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Overfare of harmonic one-forms on Riemann surfaces

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ABSTRACT. This is the second in a series of four papers developing a scattering theory for harmonic one-forms on Riemann surfaces. In this paper we develop a conformally invariant characterization of the Sobolev space $H^{-1/2}(\Gamma)$ where Γ is a border of a Riemann surface which is homeomorphic to the circle. We show that the boundary values of L^2 harmonic one-forms are in $H^{-1/2}(\Gamma)$. Also, let Σ be a Riemann surface with a finite number of borders homeomorphic to the circle. We show that the Dirichlet problem on a Riemann surface Σ with border $\partial\Sigma$ for one-forms with boundary values in $H^{-1/2}(\partial\Sigma)$ and suitable cohomological data is well-posed.

Furthermore, we prove the following "overfare" result. Let $\mathscr{R}$ be a compact Riemann surface split into two surfaces Σ_1 and Σ_2 by a complex of quasicircles. Given an L^2 harmonic one-form α_1 on Σ_1, there is a unique L^2 harmonic one-form α_2 on Σ_2 with the same boundary values in the above sense.

1. Introduction

1.1. Results. This paper is one part of a longer work [17] establishing a scattering theory of one-forms on Riemann surfaces, which we have divided into four parts. A non-technical exposition of some aspects of this scattering theory

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can be found in [21]. The scattering process starts by dividing a Riemann surface into two pieces (which themselves may not be connected) by a collection of Jordan curves. Alternatively, we can think of a compact surface obtained by sewing together several surfaces, and the Jordan curves are the seams. Then one considers functions or one-forms which are separately harmonic on the pieces, and share boundary values on the seams. The function (or one-form) on one side of the surface is obtained from the function (or one-form) on the other side using a mapping which we refer to as "overfare", which is the backbone of this particular kind of scattering theory. In [18] it was shown that the overfare process is well-defined and bounded for quasicircles. Furthermore, in order that the results be applicable to Teichmüller theory and conformal field theory, it is necessary that these seams can be quasicircles. Much evidence exists that quasicircles are analytically natural for the scattering theory [16].

In [18] we established the analytic theory of harmonic functions sharing boundary values on the seams. In this second paper, we develop the shared boundary value theory for one-forms, addressing analytic and geometric (cohomological) challenges which arise. Two further papers [19, 20] from this longer work will apply these results to the global analysis and geometry of the scattering process via Schiffer integral operators, including index theorems for these operators, as well as unitarity of the scattering process.

We now describe the specific results that are obtained in the present paper, building on [18]. Let $\mathscr{R}$ be a Riemann surface split into two pieces Σ_1 and Σ_2 by a complex of quasicircles. In [18], we showed that given a harmonic function with L^2 derivatives on one of the pieces Σ_1 (a Dirichlet harmonic function), there is a harmonic function on the other piece Σ_2 with the same boundary values as the original function. We call this process "overfare". This is well-defined and bounded. Because the seams are quasicircles, which in general can be highly irregular, there were many analytic obstacles to overcome.

In this paper we develop an overfare process for L^2 harmonic one-forms. That is, given an L^2 harmonic one-form on Σ_1, there is an L^2 harmonic one-form on Σ_2 with the same boundary values, and with specified cohomology. In order to define this process, this requires a notion of boundary values. As in the first paper, we treat the boundary of a surface — say Σ_1 — as an analytic curve, by viewing it as an ideal boundary or as an analytic curve in the double. This is an *intrinsic* point of view, which does not refer in any way to the ambient compact surface $\mathscr{R}$ — within which the boundary is quite irregular, a fact to which we will return shortly. On the other hand, overfare does depend on the regularity of the curves, and thus is *extrinsic*.

Let Γ be a border of a Riemann surface, for example Σ_1 in the description above, which is homeomorphic to the circle. The **first main result** is a new characterization of the Sobolev space $H^{-1/2}(\Gamma)$ as equivalence classes of harmonic one-forms defined on collar neighbourhoods of Γ in the surface Σ_1 (and similarly for Σ_2). That is, we show that elements of $H^{-1/2}$ can be represented by equivalence classes of L^2 harmonic one-forms defined in collar neighbourhoods. Such a characterization is also given in the case of the homogeneous Sobolev space $\dot{H}^{-1/2}(\Gamma)$. This new characterization of $H^{-1/2}(\Gamma)$ is conformally invariant. It is also fundamentally different from the usual distributional formulations of the $H^{-1/2}$ Sobolev space — one can represent *every* distribution with an actual harmonic one form in a neighbourhood of the boundary.

This is accomplished using limiting integrals approaching the boundary together with the fact that $H^{-1/2}$ is the dual space to $H^{1/2}$, to show that there is a one-to-one correspondence between elements of $H^{-1/2}$ and such equivalence classes. It turns out that anti-derivatives of such forms have well-defined boundary values in the conformally nontangential sense, which after removing a period, can be identified with elements of $H^{1/2}$. This then allows us to use the theory of conformally nontangential boundary values developed in [18] to develop the new representation. In this context, what we call the *Anchor Lemmas* (Lemmas 2.26 and 2.27) are of fundamental importance. These two lemmas say that the limiting integral of $f \in H^{1/2}$ against any L^2 harmonic one-form α on a collar neighbourhood A of the boundary exists, and depends only on the CNT boundary values of f.

The **second main result** is to show that for a Riemann surface Σ bordered by n boundary curves homeomorphic to the circle, the Dirichlet problem for one-forms with $H^{-1/2}(\partial\Sigma)$ boundary values is well-posed, with solutions in $L^2(\Sigma)$. In this Dirichlet problem cohomological data must be specified in order to have a unique solution. This well-posedness does not exist in the literature. It also leads to a new direct characterization of the boundary values of L^2 one-forms.

A classical formulation of the Dirichlet problem on Riemannian manifolds with smooth boundary is as follows. Let M be a smooth, connected, compact, Riemannian manifold of real dimension m and consider some arbitrary smooth domain $\Omega \subseteq M$ with non-empty boundary. Assume that $f \in L^2\left(\partial\Omega, \wedge^k TM\right)$, $0 \leq k \leq m$, where $L^2\left(\partial\Omega, \wedge^k TM\right)$ denotes the space of k-forms which are L^2 on the boundary of Ω. Denoting the Hodge Laplacian by $\Delta = d\delta + \delta d$ (where d is the exterior differentiation and δ its adjoint with respect to the Riemannian metric of M), the Dirichlet boundary value problem with boundary data f is

$$\begin{cases} u \in C\left(\Omega, \wedge^k TM\right) \\ \Delta u = 0 \text{ in } \Omega \\ u|_{\partial\Omega} = f \text{ on } \partial\Omega \end{cases} \tag{1.1}$$

For $0 \leq k \leq m$, this problem was studied by G. Duff and D. Spencer [6], [7], [5], [25], C. Morrey and J. Eells [12], [13], and G. Schwarz [24]. Through these investigations, it is known that for any $f \in L^2\left(\partial\Omega, \wedge^k TM\right)$ the Dirichlet problem has a unique solution $u \in H^{1/2}\left(\Omega, \wedge^k T\mathcal{M}\right)$ (Sobolev $\frac{1}{2}$-space), and moreover there exists $C > 0$ independent of f such that

$$\|u\|_{H^{1/2}(\Omega,\wedge^k TM)} \leq C\|f\|_{L^2(\partial\Omega,\wedge^k TM)}. \tag{1.2}$$

Another well-known fact is that if $k = 0$, $u \in L^2(\Omega)$ and $\Delta u \in L^2(\Omega)$ then $u|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$.

As described above we prove well-posedness of (1.1) when $k = 1$ and f in the Sobolev space of forms $H^{-1/2}\left(\partial\Sigma, \wedge^k T\Sigma\right)$, where Σ is a bordered Riemann surface. That is, for an element of $f \in H^{-1/2}$ together with sufficient cohomological data, there always exists a unique L^2 harmonic one-form u on Σ with boundary value f, and that u depends continuously on f, i.e. the analogue of (1.2) is valid in this setting. The problem for $H^{-1/2}$ boundary values is solved using the conformally invariant characterization of the $H^{-1/2}$-space (the first result above), together with the theory of conformally non-tangential boundary values developed in the first paper.

The **third main result** is as follows. Some analytic subtleties are suppressed for the moment. Let $\mathscr{R}$ be a compact surface separated into pieces Σ_1 and Σ_2 by a complex of quasicircles as above. The complex of quasicircles is the border of both Σ_1 and Σ_2. We show that for any L^2 harmonic one-form α_2 on Σ_2, there is a unique L^2 one-form α_1 on Σ_1 whose boundary values in $H^{-1/2}$ agree with those of α_2, up to specification of cohomological data (Theorem 4.11). That is, overfare of one-forms exists and is unique. Boundedness of overfare of forms will be dealt with in [20], using a different way to specify the cohomological data.

We also prove a local overfare result, in a single curve. Namely, fix a quasicircle Γ in the complex of quasicircles separating Σ_1 and Σ_2. It can be viewed as a border Γ_1 of Σ_1 or as a border Γ_2 of Σ_2. However, the Sobolev spaces $H^{-1/2}(\Gamma_1)$ and $H^{-1/2}(\Gamma_2)$ are entirely different a priori, and similarly for $H^{1/2}(\Gamma_1)$ and $H^{1/2}(\Gamma_2)$. We show that there is a local overfare taking elements of $H^{-1/2}(\Gamma_2)$ to elements of $H^{-1/2}(\Gamma_1)$, and similarly for the $H^{1/2}$ spaces. Furthermore, simultaneous overfare of elements of $H^{1/2}(\Gamma_1)$ and elements of $H^{-1/2}(\Gamma_1)$ respects the dual pairing and is bounded for a subclass of quasicircles which we call bounded zero mode quasicircles (which includes Weil-Petersson class quasicicles). For the homogeneous Sobolev spaces, this result holds for general quasicircles. These results are given in Proposition 4.10.

For applications to Teichmüller theory and conformal field theory, it was necessary to extend the overfare results for functions to more general configurations of sewn surfaces which arise there – one requires for example surfaces with many boundary curves and disks sewn on; self-sewn surfaces; and surfaces sewn along many curves. In [18], we proved overfare results sufficient to apply to these general cases. In the present paper, we obtain overfare theorems for one-forms for these general configurations; however, many of the results here are new even in the plane. In [19, 20], the results of the present paper together with [18] are applied to derive a unitary scattering theory; characterize solutions to the holomorphic boundary value problems for one-forms; generalize the Grunsky inequalities to collections of maps into compact Riemann surfaces of genus g; and derive a generalized period map for bordered Riemann surfaces which unifies the classical period map for compact surfaces with the Kirillov–Yuri'ev–Nag–Sullivan period mapping of universal Teichmüller space. The results of these sequels to this paper have also been used in approximation theory of holomorphic one-forms on Riemann surfaces [22].

2. Preliminaries

2.1. About this section. In this section we gather the basic material that is used in the paper. This material is taken largely (but not entirely) from [18] where proofs can be found. This material consist of notions of bordered surfaces, collar charts, conformally nontangential boundary values, null sets, Bergman, Dirichlet, and Sobolev spaces, harmonic measure, and finally our so-called Anchor Lemmas. These are all recalled here (together with the relevant propositions, lemmas and theorems, mostly without proofs) for the convenience of the reader, in order to make our presentation self-contained.

2.2. Bordered surfaces, collar charts and CNT boundary values. We will briefly recall the definition of a bordered surface in order to remove any ambiguity. See for example [1] for a complete treatment.

In what follows we denote by $\mathbb{A}_{a,b}$ the annulus $\{z;\ a < |z| < b\}$. Let $\mathbb{C}$ denote the complex plane, let $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ denote the upper half plane, and let $cl\,(\mathbb{H})$ denote its closure (we will let cl denote closure throughout). Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disk, and $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ denote the unit circle.

Definition 2.1. We say that a connected Hausdorff topological space $\hat{\Sigma}$ is a *bordered Riemann surface* if there is an atlas of charts $\phi : U \to \mathrm{cl}\,(\mathbb{H})$ with the following properties.

(1) Each chart is a local homeomorphism with respect to the relative topology;
(2) Every point in $\hat{\Sigma}$ is contained in the domain of some chart;
(3) Given any pair of charts $\phi_k : U_k \to \mathrm{cl}\,(\mathbb{H})$, $k = 1, 2$, if $U_1 \cap U_2$ is non-empty, then $\phi_1 \circ \phi_2^{-1}$ is a biholomorphism on $U_1 \cap U_2 \cap \mathbb{H}$.

One of our main objects of study is a particular type of bordered Riemann surface which is defined as follows:

Definition 2.2. We say that Σ is a *bordered Riemann surface of type* (g, n), if it is bordered (in the sense Definition 2.1), the border has n connected components, each of which is homeomorphic to $\mathbb{S}^1$, and its double Σ^d is a compact surface of genus $2g + n - 1$.

The connected components of $\partial\Sigma$ will be labelled $\partial_k\Sigma$ for $k = 1, \dots, n$.

Definition 2.3. We say that a homeomorphic image Γ of $\mathbb{S}^1$ is a *strip-cutting Jordan curve* if it is contained in an open set U and there is a biholomorphism $\phi : U \to \mathbb{A}_{r,R}$ for some annulus

$$\mathbb{A}_{r,R} \subset \mathbb{C}, \quad r < 1 < R,$$

in such a way that $\phi(\Gamma)$ is isotopic to the circle $|z| = 1$. We call U a doubly-connected neighbourhood of Γ and ϕ a doubly-connected chart.

We also define a kind of chart on bordered surfaces near the boundary, which we call a collar chart.

Definition 2.4. Let Σ be a bordered Riemann surface and Γ a border which is homeomorphic to $\mathbb{S}^1$. A biholomorphism $\phi : U \to \mathbb{A}_{r,1}$ is called a *collar chart* of Γ (for some fixed k) if U is an open set in Σ bounded by two Jordan curves Γ and γ, such that γ is isotopic to Γ within the closure of U. A domain U is a collar neighbourhood of Γ if it is the domain of some collar chart.

We will adhere to the convention that the continuous extension of a collar chart to the border maps the border Γ onto the unit circle $\mathbb{S}^1$. By Carathéodory's theorem, such a continuous extension exists.

Theorem 2.5. *Let Σ be a bordered surface and Γ be a component of the border which is homeomorphic to $\mathbb{S}^1$. If $\phi : U \to \mathbb{A}$ is a collar chart of Γ, then ϕ extends continuously to Γ. The extension is a homeomorphism of Γ onto $\mathbb{S}^1$.*

To keep the notation simple, we will also denote the continuous extension by ϕ.

Finally, we have the following useful facts.

Proposition 2.6. *Let Σ be a Riemann surface with border Γ homeomorphic to $\mathbb{S}^1$, and let U and V be collar neighbourhoods of Γ. There is a collar chart $\phi : W \to \mathbb{A}_{r,1}$, where $0 < r < 1$, such that $W \subseteq U \cap V$. Moreover r can be chosen so that the inner boundary of W is contained in $U \cap V$.*

Proposition 2.7. *Let Σ be a type (g, n) surface. Then every boundary curve $\partial_k \Sigma$ has a collar chart.*

There is a natural notion of non-tangential limit on the border of a Riemann surface which we refer to as the CNT limit and which is defined as follows.

Definition 2.8. Let Γ be a border of Σ with a collar chart ψ of Γ in Σ. The conformally non-tangential limit (denoted henceforth by CNT limit) of a function $h : \Sigma \to \mathbb{C}$ at $p \in \Gamma$ is ζ if $h \circ \psi^{-1}$ has a non-tangential limit of ζ at $\psi(p)$.

The CNT limit has the following three basic properties (see [18]):

(1) Its existence and value is independent of the choice of ψ.
(2) It is the same as that obtained by treating Γ as the abstract border of the Riemann surface Σ.

(3) It is conformally invariant, meaning that if $F : \Sigma_1 \to \Sigma$ is a conformal map, then h has a CNT limit of ζ at p if and only if $h \circ F$ has a CNT limit of ζ at $F^{-1}(p)$.

Next, we define a potential-theoretically negligible set on the border which we call a null set.

