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Reciprocity in the Hecke groups

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ABSTRACT. An element g in a group G is called *reciprocal* if there exists $h \in G$ such that $g^{-1} = hgh^{-1}$. The reciprocal elements are also known as 'real elements' or 'reversible elements' in the literature. In this paper, we consider the Hecke groups, which are Fuchsian groups of the first kind and generalizations of the modular group. We have classified and parametrized the reciprocal classes in the Hecke groups. This generalizes a result by Sarnak on the reciprocal elements in the modular group.

CONTENTS

1. Introduction

An element g in a group G is called *reciprocal* if there exists $h \in G$ such that $g^{-1} = hgh^{-1}$. The reciprocal elements are also known as 'real elements' and 'reversible elements'. We will call the conjugating element h a *reciprocator* of g. A reciprocator is also known as a *reverser* or a *reversing symmetry*. For a reciprocal element g, the set of all reciprocators is an index two extension of the centralizer of g, see [BR]. We call an element g in G *strongly reciprocal* if h is an involution, i.e., $h^2 = 1$. Equivalently, an element g is strongly reciprocal if it is a product of (at most) two involutions. Such elements are also known as 'bi-reflectional', 'strongly real', or 'strongly reversible'. A strongly reciprocal element is reciprocal, but the converse is not generally true. The reciprocal and strongly reciprocal elements appear at several places in the literature, e.g. [FZ],

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[La], [LR], [OS], [Sa] and the references therein. In a group G, given an infinite order strongly reciprocal element g and its reciprocator (or reverser) r, the group $\langle g, r\rangle$ is an infinite dihedral subgroup. When G is a Fuchsian group, the conjugacy classes of maximal infinite dihedral subgroups are in one-to-one correspondence with primitive reciprocal geodesics in the corresponding surface. Here, a geodesic is reciprocal if it is equivalent to itself with orientation reversed.

Sarnak [Sa] parametrized and counted reciprocal classes for the modular group. Reciprocity in the modular group $\mathrm{PSL}(2, \mathbb{Z})$ has been understood from many other independent viewpoints as well. For a survey see [OS, Chapter 7] and the references therein.

The Hecke groups are generalization of the modular group. These are Fuchsian groups of the first kind. Our main aim in this paper is to classify the reciprocal elements in the Hecke groups. It is also a natural problem to ask about reversibility in an arbitrary Fuchsian group. In the following, we first answer this question.

Theorem 1.1. *Let Γ be a Fuchsian group.*

(1) *An element g in Γ is reciprocal if and only if it is strongly reciprocal. Further, the reciprocators of a reciprocal element are all involutions.*
(2) *The only possible reciprocal elements in Γ are either the hyperbolic elements or the involutions.*

The above theorem gives an answer to an open problem in [OS, p. 104] when asking it for Fuchsian groups. By the above theorem, any reciprocator of a reciprocal element is an involution. So, we will not prefix 'strongly' anymore before reciprocal elements. The above result might have been known to experts, eg. [FK]. However, we do not know of any literature where this has been stated explicitly as in the above theorem.

Now, recall the notion of Hecke groups.

Definition 1.2. The Hecke group Γ_p is a Fuchsian group which is generated by the maps

$$\iota : z \mapsto -\frac{1}{z} \text{ and } \alpha_p : z \mapsto z + \lambda_p, \text{ where, } p \geq 3 \text{ and } \lambda_p = 2\cos\frac{\pi}{p}.$$

Geometrically, the Hecke group can be identified with $\mathbb{Z}_2 * \mathbb{Z}_p$. Let $\gamma_p = \iota\alpha_p$. Here γ_p is an element of order p (see [DKS]). If $p = 2m$, then γ_p^m is an involution. In this group $\Gamma_p = \langle \iota, \gamma_p \mid \iota^2, \gamma_p^p\rangle$ involutions are conjugate to either ι or γ_p^m (when p is even) by [MKD, Corollary 4.1.4].

Note that the Hecke group Γ_3 is the modular group $\mathrm{PSL}(2, \mathbb{Z})$. In [OS, p.143], the reciprocity in Γ_p, for $p \geq 4$, has been listed as an open problem. Our work gives a complete solution to this problem. To state our result, we first define the following.

Definition 1.3. Let A be a hyperbolic element in $\mathrm{PSL}(2,\mathbb{R})$, represented by the matrix $\tilde{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We define the *fixed-point ratio* of A by

$$\theta_A = \begin{cases} \dfrac{b-c}{a-d}, & \text{where, } a \neq d. \\ \infty, & \text{where, } a = d,\ b \neq c. \\ 1, & \text{where, } a = d,\ b = c. \end{cases}$$

The fixed-point ratio is an invariant of the cyclic group generated by A. When $a \neq d$ or, $a = d,\ b \neq c$, the fixed-point ratio is given by $\theta_A = -\dfrac{1+u_1u_2}{u_1+u_2}$, where u_1 and u_2 are the fixed-points of A. We would like to note that the degenerate case $a = d$ may arise in Hecke groups. In this case, any non-zero powers of A possess 'symmetric' fixed points of the form $\{p, -p\}$, where $p \in \mathbb{R}$. In the particular case when $a = d$ and $b = c$, the fixed point set is $\{-1, 1\}$, and the fixed-point ratio is defined to be one of the fixed points. Such elements assume a special role in the Hecke group Γ_p when p is even. We elaborate the above comments in Section 4.

The following theorem classifies reciprocal elements in the Hecke groups. It also generalizes the corresponding result for the modular group Γ_3 in [Sa], [OS, Proposition 7.30].

Theorem 1.4. *Let Γ_p be the Hecke group for $p \geq 3$ and let $g \in \Gamma_p$. Then the following are equivalent.*

(1) *g is reciprocal.*
(2) *For p odd, g is either an involution or conjugate to a hyperbolic element h such that the lifts of h in* $\mathrm{SL}(2,\mathbb{R})$ *are symmetric matrices. For p even, g is either an involution or conjugate to a hyperbolic element h such that the fixed-point ratio of h is either zero,* 1*, or,* $\cos \pi/p$.

The next question about reciprocity in the Hecke groups Γ_p would be to parametrize the reciprocal classes. For this, we will consider only the hyperbolic reciprocal classes. Given a reciprocal element A, one needs to know the number of elements in the reciprocal class of A with the same fixed-point ratio θ_A. The following theorem answers this question. Sarnak [Sa] proved a version of this theorem for the modular group. Our proof of the following theorem follows similar ideas for the Hecke group. To state the theorem, we first define the following.

Definition 1.5. A reciprocal element M in Γ_p is said to be **symmetric** if the fixed point ratio θ_M is either 0 or 1.

Definition 1.6. A reciprocal element M in Γ_p is said to be p-**reciprocal** if the fixed point ratio θ_M is either $\cos \pi/p$ or, 1.

Definition 1.7. A reciprocal element M in Γ_p is said to be **symmetric** p-**reciprocal** if the fixed point ratio θ_M is 1.

As we shall see in Section 6 that equivalently, symmetric and p-reciprocal elements respectively have ι and γ_p^m as reciprocators. Also symmetric p-reciprocal elements have reciprocators γ_p^m, as well as ι.

Theorem 1.8. *For any reciprocal hyperbolic element in Γ_p, the following possibilities can occur :*

(1) *If p is odd, every reciprocal class has exactly four symmetric elements.*
(2) *If p is even.*
    (a) *Suppose in the reciprocal class, the reciprocators are conjugate to ι. Then, there are exactly four symmetric elements in that class.*
    (b) *Suppose in the reciprocal class, the reciprocators conjugate to γ_p^m. Then, there are exactly $2p$ p-reciprocal elements.*
    (c) *Suppose in the reciprocal class, each reciprocal element has two types of reciprocators, one type of reciprocators conjugate to ι and another type of reciprocators conjugate to γ_p^m.*
        (i) *If the reciprocal class does not contain any non-zero power of $\iota\gamma_p^m$, then the class has exactly two symmetric elements and p p-reciprocal elements.*
        (ii) *If the reciprocal class contain any non-zero power of $\iota\gamma_p^m$, then the class has exactly two symmetric p-reciprocal elements and $p-2$ p-reciprocal elements.*

The above theorem parametrizes the reciprocal classes as follows. Let ρ be the set of the conjugacy classes of reciprocal elements in the Hecke group Γ_p other than the involution classes.

