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Twisted analogue of the Kummer-Leopoldt constant

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Abstract. Let F be a number field and let p be an odd prime. Denote by S the set of p-adic and infinite places of F. We study a generalization to K-theory of the Kummer-Leopoldt constant for the S-units introduced in [7, Section 4]. We express in particular its value as the exponent of some Galois module. As an application, we give a new characterization of (p, i)-regular quadratic number fields.

Contents

1. Introduction

Let p be an odd prime and let F be a number field. The Kummer-Leopoldt constant [7, Definiton 1] $\varkappa(F)$ is the smallest integer c satisfying the following property: if n is sufficiently large and u is a unit of F that is a p^{n+c}-th power locally at all primes dividing p, then u is a global p^n-th power. This constant exists when the couple (F, p) satisfies Leopoldt's conjecture. Given this definition, Kummer's lemma states that if p is a regular prime number and F is the p-th cyclotomic field then $\varkappa(F)$ is zero. Kummer's lemma has been generalized by several authors to p^n-th cyclotomic fields, $n \geq 1$ [33], [32], or to totally real number fields [27]. In [33, 32, 27], the authors give an upper bound for the Kummer-Leopoldt constant in terms of special values of the associated p-adic L-function.

More generally, for an arbitrary number field F, the quantity $p^{\varkappa(F)}$ is the exponent of the Galois group $\mathrm{Gal}(\mathrm{F}^{\mathrm{BP}}/\widetilde{F}L_F)$ [7, Théorème 1], where F^{BP} is the

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Bertrandias-Payan field of F [9, 25], $\widetilde{F}$ is the composite of all $\mathbb{Z}_p$-extensions of F and L_F is the maximal abelian unramified p-extension of F.

The Bertrandias-Payan field F^{BP} is contained in $\widehat{F}$, the maximal abelian pro-p-extension of F which is unramified outside the p-adic primes. In particular, the Kummer-Leopoldt constant $\varkappa(F)$ is trivial if $\widehat{F} = \widetilde{F}$. Number fields with $\widehat{F} = \widetilde{F}$ and satisfying Leopoldt's conjecture are called p-rational fields [20]. Obviously, $\varkappa(F)$ is trivial if the field F is p-rational. This can be considered as a generalization of Kummer's lemma since the field $\mathbb{Q}(\mu_p)$ is p-rational precisely when p is regular, μ_p being the group of p-th roots of unity.

Let S be the set of p-adic and infinite places of F and let U be the group of S-units of F. In [7, Section 4], the authors define also a Kummer-Leopoldt constant for the S-units as the smallest integer c having the following property:

$$\forall n \gg 0, \forall u \in U, \, (u \in {F_v}^{p^{c+n}}, \forall v \mid p) \Longrightarrow u \in U^{p^n},$$

where for $v \mid p$, F_v is the completion of F at v.

Denote by $\widehat{U}$ and $\widehat{F_v}$, respectively, the pro-p-completion of U and F_v. Let $G_S(F)$ be the Galois group over F of the maximal algebraic extension which is unramified outside S. Then

$$\widehat{U} \cong H^1(G_S(F), \mathbb{Z}_p(1)) \text{ and } \widehat{F_v} \cong H^1(F_v, \mathbb{Z}_p(1)).$$

For an integer i, we have a natural localization map

$$\begin{array}{rcccc} \alpha^{(i)} = \bigoplus_{v|p} \alpha_v^{(i)} : & H^1(G_S(F), \mathbb{Z}_p(i)) & \longrightarrow & \bigoplus_{v|p} H^1(F_v, \mathbb{Z}_p(i)) \\ & x & \longmapsto & (\alpha_v^{(i)}(x))_v \end{array}$$

