_2(2s)$ are defined by $b_3(u,v) = (\overline{u}, \overline{v})$ and $c_3(u,v) = (u, -\overline{v})$ respectively. Moreover, $V_2(2s)/\langle b_3\rangle = (B0, 2s)$ and c_4 is induced on $(B0, 2s)$. Consequently, we obtain $(B0, 2s)/\langle c_4\rangle = (B8, 2s) = V_2/\langle a_1, b_1, c_1\rangle$. Let $m = 2s$.

As a summary based upon the above calculations, we define homeomorphisms a_1, b_1, c_1 on $\mathbb{RP}^3 = V_1 \cup_\alpha V_2$ as follows:

$$a_1(u,v) = \begin{cases} (u, ve^{\frac{2\pi i}{n}}), & \text{if } (u,v) \in V_1 \\ (ue^{\frac{4\pi i}{n}}, ve^{\frac{2\pi i}{n}}), & \text{if } (u,v) \in V_2 \end{cases}$$

$$b_1(u,v) = \begin{cases} (\overline{u}, \overline{v}), & \text{if } (u,v) \in V_1 \\ (\overline{u}, \overline{v}), & \text{if } (u,v) \in V_2 \end{cases}$$

$$c_1(u,v) = \begin{cases} (\overline{u}e^{\frac{2\pi i}{m}}, \overline{u}ve^{\frac{\pi i}{m}}), & \text{if } (u,v) \in V_1 \\ (u, u\overline{v}e^{\frac{\pi i}{m}}), & \text{if } (u,v) \in V_2. \end{cases}$$

The group generated by a_1, b_1, c_1 is G_7. This defines an action $\varphi_7 : G_7 \to \mathrm{Homeo}_{PL}(\mathbb{RP}^3)$ where $\mathbb{RP}^3/\varphi_7 = (B1,n) \cup_{h_7'} (B8,m) = O_{h_7'}((B1,n),(B8,m))$ for some homeomorphism $h_7' : \partial(B1,n) \to \partial(B8,m)$. It follows by Lemma 29 in [KO21b] that $O_{h_7'}((B1,n),(B8,m)) \simeq O_{h_7}((B1,n),(B8,m))$. Composing with a fixed homeomorphism, we obtain a fixed covering map $\nu_7 : \mathbb{RP}^3 \to O_{h_7}((B7,n),(B7,m))$ and note that $\varphi_7(G_7)$ is the group of covering translations. We call φ_7 a *Standard Quotient Type 7 Non-Abelian Action* on $\mathbb{RP}^3$ with quotient type $O_{h_7}((B1,n),(B8,m))$.

Proposition 3.20. *Let $\nu : \mathbb{RP}^3 \to O_{h_7}((B1,n),(B8,m))$ be a regular covering map such that $\pi_1(O_{h_7}((B1,n),(B8,m)))/\nu_*(\pi_1(\mathbb{RP}^3))$ is not abelian. Then $\nu_*(\pi_1(\mathbb{RP}^3)) = \langle a^{\frac{n}{2}}(cb)^m\rangle$ where n and m are both even.*

Proof. If $\nu : \mathbb{RP}^3 \to \mathbb{RP}^3/\varphi = O_{h_7}((B1,n),(B8,m))$ is the covering map, then $\nu_*(\pi_1(\mathbb{RP}^3)) = H$ is a $\mathbb{Z}_2$ normal subgroup of $\pi_1(O_{h_7}((B1,n),(B8,m)))$. We will show that 1) and 2) in Proposition 3.19 are eliminated. If $H = \langle a^s\rangle$, then $a(t,v) = (t, ve^{\frac{2\pi i}{n}})$; and if $H = \langle (cb)^m\rangle$, then $(cb)^m$ is identified with x^m which is defined by $x^m(t,v) = (t, -v)$ operating on the universal covering space of $(B8,m)$. Therefore, the maps a and $(cb)^m$ have a fixed point in the universal covering space of $(B1,n)$ and $(B8,m)$ respectively. Thus, the covering corresponding to each H is not a manifold by Corollary 5.2 in [KO22]. Consequently, 3) in Proposition 3.19 applies and $H = \langle a^{\frac{n}{2}}(cb)^m\rangle = \nu_*(\pi_1(\mathbb{RP}^3))$, $\square$

Theorem 3.21. *Let* $\varphi : G \to Homeo_{PL}(\mathbb{RP}^3)$ *be a finite non-abelian action such that the quotient space* $\mathbb{RP}^3/\varphi = O_{h_7}((B1,n),(B8,m))$. *Then, both m and n are even integers and the following is true:*