$$\mathcal{C}' = \{(a,b,c,t) \in \mathbb{Z}[\lambda_p]^4 \mid a,c,t > 0,\ d = 4ac + b^2,\ t^2 - d = 4,$$
$$((t-b)/2, c)_p = ((t+b)/2, a)_p = 1,\ a = c,\ \text{or}\ \frac{c-a}{b} = \cos\pi/p\}$$

where, $(.,.)_p$ is defined in Subsubsection 2.1.1. Define $\phi' : \mathcal{C}' \mapsto \rho$:

$$(a,b,c,t) \longmapsto \left\{\begin{bmatrix} (t-b)/2 & a \\ c & (t+b)/2 \end{bmatrix}\right\}_{\Gamma_p}. \tag{1.1}$$

Corollary 1.9. *There is a parametrization of the reciprocal classes in Γ_p as follows.*

(1) *If p is odd. Then, every preimage of ϕ' of a reciprocal class has two elements.*
(2) *If $p = 2m$ for $m \in \mathbb{Z}$. Then one of the following cases will occur:*
    (a) *A reciprocal class has reciprocators conjugate to ι. Then the preimage of the reciprocal class under ϕ' of contains two elements.*
    (b) *A reciprocal class has reciprocators conjugate to γ_p^m. Then the preimage of the reciprocal class under ϕ' contains p elements.*
    (c) *A reciprocal class has two types of reciprocators, some are conjugate to ι, and others conjugate to γ_p^m. If the class does not contain any power of $\iota\gamma_p^m$, then the preimage of ϕ' of the reciprocal class contains*

$m+1$ elements. If the reciprocal class contains a power of $\iota\gamma_p^m$, then the preimage of the reciprocal class under ϕ' contains exactly m elements.

When p is odd, we can further modify the parametrization as follows. Let

$$\mathcal{C} = \{(a,b,t) \in \mathbb{Z}[\lambda_p]^3 \mid a,t > 0, d = 4a^2 + b^2, t^2 - d = 4,$$
$$((t-b)/2, a)_p = ((t+b)/2, a)_p = 1\},$$

where $\lambda_p = 2\cos\pi/p$. Define $\phi : \mathcal{C} \mapsto \rho$:

$$(a,b,t) \longmapsto \left\{\begin{bmatrix} (t-b)/2 & a \\ a & (t+b)/2 \end{bmatrix}\right\}_{\Gamma_p}, \tag{1.2}$$

where $\{\gamma\}_{\Gamma_p}$ is the conjugacy class of γ in Γ_p.

Corollary 1.10. *There is a map ϕ from $\mathcal{C}$ to ρ where every fiber has two elements.*

As it is known, there are several ways to understand the reciprocity in the modular group. We expect that the same should be true for the Hecke groups as well, and there should be several other methods to understand reciprocity in the Hecke groups.

Finally consider the family of Fuchsian groups $\Lambda_{p,q}$, where $p, q > 0$ are distinct integers such that $1/p + 1/q < 1/2$. The group $\Lambda_{p,q}$ is generated by the maps

$$\eta_p : z \longmapsto -\frac{1}{z + 2\cos\pi/p}, \text{ and } \eta_q : z \longmapsto -\frac{1}{z + 2\cos\pi/q}.$$

The group $\Lambda_{p,q}$ is finitely generated and may be identified with $\mathbb{Z}_p * \mathbb{Z}_q$. Following the proof of Theorem 1.4, it is easy to classify the reciprocal elements in $\Lambda_{p,q}$. We observe the following in this regard.

Corollary 1.11. *If both p and q are odd numbers, then there is no reciprocal element in $\Lambda_{p,q}$.*

If both of p and q are not odd, then for an element g in $\Lambda_{p,q}$ the following are equivalent.

(1) *g is reciprocal.*
(2) *g is either an involution or conjugate to a hyperbolic element h such that*

$$\theta_h = \begin{cases} 1 \text{ or, } \cos\pi/p & \text{if } p \text{ is even;} \\ 1 \text{ or, } \cos\pi/q & \text{if } q \text{ is even;} \\ 1 \text{ or, } \cos\pi/p \text{ or, } \cos\pi/q & \text{if both } p \text{ and } q \text{ are even.} \end{cases}$$

Structure of the paper. In Section 2, some basic notions are recalled. Theorem 1.1 has been proved in Section 3. The fixed-point ratio and its relationship with reciprocity has been noted in Section 4. Proof of Theorem 1.4 has been split into two sections. Section 5 contains the proof for p odd, and Section 6 contains the proof for p even. Section 7 is devoted to the proof of Theorem 1.8. Proof of the subsequent corollary has also been given in this section.

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2. Preliminaries

Throughout the following, we consider the upper-half space model of the hyperbolic plane $\mathbf{H}^2$. The boundary $\partial\mathbf{H}^2$ may be identified with the circle $\mathbb{R} \cup \{\infty\}$. The group $\mathrm{SL}(2,\mathbb{R})$ acts on $\mathbf{H}^2$ by the Möbius transformations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} : z \mapsto \frac{az+b}{cz+d}.$$

The isometry group of $\mathbf{H}^2$ can be identified with $\mathrm{PSL}(2,\mathbb{R})$. For an element g in $\mathrm{PSL}(2,\mathbb{R})$, we denote by $\tilde{g}$ as one of the lifts of g in $\mathrm{SL}(2,\mathbb{R})$. And a *symmetric* element in $\mathrm{PSL}(2,\mathbb{R})$ is an element that is presented by a symmetric matrix in $\mathrm{SL}(2,\mathbb{R})$. For basic information on Fuchsian groups, we refer to the text [Ka].

Recall that every isometry g of $\mathbf{H}^2$ has at least one fixed point on $\mathbf{H}^2 \cup \partial\mathbf{H}^2$. The isometry g is *elliptic* if it has a fixed point on $\mathbf{H}^2$, equivalently if $|\mathrm{trace}(\tilde{g})| < 2$; it is *parabolic* if g has a unique fixed point on $\partial\mathbf{H}^2$, equivalently if $|\mathrm{trace}(\tilde{g})| = 2$; g is *hyperbolic* if it has exactly two fixed points on $\partial\mathbf{H}^2$, equivalently, if $|\mathrm{trace}(\tilde{g})| > 2$.

A Fuchsian group is elementary if it has a finite orbit in $\mathbf{H}^2 \cup \partial\mathbf{H}^2$. We note the following result that will be used later.

Theorem 2.1. [Ka, Theorem 2.4.3] *Any elementary Fuchsian group is either cyclic or is conjugate in* $\mathrm{PSL}(2,\mathbb{R})$ *to a group generated by* $g(z) = kz$, $k > 1$, *and* $h(z) = -1/z$.

Lemma 2.2. *Let g be an element in* $\mathrm{PSL}(2,\mathbb{R})$. *Let* $g^{-1} = hgh^{-1}$ *for some h in* $\mathrm{PSL}(2,\mathbb{R})$. *Then h keeps the fixed point set F of g invariant, i.e.* $h(F) = F$.

Proof. Let $x \in F$. Then $hgh^{-1}(h(x)) = g^{-1}(h(x)) \Rightarrow g(h(x)) = h(x)$. Thus $h(x)$ is also a fixed point of g. This proves $F \subset h(F)$ and that the converse case is analogous. □

2.1. Hecke groups. Let $p \geq 3$ be an integer, the Hecke group Γ_p is generated by ι and α_p. The lifts of ι and α_p in $\mathrm{SL}(2,\mathbb{R})$ are $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & \lambda_p \\ 0 & 1 \end{bmatrix}$ resp. where $\lambda_p = 2\cos\pi/p$. So, we can think of the Hecke group as a subgroup of $\mathrm{PSL}(2,\mathbb{Z}[\lambda_\mathrm{p}])$. This group, however, is not itself a discrete group for $p \geq 4$. It is possible to know when an element of $\mathrm{PSL}(2,\mathbb{Z}[\lambda_\mathrm{p}])$ belongs to Γ_p with the help of the paper [LL].

2.1.1. Pseudo Euclidean algorithm. [LL] Let $a, b\ (\neq 0) \in \mathbb{Z}[\lambda_p]$. Then there exists an integer n such that $a = b(n\lambda_p) + r$ where $-|b\lambda_p|/2 \leq r \leq |b\lambda_p|/2$. Now repeat the procedure as follows.

$$\begin{aligned} a &= b(n_0\lambda_p) + r_1, \\ b &= r_1(n_1\lambda_p) + r_2, \\ &\vdots \\ r_{k-1} &= r_k(n_2\lambda_p) + r_{k+1}, \\ &\vdots \\ r_{m-1} &= r_m(n_2\lambda_p) + r_{m+1}. \end{aligned}$$

If this algorithm terminates at the m^{th}-step, that is $r_{m+1} = 0$ for some $m \in \mathbb{N}$, then $(a, b)_p := |r_m|$.

Some properties of $(.,.)_p$ are the following:

(1) $(a, b)_p = (b, a)_p$.
(2) $(a, b)_p = (-a, -b)_p = (-a, b)_p = (a, -b)_p$.