Definition 2.9. We say that a set $I \subset \Gamma$ is null if it is a Borel set and $\psi(I)$ has logarithmic capacity zero in $\mathbb{S}^1$.

This definition is independent of the choice of ψ.

Finally we make the following definition.

Definition 2.10. Let Σ be a Riemann surface and let $\partial_k \Sigma$ be a component of the border of Σ homeomorphic to $\mathbb{S}^1$. Given functions $h_j : \partial_k \Sigma \backslash I_j \to \mathbb{C}$, $j = 1, 2$, where I_1 and I_2 are null sets, we say that $h_1 \sim h_2$ if h_1 and h_2 are both defined on $\partial_k \Sigma \backslash I$ for some null set I and $h_1 = h_2$ on $\partial_k \Sigma \backslash I$.

2.3. Function spaces.

Definition 2.11. We say a complex-valued function f on an open set U is *harmonic* if it is C^2 on U and $d * df = 0$. We say that a complex one-form α is harmonic if it is C^1 and satisfies both $d\alpha = 0$ and $d * \alpha = 0$, where on any Riemann surface, the dual of the almost complex structure $*$ is defined in local coordinates $z = x + iy$ by

$$* \, (a \, dx + b \, dy) = a \, dy - b \, dx.$$

Denote complex conjugation of functions and forms with a bar, e.g. $\overline{\alpha}$. A holomorphic one-form is one which can be written in coordinates as $h(z) \, dz$ for a holomorphic function h, while an anti-holomorphic one-form is one which can be locally written $\overline{h(z)} \, d\bar{z}$ for a holomorphic function h.

Denote by $L^2(U)$ the set of one-forms ω on an open set U which satisfy

$$\iint_U \omega \wedge * \, \overline{\omega} < \infty.$$

For the choices of U in this paper, this is a Hilbert space with respect to the inner product

$$(\omega_1, \omega_2) = \iint_U \omega_1 \wedge * \, \overline{\omega_2}. \tag{2.1}$$

Definition 2.12. The *Bergman space of holomorphic one-forms* is

$$\mathcal{A}(U) = \{\alpha \in L^2(U) \ : \ \alpha \text{ holomorphic}\}. \tag{2.2}$$

The anti-holomorphic Bergman space is denoted $\overline{\mathcal{A}(U)}$. We will also denote

$$\mathcal{A}_{\text{harm}}(U) = \{\alpha \in L^2(U) \ : \ \alpha \text{ harmonic}\}. \tag{2.3}$$

Observe that $\mathcal{A}(U)$ and $\overline{\mathcal{A}(U)}$ are orthogonal with respect to the inner product (2.1). In fact we have the direct sum decomposition

$$\mathcal{A}_{\text{harm}}(U) = \mathcal{A}(U) \oplus \overline{\mathcal{A}(U)}. \tag{2.4}$$

If we restrict the inner product to $\alpha, \beta \in \mathcal{A}(U)$ then since $* \, \overline{\beta} = i\overline{\beta}$, we have

$$(\alpha, \beta) = i \iint_U \alpha \wedge \overline{\beta}.$$

Denote the projections induced by this decomposition by

$$\begin{aligned} \mathbf{P}_U &: \mathcal{A}_{\text{harm}}(U) \to \mathcal{A}(U) \\ \overline{\mathbf{P}}_U &: \mathcal{A}_{\text{harm}}(U) \to \overline{\mathcal{A}(U)}. \end{aligned} \tag{2.5}$$

Let $f : U \to V$ be a biholomorphism. We denote the pull-back of $\alpha \in \mathcal{A}_{\text{harm}}(V)$ under f by $f^*\alpha$. Explicitly, if α is given in local coordinates w by $a(w)\,dw + \overline{b(w)}\,d\bar{w}$ and $w = f(z)$, then the pull-back is given by

$$f^*\left(a(w)\,dw + \overline{b(w)}\,d\bar{w}\right) = a(f(z))f'(z)\,dz + \overline{b(f(z))f'(z)}\,d\bar{z}. \tag{2.6}$$

The Bergman spaces are all conformally invariant, in the sense that if $f : U \to V$ is a biholomorphism, then $f^*\mathcal{A}(V) = \mathcal{A}(U)$ and f^* preserves the inner product. The same holds for the anti-holomorphic and harmonic spaces.

Definition 2.13. We define the space $\mathcal{A}^{\text{e}}_{\text{harm}}(U)$ as the subspace of exact elements of $\mathcal{A}_{\text{harm}}(U)$, and similarly for $\mathcal{A}^{\text{e}}(\Sigma)$ and $\overline{\mathcal{A}^{\text{e}}(\Sigma)}$.

Definition 2.14. Let Σ be a bordered surface of type (g, n). We say that an L^2 one-form $\alpha \in \mathcal{A}_{\text{harm}}(\Sigma)$ is semi-exact if for any simple closed curve γ isotopic to a boundary curve $\partial_k\Sigma$,

$$\int_\gamma \alpha = 0.$$

The class of semi-exact one-forms on Σ is denoted $\mathcal{A}^{\text{se}}_{\text{harm}}(\Sigma)$. The holomorphic and anti-holomorphic semi-exact one-forms are denoted by $\mathcal{A}^{\text{se}}(\Sigma)$ and $\overline{\mathcal{A}^{\text{se}}(\Sigma)}$ respectively.

The following spaces also play significant roles in this paper.

Definition 2.15. The *Dirichlet spaces of functions* are defined by

$$\begin{aligned} \mathcal{D}_{\text{harm}}(U) &= \{f : U \to \mathbb{C}, f \in C^2(U) : df \in \mathcal{A}_{\text{harm}}(U)\}, \\ \mathcal{D}(U) &= \{f : U \to \mathbb{C} : df \in \mathcal{A}(U)\}, \text{ and} \\ \overline{\mathcal{D}(U)} &= \{f : U \to \mathbb{C} : df \in \overline{\mathcal{A}(U)}\}. \end{aligned}$$

We can define a degenerate inner product on $\mathcal{D}_{\mathrm{harm}}(U)$ by

$$(f,g)_{\mathcal{D}_{\mathrm{harm}}(U)} = (df,dg)_{\mathcal{A}_{\mathrm{harm}}(U)},$$

where the right hand side is the inner product (2.1) restricted to elements of $\mathcal{A}_{\mathrm{harm}}(U)$. The inner product can be used to define a seminorm on $\mathcal{D}_{\mathrm{harm}}(U)$, by letting

$$\|f\|^2_{\mathcal{D}_{\mathrm{harm}}(U)} := (df,df)_{\mathcal{A}_{\mathrm{harm}}(U)}.$$

We note that if one defines the *Wirtinger operators* via their local coordinate expressions

$$\partial f = \frac{\partial f}{\partial z}\,dz, \quad \overline{\partial} f = \frac{\partial f}{\partial \bar{z}}\,d\bar{z},$$

then the aforementioned inner product can be written as

$$(f,g)_{\mathcal{D}_{\mathrm{harm}}(U)} = i\iint_U \left[\partial f \wedge \overline{\partial g} - \overline{\partial} f \wedge \partial \overline{g}\right]. \tag{2.7}$$

Although this implies that $\mathcal{D}(U)$ and $\overline{\mathcal{D}(U)}$ are orthogonal, there is no direct sum decomposition into $\mathcal{D}(U)$ and $\overline{\mathcal{D}(U)}$. This is because in general there exist exact harmonic one-forms whose holomorphic and anti-holomorphic parts are not exact.

Observe that the Dirichlet spaces are conformally invariant in the same sense as the Bergman spaces. That is, if $f : U \to V$ is a biholomorphism then

$$\mathbf{C}_f h = h \circ f$$

satisfies

$$\mathbf{C}_f : \mathcal{D}(V) \to \mathcal{D}(U)$$

and this is a seminorm preserving bijection. Similar statements hold for the anti-holomorphic and harmonic spaces.

The Sobolev space $H^s(\mathbb{S}^1)$, $s \geq 0$, will also play an important role in our investigations, whose definition we also recall. Given $f \in L^2(\mathbb{S}^1)$ one defines the Fourier coefficients and the Fourier series associated to f by

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-in\theta}\,d\theta, \quad f = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta}, \tag{2.8}$$

where the convergence of the series is both in the L^2-norm and also pointwise almost everywhere. The Sobolev space $H^s(\mathbb{S}^1)$ is defined by

$$H^s(\mathbb{S}^1) = \left\{ f \in L^2(\mathbb{S}^1) : \sum_{n=-\infty}^{\infty} \left(1+|n|^2\right)^s |\hat{f}(n)|^2 < \infty \right\}. \tag{2.9}$$

Like all other L^2-based Sobolev spaces, $H^s(\mathbb{S}^1)$ is a Hilbert space and given f, $g \in H^s(\mathbb{S}^1)$ their scalar product is given by

$$\langle f, g\rangle_{H^s(\mathbb{S}^1)} = \sum_{n=-\infty}^{\infty} \left(1+|n|^2\right)^s \hat{f}(n)\overline{\hat{g}(n)}, \tag{2.10}$$

and so

$$\|f\|_{H^s(\mathbb{S}^1)} = \left(\sum_{n=-\infty}^{\infty} \left(1+|n|^2\right)^s |\hat{f}(n)|^2\right)^{1/2}. \tag{2.11}$$

Of particular interest in this paper, are the functions in the Sobolev space $H^{1/2}(\mathbb{S}^1)$ and in the homogeneous space $\dot{H}^{1/2}(\mathbb{S}^1)$ i.e.

$$\|f\|_{H^{1/2}(\mathbb{S}^1)} := \Big(\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{|f(z)-f(\zeta)|^2}{|z-\zeta|^2}\,|dz|\,|d\zeta| + \|f\|^2_{L^2(\mathbb{S}^1)}\Big)^{1/2}. \tag{2.12}$$

$$\|f\|_{\dot{H}^{1/2}(\mathbb{S}^1)} := \Big(\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{|f(z)-f(\zeta)|^2}{|z-\zeta|^2}\,|dz|\,|d\zeta|\Big)^{1/2}. \tag{2.13}$$

As was shown by J. Douglas [4], for a function $F \in \mathcal{D}_{\mathrm{harm}}(\mathbb{D})$, the restriction of F to $\mathbb{S}^1$ is in $H^{1/2}(\mathbb{S}^1)$ and if the boundary value of F is denoted by f then one has that

$$\|F\|^2_{\mathcal{D}_{\mathrm{harm}}(\mathbb{D})} = \pi \int_0^{2\pi}\int_0^{2\pi} \frac{|f(z)-f(\zeta)|^2}{|z-\zeta|^2}\,|dz|\,|d\zeta|. \tag{2.14}$$

The dual of $H^{1/2}(\mathbb{S}^1)$, identified with $H^{-1/2}(\mathbb{S}^1)$, consists of linear functionals L on $H^{1/2}(\mathbb{S}^1)$ with the property that if $\alpha_n := L(e^{in\theta})$ (this is the action of the functional L on the function $e^{in\theta}$), then

$$\sum_{n=-\infty}^{\infty} \frac{|\alpha_n|^2}{(1+|n|^2)^{1/2}} < \infty. \tag{2.15}$$

Moreover one has

$$\|L\|_{H^{-1/2}(\mathbb{S}^1)} = \sup_{\|g\|_{H^{1/2}(\mathbb{S}^1)}=1} \left|\sum_{n=-\infty}^{\infty} \alpha(n)\overline{\hat{g}(n)}\right|. \tag{2.16}$$

We shall also recall the following useful result.

Corollary 2.16. *Let $F \in \mathcal{D}_{\mathrm{harm}}(\mathbb{D})$and let f denote the boundary value of F. Then we have the following equivalence of norms:*

$$\|f\|_{H^{1/2}(\mathbb{S}^1)} \approx |F(0)| + \|F\|_{\mathcal{D}_{\mathrm{harm}}(\mathbb{D})}. \tag{2.17}$$

Now regarding Sobolev spaces on manifolds, we first recall the definition of Sobolev $H^s(M)$, $s \in \mathbb{R}$ for compact manifolds M, see e.g. [3].

Definition 2.17. Let M be an $n-$dimensional smooth compact manifold without boundary, with the smooth atlas (ϕ_j, U_j) and the corresponding smooth partition of unity ψ_j with $\psi_j \geq 0$, $\operatorname{supp}\psi_j \subset U_j$ and $\sum_j \psi_j = 1$. Given $s \geq 0$, the Sobolev spaces $H^s(M)$ are the space of complex-valued L^2 functions on M for which

$$\|f\|_{H^s(M)} := \sum_j \|(\psi_j f)\circ\phi_j^{-1}\|_{H^s(\mathbb{R}^n)} < \infty, \tag{2.18}$$

where for $s \in \mathbb{R}$, the *Sobolev space* $H^s(\mathbb{R}^n)$, consists of tempered distributions f such that

$$\|f\|^2_{H^s(\mathbb{R}^n)} := \|(1-\Delta)^{s/2} f\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\widehat{f}(\xi)|^2\, d\xi < \infty,$$

where $\widehat{f}(\xi)$ is the Fourier transform of f defined by $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-ix\cdot\xi}\, dx$, Δ is the Laplace operator, and

$$(1-\Delta)^{s/2} f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (1+|\xi|^2)^{s/2}\, \widehat{f}(\xi)\, e^{ix\cdot\xi}\, d\xi.$$

The homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^n)$, is the space of tempered distributions such that $\int_{\mathbb{R}^n} |\xi|^{2s}\, |\hat{u}(\xi)|^2\, d\xi < \infty$. The preceding definition of Sobolev spaces on manifolds extends to $s < 0$ by duality, and the homogeneous Sobolev space $\dot{H}^s(M)$ is defined using (2.18) by substituting $H^s(\mathbb{R}^n)$ with $\dot{H}^s(\mathbb{R}^n)$.

It is also well-known that different choices of the atlas and its corresponding partition of unity produce norms that are equivalent with (2.18).

In general, we define the Sobolev space of a bordered surface Σ in the following way. We treat Σ as a subset of its compact double Σ^d, so that the borders are analytic curves and in particular smooth. By the Uniformization Theorem, the double has a constant curvature Riemannian metric compatible with the complex structure. The Sobolev space $H^s(\Sigma)$ consists of restrictions of $H^s(\Sigma^d)$ to Σ. We can similarly define $H^s(\partial\Sigma)$ in the standard way. One also has that for manifolds X with boundary $\mathrm{bd}(X)$ the *trace map*

$$\mathrm{Tr} : u \mapsto u|_{\mathrm{bd}(X)}$$

from $H^s(X) \to H^{s-\frac{1}{2}}(\mathrm{bd}(X))$ is continuous for $s > \frac{1}{2}$.

We will also use the invariance of the Sobolev space H^s under diffeomorphisms. We state this below as a lemma whose proof could be found in Lemma 1.3.3 in [8].

Lemma 2.18. *Let $s \in \mathbb{R}$ and ψ be a diffeomorphism of a bounded open set $U_1 \subset \mathbb{R}^n$ onto another bounded open set $U_2 \subset \mathbb{R}^n$ such that $\psi \in \mathcal{C}^\infty(\mathrm{cl}(U_1))$ and $\psi^{-1} \in \mathcal{C}^\infty(\mathrm{cl}(U_2))$. Then one has*

$$\|f\circ\psi\|_{H^s(U_1)} \approx \|f\|_{H^s(U_2)}.$$

In order to define conformally invariant Sobolev H^1–spaces we use harmonic measure on bordered Riemann surfaces. We can also use Green's functions to give an equivalent definition of these spaces. First we recall the notion of harmonic measure in the context of bordered Riemann surfaces.

Definition 2.19. Let Σ be a bordered surface of type (g, n). Let ω_k, $k = 1, \dots, n$ be the unique harmonic function which is continuous on the closure of Σ and which satisfies

$$\omega_k = \begin{cases} 1 & \text{on } \partial_k \Sigma \\ 0 & \text{on } \partial_j \Sigma, \quad j \neq k. \end{cases}$$

The one-forms $d\omega_k$ are the *harmonic measures.*

We denote the complex linear span of the harmonic measures by $\mathcal{A}_{\mathrm{hm}}(\Sigma)$. Moreover we define $* \mathcal{A}_{\mathrm{hm}}(\Sigma) = \{* \alpha : \alpha \in \mathcal{A}_{\mathrm{hm}}(\Sigma)\}$. One has the following.