1) $G_7 = \langle a_1, b_1, c_1 \mid a_1^n = b_1^2 = c_1^2 = [a_1, c_1] = (c_1 b_1)^{2m} = a_1^{\frac{n}{2}}(c_1 b_1)^m = 1, b_1 a_1 b_1^{-1} = a_1^{-1}\rangle$
$= \langle a_1\rangle \circ (\langle c_1 b_1\rangle \circ_{-1} \langle c_1\rangle)/\sim = \mathbb{Z}_n \circ Dih(\mathbb{Z}_{2m})/\sim$.
2) The action φ *is conjugate to* φ_7, *the Standard O.R. Quotient Type 7-Action.*

Proof. By using Proposition 3.20, its proof is similar to the one in Theorem 3.3. □

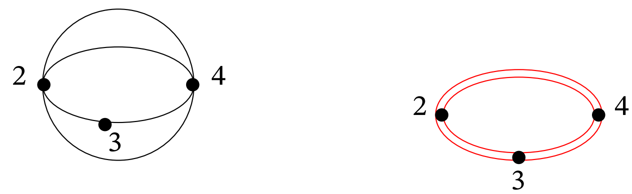
4. Examples

In this section, we will provide examples of non-abelian actions on $\mathbb{RP}^3$ which do not respect a genus 1 Heegaard decomposition. All these actions will leave a one-sided projective plane invariant. We will begin by describing the twisted I-bundle W over the projective plane $\mathbb{P}^2$. More details concerning finite group actions on W may be found in [KO18]. It will be convenient to use $W(X;Y)$ to denote an orbifold which is a twisted I-bundle over X with boundary Y, and therefore $W = W(\mathbb{P}^2;\mathbb{S}^2)$.

For $\mathbb{S}^2 \times I$, define a fixed-point free orientation preserving involution $\alpha : \mathbb{S}^2 \times I \to \mathbb{S}^2 \times I$ by $\alpha(z,t) = (i(z), 1-t)$. The map $i : \mathbb{S}^2 \to \mathbb{S}^2$ is the antipodal map. The quotient manifold $\mathbb{S}^2 \times I/\langle\alpha\rangle = W$ is a *twisted I-bundle over the one-sided projective plane* $\mathbb{P}^2$. Let $\nu : \mathbb{S}^2 \times I \to W$ be the covering map and note that $\nu(\mathbb{S}^2 \times \{1/2\}) = \mathbb{P}^2$ is a one-sided projective plane and $\nu(\mathbb{S}^2 \times \{0,1\})$ is the 2-sphere boundary of W. The levels of W are $\nu(\mathbb{S}^2 \times \{t\})$, and a homeomorphism h of W is *level preserving* if $h(\nu(\mathbb{S}^2 \times \{t\})) = \nu(\mathbb{S}^2 \times \{t\})$. We may view W as the set of equivalence classes $\{[z,t] \mid (z,t)$ is equivalent to $(i(z), 1-t)\}$. If B^3 is a 3-ball, then the manifold obtained by identifying ∂B^3 to ∂W is $\mathbb{RP}^3$, and we decompose $\mathbb{RP}^3 = B^3 \cup W$. We will first construct actions on W, and then extend them to B^3.

Example 4.1. S_4 *and* $S_4 \times \mathbb{Z}_2$ *actions on* $\mathbb{RP}^3$.

The octahedral group $O = S_4$, which is the symmetric group on 4 elements with presentation $\langle a, b \mid a^2 = b^3 = (ab)^4 = 1\rangle$. Now S_4 acts on $\mathbb{S}^2$ and commutes with the antipodal map i with quotient space the 2-orbifold $\Sigma(2,3,4) = \mathbb{S}^2/S_4$. The underlying space of $\Sigma(2,3,4)$ is a 2-sphere with cone points of orders 2, 3 and 4. The map i induces an orientation reversing involution (a reflection) $\bar{i}$ on $\Sigma(2,3,4)$ whose quotient space is the orbifold O^h (See Figure 3).



$\Sigma(2,3,4)$ O^h

Figure 3

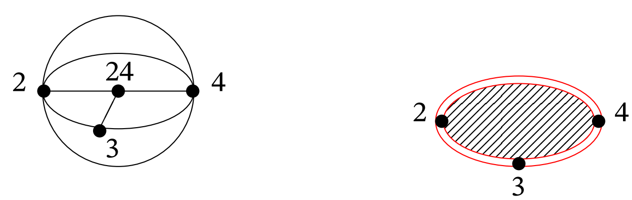
Define maps $\tilde{A}$, $\tilde{B}$ and $\tilde{\rho}$ on $\mathbb{S}^2 \times I$ as follows: $\tilde{A}(z,t) = (a(z),t)$, $\tilde{B}(z,t) = (b(z),t)$ and $\tilde{\rho}(z,t) = (z, 1-t)$. Since all these maps commute with the covering translation α, they induce maps A, B and ρ on W defined by $A[z,t] = [a(z),t]$, $B[z,t] = [b(z),t]$ and $\rho[z,t] = [z,1-t] = [i(z),t]$. The group acting on W generated by these elements $\langle A, B, \rho\rangle = S_4 \times \mathbb{Z}_2$. The orientation preserving subgroup is S_4 since ρ is orientation reversing.