3. Proof of Theorem 1.1

The Theorem 1.1 will follow from the following lemmas.

Lemma 3.1. *The only possible reciprocal elements in a Fuchsian group are either the hyperbolic elements or the involutions.*

Proof. Let Γ be a Fuchsian group. To prove this theorem it suffices to show that elliptic elements and parabolic elements are not reciprocal elements. So, we can divide it into two cases.

Case 1. Suppose that an elliptic element γ (other than an involution) is reciprocal in Γ. Then there exists an element $g \in \Gamma$ such that

$$g\gamma g^{-1} = \gamma^{-1}.$$

Since the elliptic elements in $\mathrm{PSL}(2, \mathbb{R})$ have a unique fixed-point in $\mathbb{H}^2$, γ and γ^{-1} fix the same point $x \in \mathbb{H}^2$. This implies that $g(x) = x$ and thus, g, γ commute (from [Ka, Theorem 2.3.2]). This implies $\gamma = \gamma^{-1}$, so γ is an involution. Then, the involutions are only finite order reciprocal elements in Fuchsian groups.

Case 2. For the parabolic case, the proof is very much similar. But they are not finite order elements. Then, this is a contradiction. □

Definition 3.2. An element g of a group G is said to be a **primitive** element if there does not exist any b in G such that $g = b^k$ for some $k \in \mathbb{Z}$ and $|k| > 1$.

In general, primitive elements may not exist in every group. However, for Fuchsian groups, it is necessary for hyperbolic elements to have primitive elements within the group; otherwise, the group would no longer be discrete.

Lemma 3.3. *Let g be a primitive hyperbolic element in the Fuchsian group Γ. Then, g is reciprocal if and only if g^n is reciprocal for any $n \in \mathbb{Z} - \{0\}$. Furthermore, g and g^n have the same set of reciprocators for any $n \in \mathbb{Z} - \{0\}$.*

Proof. If g is reciprocal then it is easy to see that g^n is reciprocal. So, we will show the converse part only. Suppose, g^n is a reciprocal element in Γ with n a non-zero integer. Then, there exists h in Γ such that

$$h^{-1}g^n h = g^{-n}. \tag{3.1}$$

This implies

$$(h^{-1}gh)^n = g^{-n}.$$

We know that the fixed point set remains invariant under a change of power for the hyperbolic elements. Then,

$$Fix(h^{-1}gh) = Fix(h^{-1}g^n h) = Fix(g^{-n}) = Fix(g).$$

So, g and $h^{-1}gh$ share the same fixed point set, and g is the primitive element. Then, from [Ka, Theorem 2.3.5] we obtain that

$$h^{-1}gh = g^k$$

where $k \in \mathbb{Z} - \{0\}$. If we consider $|k| > 1$, this implies that g is not a primitive element which would be a contradiction. Therefore, $k = 1$ or -1. Suppose $k = 1$. Then, h lies in the centralizer of g. Then $g^{2n} = Id$ from Equation 3.1, i.e., g is a finite order element. This implies k must be -1. This proves our lemma. □

Lemma 3.4. *Let Γ be a Fuchsian group. An element g in Γ is reciprocal if and only if it is strongly reciprocal. In particular, the reciprocators of a reciprocal element are involutions.*

Proof. If $g \in \Gamma$ is strongly reciprocal, then it is also reciprocal. We now prove the converse.

Without loss of generality (from Lemma 3.3), assume that g is a primitive element in Γ. Then suppose g is reciprocal. This implies, there exists an element $h \in \Gamma$ such that $hgh^{-1} = g^{-1}$. So, h keeps the fixed point set of g invariant due to Lemma 2.2. In other words, $\langle g, h\rangle$ would be an elementary subgroup of Γ. If g and h have the same fixed point set, then $\langle g, h\rangle$ must be cyclic, using Theorem 2.1. Hence h would be a power of g. In this case, g would be an involution.

Suppose h interchanges the fixed points of g. Then h has a fixed point which is distinct from the fixed points of g. This would imply that h^2 fixes three distinct points. Then h^2 must be identity. Thus, this is proved. □

4. The fixed-point ratio and reciprocity

For a hyperbolic element A, let θ_A be the fixed-point ratio as in Definition (1.3).

Theorem 4.1. *Let A be a hyperbolic element in* $\mathrm{PSL}(2, \mathbb{R})$. *Then $\theta_A = \theta_{A^n}$ for all $n \in \mathbb{Z} - \{0\}$. In other words, θ_A is an invariant of the cyclic group generated by A.*

Proof. Let $\tilde{A}$ is one of the lifts of A in $\mathrm{SL}(2,\mathbb{R})$ with the matrix form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $|a+d|>2$. To prove this theorem, we divide it into three cases.

First, assume $b \neq c$ and $a \neq d$ then, $\theta_A = \dfrac{b-c}{a-d}$.
Let, $c \neq 0$. The fixed point set of A is given by

$$Fix(A) = \left\{ \frac{(a-d) \pm \sqrt{(a+d)^2-4}}{2c} \right\} = \{u_1, u_2\},$$

(say). Note that, $u_1 + u_2 = \dfrac{a-d}{c}$ and $u_1 u_2 = -\dfrac{b}{c}$. Using this we see that,

$$\theta_A = \frac{b-c}{a-d} = \frac{b/c - 1}{(a-d)/c} = -\frac{1+u_1u_2}{u_1+u_2}.$$

Since, the fixed points of A and A^n are the same for any $n \in \mathbb{Z} - \{0\}$ i.e., $\theta_A = \theta_{A^n}$.
When, $c=0$, we have $Fix(A) = \{\infty, \dfrac{-b}{a-d}\}$ and $\theta_A = \dfrac{b}{a-d}$. So, by the same argument as the previous case, the theorem holds for this case also.

Now consider $a = d$ and $b \neq c$. In this case, c must be non-zero. For otherwise, A would be parabolic. So, we have $u_1 + u_2 = 0$ and $\theta_A = -\dfrac{1-u_1^2}{u_1+u_2}$. Accordingly $\theta_A = \infty$. Since, u_1 and u_2 are also fixed points of A^n for any $n \in \mathbb{Z} - \{0\}$, this implies $\theta_A = \theta_{A^n}$.

In this remaining case, when $a = d$, $b = c$, the fixed points are 1 and -1, and by definition $\theta_A = 1$. Since the fixed points do not change under iteration of A, θ_A remains unchanged. □

Corollary 4.2. *Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix with $|a+d|>2$ in* $\mathrm{SL}(2,\mathbb{R})$. *Then, $\theta_A = \theta_{A^n}$ for any non-zero integer n.*

Proof. The proof follows from Theorem 4.1 and the fact that $\theta_A = \theta_{rA}$ for any real number r. □

Lemma 4.3. *Let A be a hyperbolic element in* $\mathrm{PSL}(2,\mathbb{R})$ *such that $|\theta_A| = \eta < 1$. Then A is a reciprocal element with a reciprocator*

$$\Phi = \frac{1}{\sqrt{1-\theta_A^2}} \begin{bmatrix} \theta_A & 1 \\ -1 & -\theta_A \end{bmatrix}.$$

Proof. Let $\tilde{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$a\theta_A + c = b + d\theta_A.$$

It is easy to see by direct computation that $\Phi A \Phi^{-1} = A^{-1}$ in $\mathrm{PSL}(2,\mathbb{R})$. This implies, A is reciprocal with reciprocator Φ in $\mathrm{PSL}(2,\mathbb{R})$. □

We know that every hyperbolic element in $\mathrm{PSL}(2,\mathbb{R})$ is a product of two involutions. By the previous corollary, we can determine the involutions in the product which are given by

$$\frac{1}{\sqrt{1-\theta_A^2}}\begin{bmatrix}\theta_A & 1\\ -1 & -\theta_A\end{bmatrix} \text{ and } \frac{1}{\sqrt{1-\theta_A^2}}\begin{bmatrix}\theta_A & 1\\ -1 & -\theta_A\end{bmatrix}A.$$

Observe that for a hyperbolic element A in $\mathrm{PSL}(2,\mathbb{R})$ with $\theta_A \neq 1, \infty$,

$$\begin{aligned}1-\theta_A^2 &= 1-\left(\frac{1+u_1u_2}{u_1+u_2}\right)^2\\ &= \frac{(u_1+u_2)^2-(1+u_1u_2)^2}{(u_1+u_2)^2}\\ &= \frac{u_1^2+u_2^2-1-u_1^2u_2^2}{(u_1+u_2)^2}\\ &= \frac{(u_1^2-1)-u_2^2(u_1^2-1)}{(u_1+u_2)^2}\\ &= \frac{(u_1^2-1)(1-u_2^2)}{(u_1+u_2)^2}.\end{aligned}$$

Thus, for $|\theta_A| < 1$, one fixed point lie in the interval $(-1,1)$ and the other lie outside $(-1,1)$.