Proposition 2.20. *Let Σ be a bordered surface of type (g, n). Then $\mathcal{A}_{\mathrm{hm}}(\Sigma) \subseteq \mathcal{A}^{\mathrm{e}}_{\mathrm{harm}}(\Sigma)$ and $* \mathcal{A}_{\mathrm{hm}}(\Sigma) \subseteq \mathcal{A}_{\mathrm{harm}}(\Sigma)$.*

Definition 2.21. The *boundary period matrix* Π_{jk} of a non-compact surface Σ of type (g, n) is defined by

$$\Pi_{jk} := \int_{\partial\Sigma} \omega_j * d\omega_k = \int_{\partial_j \Sigma} * d\omega_k.$$

Theorem 2.22. *If we let j, k run from 1 to n, omitting one fixed value m say, then the resulting matrix Π_{jk} is symmetric and positive definite.*

See [18, Theorem 2.36] for a proof.

Thus Π_{jk}, $j, k = 1, \dots \hat{m}, \dots, n$ is an invertible matrix, and we can specify $n - 1$ of the boundary periods of elements of $* \mathcal{A}_{\mathrm{hm}}(\Sigma)$. Since $\sum_{j=1,\dots,n} \omega_j$ is identically equal to one, applying Stokes' theorem to the definition of Π_{jk} we see that Π_{mk} is determined by the remaining values Π_{jk}.

This leads to the following consequence.

Corollary 2.23. *Let Σ be of type (g, n) and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ be such that $\lambda_1 + \cdots + \lambda_n = 0$. Then there is an $\alpha \in * \mathcal{A}_{\mathrm{hm}}(\Sigma)$ such that*

$$\int_{\partial_k \Sigma} \alpha = \lambda_k \tag{2.19}$$

for all $k = 1, \dots, n$.

Another basic notion which is of fundamental importance in our investigations is that of Green's functions.

Definition 2.24. Let Σ be a type (g, n) surface. For fixed $z \in \Sigma$, we define the *Green's function of* Σ to be a function $G_\Sigma(w; z)$ such that

(1) for a local coordinate ϕ vanishing at z the function $w \mapsto G_\Sigma(w; z) + \log |\phi(w)|$ is harmonic in an open neighbourhood of z;
(2) $\lim_{w \to \zeta} G_\Sigma(w; z) = 0$ for any $\zeta \in \partial\Sigma$.

We shall now use the harmonic measure and Green's function to define a conformally invariant Sobolev space that will be of great significance in our investigations. Let $d\omega_k$ be the harmonic measures given in Definition 2.19. For a collar neighbourhood U_k of $\partial_k\Sigma$ and $h_k \in \mathcal{D}_{\text{harm}}(U_k)$, we can fix a simple closed analytic curve γ_k which is isotopic to $\partial_k\Sigma$, and define

$$\int_{\partial_k\Sigma} h_k \ast d\omega_k := \iint_{V_k} dh_k \wedge \ast\, d\omega_k + \int_{\gamma_k} h_k \ast d\omega_k \tag{2.20}$$

where V_k is the region bounded by $\partial_k\Sigma$ and γ_k. Here $\partial_k\Sigma$ is oriented positively with respect to Σ and γ_k has the same orientation as $\partial_k\Sigma$ (this is independent of γ_k). Equivalently, for the curves $\Gamma_r = \phi^{-1}(|z| = r)$ defined by a collar chart ϕ,

$$\int_{\partial_k\Sigma} h_k \ast d\omega_k = \lim_{r\nearrow 1} \int_{\Gamma_r} h_k \ast d\omega_k.$$

Now given $h_k \in \mathcal{D}_{\text{harm}}(\Sigma)$ we set

$$\mathscr{H}_k := \int_{\partial_k\Sigma} h_k \ast d\omega_k. \tag{2.21}$$

Definition 2.25. Let Σ be a bordered surface of type (g, n) and let $U_k \subseteq \Sigma$ be collar neighbourhoods of $\partial_k\Sigma$ for $k = 1, \dots, n$. Set $U = U_1 \cup \cdots \cup U_n$. By $H^1_{\text{conf}}(U)$ we denote the harmonic Dirichlet space $\mathcal{D}_{\text{harm}}(U)$ endowed with the norm

$$\|h\|_{H^1_{\text{conf}}(U)} := \left(\|h\|^2_{\mathcal{D}_{\text{harm}}(U)} + \sum_{k=1}^{n} |\mathscr{H}_k|^2 \right)^{\frac{1}{2}} \tag{2.22}$$

for $n > 1$. In the case that $n = 1$, fix a point $p \in \Sigma \setminus U_1$ and define instead

$$\mathscr{H}_1 := \int_{\partial_1\Sigma} h_1 \ast dG_\Sigma(w, p), \tag{2.23}$$

where $G_\Sigma(w, p)$ is Green's function of Σ.

For the Riemann surface Σ, assuming that Σ is connected, we need only one boundary integral to obtain a norm. If $n > 1$, we can choose any fixed boundary curve $\partial_n\Sigma$ say, and define the norm

$$\|h\|_{H^1_{\text{conf}}(\Sigma)} := \left(\|h\|^2_{\mathcal{D}_{\text{harm}}(\Sigma)} + |\mathscr{H}_n|^2 \right)^{1/2}, \tag{2.24}$$

where any of the $\mathscr{H}_k$ could in fact be used in place of $\mathscr{H}_n$. In the case that $n = 1$ we use (2.23) to define $\mathscr{H}_1$.

The notation H^1_{conf} is meant to indicate the fact that the norms are equivalent to the restriction of the Sobolev space norm to harmonic functions. The 1 means that the first derivative is in L^2, and the subscript conf means that the norm is conformally invariant.

We close this subsection by recalling the two "Anchor Lemmas" which were proven in [18, Lemma 3.14 and Lemma 3.15]

Lemma 2.26 (First Anchor Lemma)**.** *Let Σ be a bordered Riemann surface of type (g, n). Let $\phi : U \to \mathbb{A}_{r,1}$ be a collar chart of $\partial_k\Sigma$. Let $\alpha \in \mathcal{A}(U)$. For any $h \in \mathcal{D}_{\text{harm}}(U)$*

$$\lim_{r \nearrow 1} \int_{\Gamma_k^r} \alpha(w) h(w)$$

exists, where $\Gamma_k^r = \phi^{-1}(|z| = r)$. Furthermore, this quantity is independent of the collar chart.

Lemma 2.27 (Second Anchor Lemma)**.** *Let Σ be a bordered Riemann surface of type (g, n). Let U be a collar neighbourhood of $\partial_k\Sigma$ in Σ for some $k \in \{1, \dots, n\}$. If h_1 and h_2 are any two elements of $\mathcal{D}_{\text{harm}}(U)$ with the same* CNT *boundary values on $\partial_k\Sigma$ up to a null set, then for any $\alpha \in \mathcal{A}(U)$*

$$\int_{\partial_k\Sigma} \alpha(w) h_1(w) = \int_{\partial_k\Sigma} \alpha(w) h_2(w).$$

2.4. Quasisymmetries on Riemann surfaces and sewing. In this section we recall the important notions of quasisymmetries on Riemann surfaces, sewing (where the seams are quasicircles) and the procedure of sewing caps on the surfaces.

Definition 2.28. An orientation-preserving homeomorphism h of $\mathbb{S}^1$ is called an *orientation-preserving quasisymmetric mapping* if there is a constant $k > 0$, such that for every θ, and every ψ not equal to an integer multiple of 2π, the inequality

$$\frac{1}{k} \leq \left| \frac{h(e^{i(\theta+\psi)}) - h(e^{i\theta})}{h(e^{i\theta}) - h(e^{i(\theta-\psi)})} \right| \leq k$$

holds. We say that h is an orientation-reversing quasisymmetry if $h \circ s$ is an orientation-preserving quasisymmetry where $s(e^{i\theta}) = e^{-i\theta}$.

It is a well-known fact (e.g. by theorems of Ahlfors-Beurling or Douady-Earle [11]) that a quasisymmetric homeomorphism of $\mathbb{S}^1$ can be extended (non-uniquely) to a quasiconformal self-map of the unit disk (similar extensions are available for quasisymmetries of the line, yielding quasiconformal self-maps of the half-plane).

The sewing process results in curves called quasicircles, which we now define.

Definition 2.29. We say that a simple closed curve in the plane $\mathbb{C}$ is a *quasicircle* if it is the image of $\mathbb{S}^1$ under a quasiconformal map of the plane.

A simple closed curve Γ in a Riemann surface $\mathscr{R}$ is a quasicircle if there is an open set U containing Γ and a biholomorphism $\phi : U \to \mathbb{A}$ where $\mathbb{A}$ is an annulus in $\mathbb{C}$, such that $\phi(\Gamma)$ is a quasicircle.

Remark 2.30. One might ask whether the object defined above deserves the label "quasicircle". Riemann surfaces of type (g, n) do not contain holomorphic injective images of the plane unless $g = n = 0$, one cannot use the definition in the plane directly. Definition 2.29 weakens this to the image of a circle under an injective holomorphic map of an annulus, but one might instead define a quasicircle in $\mathscr{R}$ to be the image of an analytic simple closed curve in a Riemann surface under a bijective quasiconformal map onto $\mathscr{R}$. It can be shown using extension theorems for quasiconformal maps (with some effort) that the definition given here is equivalent. For example we showed that Definition 2.29 implies this alternate definition [15, Lemma 3.2.6]; in fact, the quasiconformal map can be taken to be conformal on one side of the curve. The converse can be established through a similar argument, but we will not pursue this. We chose Definition 2.29 because it allows application of sewing arguments with minimal preamble, and avoids overburdening the paper.

Definition 2.31. Fix $k \in \{1, \dots, n\}$. Let $\tau : \mathbb{S}^1 \to \partial_k \Sigma$ be a homeomorphism. We say that τ is a *quasisymmetry* if there is a collar chart $\phi : U \to \mathbb{A}_{r,1}$ of $\partial_k \Sigma$ such that $\phi \circ \tau$ is a quasisymmetry in the sense of Definition 2.28. We say that τ is orientation-preserving (resp. orientation-reversing) when $\phi \circ \tau$ is orientation-preserving (resp. orientation-reversing).

Using the quasisymmetric homeomorphisms above, one can define a sewing operation between two bordered Riemann surfaces as follows.

Definition 2.32. Let Σ_1 and Σ_2 be bordered surfaces of type (g_1, n_1) and (g_2, n_2) respectively. Let $\tau_1 : \mathbb{S}^1 \to \partial_{k_1} \Sigma_1$ and $\tau_2 : \mathbb{S}^1 \to \partial_{k_2} \Sigma_2$ be orientation-reversing quasisymmetries. We can *sew* these surfaces to get a new topological space Σ defined by the equivalence relation

$$q_1 \sim q_2 \Leftrightarrow q_2 = \tau_2 \circ \tau_1^{-1}(q_1)$$

for q_1, q_2 in $\partial_{k_1} \Sigma$, $\partial_{k_2} \Sigma$ respectively. We call the set of points in Σ corresponding to the boundaries *the seam*.

In this connection we have the following:

Theorem 2.33 ([14])**.** *The surface Σ in* Definition 2.32 *has a complex structure which is compatible with that of Σ_1 and Σ_2. This complex structure is unique. The seam is a quasicircle. If τ_1 and τ_2 are analytic then the seam is an analytic Jordan curve.*

Recall that analytic Jordan curves are strip-cutting by definition.

Corollary 2.34. *Let Σ be a bordered surface of type (g, n). There is a compact surface $\mathscr{R}$ and an inclusion $\iota : \Sigma \to \mathscr{R}$ which is a biholomorphism onto its image, which extends continuously to a homeomorphism of the boundary curves of Σ into n disjoint quasicircles in $\mathscr{R}$, such that $\mathscr{R} \backslash \mathrm{cl}\,(\Sigma)$ consists of n open regions biholomorphic to $\mathbb{D}$. If desired, the quasicircles can be chosen to be analytic curves.*

Proof. Let $\tau_k : \mathbb{S}^1 \to \partial_k \Sigma$ be orientation-reversing quasisymmetries for $k = 1, \dots, n$. Using τ_k, sew on n copies of $\mathbb{D}$ to Σ. The claim follows from Theorem 2.33. □

Definition 2.35. We refer to this procedure as *sewing caps on* Σ, where a *cap* is a connected component of $\mathscr{R}\backslash\Sigma$.

In relation to the forthcoming investigation of the overfare of harmonic one-forms in Section 4, we also state the following definition.

Definition 2.36. Let $\mathscr{R}$ be a compact Riemann surface, and let $\Gamma_1, \dots, \Gamma_m$ be a collection of quasicircles in $\mathscr{R}$. Denote $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$. We say that Γ *separates* $\mathscr{R}$ into Σ_1 and Σ_2 if

(1) there are doubly-connected neighbourhoods U_k of Γ_k for $k = 1, \dots, n$ such that $U_k \cap U_j$ is empty for all $j \neq k$,
(2) one of the two connected components of $U_k \backslash \Gamma_k$ is in Σ_1, while the other is in Σ_2;
(3) $\mathscr{R}\backslash\Gamma = \Sigma_1 \cup \Sigma_2$;
(4) $\mathscr{R}\backslash\Gamma$ consists of finitely many connected components;
(5) Σ_1 and Σ_2 are disjoint.

As was shown in in [18, Proposition 3.33], it turns out that one can identify $\partial\Sigma_1$ and $\partial\Sigma_2$ pointwise with Γ.

3. Dirichlet problem for L^2 harmonic one-forms

3.1. Assumptions throughout this section. In this section, we consider a bordered Riemann surface Σ of type (g, n) for $g \geq 0$ and $n > 0$.

3.2. About this section. In this section, we give a complete theory and solution of the Dirichlet problem for L^2 one-forms. This includes developing a theory of their boundary values, which we show can be identified with the Sobolev space $H^{-1/2}(\partial\Sigma)$. Given an element of $H^{-1/2}(\partial\Sigma)$ together with sufficient cohomological data, there is a unique L^2 harmonic one-form on Σ with those boundary values. Furthermore, the solution depends continuously on the data.

We also characterize the boundary values in terms of equivalence classes of L^2 harmonic one-forms defined in collar neighbourhoods. We show that there is a one-to-one correspondence between elements of $H^{-1/2}$ and such equivalence classes, and this allows us to use the theory of CNT boundary values developed in [18] to solve the problem. This is because anti-derivatives of such forms have well-defined CNT boundary values, which can be identified with elements of $H^{1/2}$ (after removing a period). This reflects the fact that $H^{-1/2}$ is the set of distributional derivatives of elements of $H^{1/2}$.

We outline the approach. In Section 3.3 we give the routine solution to the Dirichlet problem for smooth boundary values. This section does not contain any original material, but rather serves to outline how the cohomological data is dealt with without the distraction of analytic complications. In particular it establishes the cohomological preliminaries used in the proof of the general case. In Section 3.4, we show the equivalence between the CNT and $H^{-1/2}$ boundary

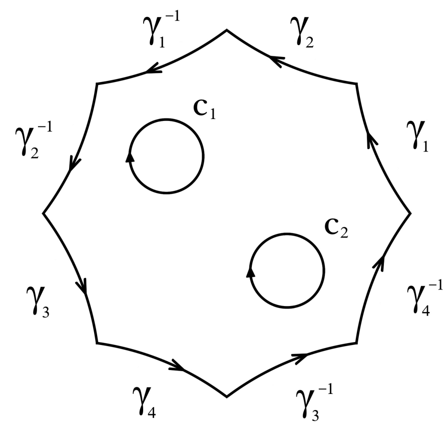

FIGURE 3.1. Polygononal decomposition of the bordered surface

values of one-forms. The bulk of the main results, namely the proof of the well-posedness of the Dirichlet problem for CNT boundary values, is given in Section 3.5. Finally, in Section 3.6 we use the equivalence between $H^{-1/2}$ and CNT boundary values of one-forms, together with the solution to the CNT boundary value problem given in Section 3.5, to solve the $H^{-1/2}$ Dirichlet problem for L^2 one-forms.