We first consider the orientation preserving subgroup S_4 acting on W. The orbifold $W(O^h; \Sigma(2,3,4)) = W/S_4$ is $\Sigma(2,3,4) \times [0,1/2]/(z,1/2) \simeq (\bar{i}(z), 1/2)$, which is an orientable twisted I-bundle over O^h with boundary $\Sigma(2,3,4)$. This can be seen by the following diagram:

$$\begin{array}{ccc} \mathbb{S}^2 \times I & \longrightarrow & \mathbb{S}^2 \times I/\langle \tilde{A}, \tilde{B}\rangle = \Sigma(2,3,4) \times I \\ \downarrow & & \downarrow \\ W = \mathbb{S}^2 \times I/\langle\alpha\rangle & \longrightarrow & W/\langle A, B\rangle = W(O^h; \Sigma(2,3,4)) = \Sigma(2,3,4) \times I/\langle\overline{\alpha}\rangle \end{array}$$

The map $\overline{\alpha} : \Sigma(2,3,4) \times I \to \Sigma(2,3,4) \times I$ is defined by $\overline{\alpha}(z,t) = (\bar{i}(z), 1-t)$.

Next, we extend the S_4 action to the 3-ball B^3 by coning. The quotient space B^3/S_4 is the orbifold $B^3(2,3,4)$. The underlying space of $B^3(2,3,4)$ is a 3-ball whose exceptional set is a graph with three edges labeled with the numbers 2, 3, and 4 meeting at a single vertex labeled with the number 24 (see Figure 4).



$B(2,3,4)$ mO^h

Figure 4

It follows that $\mathbb{RP}^3/S_4 = B(2,3,4) \cup W(O^h; \Sigma(2,3,4))$.

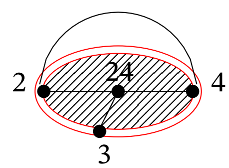
By [KM91], any orientation preserving action on a solid torus is isomorphic to either $\mathbb{Z}_m \times \mathbb{Z}_l$ or $\mathrm{Dih}(\mathbb{Z}_m \times \mathbb{Z}_l)$. Since the symmetric group S_4 is not isomorphic to either of these two groups, we have the following lemma.

Lemma 4.2. *The symmetric group S_4 acts on $\mathbb{RP}^3$ and does not preserve a genus 1 Heegaard decomposition. The quotient space*

$$\mathbb{RP}^3/S_4 = B(2,3,4) \cup W(O^h; \Sigma(2,3,4)).$$

We consider the entire action of $S_4 \times \mathbb{Z}_2$ on $\mathbb{RP}^3$ by first considering the action on W. For the covering map $\mathbb{S}^2 \times I \to \mathbb{S}^2 \times I/\langle \tilde{A}, \tilde{B}\rangle = \Sigma(2,3,4) \times I$ the induced maps ρ_1 and $\overline{\alpha}$ on $\Sigma(2,3,4) \times I$ are defined by $\rho_1(z,t) = (z, 1-t)$ and $\overline{\alpha}(z,t) = (\bar{i}(z), 1-t)$. We obtain a covering map $\Sigma(2,3,4) \times I \to \Sigma(2,3,4) \times I/\langle \rho_1 \overline{\alpha}\rangle = O^h \times I$. Finally, the induced map $\overline{\rho}$ on $O^h \times I$ is defined by $\overline{\rho}(z,t) = (z, 1-t)$, and $O^h \times I/\langle \overline{\rho}\rangle = W/S_4 \times \mathbb{Z}_2$ is homeomorphic to the orbifold $O^h \times I$ with one boundary component $O^h \times \{1\}$, and the mirrored 2-orbifold mO^h equaling $O^h \times \{0\}$ (see Figure 4). Denote this orbifold by $m(O^h \times I)$.

If $\widehat{\rho}$ is the induced action on $\mathbb{RP}^3/S_4 = B(2,3,4) \cup W(O^h; \Sigma(2,3,4))$, then by the previous paragraph $W(O^h; \Sigma(2,3,4))/\langle \widehat{\rho}\rangle = m(O^h \times I)$. It follows that $\widehat{\rho}|_{B(2,3,4)}$ is a reflection. The orbifold $B(2,3,4)/\langle \widehat{\rho}\rangle$ has underlying space a 3-ball, with half the boundary a mirrored disk containing a graph of singular points with three edges labeled with the numbers 2, 3 and 4 meeting at a single vertex labeled with the number 24. The other half of the boundary is O^h (see Figure 5). Denote this orbifold by $mB^3(2,3,4)$, and note that $\partial(mB^3(2,3,4)) = O^h$.