where, for each prime v above p, $\alpha_v^{(i)} : H^1(G_S(F), \mathbb{Z}_p(i)) \longrightarrow H^1(F_v, \mathbb{Z}_p(i))$ is the restriction homomorphism. For simplicity, if $x \in H^1(G_S(F), \mathbb{Z}_p(i))$, we keep the notation $x := \alpha_v^{(i)}(x) \in H^1(F_v, \mathbb{Z}_p(i))$. Then, we ask the following natural question: *Is there a positive integer c_i such that for all $n \gg 0, x \in H^1(G_S(F), \mathbb{Z}_p(i))$*

$$(x \in H^1(F_v, \mathbb{Z}_p(i))^{p^{c_i+n}}, \forall v \mid p) \Longrightarrow x \in H^1(G_S(F), \mathbb{Z}_p(i))^{p^n}?$$

In this article, we show that such an integer exists when the field F satisfies a twisted Leopoldt's conjecture (Conjecture 2.1), and we define the twisted analogue of the Kummer-Leopoldt constant $\varkappa_i(F)$ to be the smallest value of c_i satisfying this property. The study of the twisted Kummer-Leopoldt constant leads us to define some Galois extensions, in particular we construct a twisted analogue of the Bertrandias-Payan field and an étale analogue of the Hilbert class field (see §2). Using these definitions we express the twisted Kummer-Leopoldt constant as the exponent of a certain Galois group inside the twisted Bertrandias-Payan module (Theorem 3.8).

By the Quillen-Lichtenbaum conjecture, which is now a theorem thanks to the work of Voevodsky and Rost on the Bloch-Kato conjecture, the p-adic cohomology group $H^1(G_S(F), \mathbb{Z}_p(i))$ is isomorphic to the pro-p-completion of the

K-theory group $K_{2i-1}F$ [17, Theorem 5.6.8]. Hence for $i \geq 2$, the constant $\varkappa_i(F)$ can be considered as a generalization to K-theory of the Kummer-Lepoldt constant.

In the last section of this paper, we study the vanishing of the twisted Kummer-Leopolodt constant. We show, in particular, that $\varkappa_{1-i}(F) = 0$ if F is a (p,i)-regular number field in the sense of [2]. Furthermore, we give a new characterization of (p,i)-regular number fields in terms of the triviality of $\varkappa_{1-i}(F)$. More precisely, we prove the following theorem:

Theorem. *Let $i \neq 0, 1$ be an integer and let F be a number field satisfying the twisted Leopoldt's conjecture. Then F is (p,i)-regular if and only if the following three conditions hold:*

1. $\varkappa_{1-i}(F) = 0$*;*
2. *The natural injective map*

$$H^0(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \longrightarrow \bigoplus_{v|p} H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))$$

   is an isomorphism;
3. $H^{(i)} \subset \widetilde{F}^{(i)}$*, where the fields $\widetilde{F}^{(i)}$ and $H^{(i)}$ are defined in Definitions 2.5 and 2.9, respectively.*

As an application we get a characterization of (p,i)-regular quadratic number fields in the spirit of [12, §4.1], (see Propositions 4.6 and 4.7 below).
Notation: For a number field F, and an odd prime number p, we adopt the following notation throughout this paper:

O_F	the ring of integers of F;
μ_p	the group of p-th roots of the unity;
E	the composite of F and the p-th cyclotomic field i.e., $E = F(\mu_p)$;
S	the set of p-adic and infinite places;
U	the group of S-units in F;
$\widehat{U}$	the pro-p-completion of U;
F_v	the completion of F at a prime v of F;
U_v	the group of local units of F at a prime v of F;
$\widehat{F}_v$	the pro-p-completion of F_v;
F_∞	the cyclotomic $\mathbb{Z}_p$-extension of F;
Γ	the Galois group $\mathrm{Gal}(F_\infty/F)$;
F_n	the unique subfield of F_∞ such that $[F_n : F] = p^n$;
Γ_n	the Galois group $\mathrm{Gal}(F_\infty/F_n)$;
$\Lambda = \mathbb{Z}_p[[\Gamma]]$	the Iwasawa algebra associated to Γ;
E_∞	the cyclotomic $\mathbb{Z}_p$-extension of E;
G_∞	the Galois group $\mathrm{Gal}(E_\infty/F)$;
F_S	the maximal algebraic extension of F which is unramified outside S;
$\widehat{F}$	the maximal abelian pro-p-extension of F which is

