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Correction to "On BT_1 group schemes and Fermat curves"

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Abstract. We correct an error in Proposition 5.6(3) of [PU21] and revise other statements in the paper accordingly.

1. Corrected $u_{1,1}$-numbers

The calculation of $u_{1,1}$-numbers in part (3) of Proposition 5.6 in Section 5.3 of [PU21] is incorrect. In this section, we give more details on part (2) of Proposition 5.6 and a corrected statement and proof of part (3).

Before stating the result, we make the following definitions. Assume that w is a primitive word of length $\lambda > 2$, and rotate w so that it begins with f and ends with v. Define $d(w)$ and $u(w)$ as follows: each subword of w of the form $f^2(vf)^e v^2$ (where $e \geq 0$) contributes 1 to $d(w)$ and $e+1$ to $u(w)$. Examples:

$$d(f^3v^2) = 1, \quad u(f^3v^2) = 1, \quad d(f^4vf^2v) = 0, \quad u(f^4vf^2v) = 0,$$
$$d(fvf^2vfv^3fv) = 1, \quad u(fvf^2vfv^3fv) = 2,$$
$$d(f^2v^2f^2vfv^2) = 2, \quad u(f^2v^2f^2vfv^2) = 3.$$

The invariant d defined here turns out to be the same as the u of Proposition 5.6.

Also, as in Subsection 3.2, let r be the integer such that (up to rotation) w can be written in the form

$$w = v^{n_r} f^{m_r} \cdots v^{n_1} f^{m_1}$$

where all m_i and n_i are ≥ 1.

The following replaces parts (2) and (3) of [PU21, Proposition 5.6].

Proposition. *Let w be a primitive word of length $\lambda > 2$.*

(1) *There is a bijection*

$$\mathrm{Hom}_{\mathbb{D}_k}(M(w), M_{1,1}) \cong k^{d(w)+r}.$$

(2) *The $u_{1,1}$-number of $M(w)$ is $u(w)$.*

Proof. For (1), we use Lemma 3.1 to present $M(w)$ with generators $E_0, \ldots, E_{r-1}$ (with indices taken modulo r) and relations $V^{n_i}E_i = F^{m_i}E_{i-1}$. Let z_0, z_1 be a k-basis of $M_{1,1}$ with $Fz_0 = Vz_0 = z_1$ and $Fz_1 = Vz_1 = 0$. Then a homomorphism $\psi : M(w) \to M_{1,1}$ is determined by its values on the generators E_i. Write

$$\psi(E_i) = a_{i,0}z_0 + a_{i,1}z_1.$$

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Then ψ is a $\mathbb{D}_k$-module homomorphism if and only if $V^{n_i}\psi(E_i) = F^{m_i}\psi(E_{i-1})$ for $i = 1, \dots, r$.

This leads to the system of equations:

$$\left.\begin{matrix} a_{i,0}^{1/p} & \text{if } n_i = 1 \\ 0 & \text{if } n_i > 1 \end{matrix}\right\} = \left\{\begin{matrix} a_{i-1,0}^{p} & \text{if } m_i = 1 \\ 0 & \text{if } m_i > 1 \end{matrix}\right. \qquad (*)$$

for $i \in \mathbb{Z}/r\mathbb{Z}$. Note that the $a_{i,1}$ are all unconstrained, and this accounts for the factor k^r on the right hand side of the display in part (1).

Since w is primitive of length > 2, we may rotate w so that $m_1 > 1$ or $n_r > 1$ (or both). First we deal with the case where all of the $m_i = 1$ and $n_r > 1$. The definitions above give $d(w) = u(w) = 0$ in this case. On the other hand, the system of equations for the $a_{i,0}$ reads

$$0 = a_{r-1,0}^{p}$$
$$\left.\begin{matrix} a_{r-1,0}^{1/p} & \text{if } n_{r-1} = 1 \\ 0 & \text{if } n_{r-1} > 1 \end{matrix}\right\} = a_{r-2,0}^{p}$$
$$\vdots$$
$$\left.\begin{matrix} a_{1,0}^{1/p} & \text{if } n_1 = 1 \\ 0 & \text{if } n_1 > 1 \end{matrix}\right\} = a_{0,0}^{p}.$$

Clearly the only solution is $a_{0,0} = \cdots = a_{r-1,0} = 0$, and this shows that $\operatorname{Hom}_{\mathbb{D}_k}(M(w), M_{1,1}) \cong k^r$ and that none of these homomorphisms are surjective, in agreement with the calculations $d(w) = u(w) = 0$.