$mB(2,3,4)$
Figure 5

This shows that $\mathbb{RP}^3/S_4 \times \mathbb{Z}_2 = mB(2,3,4) \cup m(O^h \times I)$, and we have the following corollary.

Corollary 4.3. *The symmetric group $S_4 \times \mathbb{Z}_2$ acts on $\mathbb{RP}^3$ and does not preserve a genus 1 Heegaard decomposition. The quotient space $\mathbb{RP}^3/S_4 \times \mathbb{Z}_2 = mB(2,3,4) \cup m(O^h \times I)$.*

Example 4.4. *S_4 action on $\mathbb{RP}^3$.*

The octahedral group $O = S_4 = \langle ai, b\rangle$ where i is the antipodal map which acts on $\mathbb{S}^2$ and reverses orientation. The quotient space $\mathbb{S}^2/\langle ai, b\rangle$ is the orbifold T^h (See Figure 6), and by [KO18] the induced map $\bar{i}_1 : T^h \to T^h$ is a reflection that exchanges the two cone points of order 3 and fixes the cone point of order 2. It follows that the quotient space $T^h/\langle \bar{i}_1\rangle = O^h$.

Define maps $\tilde{A}_1$ and $\tilde{B}_1$ on $\mathbb{S}^2 \times I$ by $\tilde{A}_1(z,t) = (ia(z), t)$ and $\tilde{B}_1(z,t) = (b(z), t)$. The quotient space $\mathbb{S}^2 \times I/\langle \tilde{A}_1, \tilde{B}_1\rangle = T^h \times I$. Recall that $\alpha(z,t) = (i(z), 1-t)$, and the induced map $\overline{\alpha}_1 : T^h \times I \to T^h \times I$ is defined by $\overline{\alpha}_1(z,t) = (\bar{i}_1(z), 1-t)$. The orbifold quotient space $W(O^h; T^h) = T^h \times I/\langle \overline{\alpha}_1\rangle$ is $T^h \times [0, 1/2]/(z, 1/2) \simeq$

$(\bar{i}_1(z), 1/2)$, which is a twisted I-bundle over O^h with boundary T^h. Let A_1 and B_1 be the induced maps on W. We obtain the following commutative diagram:

$$\begin{array}{ccc} \mathbb{S}^2 \times I & \longrightarrow & \mathbb{S}^2 \times I/\langle \tilde{A}_1, \tilde{B}_1 \rangle = T^h \times I \\ \downarrow & & \downarrow \\ W = \mathbb{S}^2 \times I/\langle \alpha \rangle & \longrightarrow & W/\langle A_1, B_1 \rangle = W(O^h; T^h) = T^h \times I/\langle \overline{\alpha}_1 \rangle \end{array}$$

This shows that $S_4 = \langle A_1, B_1 \rangle$ acts on W with quotient space $W/\langle A_1, B_1 \rangle = W(O^h; T^h)$ being a twisted I-bundle over O^h with boundary T^h.

We now extend the action S_4 over the ball B^3 by coning. The orbifold quotient B^3/S_4 is a cone over the orbifold T^h. In other words, $B^3/S_4 = T^h \times I/(T^h \times \{1\}) \simeq * = C(T^h)$ (see Figure 6).

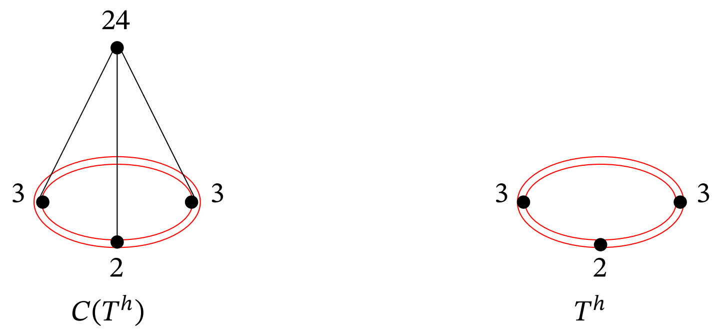


Figure 6

Therefore, we obtain the orientation reversing S_4-sction on $\mathbb{RP}^3$ such that $\mathbb{RP}^3/S_4 = C(T^h) \cup W(O^h; T^h)$. Thus, we have the following lemma.

Lemma 4.5. *The symmetric group S_4 acts on $\mathbb{RP}^3$ reversing orientation and not preserving any genus 1 Heegaard decomposition. The quotient space $\mathbb{RP}^3/S_4 = C(T^h) \cup W(O^h; T^h)$.*

The groups A_5, $A_5 \times \mathbb{Z}_2$, A_4 and $A_4 \times \mathbb{Z}_2$ can also be constructed as above to act on $\mathbb{RP}^3$. It follows by [KO18] that the alternating groups A_5 and A_4 each acts on $\mathbb{P}^2$ with quotient spaces $\mathbb{P}^2/A_5 = I^h$ and $\mathbb{P}^2/A_4 = T^v$ (see Figure 7).