	unramified outside S;
E_∞^{ab}	the maximal abelian pro-p-extension of E_∞ which is unramified outside S;
$L_\infty^{'}$	the maximal abelian unramified pro-p-extension of E_∞ which splits completely at p-adic primes of E_∞;
$X_\infty^{'}$	the Galois group $\mathrm{Gal}(L_\infty^{'}/E_\infty)$;
$G_S(K)$	the Galois group $\mathrm{Gal}(F_S/K)$, for an arbitrary field K inside F_S/F;
$M(i)$	the i-th Tate twist of a $G_S(F)$-module M $(i \in \mathbb{Z})$;
$M[p^n]$	the kernel of the multiplication by p^n;
M/p^n	the co-kernel of the multiplication by p^n;
$H^n(G_S(F), M)$	the n-th continuous cohomology group of $G_S(F)$ with coefficients in M;
$Ш^n(G_S(F), M)$	the localization kernel $\ker(H^n(G_S(F), M) \to \bigoplus_{v\in S} H^n(F_v, M))$;
$M^\vee$	$= \mathrm{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p)$, the Pontryagin dual of M;

For a group G and a commutative ring R, let I_G be the augmentation ideal of the group ring $R[G]$; it is the ideal generated by $\{\sigma - 1, \sigma \in G\}$. Unless otherwise stated, $R = \mathbb{Z}_p$.

2. On certain Galois extensions

Let F be a number field and let p be an odd prime number. We denote by F_S the maximal algebraic extension of F which is unramified outside the set S of p-adic and infinite places of F. For a subfield K of F_S containing F, we denote by $G_S(K)$ the Galois group $\mathrm{Gal}(F_S/K)$. The p-ramified Iwasawa module $\mathcal{X}_K$ is the Galois group over K of the maximal abelian pro-p-extension which is unramified outside S. In terms of homology groups, we have $\mathcal{X}_K \simeq H_1(G_S(K), \mathbb{Z}_p)$. Indeed, using the cohomology-homology duality, we have:

$$\begin{aligned} H_1(G_S(K), \mathbb{Z}_p) &\simeq H^1(G_S(K), \mathbb{Q}_p/\mathbb{Z}_p)^\vee \\ &\simeq \mathrm{Hom}(G_S(K), \mathbb{Q}_p/\mathbb{Z}_p)^\vee \\ &\simeq \mathcal{X}_K. \end{aligned}$$

For an integer i, denote by $\mathcal{X}_K^{(i)}$ the first homology group $H_1(G_S(K), \mathbb{Z}_p(-i))$ which can then be considered as a twisted analogue of the p-ramified Iwasawa module $\mathcal{X}_K$. The module $\mathcal{X}_K^{(i)}$ has been studied by several authors in the case where K is a multiple $\mathbb{Z}_p$-extension of F. For example, [14, 11] for $i = 0$ and [4] for $i \neq 0$. Returning to the case $K = F$, the $\mathbb{Z}_p$-rank of the p-ramified Iwasawa module $\mathcal{X}_F$ is conjecturally equal to $r_2 + 1$, where r_2 is the number of complex places of F (Leopoldt's conjecture). There are many equivalent formulations of this conjecture. In terms of cohomology, it is equivalent to the triviality of the second cohomology group $H^2(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p)$ (e.g., [24, Proposition 12]). More generally, we have the following conjecture (Greenberg [10], Schneider [28], ...)