3.3. Formulation of the regular Dirichlet problem. Let Σ be a Riemann surface of type (g, n). We describe a network of smooth curves on Σ. By Corollary 2.34 we can treat Σ as a subset of a compact Riemann surface $\mathscr{R}$ obtained by either sewing on caps, or as a subset of the double. In the latter case, the boundary curves are analytic, and in the former, the boundary curves can be taken to be analytic, if one sews on caps via analytic parametrizations.

For the moment, let $\gamma_1, \dots, \gamma_{2g}$ be specific simple smooth closed curves which are generators of the homology of the surface $\mathscr{R}$ obtained by sewing on caps. We choose these curves such that they lie in Σ, and furthermore, such that when $\mathscr{R}$ is cut along these curves we obtain a polygonal decomposition of $\mathscr{R}$ in the standard way. Let c_k be smooth curves in Σ which are isotopic to the boundaries $\partial_k \Sigma$ for $k = 1, \dots, n$; we assume that these are non-intersecting. See Figure 3.1 for a picture of the polygonal decomposition of $\mathscr{R}$ together with the curves c_k. For any $\alpha \in \mathcal{A}_{\mathrm{harm}}(\Sigma)$, we have that $\int_\gamma \alpha$ depends only on the homotopy class of γ, so we can define

$$\int_{\partial_k \Sigma} \alpha = \int_{c_k} \alpha,$$

and with this definition, it is clear that if we let γ denote the boundary of the polygon, then

$$\sum_{k=1}^{n} \int_{\partial_k \Sigma} \alpha = - \int_{\gamma} \alpha = 0 \tag{3.1}$$

for any $\alpha \in \mathcal{A}_{\text{harm}}(\Sigma)$.

We also need the following facts regarding the double Σ^d of Σ. Observe that each handle of Σ has a duplicate, and if there are n boundary curves for $n \geq 1$, then the double has $n-1$ additional handles. There is a basis of simple closed curves $\{\Gamma_1, \ldots, \Gamma_{4g+2n-2}\}$ for the homology of Σ^d so that $\Gamma_k = \gamma_k$ for $k = 1, \ldots, 2g$. In addition if $n \geq 2$, we can choose the basis such that $\Gamma_k = \partial_k \Sigma$ for $k = 2g+1, \ldots, 2g+n-1$.

Now let $\{\varepsilon_1, \ldots, \varepsilon_{4g+2n-2}\}$ be a dual basis of closed one-forms on Σ^d. By the Hodge decomposition theorem these can be chosen to be harmonic. We thus have

$$\int_{\Gamma_k} \varepsilon_j = \delta_j^k, \quad j, k = 1, \ldots, 4g+2n-2 \tag{3.2}$$

where δ_j^k is the Kronecker delta.

Our data in the Dirichlet problem will consist of continuous one-forms on the boundary curves together with specified period information. Since Σ is a bordered surface, the notion of continuous or smooth one-forms is well-defined; explicitly, α is a continuous or smooth one-form respectively, if for some collar chart ϕ of $\partial_k \Sigma$, setting $\psi = \phi|_{\partial_k \Sigma}$ its expression in coordinates is $\psi^* \alpha = h(e^{i\theta})\, d\theta$ for some continuous or smooth function h respectively.

The $\mathcal{C}^\infty$ Dirichlet problem for one-forms is as follows. Let Σ be a Riemann surface of type (g, n). We refer to the following data as smooth Dirichlet data for forms on a Riemann surface:

i. $\mathcal{C}^\infty$ one-forms β_k on $\partial_k \Sigma$ for each $k = 1, \ldots, n$, satisfying

$$\int_{\partial_1 \Sigma} \beta_1 + \cdots + \int_{\partial_n \Sigma} \beta_n = 0;$$

ii. constants $\rho_1, \ldots, \rho_n \in \mathbb{C}$ satisfying

$$\rho_1 + \cdots + \rho_n = 0;$$

and

iii. constants $\sigma_1, \ldots, \sigma_{2g} \in \mathbb{C}$.

Definition 3.1. We say that a harmonic one-form α on Σ solves the Dirichlet problem with data (β, ρ, σ) if α extends smoothly to $\partial \Sigma$ and

(1) for any tangent vector v_p to $\partial_k \Sigma$ at any point $p \in \partial_k \Sigma$, $\alpha(v_p) = \beta_k(v_p)$;

(2) for all $k = 1, \dots, n$

$$\int_{\partial_k \Sigma} * \alpha = \rho_k;$$

and

(3) for all $k = 1, \dots, 2g$

$$\int_{\gamma_k} \alpha = \sigma_k.$$

Note that since the one-forms β_k specify the boundary values, in particular they specify the periods around the boundary curves $\partial_k \Sigma$. Condition (2) is motivated as follows. For any harmonic measure $\sum_k d\omega_k$ and any solution α, the form $\alpha + \sum_k d\omega_k$ still satisfies (1) and (3), because $\sum_k d\omega_k$ is exact and $\sum_k d\omega_k = 0$ along $\partial\Sigma$. In fact, this is the only indeterminacy and the condition (2) uniquely determines the solution.

In fact, up to the cohomological data, the smooth Dirichlet problem for one-forms is essentially a smooth Dirichlet problem for functions. To solve the Dirichlet problem for forms, one simply subtracts off forms whose periods match the data, so that one obtains boundary values of exact forms. One then solves the Dirichlet problem for functions with respect to the primitive on the boundary. The solution to the problem for functions is well-known:

Theorem 3.2. *Let X be a compact Riemannian manifold with boundary ∂X, and Δ is the Laplacian on X. Then the Dirichlet problem*

$$\begin{cases} \Delta u = 0 \\ u|_{\partial X} = f \in \mathcal{C}^\infty(\partial X) \end{cases} \tag{3.3}$$

has a unique solution $u \in \mathcal{C}^\infty(X)$.

For the proof see e.g. [9] page 264 Example 1.

Theorem 3.3. *For smooth Dirichlet data (β, ρ, σ) there exists an $\alpha \in \mathcal{C}^\infty(\mathrm{cl}(\Sigma))$ which solves the smooth Dirichlet problem.*

Proof. We assume that Σ is included in its double, so that the boundary curves are analytic. Setting

$$\lambda_k = \int_{\partial_k \Sigma} \beta_k \tag{3.4}$$

for $k = 1, \dots, n$, by Corollary 2.23 there is a $\mu \in * \mathcal{A}_{\mathrm{hm}}$ such that

$$\int_{\partial_k \Sigma} \mu = \lambda_k \tag{3.5}$$

for every k and a harmonic one-form η in the span of $\{\varepsilon_1, \dots, \varepsilon_{2g}\}$, which were defined in connection to (3.2), such that

$$\int_{\gamma_j} \eta = \sigma_j - \int_{\gamma_j} \mu$$

for $j = 1, \dots, 2g$. By definition of the basis $\{\varepsilon_1, \dots, \varepsilon_{4g+2n-2}\}$

$$\int_{\partial_k \Sigma} \eta = 0 \tag{3.6}$$

for $k = 1, \dots, n$. Observe that the one-forms η and μ are smooth on $\partial\Sigma$.

Define functions h_k on the boundary curves $\partial_k \Sigma$ as follows. Each h_k is the anti-derivative of $\beta_k - \mu - \eta$ on c_k, that is, for any tangent vector v to the boundary c_k

$$dh_k(v) = \beta_k(v) - \mu(v) - \eta(v).$$

Note that each anti-derivative is single-valued by (3.4) and the definition of ε_k. By Theorem 2.22 we can add a suitable harmonic measure $d\omega \in \mathcal{A}_{\mathrm{hm}}(\Sigma)$ (which is exact and does not change the periods) in order to ensure that condition (2) in Definition 3.1 is satisfied. Solving now the ordinary Dirichlet problem with smooth data $h_1, \dots, h_n$ on the boundary curves using Theorem 3.3, we obtain a smooth $h \in \mathcal{D}_{\mathrm{harm}}(\Sigma)$. Then

$$\alpha = dh + \mu + \eta$$

is the desired solution to the problem. It is not hard to show that the solution is unique by keeping track of the periods and using uniqueness in Theorem 3.2. □

3.4. Boundary values of L^2 forms and $H^{-1/2}$. In this section, we will show that $H^{-1/2}(\partial_k \Sigma)$ of a boundary curve $\partial_k \Sigma$ can be identified with an equivalence class of harmonic one-forms defined in a collar neighbourhood. The idea is fairly simple, and we give a sketch in the case of the circle $\mathbb{S}^1$ before launching into the details. We can think of smooth one-forms $h(\theta)d\theta$ on the circle as dual to functions on the circle via the pairing

$$L_{hd\theta}(f) = \int_{\mathbb{S}^1} f \cdot hd\theta.$$

Of course if $hd\theta$ is in $H^{-1/2}(\mathbb{S}^1)$ and $f \in H^{1/2}(\mathbb{S}^1)$, then this only makes sense distributionally.

On the other hand, given an $\alpha \in \mathcal{A}_{\mathrm{harm}}(\mathbb{A}_{r,1})$ for an annulus $\mathbb{A}_{r,1}$, by the First Anchor Lemma 2.26 one can define a pairing

$$\lim_{r \nearrow 1} \int_{|z|=r} f\alpha. \tag{3.7}$$

If α were smooth, we could identify this integral with

$$\int_{\mathbb{S}^1} f\alpha.$$

In the general case that f is in $H^{1/2}(\mathbb{S}^1)$, it turns out that the pairing makes sense, and in fact all elements of $H^{-1/2}(\mathbb{S}^1)$ can be represented this way. The same idea works for the border of a Riemann surface, provided that we treat it as an analytic curve (see [18, Remark 2.31]).

The remainder of this section is dedicated to filling in the details of this sketch. The payoff of this approach is that it makes it possible to use the machinery of CNT boundary values to solve the Dirichlet problem for one-forms with $H^{-1/2}$ boundary data. In this way one obtains a complete theory of the boundary values of L^2 harmonic one-forms for bordered surfaces.

We begin by defining an equivalence relation, such that the equivalence classes represent the boundary values of the one-form. Later we will see that each equivalence class can be identified with a unique element of $H^{-1/2}$, and vice-versa.

Definition 3.4 (Equivalence relation for CNT Dirichlet boundary values of one-forms)**.** For collar neighbourhoods A and B of $\partial_k\Sigma$, let $\alpha \in \mathcal{A}_{\text{harm}}(A)$ and $\beta \in \mathcal{A}_{\text{harm}}(B)$. We say that $\alpha \sim \beta$ if there is a $\delta \in \mathcal{A}_{\text{harm}}(U_k)$ for some collar neighbourhood $U_k \subseteq A \cap B$ of $\partial_k\Sigma$, such that

(1) $\alpha - \delta, \beta - \delta \in \mathcal{A}^{\text{e}}_{\text{harm}}(U_k)$;
(2) if $f, g \in \mathcal{D}_{\text{harm}}(U_k)$ are such that $df = \alpha - \delta$ and $dg = \beta - \delta$, then the CNT boundary values of $f - g$ are constant on $\partial_k\Sigma$ up to a null set.

In brief, α and β are equivalent if their multi-valued primitives agree on the boundary up to an integration constant. When the boundary curve is not clear from context, we will say "$\alpha \sim \beta$ on $\partial_k\Sigma$". Denote the equivalence class of a one-form α by $[\alpha]$.

To show that it is an equivalence relation, we need the following fact. If $\alpha \sim \beta$ via some δ, then any one-form $\delta' \in \mathcal{A}_{\text{harm}}(U')$ satisfying (1) also satisfies (2). To see this, choose a collar neighbourhood $V \subset U \cap U'$, which exists by Proposition 2.6. Observe that $\delta' - \delta = (\alpha - \delta) - (\alpha - \delta')$ has a primitive h on V. So if f and g are the primitives of $\alpha - \delta$ and $\beta - \delta$ respectively, then $f - h$ and $g - h$ are the unique primitives of $\alpha - \delta'$ and $\beta - \delta'$ up to constants. But $(f - h) - (g - h) = f - g$ has constant CNT boundary values on $\partial_k\Sigma$ up to a null set, which proves the claim. With this fact in hand, it is routine to verify that $\sim$ is an equivalence relation.

Definition 3.5. [CNT Dirichlet boundary values for one-forms] For each fixed $k = 1, \ldots, n$ let $\mathcal{U}_k$ denote the collection of collar neighbourhoods of $\partial_k\Sigma$. Define

$$\mathcal{H}'(\partial_k\Sigma) = \{\alpha \in \mathcal{A}_{\text{harm}}(U_k);\ U_k \in \mathcal{U}_k\}/\sim .$$

We also denote

$$\mathcal{H}'(\partial\Sigma) = \{([\alpha_1], \ldots, [\alpha_n]);\ \alpha_k \in \mathcal{H}'(\partial_k\Sigma)\}.$$

If $\alpha \in \mathcal{A}_{\text{harm}}(U)$ where U contains a collar neighbourhood U_k of each boundary, then we set

$$[\alpha] := ([\alpha_1], \ldots, [\alpha_n]),$$

where $\alpha_k = \alpha|_{U_k}$.

For fixed k, any equivalence class $[\beta] \in \mathcal{H}'(\partial_k\Sigma)$ has a well-defined boundary period. To see this, given $[\beta]$ and a representative $\beta \in \mathcal{A}_{\text{harm}}(U_k)$ for some collar neighbourhood U_k, let c_k be a smooth closed curve in U_k which is homotopic to $\partial_k\Sigma$, and define

$$\int_{\partial_k\Sigma}[\beta] = \int_{c_k}\beta.$$

To see that this is well-defined, let $\beta' \in \mathcal{A}_{\mathrm{harm}}(U_k')$ be another representative of $[\beta]$ and c_k' be another such curve. By Proposition 2.6 there is a canonical collar chart $\phi_{k,r} : U_{k,r} \to \mathbb{A}_{r,1}$ such that the inner boundary Γ of $U_{k,r}$ is contained in $U_k \cap U_k'$ and $\phi_{k,r}$ extends analytically to Γ. Since Γ is isotopic to $\partial_k\Sigma$, it is isotopic to c_k in U_k and isotopic to c_k' in U_k'. Thus

$$\int_{c_k'}\beta' = \int_\Gamma \beta = \int_{c_k}\beta,$$

proving the claim.

It also follows directly from the definition of the equivalence classes that $\mathcal{H}'(\partial\Sigma)$ is conformally invariant in the following sense.

Proposition 3.6. *Let Σ_1 and Σ_2 be bordered surfaces and fix borders $\partial_{k_1}\Sigma_1$ and $\partial_{k_2}\Sigma_2$ which are homeomorphic to $\mathbb{S}^1$. Let U and V be collar neighbourhoods of $\partial_{k_1}\Sigma_1$ and $\partial_{k_2}\Sigma_2$ respectively, and let $f : U \to V$ be a conformal map. Then for any two representatives α and β of $[\alpha] \in \mathcal{H}'(\partial_{k_2}\Sigma_2)$ we have*

$$[f^*\alpha] = [f^*\beta].$$

In particular, we have a well-defined pull-back map

$$\begin{aligned} f^* : \mathcal{H}'(\partial_{k_2}\Sigma_2) &\to \mathcal{H}'(\partial_{k_1}\Sigma_1) \\ [\alpha] &\mapsto [f^*\alpha]. \end{aligned}$$

We will require the following elementary lemma, in order to define a norm on $\mathcal{H}'(\partial_k\Sigma)$.

Lemma 3.7. *Let $[\alpha] \in \mathcal{H}'(\mathbb{S}^1)$ where $\mathbb{S}^1$ is treated as the border of the disk $\mathbb{D}$. Then α has a unique representative of the form*

$$\alpha = f(z)dz + \overline{g(z)}d\bar{z} + \delta$$

where $f(z), g(z) \in \mathcal{D}(\mathbb{D})$ have the form

$$f(z) = \sum_{n=1}^{\infty} f_n z^n, \quad \overline{g(z)} = \sum_{n=1}^{\infty} \overline{g_n}\,\bar{z}^n,$$

and

$$\delta = \frac{a}{4\pi i}\left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}}\right).$$

for some constant $a \in \mathbb{C}$.