Figure 7

As above, we let $W(I^h; \Sigma(2,3,5))$ and $W(T^v; \Sigma(2,3,3))$ be twisted I-bundles over I^h and T^v with boundaries $\Sigma(2,3,5)$ and $\Sigma(2,3,3)$ respectively. The orbifolds $m(I^h \times I)$ and $m(T^v \times I)$ are quotients of $I^h \times I$ and $T^v \times I$ respectively with $\partial(m(I^h \times I)) = I^h$ and $\partial(m(T^v \times I)) = T^v$. Defined similarly as $B(2,3,4)$, we have the orbifolds $B(2,3,5)$ and $B(2,3,3)$. Their quotients by involutions are $mB(2,3,5)$ and $mB(2,3,3)$ having boundaries I^h and T^v respectively. We obtain the following lemma.

Lemma 4.6. *The groups A_5, $A_5\times\mathbb{Z}_2$, A_4 and $A_4\times\mathbb{Z}_2$ act on $\mathbb{RP}^3$, do not preserve any Heegaard torus, and have the following quotient spaces:*

1) $\mathbb{RP}^3/A_5 = W(I^h;\Sigma(2,3,5))\cup B(2,3,5)$.

2) $\mathbb{RP}^3/A_5\times\mathbb{Z}_2 = m(I^h\times I)\cup mB(2,3,5)$.

3) $\mathbb{RP}^3/A_4 = W(T^v;\Sigma(2,3,3))\cup B(2,3,3)$.

4) $\mathbb{RP}^3/A_4\times\mathbb{Z}_2 = m(T^v\times I)\cup mB(2,3,3)$.

5. Main results

In this section we state and prove the main result. We recall the definitions of the following groups:

$$G_1 = \langle w_1, a_1 \mid a_1w_1a_1^{-1} = w_1^{-1}, w_1^n = a_1^{2m} = 1, w_1^{\frac{n}{2}}a_1^m = 1\rangle$$
$$=(\langle w_1\rangle\circ_{-1}\langle a_1\rangle)/\sim = (\mathbb{Z}_n\circ_{-1}\mathbb{Z}_{2m})/\sim.$$

$G_2 = \langle a_1, b_1, c_1 \mid a_1^n = b_1^2 = c_1^{2m} = [a_1,c_1] = 1, b_1a_1b_1^{-1} = a_1^{-1}, c_1b_1c_1^{-1} = b_1a_1, (b_1c_1)^nc_1^{m-n} = 1\rangle = (\langle a_1\rangle\circ_{-1}\langle b_1\rangle)\circ\langle c_1\rangle/\sim = \mathrm{Dih}(\mathbb{Z}_n)\circ\mathbb{Z}_{2m}/\sim$ if n, m are even.

$G_2 = \langle a_1, b_1, c_1 \mid a_1^n = b_1^2 = c_1^{2m} = [a_1,c_1] = 1, b_1a_1b_1^{-1} = a_1^{-1}, c_1b_1c_1^{-1} = b_1a_1, (b_1c_1)^nc_1^{m-n} = 1\rangle = \langle a_1\rangle\circ_{-1}\langle c_1\rangle = \mathbb{Z}_n\circ_{-1}\mathbb{Z}_{2m}$ if n, m are odd.

$$G_3 = \langle a_1, b_1, c_1 \mid a_1^m = b_1^n = c_1^2 = [a_1,b_1] = [a_1,c_1] = 1, c_1b_1c_1^{-1} = b_1^{-1},$$
$$b_1^{\frac{n}{2}}a_1^{\frac{m}{2}} = 1\rangle = (\langle b_1\rangle\circ_{-1}\langle c_1\rangle)\times\langle a_1\rangle/\sim = \mathrm{Dih}(\mathbb{Z}_n)\times\mathbb{Z}_m/\sim.$$

$$G_4 = \langle a_1, b_1 \mid a_1^{2n} = b_1^2 = (a_1b_1)^{2m} = 1, b_1a_1^2b_1^{-1} = a_1^{-2}, a_1^n(a_1b_1)^m = 1\rangle$$
$$= (\langle a_1^2\rangle\circ_{-1}\langle a_1b_1\rangle)\circ\langle b_1\rangle/\sim = (\mathbb{Z}_n\circ_{-1}\mathbb{Z}_{2m})\circ\mathbb{Z}_2/\sim.$$

$$G_5 = \langle a_1, b_1, c_1, d_1 \mid a_1^2 = b_2^2 = c_1^2 = (b_1c_1)^n = d_1^2 = (a_1d_1)^m = 1,$$
$$[a_1,b_1] = [a_1,c_1] = [b_1,d_1] = [c_1,d_1] = 1, (b_1c_1)^{\frac{n}{2}}(a_1d_1)^{\frac{m}{2}} = 1\rangle$$
$$= (\langle b_1c_1\rangle\circ_{-1}\langle c_1\rangle)\times(\langle a_1d_1\rangle\circ_{-1}\langle a_1\rangle)/\sim = \mathrm{Dih}(\mathbb{Z}_n)\times\mathrm{Dih}(\mathbb{Z}_m)/\sim.$$