Now we assume that at least one of the $m_i > 1$, we rotate w so that m_1 is one of them, and we write $1 = i_1 < i_2 < \cdots$ for the set of indices such that $m_{i_j} > 1$. Then the system $(*)$ breaks up into subsystems involving the variables $a_{i_j,0}, \dots, a_{i_{j+1}-1,0}$ and “controlled” by the subwords $s = v^{n_{i_{j+1}-1}} f \dots v^{n_{i_j}} f^{m_{i_j}}$. (All the exponents of f in this subword except m_{i_j} are 1.) If none of the exponents of v are > 1, then an argument similar to that in the previous paragraph shows that the only solution has $a_{i_j,0} = \cdots = a_{i_{j+1}-1,0} = 0$.

For the main case, continue to focus on a subword

$$s = v^{n_{i_{j+1}-1}} \cdots f^{m_{i_j}}$$

and assume that some exponent of v in s is > 1. To streamline notation, rewrite s in the form

$$s = v^{\nu_t} \cdots f^{\mu_1} = (vf)^e v^{\nu_{t-e}} \cdots f^{\mu_1}$$

where $e \geq 0$ and we write $\nu_\cdot$ for $n_{i_j+\cdot-1}$ and $\mu_\cdot$ for $m_{i_j+\cdot-1}$. Note that we have assumed that $\nu_{t-e} > 1$ and all $\mu_i = 1$ except μ_1. Writing $a_\cdot$ for $a_{m_{i_j}+\cdot-1,0}$, the relevant part of $(*)$ reads

$$\begin{aligned} a_t^{1/p} &= a_{t-1}^p \\ a_{t-1}^{1/p} &= a_{t-2}^p \\ &\vdots \\ a_{m_{t-e+1}}^{1/p} &= a_{t-e}^p \\ 0 &= a_{t-e-1}^p \\ \left.\begin{matrix} a_{t-e-1}^{1/p} & \text{if } \nu_{t-e-1} = 1 \\ 0 & \text{if } \nu_{t-e-1} > 1 \end{matrix}\right\} &= a_{t-e-2,0}^p \\ \left.\begin{matrix} a_{t-e-2}^{1/p} & \text{if } \nu_{t-e-1} = 1 \\ 0 & \text{if } \nu_{t-e-1} > 1 \end{matrix}\right\} &= a_{t-e-3,0}^p \\ &\vdots \\ \left.\begin{matrix} a_1^{1/p} & \text{if } \nu_1 = 1 \\ 0 & \text{if } \nu_1 > 1 \end{matrix}\right\} &= 0. \end{aligned}$$

The general solution of this system is given by choosing a_t arbitrarily in k and letting

$$a_t = a_{t-1}^{p^2} = \cdots = a_{t-e}^{p^{2e}} \quad \text{and} \quad a_{t-e-1} = \cdots = a_1 = 0. \tag{**}$$

This shows that there is one free parameter in the general solution of $(*)$ for each subword s satisfying the hypotheses of this paragraph, and the general solution involves (a highly non-linear!) combination of $e+1$ non-zero values.

To make the connection with the definitions of $d(w)$ and $u(w)$, note that the number of subwords of $w = v^{n_r} \cdots f^{m_1}$ of the form $(vf)^e v^{>1} \cdots f^{>1}$ is the same as the number of subwords of the rotation $f^{m_1} v^{n_r} \cdots v^{n_1}$ of the form $f^2(vf)^e v^2$. Thus the general solution of $(*)$ depends on exactly $d(w) + r$ free parameters from k. This completes the proof of part (1) of the proposition.

Turning to part (2), take an element $\phi \in \mathrm{Hom}_{\mathbb{D}_k}(M(w), M_{1,1}^{\mathfrak{u}})$ for some integer $\mathfrak{u} > 0$. The proof of part (1) gives explicit information about the matrix of ϕ (as a k-linear map) with respect to a suitable basis which we now record. For an ordered basis of $M(w)$, we take

$$E_1, \ldots, E_r, FE_1, \ldots, FE_r, VE_1, \ldots, VE_r, \ldots$$

where we omit VE_i if $m_i = n_i = 1$ (since in this case this element has already appeared as FE_i) and the final ... stands for higher powers of F or V applied to the E_i. As a basis of $M_{1,1}^{\mathfrak{u}}$, we use $\mathfrak{u}$ copies of z_0 followed by $\mathfrak{u}$ copies of z_1.