In $\mathrm{PSL}(2,\mathbb{R})$, we have some hyperbolic elements which are reciprocal in $\mathrm{PSL}(2,\mathbb{R})$, but the invariant is greater than 1. For example, $B = \begin{bmatrix}3 & 5\\ 1 & 2\end{bmatrix}$ gives a hyperbolic element in $\mathrm{PSL}(2,\mathbb{R})$ that is reciprocal but $|\theta_B| > 1$.

For such B whose $|\theta_B| > 1$, the fixed points are either inside of $(-1,1)$ or both are outside $(-1,1)$. By conjugation, we can move the fixed points such that one will lie inside $(-1,1)$. Then, we can do the rest as we have done in Lemma 4.3. In that case, the reciprocators are conjugate to an element of the form Φ.

If we consider $\theta_A = 1$, i.e., $a = d, c = b$ for the hyperbolic element A, the hyperbolic axis I of A is the semi-circle of (Euclidean) radius one which is already discussed in Introduction. We may obtain the reciprocator Φ as

$$\frac{1}{\sin t}\begin{bmatrix}\cos t & 1\\ -1 & -\cos t\end{bmatrix},$$

where $t \in \mathbb{R}$. The fixed points of the involution Φ lie on that axis I. Then, Φ interchanges the fixed points $\{-1,1\}$. This implies that Φ will be a reciprocator for A.

The following Corollary 4.4 follows from the above discussion.

Corollary 4.4. *Let f be a hyperbolic element in* $\mathrm{PSL}(2,\mathbb{R})$. *The fixed point set of f is* $\{-1,1\}$. *Then f is a strongly reciprocal element and the reciprocators of f are of the form e^{it} where $t \in \mathbb{R}$.*

Recall Definition (1.2) for the definition of ι.

Corollary 4.5. *If A is conjugate to B by ι in* $\mathrm{PSL}(2,\mathbb{R})$ *with $\theta_A \neq 1$, then $\theta_A = -\theta_B$.*

Corollary 4.6. *Let A be a reciprocal element in the Hecke group Γ_p with reciprocator ι. Then, $\theta_A = 0$ or 1, i.e., every lift of A in* $\mathrm{SL}(2,\mathbb{R})$ *is a symmetric matrix.*

The converse is also true, and that follows from Lemma 5.2. Finally, we note the following lemma.

Lemma 4.7. *Let A be a hyperbolic element in* $\mathrm{PSL}(2,\mathbb{R})$ *then, $\tilde{A}^k$ is symmetric if and only if $\tilde{A}$ is symmetric for any non-zero integer k.*

Proof. Let $\tilde{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix in $\mathrm{SL}(2,\mathbb{R})$, and $Fix(A) = Fix(A^k)$. The fixed points, say z_1 and z_2, of A satisfy this equation,

$$cz^2 - (a-d)z - b = 0. \tag{4.1}$$

If $\tilde{A}$ is a symmetric matrix, then $b = c$ i.e., $z_1 z_2 = -1$. Since, $Fix(A) = Fix(A^k)$, and $\tilde{A^k} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$. Then z_1, z_2 both are roots of the equation,

$$rz^2 - (p-s)z - q = 0. \tag{4.2}$$

Then, $z_1 z_2 = -q/r \Rightarrow -1 = -q/r \Rightarrow q = r$. Similarly, the converse is also true. □

5. Proof of Theorem 1.4 for p odd

We note the following observations.

Lemma 5.1. *Let $g \in$* $\mathrm{PSL}(2,\mathbb{R})$ *be represented by a symmetric matrix $\tilde{g} \in$* $\mathrm{SL}(2,\mathbb{R})$. *The map $\iota : z \mapsto -\dfrac{1}{z}$ keeps the fixed point set of g invariant under ι.*

Proof. Let $\tilde{g}$, a lift of g of $\mathrm{PSL}(2,\mathbb{R})$ represents by the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\mathrm{SL}(2,\mathbb{R})$. Recall that the fixed point set of g is given by the equation

$$cz^2 + (d-a)z - b = 0.$$

If A is symmetric, then $b = c$, and hence the above equation is invariant under the map ι. □

Lemma 5.2. *Let a hyperbolic element g in Γ_p be such that g is conjugate to an element in Γ_p that has a symmetric matrix representation in* $\mathrm{SL}(2,\mathbb{R})$. *Then g is reciprocal.*

Proof. Let g be a hyperbolic element in Γ_p. Then, without loss of generality, we show that if $\tilde{g}$ is a symmetric matrix in $\mathrm{SL}(2,\mathbb{R})$, then g is reciprocal with reciprocator ι. Also, without loss of generality (see Lemma 4.7), we assume that g is a primitive element in Γ_p. By Lemma 5.1, the fixed point set of g is invariant under ι. So, g and $\iota g\iota$ have the same set of fixed points. Thus, $\langle g, \iota g\iota\rangle$ is an elementary two-generator subgroup of Γ_p. Using arguments as in the proof of Lemma 3.4, it follows that

$$\iota g\iota = g^p \text{ for some } p \in \mathbb{Z} - \{0\}.$$

But, $g = \iota g^p \iota$, i.e. $g = (\iota g\iota)^p$. That means p must be 1 or -1; otherwise, it will be a contradiction to the assumption that g is primitive. If $p = 1$, then g must lie in the centralizer of ι in Γ_p, so g and ι have same fixed point. Then, g has finite order, which is the contradiction. Hence p must be -1, and hence g is reciprocal. □

Proposition 5.3. *The symmetric hyperbolic elements in* PSL(2, ℝ) *are reciprocal, and the reverser is* ι.

Proof. The proof follows from the Lemma 5.2 and Lemma 4.7. □

Lemma 5.4. *A hyperbolic element g in Γ_p, is reciprocal if and only if $\tilde{g}$ is conjugate to an element that has a symmetric matrix representation in* SL(2, ℝ).

Proof. The necessary condition follows from Lemma 5.2. We now prove the converse.

Suppose that g is a reciprocal and primitive element in the Hecke group Γ_p. Then there is an involution h in Γ_p such that $hgh^{-1} = g^{-1}$. Since p is odd, up to conjugacy, ι is the only involution in Γ_p. Let $m \in \Gamma_p$ be an element such that $h = m\iota m^{-1}$. Now we see that,

$$hgh^{-1} = g^{-1} \Rightarrow m\iota m^{-1} g m\iota m^{-1} = g^{-1}$$
$$\Rightarrow \iota(m^{-1}gm)\iota = m^{-1}g^{-1}m$$

Let $k = m^{-1}gm$. Then ι conjugates k to k^{-1}. This implies, $m^{-1}gm$ has symmetric matrix representation due to Corollary 4.6. Hence g is conjugate to a symmetric matrix. □

5.1. Proof of Theorem 1.4 for odd p. Combining Lemma 3.4 and Lemma 3.1, we obtain the equivalence of (1) and (2). Equivalence between (2) and (3) follows from the above two lemmas. So, the theorem is proved for the odd case. □

6. Proof of Theorem 1.4 for *p* even

We have seen reciprocity in Γ_p for $p \geq 3$ odd. But for p even, some questions appear.

(1) Is there any other involution not conjugate to ι?
(2) If it is, then does it work as a reciprocator?

The answers to these questions are yes. For the first question, the other involutions are conjugate to $(\iota\alpha_p)^m$ where $2m = p$. We will denote $\iota\alpha_p$ by γ_p. And for the second, it suffices to show the existence of reciprocator other than ι.

6.1. Suppose p is even, $p = 2m$, $m \geq 2$. The element γ_p defined by $\gamma_p : z \mapsto -\dfrac{1}{z+\lambda_p}$, is of order p. Then γ_p^m is an involution in Γ_p, see [DKS]. Lifts of γ_p in $\mathrm{SL}(2,\mathbb{R})$ are $\pm\begin{bmatrix} 0 & -1 \\ 1 & \lambda_p \end{bmatrix}$. To determine the involution γ_p^m, we need the following lemma.