$G_6 = \langle a_1, b_1, c_1 \mid a_1^2 = b_1^{2n} = c_1^2 = [a_1,b_1^2] = [a_1,c_1] = 1, c_1b_1c_1^{-1} = b_1^{-1}, b_1^{n-m}(a_1b_1)^m = 1\rangle = ((\langle d_1\rangle\circ_{-1}\langle a_1\rangle)\circ(\langle b_1\rangle\circ_{-1}\langle c_1\rangle)/\sim = \mathrm{Dih}(\mathbb{Z}_m)\circ\mathrm{Dih}(\mathbb{Z}_{2n})/\sim$.

$G_7 = \langle a_1, b_1, c_1 \mid a_1^n = b_1^2 = c_1^2 = [a_1,c_1] = (c_1b_1)^{2m} = a_1^{\frac{n}{2}}(c_1b_1)^m = 1, b_1a_1b_1^{-1} = a_1^{-1}\rangle = \langle a_1\rangle\circ(\langle c_1b_1\rangle\circ_{-1}\langle c_1\rangle)/\sim = \mathbb{Z}_n\circ\mathrm{Dih}(\mathbb{Z}_{2m})/\sim$.

Theorem 5.1. *Let $\varphi : G \to Homeo_{PL}(\mathbb{RP}^3)$ be an orientation-reversing finite non-abelian action which preserves a genus 1 Heegaard decomposition. Then one of the following cases is true where G is isomorphic to G_i for $1\le i\le 7$:*

1) $G_1 = (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_{2m})/\sim$ *where n, m are even and*

$$\mathbb{RP}^3/\varphi \simeq O_{h_1}((A1, n), (B5, m)).$$

2) $G_2 = Dih(\mathbb{Z}_n) \circ \mathbb{Z}_m/\sim$ *if n, m are even, or* $G_2 = \mathbb{Z}_n \circ_{-1} \mathbb{Z}_{2m}$ *if n, m are both odd, and*

$$\mathbb{RP}^3/\varphi \simeq O_{h_2}((A3, n), (B4, m))$$

in both cases.

3) $G_3 = Dih(\mathbb{Z}_n) \times \mathbb{Z}_m/\sim$*, where* $n > 2$*, m are even and*

$$\mathbb{RP}^3/\varphi \simeq O_{h_3}((A2, n), (B3, m)).$$

4) $G_4 = (\mathbb{Z}_n \circ_{-1} \mathbb{Z}_{2m}) \circ \mathbb{Z}_2/\sim$*, where n, m are even and*

$$\mathbb{RP}^3/\varphi \simeq O_{h_4}((B2, n), (B2, m)).$$

5) $G_5 = Dih(\mathbb{Z}_n) \times Dih(\mathbb{Z}_m)/\sim$*, where n, m are even and*

$$\mathbb{RP}^3/\varphi \simeq O_{h_5}((B6, n), (B6, m)).$$

6) $G_6 = Dih(\mathbb{Z}_m) \circ Dih(\mathbb{Z}_{2n})/\sim$*, where* $m = n(mod\ 2)$ *and*

$$\mathbb{RP}^3/\varphi \simeq O_{h_6}((B7, n), (B7, m)).$$

7) $G_7 = \mathbb{Z}_n \circ Dih(\mathbb{Z}_{2m})/\sim$*, where m, n are even and*

$$\mathbb{RP}^3/\varphi \simeq O_{h_7}((B1, n), (B8, m)).$$

Furthermore, in each individual case i), where $1 \leq i \leq 7$*,* φ *is equivalent to* φ_i*, the Standard Quotient Type i Non-Abelian Action.*

Proof. By Propositions 34 and 35 in [KO21b] we have the following relations:

$$O_{h_1}((A1, 2), (B5, 1)) \simeq O_{h_4}((B2, 1), (B2, 1)),$$
$$O_{h_3}((A2, 2), (B3, 2)) \simeq O_{h_6}((B7, 1), (B7, 1)),$$
$$O_{h_3}((A2, 1), (B3, 2)) \simeq O_{h_7}((B1, 1), (B8, 1)),$$
$$O_{h_3}((A2, 2), (B3, 1)) \simeq O_{h_5}((B6, 1), (B6, 1)).$$

Furthermore, except for these four pairs, the non-orientable orbifolds having an Euler number zero genus 1 Heegaard decomposition are distinct up to homeomorphism. Comparing these four pairs of orbifolds with the orbifolds in the statement of the theorem, we see that the orbifolds listed in cases 1) - 7) of the theorem are distinct.