Let A be the matrix of ϕ with respect to these bases, and let A_0 be the first $\mathfrak{u}$ rows of A. Then A_0 is zero outside its first r columns, and its rows consist of zeroes and sequences $a, a^{p^2}, a^{p^4}, \ldots, a^{p^{2e}}$ as described at $(**)$ above. In particular, only $u(w)$ of the columns of A_0 may be non-zero. This implies that $u_{1,1}(M(w)) \le u(w)$.

To see the reverse inequality, we choose solutions $(**)$ so that A_0 has a block structure

$$\begin{pmatrix} 0 & B_1 & 0 & 0 & \dots \\ 0 & 0 & 0 & B_2 & \dots \\ \vdots & & & & \end{pmatrix}$$

where the B_i correspond to the subwords $f^2(vf)^e v^2$ of w and have the shape

$$\begin{pmatrix} \alpha_1 & \alpha_1^{p^2} & \alpha_1^{p^4} & \dots & \alpha_1^{p^{2e}} \\ \alpha_2 & \alpha_2^{p^2} & \alpha_2^{p^4} & \dots & \alpha_2^{p^{2e}} \\ \dots & & & & \\ \alpha_{e+1} & \alpha_{e+1}^{p^2} & \alpha_{e+1}^{p^4} & \dots & \alpha_{e+1}^{p^{2e}} \end{pmatrix}$$

Choosing the $\alpha_i \in k$ generically results in each of the B_i having maximal rank, namely $e+1$, and A_0 having rank $u(w)$.

With these choices of solutions of $(**)$, the columns $r+1, \dots, 2r$ of the bottom half of A (corresponding to the basis elements $FE_1, \dots, FE_r$ and copies of z_1) has the shape

$$\begin{pmatrix} 0 & B_1^{(p)} & 0 & 0 & \dots \\ 0 & 0 & 0 & B_2^{(p)} & \dots \\ \vdots & & & & \end{pmatrix}$$

where $B^{(p)}$ is obtained from B by taking the p-th power of each entry. It follows that A has rank $2u(w)$, so our choices of solutions to $(**)$ have produced a surjection $M(w) \twoheadrightarrow M_{1,1}^{u(w)}$, and this completes the proof that $u_{1,1}(M(w)) = u(w)$. $\square$

2. Other revisions

The correction to Proposition 5.6 requires minor revisions later in the paper:

- In Proposition 5.8 of [PU21], $u_{1,1}$ should be replaced by $\sum_w \mu_w d(w)$, where $H^1_{dR}(X) = \bigoplus_w M(w)^{\mu_w}$.
- In Proposition 5.9(4) of [PU21], the current formula for $u_{1,1}$ is

$$\sum_{j=0}^{\lfloor(\ell-4)/2\rfloor} \mu(-v^2(fv)^j f^2),$$

  and the correct formula is

$$\sum_{j=0}^{\lfloor(\ell-4)/2\rfloor} (j+1)\mu(-f^2(vf)^j v^2).$$

- In the table of examples for $g = 4$ in Section 5.6 of [PU21], the $u_{1,1}$-number in the line $[0,0,1,1]$ should be 2.

- In part (4) of Proposition 10.3 in [PU21], one should add a coefficient $(j+1)$ to the summand in the display, so the correct formula is

$$\sum_{j=0}^{\lfloor(\ell-4)/2\rfloor}(j+1)\left(\frac{p+1}{2}\right)^2\left(\frac{p-1}{2}\right)^{2j+1}\left(\frac{p^{\ell-3-2j}-1}{2}\right).$$

- Similarly, in part (4) of Proposition 11.3 in [PU21], the correct formula is

$$\sum_{j=0}^{\lfloor(\lambda-4)/2\rfloor}(j+1)\left(\frac{p+1}{2}\right)^2\left(\frac{p-1}{2}\right)^{2j+1}\left(\frac{p^{\lambda-3-2j}+1}{2}\right)$$
$$+\begin{cases}0 & \text{if } \lambda=1,\\ \left(\frac{\lambda-1}{2}\right)\left(\frac{p+1}{2}\right)^2\left(\frac{p-1}{2}\right)^{\lambda-2} & \text{if } \lambda>1 \text{ and odd,}\\ \left(\frac{\lambda}{2}\right)\left(\frac{p+1}{2}\right)\left(\frac{p-1}{2}\right)^{\lambda-1} & \text{if } \lambda \text{ even.}\end{cases}$$

References

[PU21] Pries, R. and Ulmer, D. On BT1 group schemes and Fermat curves, *New York J. Math.* **27** (2021), 705–739. MR4250272, Zbl 1471.11200. 1024, 1027, 1028

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