Lemma 6.1. *Let $A = \begin{bmatrix} 0 & -1 \\ 1 & \lambda_p \end{bmatrix}$ be the lift of γ_p in* $\mathrm{SL}(2,\mathbb{R})$*. Then, for any natural number k, $A^k = \begin{bmatrix} -a_{k-1} & -a_k \\ a_k & a_{k+1} \end{bmatrix}$ where $a_{k+1} = -a_{k-1} + \lambda_p a_k$ and $a_0 = 0, a_1 = 1$.*

Proof. We will use induction on k. For $k = 1$, the matrix A is $\begin{bmatrix} -a_0 & -a_1 \\ a_1 & a_2 \end{bmatrix}$, i.e., $\begin{bmatrix} 0 & -1 \\ 1 & \lambda_p \end{bmatrix}$. So the lemma is true for $k = 1$.

Let us consider the lemma is true for $k = n$. So, that means $A^n = \begin{bmatrix} 0 & -1 \\ 1 & \lambda_p \end{bmatrix}^n$

$$\Rightarrow A^n = \begin{bmatrix} -a_{n-1} & -a_n \\ a_n & a_{n+1} \end{bmatrix},$$

where $a_{n+1} = -a_{n-1} + \lambda_p a_n$. Then

$$A^{n+1} = \begin{bmatrix} -a_{n-1} & -a_n \\ a_n & a_{n+1} \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & \lambda_p \end{bmatrix}$$

$$= \begin{bmatrix} -a_n & -a_{n+1} \\ a_{n+1} & -a_n + \lambda_p a_{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} -a_n & -a_{n+1} \\ a_{n+1} & a_{n+2} \end{bmatrix},$$

since, $a_{k+1} = -a_{k-1} + \lambda_p a_k$. So the lemma is true for $k = n + 1$. Hence the lemma is proved. □

Lemma 6.2. *The involution γ_p^m in Γ_p lifts to the following matrices in* $\mathrm{SL}(2,\mathbb{R})$*:*

$$\pm\begin{bmatrix} -\dfrac{\cos\frac{\pi}{p}}{\sin\frac{\pi}{p}} & -\dfrac{1}{\sin\frac{\pi}{p}} \\ \dfrac{1}{\sin\frac{\pi}{p}} & \dfrac{\cos\frac{\pi}{p}}{\sin\frac{\pi}{p}} \end{bmatrix}.$$

Proof. Let us consider $p = 2m$. Let A be one of the lifts of γ_p in $\mathrm{SL}(2,\mathbb{R})$. So, $B = A^m = \begin{bmatrix} -a_{m-1} & -a_m \\ a_m & a_{m+1} \end{bmatrix}$, where $a_{m+1} = -a_{m-1} + \lambda_p a_m$, is a lift of the involution γ_p^m. Thus, then $B^2 = \pm Id$. That implies,

$$\begin{bmatrix} -a_{m-1} & -a_m \\ a_m & a_{m+1} \end{bmatrix}^2 = \pm Id$$

$$\Rightarrow \begin{bmatrix} a_{m-1}^2 - a_m^2 & a_{m-1}a_m - a_m a_{m+1} \\ -a_{m-1}a_m + a_m a_{m+1} & a_{m+1}^2 - a_m^2 \end{bmatrix} = \pm Id$$

$$\Rightarrow a_{m-1}^2 - a_m^2 = \pm 1 \ , \ a_{m-1}a_m - a_m a_{m+1} = 0 \ , \ a_{m+1}^2 - a_m^2 = \pm 1.$$

From the second equation, if we obtain two cases.

Case 1. $a_m = 0$, then from the first and third equation $a_{m+1}^2 = 1 \Rightarrow a_{m+1} = \pm 1$. Similarly $a_{m-1} = \pm 1$. Also for the determinant condition a_{m-1} and a_{m+1} have opposite signs. This implies, B must project to the identity of $\mathrm{PSL}(2, \mathbb{R})$. This contradicts our hypothesis. So, this case will not appear.

Case 2. $a_m \neq 0$, then $a_{m+1} = a_{m-1}$ then

$$a_{m+1} = -a_{m-1} + \lambda_p a_m \Rightarrow 2a_{m+1} = \lambda_p a_m$$

$$\Rightarrow a_{m+1} = \cos\left(\frac{\pi}{p}\right) a_m.$$

Therefore, $a_{m+1}^2 \leq a_m^2$ and

$$a_{m+1}^2 - a_m^2 = -1$$

$$\Rightarrow a_m^2 - \cos^2\left(\frac{\pi}{p}\right) a_m^2 = 0$$

$$\Rightarrow a_m = \pm\frac{1}{\sin\frac{\pi}{p}}.$$

So the lifts of γ_p^m are $\pm\begin{bmatrix} -\frac{\cos\frac{\pi}{p}}{\sin\frac{\pi}{p}} & -\frac{1}{\sin\frac{\pi}{p}} \\ \frac{1}{\sin\frac{\pi}{p}} & \frac{\cos\frac{\pi}{p}}{\sin\frac{\pi}{p}} \end{bmatrix}$ in $\mathrm{SL}(2, \mathbb{R})$. □

Now, we show the existence of reciprocator conjugate to γ_p^m.

Lemma 6.3. *Any reciprocal hyperbolic element in Γ_p have reciprocators conjugate to either ι or γ_p^m.*

Proof. Let us consider M to be a reciprocal element in the Hecke group Γ_p. So, by [HR, Lemma 3.1], M is conjugate to

$$W = V_{j_1} V_{j_2} \dots V_{j_n},$$

where $V_{j_k} = (U)^{j_k - 1}\alpha_p$, $U = \alpha_p \iota$, $1 \leq j_k \leq p-1$ for $1 \leq k \leq n$ and $n \in \mathbb{N}$. The product is unique up to cyclic permutation. Since reciprocity is invariant under the conjugation, by [MKD, Theorem 1.4], the reduced form of W and W^{-1} differ by a cyclic permutation. Then,

$$W^{-1} = V_{j_n}^{-1} V_{j_{n-1}}^{-1} \dots V_{j_1}^{-1}$$

$$\Rightarrow W^{-1} = \alpha_p^{-1}U^{p-j_n+1}\alpha_p^{-1}U^{p-j_{n-1}+1} \dots \alpha_p^{-1}U^{p-j_1+1}$$

$$\Rightarrow W^{-1} = \alpha_p^{-1}(\alpha_p\iota \dots \alpha_p\iota)\alpha_p^{-1}(\alpha_p\iota\alpha_p\iota \dots \alpha_p\iota) \dots \alpha_p^{-1}(\alpha_p\iota\alpha_p\iota \dots \alpha_p\iota)$$

$$\Rightarrow W^{-1} = \iota(\alpha_p\iota\alpha_p\iota \dots \alpha_p\iota)\iota(\alpha_p\iota\alpha_p\iota \dots \alpha_p\iota)\iota \dots \iota(\alpha_p\iota\alpha_p\iota \dots \alpha_p\iota)$$

$$\Rightarrow W^{-1} = \iota(\alpha_p\iota)^{p-j_n-1}\alpha_p(\alpha_p\iota)^{p-j_{n-1}-1}\alpha_p \dots (\alpha_p\iota)^{p-j_1-1}\alpha_p\iota$$

$$\Rightarrow W^{-1} = \iota V_{p-j_n}V_{p-j_{n-1}} \dots V_{p-j_1}\iota$$

$$\Rightarrow \iota W^{-1}\iota = V_{p-j_n}V_{p-j_{n-1}} \dots V_{p-j_1} = P\ (say).$$

P is a cyclically reduced word in Γ_p. Then it is a cyclic permutation of W. That implies, $RPR^{-1} = W$ where $R = (V_{j_1}V_{j_2} \dots V_{j_r})^{\pm 1}$ for $r \leq n$ and let $q = n - r$. Then either R or R^{-1} is a subword of W. Without loss of generality, let us assume R is a subword. Then, $RPR^{-1} = W \Rightarrow R\iota W^{-1}\iota R^{-1} = W$. By Lemma 3.4 it follows that $R\iota$ is an involution. Then

$$R\iota = V_{j_1}V_{j_2} \dots V_{j_r}\iota$$
$$= U^{j_1-1}\alpha_p U^{j_2-1}\alpha_p \dots U^{j_r-1}\alpha_p\iota$$
$$= U^{j_1-1}\alpha_p\iota\iota U^{j_2-1}\alpha_p\iota\iota \dots U^{j_r-1}\alpha_p\iota$$
$$= U^{j_1}\iota U^{j_2}\iota \dots U^{j_r}.$$

Therefore,

$$R\iota R\iota = Id$$
$$\Rightarrow U^{j_1}\iota U^{j_2}\iota \dots U^{j_r}U^{j_1}\iota U^{j_2}\iota \dots U^{j_r} = Id$$
$$\Rightarrow U^{j_1}\iota U^{j_2}\iota \dots U^{j_r+j_1}\iota U^{j_2}\iota \dots U^{j_r} = Id.$$