Conjecture 2.1 ($C^{(i)}$)**.** *Let F be a number field. Then for every integer $i \neq 1$, the second cohomology group $H^2(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(i))$ is trivial.*

Conjecture $C^{(0)}$ is the Leopoldt's conjecture, it holds for all F that are abelian over $\mathbb{Q}$ or over an imaginary quadratic number field. If $i \geq 2$, Conjecture $C^{(i)}$ holds for any number field F, as a consequence of the finiteness of the K-theory groups $K_{2i-2}O_F$ [30]. By a well known result on Brauer groups [13] or [28, §4, Lemma 2], there is no Conjecture $C^{(1)}$.

In the next proposition we give two equivalent formulations of the Conjecture $C^{(i)}$ that we will use in the sequel. These formulations are well known, we add here a proof for the reader's convenience.

Proposition 2.2. *Let F be a number field and let $i \neq 1$ be an integer. The following assertions are equivalent:*

1) *Conjecture $C^{(i)}$ holds for F;*
2) *the p-adic cohomology group $H^2(G_S(F), \mathbb{Z}_p(i))$ is finite;*
3) *the Galois module $X'_\infty(i-1)_{G_\infty}$ is finite.*

Proof. For $k \geq 1$, the exact sequence

$$0 \longrightarrow \mathbb{Z}_p(i) \xrightarrow{p^k} \mathbb{Z}_p(i) \longrightarrow \mathbb{Z}/p^k(i) \longrightarrow 0$$

induces in cohomology the exact sequence

$$H^n(G_S(F), \mathbb{Z}_p(i))/p^k \hookrightarrow H^n(G_S(F), \mathbb{Z}/p^k(i)) \twoheadrightarrow H^{n+1}(G_S(F), \mathbb{Z}_p(i))[p^k]$$

Passing to the direct limit on k, we obtain the exact sequence [23, (4.3.4.1)]

$$0 \longrightarrow H^n(G_S(F), \mathbb{Z}_p(i)) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow H^n(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(i)) \longrightarrow \mathrm{tor}_{\mathbb{Z}_p} H^{n+1}(G_S(F), \mathbb{Z}_p(i)) \longrightarrow 0. \quad (1)$$

In fact, by [31, Proposition 2.3], $H^n(G_S(F), \mathbb{Z}_p(i)) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ is the maximal divisible subgroup of $H^n(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(i))$.

Since the cohomological dimension $\mathrm{cd}(G_S(F)) \leq 2$, we have an isomorphism

$$H^2(G_S(F), \mathbb{Z}_p(i)) \otimes \mathbb{Q}_p/\mathbb{Z}_p \simeq H^2(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(i)) \quad (2)$$

Since the $\mathbb{Z}_p$-module $H^2(G_S(F), \mathbb{Z}_p(i))$ is finitely generated (see [23, Proposition 4.2.3]), the equivalence between 1) and 2) follows from the isomorphism (2).

Observe that if $i \neq 1$, $Ш^2(G_S(F), \mathbb{Z}_p(i)) \simeq X'_\infty(i-1)_{G_\infty}$ [28, Section 6, Lemma 1] and by the local duality theorem, we have

$$H^2(F_v, \mathbb{Z}_p(i)) \cong H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^\vee.$$

In particular, the group $\bigoplus_{v\in S} H^2(F_v, \mathbb{Z}_p(i))$ is finite. Then, the equivalence 2) $\iff$ 3) follows from the exact sequence

$$0 \to Ш^2(G_S(F), \mathbb{Z}_p(i)) \to H^2(G_S(F), \mathbb{Z}_p(i)) \to \bigoplus_{v\in S} H^2(F_v, \mathbb{Z}_p(i)).$$