Let $\varphi : G \to Homeo_{PL}(\mathbb{RP}^3)$ be an orientation-reversing finite non-abelian action which preserves a genus 1 Heegaard decomposition. We may write $\mathbb{RP}^3 = V_1' \cup_{\alpha'} V_2'$ where each V_i' for $i = 1,\ 2$ is a $\varphi(G)$-invariant solid torus and $\alpha' : \partial V_1' \to V_2'$ is a homeomorphism. By [KM91], the non-orientable 3-orbifold quotient $V_i'/\varphi(G)$ is one of the orbifolds $(A1, n)$,..., $(A3, n)$, $(B1, n)$,..., $(B8, n)$. Letting $\nu : \mathbb{RP}^3 \to \mathbb{RP}^3/\varphi(G)$ be the orbifold covering map, we see that

$\mathbb{RP}^3/\varphi(G) = O_\xi(X,Y)$ where X and Y are one of the orbifolds listed above and $\xi : \partial X \to \partial Y$ is a homeomorphism. It follows that $\pi_1(O_\xi(X,Y))$ is a finite fundamental group since $\nu_*(\pi_1(\mathbb{RP}^3)) = \mathbb{Z}_2$ has finite index equal to the order of G. By Theorem 2.1, $O_\xi(X,Y)$ is homeomorphic to one of the seven orbifolds listed in the chart and the statement of the theorem. Applying Theorems 3.3, 3.6, 3.9, 3.12, 3.15, 3.18 and 3.21 proves the result. □

References

[BM84] BASS, H.; MORGAN, J. W., EDS. The Smith Conjecture, *Pure and Appl. Math.*, Academic Press, 1984, MR0758460, Zbl 0599.57001. 282

[BMP03] BOILEAU. M.; MAILLOT, S.; PORTI, J. Three-Dimensional Orbifolds and their Geometric Structures, *Panoramas et Syntheses*, Soc. Math. de France, Paris, Vol. 15, 2003, MR2060653, Zbl 1058.57009. 285

[Cho00] CHOI, S. Geometric Structures on 2-Orbifolds: Exploration of Discrete Symmetry, *MSJ Memoirs*, Math. Soc. Japan, Tokyo, Vol. 5, 2000. MR2962023, Zbl 1260.57043. 285

[CHK12] COOPER, D.; HODGSON, C.; KERCKHOFF, S. Three dimensional Orbifolds and Cone-Manifolds, *MSJ Memoirs*, Math. Soc. Japan, Tokyo, Vol. 27, 2012, MR1778789, Zbl 0955.57014. 285, 286

[DL09] DINKELBACH, J.; LEEB, B. Equivariant Ricci flow with surgery and applications to finite group actions on geometric 3-manifolds, *Geometry and Topology*, Vol. 13, **2**, 2009, 1129 - 1173, MR2491658, Zbl 1181.57023. 283

[Dun88] DUNBAR, W. Geometric orbifolds, *Rev. Mat. Univ. Complut*, Madrid, Vol. 1, (1-3) 1988, 67 - 99, MR0977042, Zbl 0655.57008. 285

[Hal03] HALL, B. C. Lie Groups, Lie Algebras, and Representations An Elementary Introduction, *Graduate Texts in Mathematics*, Springer-Verlag, 2003, MR1997306, Zbl 1026.22001. 283

[KM91] KALLIONGIS, J.; MILLER, A. The symmetries of genus one handlebodies, *Can. J. Math.*, Vol. 2, 1991, 371 - 404, MR1113761, Zbl 0731.57009. 282, 283, 286, 287, 314, 318

[KM03] KALLIONGIS, J.; MILLER,A. Actions on lens spaces which respect a Heegaard decomposition, *Topol. and its Appl.*, Vol. 130, 2003, 19 - 55, MR1966672, Zbl 1020.57015. 282, 283

[KM08] KALLIONGIS, J.; MILLER, A. Orientation-reversing actions on lens spaces and Gaussian integers, *J. Pure and Appl. Alg.*, Vol. 212, Issue 3, 2008, 652 - 667, MR2365339, Zbl 1156.57016. 283

[KO18] KALLIONGIS, J.; OHASHI, R. Finite actions on the 2-sphere, projective plane and I-bundles over the projective plane, *ARS Math. Cont.*, Vol. 15, **2**, 2018, 297 - 321. MR3879992, Zbl 1414.57011. 282, 313, 315, 316