Proof. Given a representative $\tilde{\alpha}$ on some annulus $\mathbb{A}_{r,1}$, there is a δ of the form above such that $\tilde{\alpha} - \delta$ is exact. Thus $\alpha - \delta = dh$ for some $h \in \mathcal{D}(\mathbb{A}_{r,1})$. Then the non-tangential boundary values $\hat{h}$ of h are in $H^{1/2}(\mathbb{S}^1)$. Thus there is a harmonic function $H \in \mathcal{D}(\mathbb{D})$ with non-tangential boundary values equal to $\hat{h}$; this can be written as $F + \overline{G}$ for holomorphic F and G. Set then $f = F'$, $g = G'$. Uniqueness follows from the fact that there is only one δ with the required period, together with the fact that there is only one pair f and g with boundary values $\hat{h}$, and $\hat{h}$ is determined up to a constant by $\alpha - \delta$. □

This allows us to define a norm on $\mathcal{H}'(\mathbb{S}^1)$. Given any $[\alpha]$ let

$$\alpha = f(z)dz + \overline{g(z)}d\overline{z} + \frac{\lambda}{4\pi i}\left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}}\right)$$

be the representative given by Lemma 3.7. We define

$$\|[\alpha]\|^2_{\mathcal{H}'(\mathbb{S}^1)} = \|f(z)dz + \overline{g(z)}d\bar{z}\|^2_{\mathcal{A}_{\mathrm{harm}}(\mathbb{D})} + |\lambda|^2.$$

For any boundary curve $\partial_k\Sigma$, we define a norm on $\mathcal{H}'(\partial_k\Sigma)$ as follows. Choose a collar chart $\phi : U \to \mathbb{A}_{r,1}$ of $\partial_k\Sigma$. Implicitly using Proposition 3.6, we define

$$\|[\alpha]\|_{\mathcal{H}'(\partial_k\Sigma)} = \|\phi^*[\alpha]\|_{\mathcal{H}'(\mathbb{S}^1)}. \tag{3.8}$$

This norm of course depends on the collar chart. However, we will see ahead that different collar charts induce equivalent norms.

Given a collection $\phi = (\phi_1, \dots, \phi_n)$ of collar charts of $\partial_1\Sigma, \dots, \partial_n\Sigma$, we define a norm on $\mathcal{H}'(\partial\Sigma)$ by

$$\|([\alpha_1], \dots, [\alpha_n])\|^2_{\mathcal{H}'(\partial\Sigma)} = \|[\alpha_1]\|^2_{\mathcal{H}'(\partial_1\Sigma)} + \cdots \|[\alpha_n]\|^2_{\mathcal{H}'(\partial_n\Sigma)}. \tag{3.9}$$

Again, this norm depends on the collection of collar charts ϕ. Regarding the norm defined above, we state the following lemma which will be useful in connection to Theorem 3.11 and Lemma 3.19 ahead.

Lemma 3.8. *Let $\phi : U \to \mathbb{A}_{r,1}$ be a collar chart defined near $\partial_k\Sigma$ for fixed k. Then*

$$h \mapsto h \circ \phi$$

is a bounded isomorphism from $H^{1/2}(\mathbb{S}^1)$ to $H^{1/2}(\partial_k\Sigma)$.

Proof. By Carathéodory's theorem and the Schwarz reflection principle, ϕ extends to a conformal map from a doubly connected neighbourhood V of $\partial_k\Sigma$ to the annulus $\mathbb{A}_{r,1/r}$. The restriction of ϕ to $\partial_k\Sigma$ is thus an analytic diffeomorphism between the compact manifolds $\partial_k\Sigma$ and $\mathbb{S}^1$, so the claim follows from Lemma 2.18. □

Lemma 3.9. *Let $\varphi : C_1 \to C_2$ be a quasisymmetric mapping between the closed smooth curves C_j, $j = 1, 2$. Then φ induces an equivalence between $\dot{H}^{\frac{1}{2}}(C_1)$ and $\dot{H}^{\frac{1}{2}}(C_2)$, i.e.*

$$\|f\|_{\dot{H}^{\frac{1}{2}}(C_2)} \approx \|f\circ\varphi\|_{\dot{H}^{\frac{1}{2}}(C_1)}.$$

As a consequence, we have that if ϕ_k is a quasisymmetric map from $\mathbb{S}^1 \to \partial_k\Sigma$ then

$$\|f\|_{\dot{H}^{\frac{1}{2}}(\partial_k\Sigma)} \approx \|f\circ\varphi_k\|_{\dot{H}^{\frac{1}{2}}(\mathbb{S}^1)}.$$

Proof. This is just a special case of Theorem 5.1 in [10]. □

Let Σ be a bordered Riemann surface of type (g, n). Fixing k, we can define a pairing between elements of $H^{1/2}(\partial_k\Sigma)$ and $\mathcal{H}'(\partial_k\Sigma)$ as follows. Given $[\alpha] \in \mathcal{H}'(\partial_k\Sigma)$ and $h \in H^{1/2}(\partial_k\Sigma)$, let $\alpha \in \mathcal{A}(U)$ be a representative of $[\alpha]$ for a collar neighbourhood U of $\partial_k\Sigma$, and let $H \in \mathcal{D}_{\text{harm}}(U')$ have CNT boundary values h on some collar neighbourhood U'. There exists at least one such H, by solving the Dirichlet problem on Σ with $H = h$ on $\partial_k\Sigma$ and 0 on the other boundary curves. By Proposition 2.6 we can choose a common collar neighbourhood $V \subset U \cap U'$. Define

$$L_{[\alpha]}(h) = \int_{\partial_k\Sigma} [H\alpha] = \lim_{r\nearrow 1} \int_{\Gamma_r} H\alpha \tag{3.10}$$

for limiting curves $\Gamma_r = \phi^{-1}(|z| = r)$ approaching $\partial_k\Sigma$. We have already shown that for fixed H this is well-defined. By the second Anchor Lemma 2.27 for any two $H_m \in \mathcal{D}_{\text{harm}}(U_m)$ on collar neighbourhoods U_m for $m = 1, 2$ with the same boundary values on $\partial_k\Sigma$, we have for fixed α

$$\int_{\partial_k\Sigma} H_1\alpha = \int_{\partial_k\Sigma} H_2\alpha.$$

Thus $L_{[\alpha]}$ is well-defined.

The pairing is conformally invariant.

Proposition 3.10. *Let Σ_1 and Σ_2 be bordered surfaces and fix borders $\partial_{k_1}\Sigma_1$ and $\partial_{k_2}\Sigma_2$ which are homeomorphic to $\mathbb{S}^1$. Let U and V be collar neighbourhoods of $\partial_{k_1}\Sigma_1$ and $\partial_{k_2}\Sigma_2$ respectively, and let $f : U \to V$ be a conformal map. For any $H \in H^{1/2}(\partial_{k_2}\Sigma_2)$,*

$$\int_{\partial_{k_2}\Sigma_2} [\alpha]H = \int_{\partial_{k_1}\Sigma_1} f^*[\alpha]\, H\circ f.$$

Proof. Let $\phi : U_2 \to \mathbb{A}_{r,1}$ be a collar chart of $\partial_{k_2}\Sigma_2$. Then $\phi\circ f : U_1 \to \mathbb{A}_{r,1}$ is a collar chart of $\partial_{k_1}\Sigma$, shrinking U_2 if necessary. Let Γ_r^2 be the limiting curves $\phi^{-1}(|z| = r)$ induced by ϕ, and similarly Γ_r^1 by $\phi\circ f$ (so that $f(\Gamma_r^1) = \Gamma_r^2$).

Now choose a representative α of $[\alpha]$ and let h be an extension of H to a collar neighbourhood of $\partial_{k_2}\Sigma_2$. Then by the Anchor Lemmas 2.26 and 2.27 and a change of variables, we have

$$\int_{\partial_{k_2}\Sigma_2} [\alpha]H = \lim_{r\nearrow 1} \int_{\Gamma_r^2} \alpha h = \lim_{r\nearrow 1} \int_{\Gamma_r^1} f^*\alpha\, h\circ f$$
$$= \int_{\partial_{k_1}\Sigma_1} f^*[\alpha]H\circ f$$

where in the last equality we have also used Proposition 3.6. □

Theorem 3.11. *Let Σ be a bordered Riemann surface of type (g, n). For any fixed $k \in \{1, \dots, n\}$, the bijection*

$$\mathcal{H}'(\partial_k\Sigma) \to H^{-1/2}(\partial_k\Sigma)$$
$$[\alpha] \mapsto L_{[\alpha]}$$

is a bounded isomorphism.

We first prove surjectivity in the case of $\mathbb{S}^1$.

Theorem 3.12. *Let L be in $H^{-1/2}(\mathbb{S}^1)$. Then there is an $\alpha \in \mathcal{A}_{\text{harm}}(\mathbb{A}_{r,1})$ such that*

$$L(f) = \lim_{s\nearrow 1} \int_{|z|=s} f\alpha. \tag{3.11}$$

Proof. Since $H^{1/2}(\mathbb{S}^1)$ is a Hilbert space, the Riesz representation theorem yields that there exists a unique $F \in H^{1/2}(\mathbb{S}^1)$ such that, if $f = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta}$ and $F = \sum_{n=-\infty}^{\infty} \hat{F}(n)e^{in\theta}$ then

$$L(f) = \langle f, F\rangle_{H^{1/2}(\mathbb{S}^1)} = \sum_{n=-\infty}^{\infty} \left(1+|n|^2\right)^{1/2} \hat{f}(n)\overline{\hat{F}(n)}.$$

Moreover $\|L\|_{H^{-1/2}(\mathbb{S}^1)} = \|F\|_{H^{1/2}(\mathbb{S}^1)}$. Now by Parseval's formula we also have

$$\sum_{n=-\infty}^{\infty} \left(1+|n|^2\right)^{1/2} \hat{f}(n)\overline{\hat{F}(n)} = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})((1-\partial_\theta^2)^{1/2}F)(e^{i\theta})\, d\theta. \tag{3.12}$$

This and the requirement of harmonicity of α suggest that the desired α should be taken as the Poisson extension of $((1-\partial_\theta^2)^{1/2}F)(e^{i\theta})$ (i.e. its convolution with the Poisson kernel of the unit disk), which for $s \le 1$ yields that

$$\alpha(se^{i\theta}) = \sum_{n=-\infty}^{\infty} \left(1+|n|^2\right)^{1/2} \hat{F}(n)\, s^{|n|}\, e^{in\theta}. \tag{3.13}$$

Moreover, a calculation reveals that for $0 < r < 1$ one has

$$\|\alpha\|^2_{L^2(\mathbb{A}_{r,1})} = \pi \sum_{n=-\infty}^{\infty} (1-r^{2|n|+2})\frac{1+|n|^2}{1+|n|}|\hat{F}(n)|^2 \lesssim \sum_{n=-\infty}^{\infty} (1+|n|^2)^{1/2}|\hat{F}(n)|^2 < \infty, \tag{3.14}$$

since $F \in H^{1/2}(\mathbb{S}^1)$. Therefore $\alpha \in \mathcal{A}_{\text{harm}}(\mathbb{A}_{r,1})$, as desired. □

We now return to the proof of Theorem 3.11.

Proof of Theorem 3.11. Let $\phi : U \to \mathbb{A}_{r,1}$ be a collar chart. For any $h \in H^{1/2}(\partial_k \Sigma)$, recall that we have

$$\int_{\partial_k \Sigma} [\alpha] h = \int_{\mathbb{S}^1} \phi^* [\alpha] h \circ \phi \tag{3.15}$$

by Proposition 3.10. Thus, by Lemma 3.8 and recalling the definition (3.8) of the chart-dependent norm, it is enough to prove the claim on $\mathcal{H}'(\mathbb{S}^1)$.

We first need to show that for any given $[\alpha] \in \mathcal{H}'(\mathbb{S}^1)$, the linear functional $L_{[\alpha]}$ is bounded, and hence in $H^{-1/2}(\mathbb{S}^1)$. To see this, let α be a representative as in Lemma 3.7, so that $\alpha - \delta$ is exact where

$$\delta = \frac{a}{4\pi i}\left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}}\right)$$

for some $a \in \mathbb{C}$, namely

$$a = \int_{\mathbb{S}^1} [\alpha].$$

For any $h \in H^{1/2}(\mathbb{S}^1)$ let H be its unique harmonic extension in $\mathcal{D}_{\text{harm}}(\mathbb{D})$, and write $H(z) = H_1(z) + H(0)$ where $H_1(0) = 0$. Recall that

$$\|h\|^2_{H^{1/2}(\mathbb{S}^1)} = |H(0)|^2 + \|dH\|^2_{\mathcal{D}_{\text{harm}}(\mathbb{D})}.$$

By the mean-value theorem for the harmonic function H_1, one has

$$\begin{aligned}
\lim_{r \nearrow 1} \int_{|z|=r} \alpha(z) H(z) &= \lim_{r \nearrow 1} \int_{|z|=r} H(0)\alpha(z) + \lim_{r \nearrow 1} \int_{|z|=r} \alpha H_1(z) \\
&= \lim_{r \nearrow 1} \int_{|z|=r} H(0)\delta(z) + \lim_{r \nearrow 1} \int_{|z|=r} (\alpha - \delta) H_1(z) \\
&= aH(0) - \iint_{\mathbb{D}} (\alpha - \delta) \wedge_w dH_1(z),
\end{aligned}$$

so the Cauchy-Schwarz inequality and Corollary 2.16 yield

$$\begin{aligned}
\left| \lim_{r \nearrow 1} \int_{|z|=r} \alpha(z) H(z) \right| &\le |aH(0)| + \|\alpha - \delta\|_{\mathcal{A}_{\text{harm}}(\mathbb{D})} \|H\|_{\mathcal{D}_{\text{harm}}(\mathbb{D})} \\
&\le C \|\alpha\|_{\mathcal{H}'(\mathbb{S}^1)} \|h\|_{H^{1/2}(\mathbb{S}^1)},
\end{aligned}$$

for some constant C. Thus $L_{[\alpha]} \in H^{-1/2}(\mathbb{S}^1)$. The same inequality also shows that the map $[\alpha] \to L_{[\alpha]}$ is bounded.

The map $[\alpha] \to L_{[\alpha]}$ is surjective by Theorem 3.12, so it remains to show that it is injective. Assume that $L_{[\alpha]} h = 0$ for all $h \in H^{1/2}(\mathbb{S}^1)$. Let $\alpha, \delta, f, \overline{g}$ be as in Lemma 3.7. Since

$$0 = L_{[\alpha]}(1) = \lim_{r \nearrow 1} \int_{|z|=r} \alpha = a$$

we must have $a = 0$. Similarly using $0 = L_{[\alpha]}(z^n) = L_{[\alpha]}(\bar{z}^n)$ for all $n \in \mathbb{N}$ shows that $f = \overline{g} = 0$, so $\alpha = 0$. Thus $[\alpha] = 0$. □

This also shows that different collar charts induce equivalent norms, as promised.

Corollary 3.13. *For any fixed k, and any pair of collar charts ϕ, ψ near $\partial_k\Sigma$, the norm induced on $\mathcal{H}'(\partial_k\Sigma)$ by ϕ and ψ are equivalent.*

Similarly, for any two collections of collar charts $\phi = (\phi_1, \dots, \phi_n)$ and $\psi = (\psi_1, \dots, \psi_n)$ of the boundaries $\partial_1\Sigma, \dots, \partial_n\Sigma$, the norms induced on $\mathcal{H}'(\partial\Sigma)$ by ϕ and ψ are equivalent.