Since every $j_k \geq 1$, then the above equation will hold if the following conditions are satisfied: $j_r + j_1 = p, j_{r-1} + j_2 = p, \dots, j_{r-i} + j_{1+i} = p, \dots, j_1 + j_r = p$. Therefore,

$$P = R^{-1}WR$$
$$\Rightarrow V_{p-j_n}V_{p-j_{n-1}} \dots V_{p-j_1} = V_{j_{r+1}}V_{j_{r+2}} \dots V_{j_{r+q}}V_{j_1}V_{j_2} \dots V_{j_r}$$
$$\Rightarrow U^{p-j_n-1}\alpha_p U^{p-j_{n-1}-1}\alpha_p \dots U^{p-j_1-1}\alpha_p =$$
$$U^{j_{r+1}-1}\alpha_p U^{j_{r+2}-1}\alpha_p \dots U^{j_{r+q}-1}\alpha_p U^{j_1-1}\alpha_p \dots U^{j_r-1}\alpha_p$$
$$\Rightarrow U^{p-j_n-1}\alpha_p\iota\iota U^{p-j_{n-1}-1}\alpha_p\iota\iota \dots U^{p-j_1-1}\alpha_p\iota\iota$$
$$= U^{j_{r+1}-1}\alpha_p\iota\iota U^{j_{r+2}-1}\alpha_p\iota\iota \dots U^{j_{r+q}-1}\alpha_p\iota\iota U^{j_1-1}\alpha_p\iota\iota \dots U^{j_r-1}\alpha_p\iota\iota$$
$$\Rightarrow U^{p-j_n}\iota U^{p-j_{n-1}}\iota \dots U^{p-j_1}\iota = U^{j_{r+1}}\iota U^{j_{r+2}}\iota \dots U^{j_{r+q}}\iota U^{j_1}\iota \dots U^{j_r}\iota.$$

This implies, $j_n + j_{r+1} = p, j_{n-1} + j_{r+2} = p, \dots, j_{n-k+1} + j_{r+k} = p, \dots, j_{n-q+1} + j_{r+q} = p$. Let $\sigma(x)$ denote a cyclically reduced form of x, i.e., $\sigma(x^{-1}yx) = \sigma(y)$ for x, y are elements of Γ_p.

Then the cyclic reduced form of $R\iota$ i.e. $\sigma(R\iota) = \sigma(U^{j_1}\iota U^{j_2}\iota \dots U^{j_r})$ is either ι or U^m where $2m = p$. Also U^m is conjugate to γ_p^m. That means $R\iota$ is conjugate to ι or γ_p^m depending on r is even or odd, respectively. □

Remark 6.4. In the proof of the Lemma 6.3, we just show that γ_p^m is a reciprocator for some reciprocal elements. But, eventually it provides more information. Consider $W = RQ$ where R, Q are subwords of W. We know that $R\iota$ is an involution and a reciprocator for W. Then, $\iota R^{-1} = R\iota$. Also, $R\iota W$ will be an involution. Then,

$$W = RQ = R\iota\iota Q$$
$$\Rightarrow R\iota W = (R\iota)^2\iota Q = \iota Q.$$

That means ιQ is an involution. Then, ιQ is also a reciprocator of W.

$$Q = V_{j_{r+1}} V_{j_{r+2}} \dots V_{j_{r+q}},$$

where, $r + q = n$. This implies,

$$\begin{aligned}\iota Q &= \iota U^{j_{r+1}-1}\alpha_p U^{j_{r+2}-1}\alpha_p \dots U^{j_{r+q}-1}\alpha_p \\ &= \iota U^{j_{r+1}-1}\alpha_p\iota\iota U^{j_{r+2}-1}\alpha_p\iota\iota \dots U^{j_{r+q}-1}\alpha_p\iota\iota \\ &= \iota U^{j_{r+1}}\iota U^{j_{r+2}}\iota \dots U^{j_{r+q}}\iota.\end{aligned}$$

So, $\sigma(\iota Q) = \sigma(\iota U^{j_{r+1}}\iota U^{j_{r+2}}\iota \dots U^{j_{r+q}}\iota)$.

If q is even, then, $\sigma(\iota Q) = \sigma(\iota)$. If q is odd, then $\sigma(\iota Q) = \sigma(\gamma_p^m)$. So, the conclusion is that the reciprocator of W depends on the value of n. If n is odd, then the reciprocators of a reciprocal element conjugate to ι and γ_p^m respectively. If n is even, then the reciprocator conjugate either to ι (if r and q are both even), or γ_p^m (if r and q are both odd). For example, if we choose

$$W = V_1 V_{p-1} V_m$$

where, $V_j = U^{j-1}\alpha_p$. Then we see that

$$\begin{aligned}W = \alpha_p(\alpha_p\iota)^{p-2}\alpha_p(\alpha_p\iota)^{m-1}\alpha_p &= \alpha_p\iota\iota(\alpha_p\iota)^{p-2}\alpha_p\iota\iota(\alpha_p\iota)^{m-1}\alpha_p\iota\iota \\ &= (\alpha_p\iota)\iota(\alpha_p\iota)^{p-1}\iota(\alpha_p\iota)^m\iota \\ &= \iota(\iota\alpha_p)\iota(\iota\alpha_p)^{p-1}\iota(\iota\alpha_p)^m \\ &= \iota(\gamma_p)\iota\gamma_p^{p-1}\iota\gamma_p^m.\end{aligned}$$

So, one of the lifts of W is

$$\frac{1}{\sqrt{1-(\frac{\lambda_p}{2})^2}}\begin{bmatrix}(3\lambda_p^2+2)/2 & (\lambda_p^3+3\lambda_p)/2 \\ 3\lambda_p/2 & (\lambda_p^2+2)/2\end{bmatrix}.$$

We notice that $\theta_W = \lambda_p/2 = \cos\pi/p$. If we take the conjugate element of W, $\gamma_p^{p-1}\iota W\iota\gamma_p$, then one of the lifts of this element is

$$\frac{1}{\sqrt{1-(\frac{\lambda_p}{2})^2}}\begin{bmatrix}1 & \frac{3\lambda_p}{2} \\ \frac{3\lambda_p}{2} & 2\lambda_p^2+1\end{bmatrix},$$

which is a symmetric matrix with fixed point ratio 0.

Corollary 6.5. *Some infinite order reciprocal elements have two types of reciprocators one conjugates to ι, and the other one conjugates to γ_p^m.*

Proof of the Corollary 6.5 follows from the previous remark.

Proposition 6.6. *Let A be a hyperbolic element in Γ_p with $\theta_A = \cos\frac{\pi}{p}$. Then A is a reciprocal element in Γ_p with reciprocator γ_p^m.*

Proof. The proof of the proposition follows from the Lemma 4.3. □

Lemma 6.7. *Let g be a hyperbolic reciprocal element in Γ_p with reciprocator h, conjugate to γ_p^m. Then, it is conjugate to an element A in Γ_p with either $\theta_A = \cos\frac{\pi}{p}$ or, $\theta_A = 1$.*

Proof. g is a hyperbolic reciprocal element in Γ_p and

$$hgh^{-1} = g^{-1}.$$

So, h must be conjugate to ι or γ_p^m. From the hypothesis consider, h is conjugate to γ_p^m then, there exists $t \in \Gamma_p$ such that,

$$tht^{-1} = \gamma_p^m.$$

This implies,

$$tht^{-1}tgt^{-1}tht^{-1} = tg^{-1}t^{-1}.$$

Let us consider tgt^{-1} as a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\mathrm{SL}(2,\mathbb{R})$ and B is one of the lifts of γ_p^m from Lemma 6.2. Then

$$B\begin{bmatrix} a & b \\ c & d \end{bmatrix}B^{-1} = \pm\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$$

$$\Rightarrow B\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \pm\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} B$$

$$\Rightarrow \pm\frac{1}{\sin\frac{\pi}{p}}\begin{bmatrix} -\cos\frac{\pi}{p} & -1 \\ 1 & \cos\frac{\pi}{p} \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \pm\frac{1}{\sin\frac{\pi}{p}}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} -\cos\frac{\pi}{p} & -1 \\ 1 & cos\frac{\pi}{p} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -\frac{\lambda_p}{2} & -1 \\ 1 & \frac{\lambda_p}{2} \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \pm\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} -\frac{\lambda_p}{2} & -1 \\ 1 & \frac{\lambda_p}{2} \end{bmatrix} \text{ where, } \lambda_p = 2\cos\frac{\pi}{p}$$

$$\Rightarrow \begin{bmatrix} -a\frac{\lambda_p}{2} - c & -b\frac{\lambda_p}{2} - d \\ a + \frac{\lambda_p}{2}c & b + d\frac{\lambda_p}{2} \end{bmatrix} = \pm\begin{bmatrix} -\frac{\lambda_p}{2}d - b & -d - b\frac{\lambda_p}{2} \\ \frac{\lambda_p}{2}c + a & c + a\frac{\lambda_p}{2} \end{bmatrix}.$$