[KO21a] KALLIONGIS, J.; OHASHI, R. Cyclic p-group actions on $\mathbb{RP}^3$, *J. Alg. and its Appl.*, Vol. 20, No. 10, 2021, 1 - 29, MR4326096, Zbl 1476.57042. 282

[KO21b] KALLIONGIS, J.; OHASHI, R. Orbifolds having Euler number zero Heegaard decomposition, *Illinois J. Math.*, Vol. 65, **2**, 2021, 339 - 384, MR4278692, Zbl 1471.57019. 284, 287, 288, 291, 292, 295, 296, 298, 301, 302, 304, 306, 309, 310, 312, 318

[KO22] KALLIONGIS, J.; OHASHI, R. Orientation reversing finite abelian actions on $\mathbb{RP}^3$, *Rend. Istit. Mat. Univ. Trieste*, Vol. 54, 2022, MR4442735, Zbl 1515.57024. 282, 283, 287, 288, 291, 292, 295, 296, 298, 299, 302, 305, 306, 309, 310, 311, 312

[Kim77] KIM, P. K. Orientation-reversing periodic PL maps of lens spaces, *Proc. Amer. Math. Soc.*, Vol. 64, **2**, 1977, 351 - 356, MR0442919, Zbl 0411.57040. 283, 299

[Kwu70] KWUN, K.W. Scarcity of orientation-reversing PL involutions of lens spaces, *Michigan Math. J.*, Vol. 17, **4**, 1970, 355 - 358, MR0279814, Zbl 0191.54901. 283

[NR78] NEUMANN, W.; RAYMOND, F. Seifert manifolds, plumbing, μ-invariant and orientation reversing maps, *Algebraic and geometric topology* (Proc. Sympos., Univ. California, Santa Barbara, Calif., 1977), Springer-Verlag Lecture Notes in Math. 664, Springer, 1978, 163 - 196, MR0518415, Zbl 0401.57018. 283

[Per02] PERELMAN, G. The entropy formula for the Ricci flow and its geometric applications, 2002, Mathematics ArXiv math.DG/0211159, MR1955221, Zbl 1130.53001. 283

[Per03a] PERELMAN, G. Ricci flow with surgery on three-manifolds, 2003, Mathematics ArXiv math.DG/0303109, MR2026939, Zbl 1130.53002. 283

[Per03b] PERELMAN, G. Finite extinction time for the solutions to the Ricci flow on certain three-manifolds, 2003, Mathematics ArXiv math.DG/0307245, MR2437178, Zbl 1130.53003. 283

[Sat56] SATAKE, I. On a generalization of the notion of manifold, *Proc. Nat. Acad. Sci. U.S.A.*, Vol. 42, 1956, 359 - 363, MR0079769, Zbl 0074.18103. 285

[Sat57] SATAKE, I. On The Gauss-Bonnet Theorem for V-manifolds, *J. Math. Soc. Japan*, Vol. 9, **4**, 1957, 464 - 492. MR0095520, Zbl 0080.37403. 285

[Thu78] THURSTON, W. P. The Geometry and Topology of 3-Manifolds, American Mathematical Society, Providence, RI, Vo 4, 2022, MR4554426, Zbl 1507.57005. 285

[Yok16] YOKOYAMA, M. A general theory of orbifolds, and essential 2-orbifolds respecting graph products, *Topol. and its Appl.*, Vol. 207, 2016, 62 - 95, MR3501267, Zbl 1361.57024. 285

[Zim96] ZIMMERMANN, B. Genus actions of finite groups on 3-manifolds, *Michigan Math. J.*, Vol. 43, No. 3, 1996, 593 – 610, MR1420594, Zbl 0872.57016. 283

[Zim19] ZIMMERMANN, B. On upper and lower bounds for finite group-actions on bounded surfaces, handlebodies, closed handles and finite graphs, *Rend. Istit. Mat. Univ. Trieste*, Vol. 51, 2019, 167 - 179, MR4048837, Zbl 1459.57028. 282

[Zim20] ZIMMERMANN, B. On large orientation-reversing group-actions on 3-manifolds and equivariant Heegaard decompositions, *Monatsh. Math.*, 191, 2020, 437-444, MR4055222, Zbl 1432.57048. 283

(John Kalliongis) DEPARTMENT OF MATHEMATICS AND STATISTICS, SAINT LOUIS UNIVERSITY, 220 NORTH GRAND BOULEVARD, ST.LOUIS, MO 63103, U.S.A.
kalliongisje@slu.edu

(Ryo Ohashi) DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, KING'S COLLEGE, 133 NORTH RIVER STREET, WILKES-BARRE, PA 18711, U.S.A.
ryoohashi@kings.edu

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