Proof. It suffices to establish the case of one boundary curve. Fixing a collar chart ϕ by Theorem 3.11 the map $[\alpha] \to L_{[\alpha]}$ is a bounded isomorphism between $\mathcal{H}'(\partial_k\Sigma)$ and $H^{-1/2}(\partial_k\Sigma)$ with respect to the norm on $\mathcal{H}'(\partial_k\Sigma)$ induced by this chart. Since this is true for any collar chart, the norms induced by different collar charts must be equivalent. □

Finally, we observe that harmonic measures generate the zero equivalence class of $\mathcal{H}'(\partial_k\Sigma)$ for any $k = 1, \dots, n$.

Proposition 3.14. *For any $d\omega \in \mathcal{A}_{\mathrm{hm}}(\Sigma)$ we have*

$$[d\omega] = 0.$$

Proof. By Theorem 3.11 it suffices for us to show that $L_{[d\omega]} = 0$. Since $L_{[d\omega]}$ is bounded, it suffices to show that it is zero on the dense subset $H^1_{\mathrm{conf}}(U)$ where U is a doubly connected neighbourhood of $\partial_k\Sigma$ in the double of Σ. Observing that $d\omega$ has an extension to the double, for any such $h \in H^1_{\mathrm{conf}}(U)$ we obtain

$$L_{[d\omega]}(h) = \int_{\partial_k\Sigma} h\, d\omega$$

where the integral on the right hand side can be evaluated directly on the curve $\partial_k\Sigma$. Since $d\omega = 0$ for vectors tangent to $\partial_k\Sigma$, this completes the proof. □

A model of the homogeneous space $\dot{H}^{-1/2}(\partial_k\Sigma)$ can also be given in terms of one-forms. Consider the Sobolev space $\dot{H}^{1/2}(\partial_k\Sigma)$ as consisting of functions modulo constants. Let $\dot{H}^{-1/2}(\partial_k\Sigma)$ denote its dual space.

Assume that $[\alpha] = [\beta]$ in $\mathcal{H}'(\partial_k\Sigma)$. If $\int_{\partial_k\Sigma} \alpha = 0$, then $\int_{\partial_k\Sigma} \beta = 0$. Thus we may define

$$\dot{\mathcal{H}}'(\partial_k\Sigma) = \{[\alpha] \in \mathcal{H}'(\partial_k\Sigma) : [\alpha] \text{ has an exact representative}\}.$$

We can similarly define $\dot{\mathcal{H}}'(\partial\Sigma)$ as above.

It is easy to see that for $[\alpha] \in \dot{\mathcal{H}}'(\partial_k\Sigma)$ and for any constant function $c \in H^{1/2}(\partial_k\Sigma)$ we have

$$L_{[\alpha]}c = 0.$$

Thus, $[\alpha]$ generates a well-defined functional on $\dot{H}^{1/2}(\partial_k\Sigma)$.

Therefore we have

Theorem 3.15. *Let Σ be a bordered Riemann surface of type (g, n). For any fixed $k \in \{1, \dots, n\}$, the bijection*

$$\dot{\mathcal{H}}'(\partial_k\Sigma) \to \dot{H}^{-1/2}(\partial_k\Sigma)$$
$$[\alpha] \mapsto L_{[\alpha]}$$

is a bounded isomorphism.

3.5. Formulation and solution of the CNT Dirichlet problem for L^2 one-forms. We can now state the general Dirichlet problem for L^2 one-forms.

Definition 3.16 (CNT Dirichlet data for one-forms)**.** By CNT Dirichlet data for one-forms, we mean $([\beta], \rho, \sigma)$ where

(1) $[\beta] = ([\beta_1], \dots, [\beta_n]) \in \mathcal{H}'(\partial\Sigma)$ such that
$$\int_{\partial_1\Sigma}[\beta_1] + \cdots + \int_{\partial_n\Sigma}[\beta_n] = 0;$$
(2) $\rho = (\rho_1, \dots, \rho_n) \in \mathbb{C}^n$ satisfying
$$\rho_1 + \cdots + \rho_n = 0;$$
and
(3) $\sigma = (\sigma_1, \dots, \sigma_{2g}) \in \mathbb{C}^{2g}$.

The Dirichlet problem for this data is as follows. As in Section 3.3, $\gamma_1, \dots, \gamma_{2g}$ are a collection of simple closed curves forming a basis of the homology of the genus g surface obtained from Σ by sewing on caps.

Definition 3.17 (CNT Dirichlet problem for one-forms)**.** We say that a harmonic one-form α on Σ solves the CNT Dirichlet problem with data $([\beta], \rho, \sigma)$, if $([\beta], \rho, \sigma)$ is CNT Dirichlet data and

(1) $[\alpha] = ([\beta_1], \dots, [\beta_n])$;
(2) for all $k = 1, \dots, n$
$$\int_{\partial_k\Sigma} * \alpha = \rho_k;$$
and
(3) for all $k = 1, \dots, 2g$
$$\int_{\gamma_k} \alpha := \sigma_k.$$

Our next goal is to show that the CNT Dirichlet problem has a solution which depends continuously on the data. Before proceeding, we shall recall a couple of facts from [18] that will play an important role in what follows. In [18, Definition 3.23] one defined the so-called *bounce operator* as follows. Let Σ be a bordered surface of type (g, n) and let $U_k \subseteq \Sigma$ be collar neighbourhoods of $\partial_k\Sigma$

for $k = 1, \dots, n$. Set $U = U_1 \cup \cdots \cup U_n$ and let $h : U \to \mathbb{C}$ be the function whose restriction to U_k is h_k for each $k = 1, \dots, n$. The bounce operator is defined by

$$\begin{aligned} \mathbf{G}_{U,\Sigma} : \mathcal{D}_{\mathrm{harm}}(U) &\to \mathcal{D}_{\mathrm{harm}}(\Sigma) \\ h &\mapsto H \end{aligned}$$

where H is the unique element whose CNT boundary values agree with h. Also recall that the composition operator $\mathbf{C}_f$ is defined by $\mathbf{C}_f g = g \circ f$. By conformal invariance of CNT limits, the bounce operator is conformally invariant, that is, if $f : \Sigma \to \Sigma'$ is a biholomorphism and $f(U) = U'$, then

$$\mathbf{G}_{U,\Sigma} \mathbf{C}_f = \mathbf{C}_f \mathbf{G}_{U',\Sigma'}. \tag{3.16}$$

This operator is bounded [18, Theorem 3.24]. The bounce operator will allow us to use a cut-and-paste technique ahead.

Now we are ready to state the well-posedness result for the CNT Dirichlet problem for forms.

Theorem 3.18 (Well-posedness of Dirichlet's problem for CNT data)**.** *For* CNT *Dirichlet data* $([\beta], \rho, \sigma)$ *there exists a unique* $\alpha \in \mathcal{A}_{\mathrm{harm}}(\Sigma)$ *which solves the Dirichlet problem. Moreover, the operator*

$$\mathbf{Dir}_{\partial\Sigma,\Sigma} : \mathcal{H}'(\partial\Sigma) \oplus \mathbb{C}^{2g+n-1} \to \mathcal{A}_{\mathrm{harm}}(\Sigma)$$

taking $([\beta], \rho, \sigma)$ *to the solution is bounded.*

Here of course the entries of $\mathbb{C}^{2g+n-1}$ are

$$(\rho_1, \dots, \rho_{n-1}, \sigma_1, \dots, \sigma_{2g}).$$

Before the proof of this result, we will need some preparations and a lemma. To that end, fix $k \in \{1, \dots, n\}$ and let

$$\mathcal{H}'_{\mathrm{e}}(\partial_k\Sigma) = \left\{ [\alpha] \in \mathcal{H}'(\partial_k\Sigma) : \int_{\partial_k\Sigma} [\alpha] = 0 \right\}.$$

Let $\phi : U \to \mathbb{A}_{r,1}$ be a collar chart defined near $\partial_k\Sigma$. Define a linear map

$$\mathbf{B}(\phi) : \mathcal{H}'_{\mathrm{e}}(\partial_k\Sigma) \to H^1_{\mathrm{conf}}(\mathbb{D})$$

as follows. Given $[\alpha] \in \mathcal{H}'_{\mathrm{e}}(\partial_k\Sigma)$, choose a representative α of $[\alpha]$ and let $h \in H^1_{\mathrm{conf}}(U)$ be such that $dh = \alpha$ (shrinking U if necessary using Proposition 2.6). Now let $H = \mathbf{G}_{\mathbb{A}_{r,1},\mathbb{D}} h \circ \phi$ and observe that $H \in H^1_{\mathrm{conf}}(\mathbb{D})$ is the unique harmonic map on $\mathbb{D}$ whose CNT boundary values agree with those of $h \circ \phi$. Since h is itself only determined up to a constant, we then impose the integral condition

$$\int_{\mathbb{S}^1} H(e^{i\theta})\, d\theta = 0 \tag{3.17}$$

and set

$$\mathbf{B}(\phi)[\alpha] = H.$$

Lemma 3.19. *For a collar chart* $\phi : U \to \mathbb{A}_{r,1}$ *near* $\partial_k\Sigma$, $\mathbf{B}(\phi)$ *is bounded.*

Proof. Treating $\partial_k\Sigma$ as an analytic curve in the double, observe that ϕ has a biholomorphic extension taking a doubly-connected neighbourhood of $\partial_k\Sigma$ to $\mathbb{A}_{r,1/r}$, and $h \mapsto h\circ\phi$ is a bijection which is bounded from $\dot{H}^{1/2}(\partial_k\Sigma)$ to $\dot{H}^{1/2}(\mathbb{S}^1)$ by Lemma 3.9. Furthermore, since the extension of $h\circ\phi$ from $\dot{H}^{1/2}(\mathbb{S}^1)$ to $\mathcal{D}_{\mathrm{harm}}(\mathbb{D})$ with any choice of constant is bounded with respect to the Dirichlet norm, and since condition (3.17) yields that

$$\|H\|_{H^1_{\mathrm{conf}}(\mathbb{D})} \approx \|H\|_{\mathcal{D}_{\mathrm{harm}}(\mathbb{D})},$$

one obtains the desired boundedness result. □

Proof of Theorem 3.18. First, we show that the exact solution to the Dirichlet problem depends continuously on the data. That is, let

$$\mathcal{H}'_{\mathrm{e}}(\partial\Sigma) = \oplus_{k=1}^{n} \mathcal{H}'_{\mathrm{e}}(\partial_k\Sigma).$$

The solution to the boundary value problem for exact forms with data in $\mathcal{H}'_{\mathrm{e}}(\partial\Sigma)$ is as follows: given $([\alpha], \rho_1, \dots, \rho_{n-1}) \in \mathcal{H}'_{\mathrm{e}}(\partial\Sigma) \oplus \mathbb{C}^{n-1}$ we want a one-form $\beta \in \mathcal{A}_{\mathrm{harm}}(\Sigma)$ such that $[\beta] = [\alpha]$ and

$$\int_{\partial_k\Sigma} *\,\beta = \rho_k, \quad k = 1, \dots, n. \tag{3.18}$$

We define a map

$$\mathbf{E} : \mathcal{H}'_{\mathrm{e}}(\partial\Sigma) \oplus \mathbb{C}^{n-1} \to \mathcal{A}_{\mathrm{harm}}(\Sigma)$$

taking data to the solution as follows. We use a lemma [18, Lemma 3.17] which states that for any fixed $k = 1, \dots, n$ there is a collar chart $\psi_k : U_k \to \mathbb{A}_{r_k,1}$ such that for any $h \in \mathcal{D}_{\mathrm{harm}}(U_k)$ we have

$$\int_{\partial_k\Sigma} h * d\omega_k = \int_{\mathbb{S}^1} h\circ\psi_k(e^{i\theta})d\theta.$$

For such collar charts we have that for any $[\alpha_k] \in \mathcal{H}'_{e}(\partial_k\Sigma)$

$$\int_{\partial_k\Sigma} \mathbf{C}_{\psi_k^{-1}}\mathbf{B}(\psi_k)[\alpha_k] * d\omega_k = \int_{\mathbb{S}^1} \mathbf{B}(\psi_k)[\alpha_k]\, d\theta = 0. \tag{3.19}$$

If we set $\psi = (\psi_1, \dots, \psi_n)$ and define

$$\mathbf{B}(\psi) = \oplus_{k=1}^{n} \mathbf{B}(\psi_k) : \mathcal{H}'_{\mathrm{e}}(\partial\Sigma) \to \oplus_{k=1}^{n} H^1_{\mathrm{conf}}(\mathbb{D}),$$

by Lemma 3.19 this is bounded.

We define restriction maps from the direct product $\mathbb{D}^n = \mathbb{D} \times \cdots \times \mathbb{D}$ to $\mathbb{A}^n = \mathbb{A}_{r_1,1} \times \cdots \times \mathbb{A}_{r_n,1}$. Namely, let $\mathbf{R}^{\mathrm{h}}_{\mathbb{D}^n,\mathbb{A}^n} = \oplus_{k=1}^{n} \mathbf{R}^{\mathrm{h}}_{\mathbb{D},\mathbb{A}_{r_k,1}}$, where as in [18], $\mathbf{R}^{\mathrm{h}}_{\mathbb{D},\mathbb{A}_{r_k,1}}$ denotes the restriction map $\mathcal{A}_{\mathrm{harm}}(\mathbb{D}) \to \mathcal{A}_{\mathrm{harm}}(\mathbb{A}_{r_k,1})$. Set

$$H = \mathbf{G}_{U,\Sigma}\mathbf{C}_{\phi^{-1}}\mathbf{R}^{\mathrm{h}}_{\mathbb{D}^n,\mathbb{A}^n}\mathbf{B}(\phi)[\alpha]$$

and

$$c_k = \int_{\partial_k \Sigma} * \, dH.$$

Finally define

$$\mathbf{E}([\alpha], \rho_1, \dots, \rho_{n-1}) = dH + \sum_{m=1}^{n-1} b_m d\omega_m \tag{3.20}$$

where the b_k are defined by

$$\rho_k - c_k = \sum_{m=1}^{n-1} \Pi_{km} b_m,$$

with the help of Theorem 2.22.

We show that $\mathbf{E}([\alpha], \rho_1, \dots, \rho_{n-1})$ solves the boundary value problem. By construction $\beta = \mathbf{E}([\alpha], \rho_1, \dots, \rho_{n-1})$ satisfies

$$[\beta] = [\alpha]$$

since $[d\omega_k] = 0$ for all $k = 1, \dots, n$ by Proposition 3.14. To see that (3.18) is satisfied, we set $\beta = \mathbf{E}([\alpha], \rho_1, \dots, \rho_{n-1})$ and compute

$$\begin{aligned} \int_{\partial_k \Sigma} * \, \beta &= c_k + \sum_{m=1}^{n-1} b_m \int_{\partial_k \Sigma} * \, d\omega_m = c_k + \sum_{m=1}^{n-1} b_m \Pi_{km} \\ &= \rho_k. \end{aligned}$$

Finally we show that $\mathbf{E}$ is bounded. The boundedness of the first term follows from boundedness of the bounce operator $\mathbf{G}_{U,\Sigma}$, Lemma 3.19, and the fact that $\mathbf{C}_{\psi_k^{-1}}$ is bounded from $H^1_{\mathrm{conf}}(\mathbb{A}_{r_k,1})$ to $H^1_{\mathrm{conf}}(U_k)$ for $k = 1, \dots, n$, which is precisely the content of [18, Lemma 3.27]. To bound the second term, observe that

$$\begin{aligned} c_k = \int_{\partial_k \Sigma} * \, dH &= \int_{\partial \Sigma} \omega_k * dH \\ &= \iint_{\Sigma} d\omega_k \wedge * \, dH, \end{aligned}$$

so

$$\|(c_1, \cdots, c_{n-1})\|_{\mathbb{C}^{n-1}} \leq \|H\|_{H^1_{\mathrm{conf}}(\Sigma)} \sup_{k=1,\dots,n} \|d\omega_k\|_{\mathcal{A}_{\mathrm{harm}}(\Sigma)}.$$

This together with the facts that H is bounded by the data, and that Π is a finite matrix and therefore bounded, proves the claim.