Case 1.

$$\begin{bmatrix} -a\frac{\lambda_p}{2} - c & -b\frac{\lambda_p}{2} - d \\ a + \frac{\lambda_p}{2}c & b + d\frac{\lambda_p}{2} \end{bmatrix} = -\begin{bmatrix} -\frac{\lambda_p}{2}d - b & -d - b\frac{\lambda_p}{2} \\ \frac{\lambda_p}{2}c + a & c + a\frac{\lambda_p}{2} \end{bmatrix}.$$

Then,

$$-b\frac{\lambda_p}{2} - d = 0 \text{ and } a + \frac{\lambda_p}{2}c = 0$$

$$\Rightarrow \frac{\lambda_p}{2}(b+c) + a + d = 0.$$

Here, $b + c \neq 0$ since, $|a + d| > 2$. Also,

$$-a\frac{\lambda_p}{2} - c = \frac{\lambda_p}{2}d + b.$$

From the above equations we obtain that

$$\frac{\lambda_p}{2} = -\frac{b+c}{a+d} \text{ and } \frac{\lambda_p}{2} = -\frac{a+d}{b+c}.$$

Then, $\frac{\lambda_p^2}{4} = 1$ which is contradiction. So, this case will not appear.

Case 2.

$$\begin{bmatrix} -a\frac{\lambda_p}{2} - c & -b\frac{\lambda_p}{2} - d \\ a + \frac{\lambda_p}{2}c & b + d\frac{\lambda_p}{2} \end{bmatrix} = \begin{bmatrix} -\frac{\lambda_p}{2}d - b & -d - b\frac{\lambda_p}{2} \\ \frac{\lambda_p}{2}c + a & c + a\frac{\lambda_p}{2} \end{bmatrix}.$$

This implies,

$$-a\frac{\lambda_p}{2} - c = -\frac{\lambda_p}{2}d - b \Rightarrow (a-d)\frac{\lambda_p}{2} = b - c.$$

Subcase 1. If the lifts are symmetric matrices i.e., $b = c$ then, $a = d$. This and Proposition 5.3 implies that if the lifts have the same diagonal and off-diagonal entries, then they have reciprocators as ι and γ_p^m both and $\theta_A = 1$.

Subcase 2. If the lifts are non-symmetric matrices then

$$\frac{\lambda_p}{2} = \frac{b-c}{a-d}.$$

So, the reciprocal elements whose reciprocator is γ_p^m, conjugate to a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that either $\theta_A = \cos\frac{\pi}{p}$ or, $b = c$ and $a = d$. □

6.2. Proof of Theorem 1.4 for even *p*. Combining Lemma 3.4, Lemma 3.1 and Lemma 6.3, we obtain that the reciprocal elements of Γ_p are either involutions or hyperbolic elements with reciprocator ι or γ_p^m. If γ_p^m conjugates to a reciprocator of a reciprocal element, then by Lemma 6.7, the reciprocal element is conjugate to an element whose fixed point ratio is $\cos\frac{\pi}{p}$ or 1. If ι conjugates to a reciprocator of a reciprocal element, then by Corollary 4.6, the reciprocal element is conjugate to an element whose fixed point ratio is 0 or 1. Hence, the theorem is proved. □

7. Parametrization of the reciprocal classes in Γ_p

7.1. Proof of Theorem 1.8. Without loss of generality, consider a primitive reciprocal hyperbolic element r in Γ_p. Let S be a reciprocator for r. Then,

$$S^{-1}rS = r^{-1}. \tag{7.1}$$

This implies,

$$r^k S r^k = S \;\forall k \in \mathbb{Z}. \tag{7.2}$$

Then, by Lemma 6.3, S is conjugate to either ι or γ_p^m.

Case 1. First, suppose S is conjugate to ι. Then, let $\eta \in \Gamma_p$ be an element such that,

$$\eta^{-1}S\eta = \iota. \tag{7.3}$$

If η_S is a solution of Equation (7.3) then $S\eta_S$ is also a solution of the equation. This gives two hyperbolic symmetric elements, $\eta_S^{-1}r\eta_S$ and inverse of it.
Let us consider another symmetric element in the conjugacy class of r, say $\beta^{-1}r\beta$. Since it is symmetric, then

$$\iota\beta^{-1}r\beta\iota = \beta^{-1}r^{-1}\beta. \tag{7.4}$$

That means,

$$\begin{aligned}
&\beta\iota\beta^{-1}r\beta\iota\beta^{-1} = r^{-1}\\
&\Rightarrow (\beta\iota\beta^{-1})r(\beta\iota\beta^{-1}) = r^{-1}\\
&\Rightarrow \beta\iota\beta^{-1}r\beta\iota\beta^{-1} = S^{-1}rS\\
&\Rightarrow S\beta\iota\beta^{-1}r\beta\iota\beta^{-1}S^{-1} = r\\
&\Rightarrow (\beta\iota\beta^{-1}S^{-1})^{-1}r\beta\iota\beta^{-1}S^{-1} = r.
\end{aligned}$$

That implies, $\beta\iota\beta^{-1}S^{-1} \in C(r)$ centralizer of r. Then, $\beta\iota\beta^{-1} = bS$, where $b \in C(r)$ i.e.,

$$b = r^n \text{ for some } n \in \mathbb{Z}. \tag{7.5}$$

Suppose $S' = r^{2k}S$ for $k \in \mathbb{Z}$. This implies

$$r^{-k}S'r^k = r^k S r^k. \tag{7.6}$$

Combining (7.6) and (7.2), we obtain,

$$r^{-k}S'r^k = S. \tag{7.7}$$

Again, S' also satisfy (7.1) and (7.3). So, $\eta_{S'}$ and $S'\eta_{S'}$ are the solutions for

$$\eta^{-1}S'\eta = \iota. \tag{7.8}$$

Then, $(\eta_{S'}\eta_S^{-1})^{-1}r^k \in C(S) = \{Id, S\}$. If

$$(\eta_{S'}\eta_S^{-1})^{-1}r^k = Id.$$

Then it implies,

$$\begin{aligned}
&(\eta_{S'}\eta_S^{-1}) = r^k\\
&\Rightarrow \eta_{S'} = r^k\eta_S\\
&\Rightarrow \eta_{S'}^{-1}r\eta_{S'} = \eta_S^{-1}r\eta_S.
\end{aligned}$$

If $\eta_{S'} = r^k S \eta_S$. This implies,

$$\eta_{S'}^{-1} r \eta_{S'} = (r^k S \eta_S)^{-1} r r^k S \eta_S = \eta_S^{-1} S^{-1} r S \eta_S = \eta_S^{-1} r^{-1} \eta_S.$$

Thus, if two reciprocators differ by an even factor of r, then the pair of symmetric elements related to the respective reciprocators are the same.

Case 2. Suppose the reciprocator of r, S is conjugate to γ_p^m. Then,

$$S^{-1} r S = r^{-1}. \tag{7.9}$$

And there is an element $\eta \in \Gamma_p$ such that,

$$\eta^{-1} S \eta = \gamma_p^m \tag{7.10}$$

$$\Rightarrow S = \eta \gamma_p^m \eta^{-1} \Rightarrow S = (\eta \gamma_p \eta^{-1})^m.$$

So, $C(S) \simeq \langle \eta \gamma_p \eta^{-1} \rangle$ in Γ_p, then let α be the generator of $C(S)$, such that $S = \alpha^m$. Let η_S and η'_S be solutions for (7.10), i.e.,

$$\eta_S^{-1} S \eta_S = \gamma_p^m = (\eta'_S)^{-1} S \eta'_S.$$

This implies,

$$\eta_S (\eta'_S)^{-1} \in \langle \alpha \rangle \Rightarrow \eta_S = \alpha^i (\eta'_S)$$

where, $i = 0, 1, \dots, p-1$. Then, there are p solutions for the equation (7.10) for each S. Those are in the set $\mathcal{P} = \{\eta_S, \alpha\eta_S, \dots, \alpha^{p-1}\eta_S\}$, and they generate the p-reciprocal elements and their inverses in the same reciprocal class.