The remainder of the proof takes into account the cohomological data. We are given an arbitrary $([\beta], \rho, \sigma) \in \mathcal{H}'(\partial\Sigma) \oplus \mathbb{C}^{2g+n-1}$. Setting

$$\lambda_k = \int_{\partial_k \Sigma} [\beta_k] \tag{3.21}$$

for $k = 1, \dots, n$, by Corollary 2.23 there is a $\delta \in * \mathcal{A}_{\mathrm{hm}}(\Sigma)$ such that

$$\int_{\partial_k \Sigma} \delta = \lambda_k \tag{3.22}$$

for every k. Furthermore, there is a unique harmonic one-form η in the span of $\{\varepsilon_1, \dots, \varepsilon_{2g}\}$ such that

$$\int_{\gamma_j} \eta = \sigma_j - \int_{\gamma_j} \delta \tag{3.23}$$

for $j = 1, \dots, 2g$. We also have by definition of ε_k that

$$\int_{\partial_k \Sigma} \eta = 0, \tag{3.24}$$

for $k = 1, \dots, n$. Thus $[\beta - \delta - \eta] \in \mathcal{H}'_{\mathrm{e}}(\partial\Sigma)$.

We will require several estimates. Although the notation is involved, the reader could keep in mind that the estimates are elementary due to the fact that only finite-dimensional spaces are involved. Since δ is in the span of the finite-dimensional space $* \mathcal{A}_{\mathrm{hm}}$, and uniquely determined by $\lambda = (\lambda_1, \dots, \lambda_{n-1})$, we have that

$$\|[\delta]\|_{\mathcal{H}'(\partial\Sigma)} \leq C\|\lambda\|_{\mathbb{C}^{n-1}} \leq C\|[\beta]\|_{\mathcal{H}'(\partial\Sigma)\oplus\mathbb{C}^{2g+n-1}}. \tag{3.25}$$

Similarly

$$\|\delta\|_{\mathcal{A}_{\mathrm{harm}}(\Sigma)} \leq C\|\lambda\|_{\mathbb{C}^{n-1}} \leq C\|[\beta]\|_{\mathcal{H}'(\partial\Sigma)\oplus\mathbb{C}^{2g+n-1}}. \tag{3.26}$$

If desired, an explicit estimate could be obtained from the supremum over $k = 1, \dots, n-1$ of the norms of $* \, d\omega_k$, but this won't be needed.

Similarly, since the span of $\{\varepsilon_1, \dots, \varepsilon_{2g}\}$ is finite-dimensional, the dependence of η on the data is continuous. Observe that

$$e_j = \int_{\gamma_j} \delta, \quad j = 1, \dots, 2g$$

depend linearly on δ and hence continuously on $\|[\beta]\|_{\mathcal{H}'(\partial\Sigma)\oplus\mathbb{C}^{2g+n-1}}$. Denote $e = (e_1, \dots, e_{2g})$. Now by the definition (3.23) of η, referring to (3.23) and using the fact that η must lie in a fixed finite-dimensional space — namely the span of $\{\varepsilon_1, \dots, \varepsilon_{2g}\}$ — we obtain

$$\|[\eta]\|_{\mathcal{H}'(\partial\Sigma)} \leq C\|\sigma - e\|_{\mathbb{C}^{2g}} \leq C\|([\beta], \rho, \sigma)\|_{\mathcal{H}'(\partial\Sigma)\oplus\mathbb{C}^{2g+n-1}}. \tag{3.27}$$

Similarly

$$\|\eta\|_{\mathcal{A}_{\mathrm{harm}}(\Sigma)} \leq C\|([\beta], \rho, \sigma)\|_{\mathcal{H}'(\partial\Sigma)\oplus\mathbb{C}^{2g+n-1}}. \tag{3.28}$$

We need one further bound. Set $d = (d_1, \dots, d_{n-1})$ where

$$d_k = \int_{\partial_k \Sigma} * (\eta + \delta), \;\; k = 1, \dots, n$$

(note that d_n is $1 - d_1 - \cdots - d_{n-1}$). The d_k's depend boundedly on δ and η, so

$$\|d\|_{\mathbb{C}_{n-1}} \leq \|([\beta], \rho, \sigma)\|_{\mathcal{H}'(\partial\Sigma)\oplus\mathbb{C}^{2g+n-1}}. \tag{3.29}$$

Given the definitions of δ, η, and d, it is easily verified that the solution to the Dirichlet problem is

$$\mathbf{Dir}_{\partial\Sigma,\Sigma}([\beta],\rho,\sigma) = \mathbf{E}([\beta-\eta-\delta],\rho-d)+\eta+\delta. \tag{3.30}$$

The continuous dependence is now easily obtained: by boundedness of $\mathbf{E}$, (3.25), (3.27), and (3.29) we have

$$\begin{aligned}\|\mathbf{E}([\beta-\eta-\delta],\rho-d)\|_{\mathcal{A}_{\mathrm{harm}}(\Sigma)} &\leq \|[\beta-\eta-\delta]\|_{\mathcal{H}'(\partial\Sigma)} + \|\rho-d\|_{\mathbb{C}^{n-1}} \\ &\leq \|\beta\|_{\mathcal{H}'(\partial\Sigma)} + \|\delta\|_{\mathcal{H}'(\partial\Sigma)} + \|\eta\|_{\mathcal{H}'(\partial\Sigma)} + \|\rho-d\|_{\mathbb{C}^{n-1}} \\ &\leq C\|([\beta],\rho,\sigma)\|_{\mathcal{H}'(\partial\Sigma)\oplus\mathbb{C}^{2g+n-1}}.\end{aligned}$$

Therefore (3.30), the above bound, (3.26), and (3.28) yield that

$$\|\mathbf{Dir}_{\partial\Sigma,\Sigma}([\beta],\rho,\sigma)\|_{\mathcal{A}_{\mathrm{harm}}(\Sigma)} \leq C\|([\beta],\rho,\sigma)\|_{\mathcal{H}'(\partial\Sigma)\oplus\mathbb{C}^{2g+n-1}}.$$

It remains to show that the solution is unique. Let α' be another solution to the Dirichlet problem. Conditions (1) and (3) of Definition 3.23 imply that $\alpha'-\alpha$ is exact and has a global primitive h, which has constant CNT boundary values on $\partial\Sigma$. So h is in the linear span of the harmonic measures. Condition (2) then implies that $\alpha' = \alpha$. Summing up, we have shown that the Dirichlet problem with the aforementioned CNT data is well-posed in the spaces that are given in the statement of the theorem. □

Remark 3.20. Because of condition (1) on CNT Dirichlet boundary data, one of the constants λ_n in the $\mathcal{H}'(\partial\Sigma)$ is redundant and depends continuously on the other constants. So one constant can be removed from the norm of $\mathcal{H}'(\partial\Sigma)$ in the estimate.

Remark 3.21 (Special cases $n=1$ or $g=0$). If there is only one boundary curve $\partial_1\Sigma$, then condition (2) requires that

$$\int_{\partial_1\Sigma} *\,\alpha = 0.$$

This is true for any $*\,\alpha \in \mathcal{A}_{\mathrm{harm}}(\Sigma)$, so condition (2) may be omitted. Similarly, in condition (1) it is required that

$$\int_{\partial_1\Sigma} [\beta_1] = 0,$$

which is true for any $[\beta_1] \in \mathcal{H}'(\partial\Sigma)$, and thus this part of condition (1) can be omitted.

If the genus g of Σ is zero, then the third condition is omitted.

In either case, some steps in the proof of Theorem 3.18 can be omitted.

The following proposition verifies that the CNT Dirichlet problem is natural.

Proposition 3.22. *If the Dirichlet data* $([\beta],\rho,\sigma)$ *is such that* $[\beta]$ *has a representative on a collar neighbourhood which is* $\mathcal{C}^\infty$*, then* $\mathbf{Dir}_{\partial\Sigma,\Sigma}([\beta],\rho,\sigma)$ *is the solution to the* $\mathcal{C}^\infty$ *Dirichlet problem.*

Proof. Choose a representative $(\beta_1, \dots, \beta_n)$ of $[\beta]$ on a collection of collar neighbourhoods U_k of $\partial_k\Sigma$ for $k = 1, \dots, n$, which are smooth on $\partial_k\Sigma$. By Theorem 3.3 there is a $\mathcal{C}^\infty$ solution α to the Dirichlet problem with data given by (β, ρ, σ) with β given by the restriction of β_k to the boundaries $\partial_k\Sigma$ for $k = 1, \dots, n$.

We claim that α is the solution to the CNT Dirichlet problem. Once this is shown, the proof is complete thanks to uniqueness statement of Theorem 3.18. First, observe that since α is $\mathcal{C}^\infty$ on $\operatorname{cl}\Sigma$, it is in $\mathcal{A}_{\mathrm{harm}}(\Sigma)$. So we need only show that the CNT boundary values of the $\mathcal{C}^\infty$ solution are equal to $[\beta]$.

To see this, choose one-forms δ_k on a collar neighbourhood U_k of $\partial_k\Sigma$, which extend smoothly to $\partial_k\Sigma$ and such that

$$\int_{\partial_k\Sigma} (\alpha - \delta_k) = 0$$

for $k = 1, \dots, n$. This can be arranged for example by considering Σ to be a subset of its double. The primitive h_k of $\alpha - \delta_k$ on U_k is $\mathcal{C}^\infty$, and in particular extends continuously to $\partial_k\Sigma$ for $k = 1, \dots, n$. But the CNT boundary values must equal the continuous extension by definition. By definition of the $\mathcal{C}^\infty$ solution to the Dirichlet problem, $dh_k = \beta_k - \delta_k$ on the boundary, so $[\alpha] = [\beta]$. This completes the proof. □

3.6. Dirichlet problem for one-forms with $H^{-1/2}$ data.

The solution to the Dirichlet problem can be phrased in terms of $H^{-1/2}$ boundary data as follows.

Definition 3.23 ($H^{-1/2}$ data for one-forms)**.** By $H^{-1/2}$ data for one-forms we mean the following:

(1) $L = (L_1, \dots, L_n) \in \bigoplus_{k=1}^n H^{-1/2}(\partial_k\Sigma)$ such that
$$L_1(1) + \cdots L_n(1) = 0;$$
(2) $\rho = (\rho_1, \dots, \rho_n) \in \mathbb{C}^n$ satisfying
$$\rho_1 + \cdots + \rho_n = 0;$$
and
(3) $\sigma = (\sigma_1, \dots, \sigma_{2g}) \in \mathbb{C}^{2g}$.

In the following, recall the definition (3.10) for the element $L_{[\alpha]}$ of $H^{-1/2}(\partial_k\Sigma)$ associated to a one-form α.

Definition 3.24 ($H^{-1/2}$ Dirichlet problem for one-forms)**.** We say that a harmonic one-form α on Σ solves the $H^{-1/2}$ Dirichlet problem with $H^{-1/2}$ Dirichlet data (L, ρ, σ) if

(1) for $k = 1, \dots, n$, for any $h_k \in H^{1/2}(\partial_k\Sigma)$ we have
$$L_k(h_k) = L_{[\alpha]} h_k;$$
(2) for all $k = 1, \dots, n$
$$\int_{\partial_k\Sigma} * \alpha = \rho_k;$$
and

(3) for all $k = 1, \dots, 2g$

$$\int_{\gamma_k} \alpha := \sigma_k.$$

The CNT Dirichlet problem has a solution which depends continuously on the data.

Theorem 3.25 (Well-posedness of Dirichlet's problem for $H^{-1/2}$ data)**.** *For $H^{-1/2}$ Dirichlet data (L, ρ, σ) there exists a unique $\alpha \in \mathcal{A}_{\mathrm{harm}}(\Sigma)$ which solves the Dirichlet problem. The operator*

$$\widetilde{\mathbf{Dir}}_{\partial\Sigma,\Sigma} : \bigoplus_{k=1}^{n} H^{-1/2}(\partial_k \Sigma) \oplus \mathbb{C}^{2g+n-1} \to \mathcal{A}_{\mathrm{harm}}(\Sigma)$$

taking (L, ρ, σ) to the solution is bounded. Here the entries of $\mathbb{C}^{2g+n-1}$ are

$$(\rho_1, \dots, \rho_{n-1}, \sigma_1, \dots, \sigma_{2g}).$$

Proof. This follows immediately from Theorems 3.11 and 3.18. □

4. Overfare of harmonic one-forms

4.1. Assumptions throughout this section. The following assumptions will be in force throughout Section 4. Additional hypotheses are added to the statement of each theorem where necessary.

(1) $\mathscr{R}$ is a compact Riemann surface;
(2) $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_n$ is a collection of quasicircles;
(3) Γ separates $\mathscr{R}$ into Σ_1 and Σ_2 in the sense of Definition 2.36.

We will furthermore assume that the ordering of the boundaries of $\partial\Sigma_1$ and $\partial\Sigma_2$ is such that $\partial_k\Sigma_1 = \partial_k\Sigma_2 = \Gamma_k$ as sets for $k = 1, \dots, n$.

4.2. About this Section. In this Section, we address the problem of overfare of one-forms. Given an L^2 harmonic one-form on Σ_1, we show that there is an L^2 harmonic one-form on Σ_2 with the same boundary values. To do this, we first show that the local boundary values in $H^{-1/2}(\partial_k\Sigma_1)$ (equivalently, in $\mathcal{H}'(\partial_k\Sigma_1)$) uniquely determine boundary values in $H^{-1/2}(\partial_k\Sigma_2)$ (equivalently, in $\mathcal{H}'(\partial_k\Sigma_2)$).

Of course, to uniquely determine the one-form on Σ_2 one also needs to specify cohomological data. One way to do this is simply to specify the CNT Dirichlet data for forms on Σ_2 as in Section 3.5.

4.3. Overfare of functions. We briefly review the definitions and results for overfare of harmonic functions necessary here. The details and proofs, which are somewhat involved, can be found in [18]. Given $h_1 \in \mathcal{D}(\Sigma_1)$, there is an $h_2 \in \mathcal{D}(\Sigma_2)$ whose CNT boundary values on Γ agree with h_1 up to a null set. It should be observed that Definition 2.9 of null set depends a priori on whether the collar chart is taken in Σ_1 or Σ_2. In fact for quasicircles a set which is null when viewed from Σ_1 is also null when viewed from Σ_2 and vice versa. The fact that such an h_2 exists is precisely the content of [18, Theorem 3.40]. We call h_2 the overfare of h_1 and denote it by

$$h_2 = \mathbf{O}_{1,2} h_1.$$

If Σ_1 is connected, [18, Theorem 3.43] states that $\mathbf{O}_{1,2}$ is bounded with respect to the Dirichlet semi-norm.

4.4. Partial overfare of one-forms. In this section we define overfare of one-forms and functions, and show that it exists and is bounded.

This subsection is devoted to a kind of “partial” overfare, where only the boundary data is mapped into the new surface. We first define this for $H^{1/2}$. Recall that the Sobolev spaces are defined by treating the boundary curves of Σ_k as analytic curves in the double. Thus, we distinguish $H^{1/2}(\partial_k \Sigma_1)$ and $H^{1/2}(\partial_k \Sigma_2)$.

We define the partial overfare as follows. Let $h_1 \in H^{1/2}(\partial_k \Sigma_1)$. Let $\phi : U \to \mathbb{C}$ be a doubly-connected chart defined in a neighbourhood of $\partial_k \Sigma$, whose inner curves are analytic. For any extension $H_1 \in \mathcal{D}_{\mathrm{harm}}(U_1)$ whose CNT boundary values equal h, let $H_2 \in \mathcal{D}_{\mathrm{harm}}(U_2)$ be its overfare, and let h_2 be its CNT boundary values. We set

$$\begin{aligned} \mathbf{O}(\partial_k \Sigma_1, \partial_k \Sigma_2) : H^{1/2}(\partial_k \Sigma_1) &\to H^{1/2}(\partial_k \Sigma_2) \\ h_1 &\mapsto h_2. \end{aligned}$$

We define $\mathbf{O}(\partial_k \Sigma_2, \partial_k \Sigma_1)$ similarly.