Let $\eta'_S \in \mathcal{P}$ i.e., $\eta'_S = \alpha^i \eta_S$. So,

$$(\eta'_S)^{-1} r \eta'_S = \eta_S^{-1} \alpha^{-i} r \alpha^i \eta_S = \eta_S^{-1} \alpha^{m-i} S^{-1} r S \alpha^{-m+i} \eta_S = \eta_S^{-1} \alpha^{m-i} r^{-1} \alpha^{-m+i} \eta_S$$
$$= (\eta_S^{-1} \alpha^{m-i} r \alpha^{-m+i} \eta_S)^{-1}.$$

That means, for the reciprocator S, there are p p-reciprocal elements. If we choose another p-reciprocal element distinct from the previous ones, i.e., $\beta r \beta^{-1}$ in the reciprocal class of r. Then, from the previous equation, (7.4), we obtain similar result i.e., $\beta \gamma_p^m \beta^{-1} \in C(r)$, i.e., $\beta \gamma_p^m \beta^{-1} = bS$ for some $b \in C(r)$ where $b = r^n$ for some $n \in \mathbb{Z}$. Similar to the previous case, we suppose that $\beta \gamma_p^m \beta^{-1} = S'$ and $n = 2k$. Then, by (7.6) and (7.7), S' satisfies the equation,

$$\eta^{-1} S' \eta = \gamma_p^m \tag{7.11}$$

for some $\eta \in \Gamma_p$. Let the solutions for each S' be $\eta_{S'}$ (β is one of the solutions for the above equation). Then,

$$\eta_{S'}^{-1} r^k S r^{-k} \eta_{S'} = \eta_S^{-1} S \eta_S \Rightarrow \eta_S \eta_{S'}^{-1} r^k \in C(S) = \langle \alpha \rangle.$$

So, for some $0 \leq i \leq p-1$,

$$\eta_S \eta_{S'}^{-1} r^k = \alpha^i. \tag{7.12}$$

Therefore,

$$r^k = \eta_{S'} \eta_S^{-1} \alpha^i \Rightarrow r^k \alpha^{p-i} \eta_S = \eta_{S'} \Rightarrow \eta_{S'}^{-1} r \eta_{S'} = (\alpha^{p-i}\eta_S)^{-1} r \alpha^{p-i} \eta_S.$$

That means we get the same set of p-reciprocal elements if $S' = r^{2k} S$.

It remains to show the odd n case. First, we claim that if $S' = r^{2k+1}S$ and S', S both are conjugate to the same involution, then, $\eta_S^{-1}r\eta_S$ and $\eta_{S'}^{-1}r\eta_{S'}$ are distinct for any $\eta_S, \eta_{S'}$ satisfying (7.3), (7.10) respectively.

For a contradiction, we suppose,

$$\eta_S^{-1}r\eta_S = \eta_{S'}^{-1}r\eta_{S'} \Rightarrow \eta_S\eta_{S'}^{-1}r\eta_{S'}\eta_S^{-1} = r.$$

Then, $\eta_{S'} = b\eta_S$ where $b \in C(r)$. Therefore,

$$\eta_S^{-1}S\eta_S = \eta_{S'}^{-1}S'\eta_{S'} \tag{7.13}$$

$$\Rightarrow \eta_{S'}\eta_S^{-1}S\eta_S\eta_{S'}^{-1} = S' \Rightarrow bSb^{-1} = S'.$$

If S is an involution then, r^tS is also an involution for any $t \in \mathbb{Z}$. That means,

$$Sb^{-1} = S'b \Rightarrow Sb^{-2} = S'.$$

This implies,

$$S' = b^2S, \text{ where, } b \in C(r). \tag{7.14}$$

This contradicts our assumption.

The same claim will not work for the case where the reciprocators S and S' are the solutions of (7.3) and (7.11), respectively. To show that suppose

$$\eta_S^{-1}r\eta_S = \eta_{S'}^{-1}r\eta_{S'}. \tag{7.15}$$

That implies,

$$\eta_S^{-1}S\eta_S\eta_S^{-1}S^{-1}rS\eta_S\eta_S^{-1}S\eta_S = \eta_{S'}^{-1}S'\eta_{S'}\eta_{S'}^{-1}S'^{-1}rS'\eta_{S'}\eta_{S'}^{-1}S'\eta_{S'}$$
$$\Rightarrow \iota\eta_S^{-1}r^{-1}\eta_S\iota = \gamma_p^m\eta_{S'}^{-1}r^{-1}\eta_{S'}\gamma_p^m$$
$$\Rightarrow \gamma_p^m\iota(\eta_S^{-1}r^{-1}\eta_S)\iota\gamma_p^m = (\eta_{S'}^{-1}r^{-1}\eta_{S'})$$
$$\Rightarrow \gamma_p^m\iota(\eta_S^{-1}r^{-1}\eta_S)\iota\gamma_p^m = (\eta_S^{-1}r^{-1}\eta_S)$$
$$\Rightarrow (\eta_S^{-1}r\eta_S) = (\iota\gamma_p^m)^k \text{ where, } k \in \mathbb{Z}.$$

Since r is a primitive element, k is either 1 or -1. So, the elements conjugate to any non-zero power of $\iota\gamma_p^m$ have exactly p *p-reciprocal* elements and exactly two *symmetric p-reciprocal* elements. If there is any other *symmetric p-reciprocal* element, it must be a power of r (since r is a primitive element.). This leads to a contradiction. If possible, let there be another η such that it satisfies either (7.3) or (7.11) with (7.15). Without loss of generality, we choose η satisfying (7.11) then,

$$\eta_S^{-1}\alpha^{-i}r\alpha^i\eta_S = \eta_S^{-1}r\eta_S.$$

This implies,

$$\alpha^{-i}r\alpha^i = r$$

which is a contradiction for any non-zero i. So there are only two elements η_S and $\eta_{S'}$ such that $\eta_S^{-1}r\eta_S = \eta_{S'}^{-1}r\eta_{S'}$.

Now fix a reciprocator S. We can divide the class of reciprocators in terms of $[S]$ and $[rS]$ as reciprocators of each class will generate the same set of symmetric elements or p-reciprocal elements. If S and rS are both conjugate to either ι

or γ_p^m, the proof of part (a) and (b) of the theorem follow. If any of S, rS conjugates to ι and the other one conjugates to γ_p^m and r is a primitive element. This implies the proof of (c). □

To parametrize the reciprocal classes in proof of Corollary 1.9, we need to recall the definition of $(.,.)_p$ in Section 2. Also, we require the following proposition.

Proposition 7.1. [LL] *Let* $\tilde{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *be a lift in* $\mathrm{SL}(2,\mathbb{R})$ *of an element* A *in* $\mathrm{PSL}(2,\mathbb{Z}[\lambda_{\mathrm{p}}])$. *Then* $A \in \Gamma_p$ *if and only if* $(a,c)_p = (b,d)_p = 1$.

For the proof, we refer the Proposition 3.7 from [LL].

7.2. Proof of the Corollary 1.9. Let A be a reciprocal element in Γ_p such that A is either symmetric or p-reciprocal. Let $\tilde{A} = \begin{bmatrix} u & v \\ y & w \end{bmatrix}$ be a lift of A in $\mathrm{SL}(2,\mathbb{R})$. Then, $u, v, y, w \in \mathbb{Z}[\lambda_p]$. Also, $uw - vy = 1$ and from Proposition 7.1, $(u,v)_p = (y,w)_p = 1$. Consider,

$$uw - vy = 1$$

$$\Rightarrow 4(uw - vy) = 4$$

$$\Rightarrow (u+w)^2 - (u-w)^2 - 4vy = 4.$$

Let, $u + w = t$ and $u - w = b$ and $d = 4vy + b^2$. That means

$$t^2 - d = 4.$$

Then, we get a 4-tuple (v, b, y, t) in $\mathcal{C}'$ such that $\begin{bmatrix} (t+b)/2 & v \\ y & (t-b)/2 \end{bmatrix} = \tilde{A}$.

If A is represented by some 4-tuple in $\mathcal{C}'$ then there does not exist a 4-tuple in $\mathcal{C}'$ that represents $\tilde{A}^{-1}$. To see this, let $(a,b,c,t) \in \mathcal{C}'$. Then there is a symmetric or, a p-reciprocal element B such that

$$\tilde{B} = \begin{bmatrix} (t-b)/2 & a \\ c & (t+b)/2 \end{bmatrix}.$$

So,

$$\tilde{B}^{-1} = \begin{bmatrix} (t+b)/2 & -a \\ -c & (t-b)/2 \end{bmatrix},$$

but $(-a, b, -c, t) \notin \mathcal{C}'$. Also, in $\mathrm{PSL}(2,\mathbb{R})$, $\tilde{B}$ and $-\tilde{B}$ are same, but $(a, b, c, -t) \notin \mathcal{C}'$. So, there are at most half of the symmetric, p-reciprocal or symmetric p-reciprocal elements that can be given by $\mathcal{C}'$. Now the assertion follows from Theorem 1.8. □

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