Proposition 4.1. *Given $h \in H^{1/2}(\partial_k \Sigma_1)$, let H be any element of $\mathcal{D}_{\mathrm{harm}}(\Sigma_1)$ whose* CNT *boundary values equal h on $\partial_k \Sigma_1$. Then the boundary values of $\mathbf{O}_{1,2} H$ equal $\mathbf{O}(\partial_k \Sigma_1, \partial_k \Sigma_2) h$.*

Proof. This follows immediately from the observation that the CNT boundary values of $\mathbf{O}(\partial_k \Sigma_1, \partial_k \Sigma_2) h$ agree with those of h, and therefore with those of H. By definition of overfare, the CNT boundary values of $\mathbf{O}_{1,2} H$ agree with those of H. □

In particular, $\mathbf{O}(\partial_k \Sigma_1, \partial_k \Sigma_2)$ is independent of the choice of extension H_1 and doubly-connected chart.

Let us also recall the definition of the so-called *bounded zero mode quasicircles*, which are more regular than general quasicircles, from [18].

Definition 4.2. Let Γ be a quasicircle in $\bar{\mathbb{C}}$, and let Ω_1 and Ω_2 denote the connected components of the complement. We say that Γ is a *bounded zero mode quasicircle* (BZM for short), if the overfares $\mathbf{O}_{\Omega_1,\Omega_2}$ and $\mathbf{O}_{\Omega_2,\Omega_1}$ obtained from $\Sigma_k = \Omega_k$, $k = 1, 2$, are bounded with respect to $H^1_{\mathrm{conf}}(\Omega_k)$.

A quasicircle Γ in a Riemann surface $\mathscr{R}$ is called an BZM quasicircle if there is an open set U containing Γ and a conformal map $\phi : U \to \mathbb{A}$ onto an annulus $\mathbb{A} \subseteq \mathbb{C}$ such that $\phi(\Gamma)$ is a BZM quasicircle.

The next result states that the partial overfare is bounded.

Proposition 4.3. *The following statements hold:*

(1) $\mathbf{O}(\partial_k\Sigma_1, \partial_k\Sigma_2)$ *is bounded as a map from* $\dot{H}^{1/2}(\partial_k\Sigma_1)$ *to* $\dot{H}^{1/2}(\partial_k\Sigma_2)$.

(2) *If* $\partial_k\Sigma_1$ *is a* BZM *quasicircle, then* $\mathbf{O}(\partial_k\Sigma_1, \partial_k\Sigma_2)$ *is bounded as a map from* $H^{1/2}(\partial_k\Sigma_1)$ *to* $H^{1/2}(\partial_k\Sigma_2)$.

Proof. By Proposition 4.1 we may choose any doubly-connected chart to define the partial overfare. Choose such a chart ϕ on a doubly-connected domain U and let U_1, U_2 be collar neighbourhoods of $\partial_k\Sigma_1$ and $\partial_k\Sigma_2$. We thus obtain a pair of domains in the plane Ω_k bounded by $\gamma := \phi(\partial_k\Sigma_1) = \phi(\partial_k\Sigma_2)$, and $\mathbf{O}_{1,2}(\phi) := \phi^{-1}\circ\mathbf{O}_{\Omega_1,\Omega_2}\circ\phi$ defines a map

$$\mathbf{O}_{1,2}(\phi) : \mathcal{D}_{\text{harm}}(U_1) \to \mathcal{D}_{\text{harm}}(U_2)$$

such that $\mathbf{O}_{1,2}(\phi)h_1$ has the same CNT boundary values as h_1 for any h_1. Now Sobolev trace and extension are bounded from $H^1(\Omega_k)$ to $H^{1/2}(\gamma)$ and $\dot{H}^1(\Omega)$ to $\dot{H}^{1/2}(\gamma)$. (Note that the definition of $H^{1/2}(\gamma)$ depends on the choice of side Ω_1 or Ω_2, treating γ as an analytic curve in the double of Ω_1/Ω_2 respectively). So it suffices to show that $\mathbf{O}_{1,2}(\phi)$ is bounded in both case (1) and (2). But this is precisely the content of [18, Lemma 3.44]. □

Remark 4.4 (Unique extension from $H^{1/2}$ to $\mathcal{H}$). Let Γ be a border of a Riemann surface Σ. We treat Γ as an analytic curve in the double. We assume for simplicity that there are no other boundary points, although the discussion holds in the general case.

Elements of $H^{1/2}(\Gamma)$ which agree with each other almost everywhere are the same in that Sobolev space. On the other hand, functions in $\mathcal{H}(\Gamma)$ are the same only if they agree up to a (potential-theoretic) null set. Sets of measure zero need not be null; for example, in the circle, not every set of measure zero has logarithmic capacity zero. Thus, an element of $H^{1/2}(\Gamma)$ does not a-priori lead to a well-defined element of $\mathcal{H}(\Gamma)$.

However, given $h \in H^{1/2}(\Gamma)$, a well-defined element of $\mathcal{H}(\Gamma)$ can be obtained as follows. Let $H \in H^1(\Sigma)$ be the unique harmonic Sobolev extension of h. In particular, $H \in \mathcal{D}_{\text{harm}}(\Sigma)$ and thus has CNT boundary values $\tilde{h}$ defined except possibly on a null set. Therefore h determines a unique element of $\mathcal{H}(\Gamma)$.

Remark 4.5 (Subtlety in defining overfare on $H^{1/2}$). There is an important technical subtlety in the definition of the partial overfare. For simplicity, we assume that Σ_1 and Σ_2 have only one border $\partial\Sigma_1 = \partial\Sigma_2$ which is shared between them. As in the previous remark, the discussion here applies to the general case.

Given $h_1 \in H^{1/2}(\partial_1\Sigma)$, one might seek an element $h_2 \in H^{1/2}(\partial_2\Sigma)$ which agrees with h_1 almost everywhere. This is not even well-defined, because sets of measure zero in $\partial_1\Sigma$ are not necessarily of measure zero in $\partial_2\Sigma$.

Consider the case that Γ is a quasicircle in the plane bounding Ω_1 and Ω_2, sets of Lebesgue measure zero in Γ treated as an analytic curve in the double of Ω_1 are precisely sets of harmonic measure zero in Ω_1, and similarly treating Γ as an analytic curve in the double of Ω_2. However, sets of harmonic measure zero in Γ with respect to Ω_1 need not be harmonic measure zero with respect to Ω_2, see Beurling and Ahlfors [2]. Thus the partial overfare cannot be formulated this way, necessitating the definition above and Propositions 4.1 and 4.3.

On the other hand, using Remark 4.4 the definition of partial overfare can be stated succinctly as follows. Given $h_1 \in H^{1/2}(\partial_k \Sigma_1)$, let $\tilde{h} \in \mathcal{H}(\partial_k \Sigma_1) = \mathcal{H}(\partial_k \Sigma_2)$ be the unique element corresponding to h. Then $\tilde{h}$ agrees with a unique element $h_2 \in H^{1/2}(\partial_k \Sigma_2)$, and we can set

$$h_2 = \mathbf{O}(\partial_k \Sigma_1, \partial_k \Sigma_2) h_1.$$

Next, we will define a partial overfare of elements of $H^{-1/2}$. Again, recall that $H^{-1/2}(\partial_k \Sigma_m)$ is defined by treating $\partial_k \Sigma_m$ as an analytic curve in the double of Σ_m, and therefore we must distinguish $H^{-1/2}(\partial_k \Sigma_1)$ from $H^{-1/2}(\partial_k \Sigma_2)$.

Let $L \in H^{-1/2}(\partial_k \Sigma_1)$. We define

$$\mathbf{O}'(\partial_k \Sigma_1, \partial_k \Sigma_2) : H^{-1/2}(\partial_k \Sigma_1) \to H^{-1/2}(\partial_k \Sigma_2)$$

by

$$[\mathbf{O}'(\partial_k \Sigma_1, \partial_k \Sigma_2) L](h) = -L(\mathbf{O}(\partial_k \Sigma_2, \partial_k \Sigma_1) h) \quad \text{for all } h \in H^{1/2}(\partial_k \Sigma_1).$$

$\mathbf{O}'(\partial_k \Sigma_2, \partial_k \Sigma_1)$ is defined similarly.

Remark 4.6. The negative sign is introduced in order to take into account the change of orientation of the boundary.

We also define

$$\dot{\mathbf{O}}'_{1,2} : \dot{H}^{-1/2}(\partial_k \Sigma_1) \to \dot{H}^{-1/2}(\partial_k \Sigma_2)$$

and

$$\dot{\mathbf{O}}'_{2,1} : \dot{H}^{-1/2}(\partial_k \Sigma_2) \to \dot{H}^{-1/2}(\partial_k \Sigma_1)$$

in the obvious way. It is easily verified that these are well-defined.

Proposition 4.7. *For any* $L \in H^{-1/2}(\partial_k \Sigma)$,

$$[\mathbf{O}'(\partial_k \Sigma_1, \partial_k \Sigma_2) L](1) = -L(1).$$

Proof. This follows from the easily-verified fact that $\mathbf{O}(\partial_k \Sigma_1, \partial_k \Sigma_2) 1 = 1$. □

Proposition 4.8. *The following statements are valid:*

(1) *The partial overfare* $\dot{\mathbf{O}}'(\partial_k \Sigma_1, \partial_k \Sigma_2)$ *is bounded as a map from* $\dot{H}^{-1/2}(\partial_k \Sigma_1)$ *to* $\dot{H}^{-1/2}(\partial_k \Sigma_2)$.

(2) *If* $\partial_k \Sigma$ *is a* BZM *quasicircle, then* $\mathbf{O}'(\partial_k \Sigma_1, \partial_k \Sigma_2)$ *is bounded as a map from* $H^{-1/2}(\partial_k \Sigma_1)$ *to* $H^{-1/2}(\partial_k \Sigma_2)$.

Proof. This follows immediately from Proposition 4.3. □

The association between $H^{-1/2}(\partial_k \Sigma_m)$ and $\mathcal{H}'(\partial_k \Sigma_m)$ given by Theorem 3.11 immediately defines a bounded overfare

$$\mathbf{O}'(\partial_k \Sigma_1, \partial_k \Sigma_2) : \mathcal{H}'(\partial_k \Sigma_1) \to \mathcal{H}'(\partial_k \Sigma_2)$$

and similarly for the homogeneous spaces

$$\dot{\mathbf{O}}'(\partial_k \Sigma_1, \partial_k \Sigma_2) : \dot{\mathcal{H}}'(\partial_k \Sigma_1) \to \dot{\mathcal{H}}'(\partial_k \Sigma_2).$$

We will use the same notation for the overfares on $H^{-1/2}(\partial_k \Sigma_m)$ and $\mathcal{H}'(\partial_k \Sigma_m)$.
The partial overfare preserves periods.

Proposition 4.9. *For any $k = 1, \dots, n$ and $[\alpha] \in \mathcal{H}'(\partial_k \Sigma_1)$ we have that*

$$\int_{\partial_k \Sigma_2} \mathbf{O}'(\partial_k \Sigma_1, \partial_k \Sigma_2)[\alpha] = - \int_{\partial_k \Sigma_1} [\alpha].$$

The same claim holds with the roles of 1 and 2 switched.

Proof. This follows from Proposition 4.7 after observing that

$$L_{[\alpha]}(1) = \int_{\partial_k \Sigma_1} [\alpha].$$

□

We also have the following.

Proposition 4.10. *Let U be a doubly-connected neighbourhood of $\partial_k \Sigma_1 = \partial_k \Sigma_2$.*
(1) *For any $\alpha \in \mathcal{A}^{\mathrm{e}}_{\mathrm{harm}}(U)$ we have*

$$\mathbf{O}'(\partial_k \Sigma_1, \partial_k \Sigma_2)[\alpha] = [\alpha]$$

where the equality above is in $\dot{H}^{-1/2}(\partial_k \Sigma_2)$.
(2) *If $\partial_k \Sigma_1$ is a* BZM *quasicircle, then for any $\alpha \in \mathcal{A}_{\mathrm{harm}}(U)$ we have*

$$\mathbf{O}'(\partial_k \Sigma_1, \partial_k \Sigma_2)[\alpha] = [\alpha].$$

Proof. Denote by $L^m_{[\alpha]}$ the elements of $H^{-1/2}(\partial_k \Sigma_m)$ induced by α for $m = 1, 2$. We need to show that $L^1_{[\alpha]} = L^2_{[\alpha]}$. By Proposition 4.3 it is enough to prove this on the dense set $\mathcal{D}_{\mathrm{harm}}(U)$ in both cases (1) and (2). Let Γ^m_r denote the limiting curves $\phi^{-1}(|z| = r)$ for a collar chart ϕ, and for each such r sufficiently close to one let U_r denote the region bounded by Γ^1_r and Γ^2_r. For H in this dense set, we have

$$\begin{aligned} L^2_{[\alpha]} H - L^1_{[\alpha]} H &= \lim_{r \to 1} \left(\int_{\Gamma^2_r} \alpha H - \int_{\Gamma^1_r} \alpha H \right) \\ &= - \lim_{r \to 1} \iint_{U_r} \alpha \wedge dH. \end{aligned}$$

Therefore by the Cauchy-Schwarz inequality, for all $r < 1$ sufficiently close to 1

$$\left| L^2_{[\alpha]} H - L^1_{[\alpha]} H \right| \leq \|dH\|_{\mathcal{A}_{\mathrm{harm}}(U_r)} \cdot \|\alpha\|_{\mathcal{A}_{\mathrm{harm}}(U_r)}.$$

Letting r go to one, the claim now follows from the facts that $U_r \subset U$ for r sufficiently close to 1, $dH \in \mathcal{A}_{\mathrm{harm}}(U)$, and $\cap_r U_r$ has measure zero because quasicircles have measure zero. □

In other words, one-forms which extend harmonically across a border are their own overfare.

4.5. Overfare of one-forms. We first recall some notation and establish conventions. Assume that Σ_k are connected and have genus g_k for $k = 1, 2$. Let

$$\{\gamma_1^k, \ldots, \gamma_{2g_k}^k, \partial_1\Sigma_k, \ldots, \partial_{n-1}\Sigma_k\}$$

be a set of generators for the fundamental group of Σ_k. The generators $\partial_j\Sigma_k$ are common to both Σ_1 and Σ_2, when viewed as subsets of $\mathscr{R}$. Note that these are not the same generators as those appearing in Section 3, since $\mathscr{R}$ need not be the double of either Σ_1 or Σ_2.

In this section we show that overfare of one-forms exists and is well-defined. That is, given $\alpha_2 \in \mathcal{A}_{\mathrm{harm}}(\Sigma_2)$, we obtain a form $\alpha_1 \in \mathcal{A}_{\mathrm{harm}}(\Sigma_1)$ with the same boundary values. Needless to say, one must specify more data about α_1 to make this well-posed, as we saw in Section 3.

Theorem 4.11. *Given $\alpha_2 \in \mathcal{A}_{\mathrm{harm}}(\Sigma_2)$, $\sigma_1, \ldots, \sigma_{2g} \in \mathbb{C}$ and $\rho_1, \ldots, \rho_{n-1} \in \mathbb{C}$, there is a unique $\alpha_1 \in \mathcal{A}_{\mathrm{harm}}(\Sigma_1)$ such that*

(1)

$$\mathbf{O}(\partial_k\Sigma_2, \partial_k\Sigma_1)[\alpha_2] = [\alpha_1], \quad k = 1, \ldots, n;$$

(2)

$$\int_{\gamma_m} \alpha_1 = \sigma_m, \quad m = 1, \ldots, 2g;$$

and

(3)

$$\int_{\partial_k\Sigma_1} * \alpha_1 = \rho_k, \quad k = 1, \ldots, n-1.$$

Proof. This follows immediately from Theorems 3.18 and Proposition 4.8. □

Remark 4.12. In fact by applying Theorems 3.18 and Proposition 4.8, one sees the above result holds when Σ_1 is not connected, provided that sufficient cohomological data is provided on each connected component.

Remark 4.13. One can formulate and prove continuous dependence of α_1 on $\alpha_2, \sigma_1, \ldots, \sigma_{2g}$, and $\rho_1, \ldots, \rho_{n-1}$. However, we will take a different approach to boundedness of the overfare of forms in [20].

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