□

Remark 2.3. *For an integer i, we denote by $\mathcal{T}_F^{(i)}$ the $\mathbb{Z}_p$-torsion sub-module of $\mathcal{X}_F^{(i)}$. When the field F satisfies Conjecture* $\mathrm{C}^{(i)}$ *($i \neq 1$), the cohomology group $H^2(G_S(F), \mathbb{Z}_p(i))$ is finite. Hence the exact sequence (1) (for $n = 1$) induces by duality the following well known cohomological description of $\mathcal{T}_F^{(i)}$* [26, Lemme 4.1]

$$\mathcal{T}_F^{(i)} \simeq H^2(G_S(F), \mathbb{Z}_p(i))^\vee.$$

As in the case where $i = 0$, Conjecture $\mathrm{C}^{(i)}$ is related to the $\mathbb{Z}_p$-rank of the module $\mathcal{X}_F^{(i)}$. In [28, §4, Satz 6], the co-ranks of the groups $H^1(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(i))$ were computed. By duality,

$$\mathrm{rank}_{\mathbb{Z}_p} H_1(G_S(F), Z_p(-i)) = \mathrm{corank}_{\mathbb{Z}_p} H^1(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(i)).$$

It follows that if $i \neq 0, 1$, the field F satisfies $\mathrm{C}^{(i)}$ if and only if

$$\mathrm{rank}_{\mathbb{Z}_p} \mathcal{X}_F^{(i)} = \begin{cases} r_2 + r_1 & \text{if } i \text{ is odd;} \\ r_2 & \text{if } i \text{ is even,} \end{cases} \tag{3}$$

here, as usual, r_1 (resp. r_2) is the number of real (resp. complex) places. In the sequel we will frequently use the following well known lemma:

Lemma 2.4 (Tate's lemma)**.** *Let F be a number field and let i be a non-zero integer. Then the Galois cohomology groups $H^k(G, \mathbb{Q}_p/\mathbb{Z}_p(i))$ vanish for all $k \geq 1$, where G is either $G_\infty = \mathrm{Gal}(E_\infty/F)$ or $G_{\infty,v} = \mathrm{Gal}(E_{\infty,v}/F_v)$, v being a finite prime of F.*

As a consequence of Tate's lemma, we get that

$$H^1(\Gamma, H^0(G_S(F_\infty), \mathbb{Q}_p/\mathbb{Z}_p(i))) = 0,$$

where Γ is the Galois group $\mathrm{Gal}(F_\infty/F)$. Indeed, let Δ be the Galois group $\mathrm{Gal}(E_\infty/F_\infty)$. We have

$$\begin{aligned} H^0(G_S(F_\infty), \mathbb{Q}_p/\mathbb{Z}_p(i)) &= H^0(\Delta, H^0(G_S(E_\infty), \mathbb{Q}_p/\mathbb{Z}_p(i))) \\ &= H^0(\Delta, \mathbb{Q}_p/\mathbb{Z}_p(i)). \end{aligned} \tag{4}$$

Since $\mathrm{cd}(\Gamma) \leq 1$, the Hochschild-Serre spectral sequence associated to the group extension

$$0 \to \Delta \to G_\infty \to \Gamma \to 0$$

yields the following exact sequence

$$0 \to H^1(\Gamma, H^0(\Delta, \mathbb{Q}_p/\mathbb{Z}_p(i))) \to H^1(G_\infty, \mathbb{Q}_p/\mathbb{Z}_p(i)) \to H^1(\Delta, \mathbb{Q}_p/\mathbb{Z}_p(i))^\Gamma \to 0.$$

By Tate's Lemma, we get

$$H^1(\Gamma, H^0(\Delta, \mathbb{Q}_p/\mathbb{Z}_p(i))) = 0.$$

From the equality (4), it follows that

$$H^1(\Gamma, H^0(G_S(F_\infty), \mathbb{Q}_p/\mathbb{Z}_p(i))) = 0$$

as required.

Recall that the $\mathbb{Z}_p$-module $\mathcal{X}_F^{(0)}$ is isomorphic to $\mathcal{X}_F = \mathrm{Gal}(\widehat{F}/F)$, where $\widehat{F}$ is the maximal abelian pro-p-extension of F which is unramified outside S. When the integer i is non-zero, the $\mathbb{Z}_p$-module $\mathcal{X}_F^{(i)}$ can also be realized as a Galois group. Indeed, using Tate's lemma we get that $H^1(G_\infty, \mathbb{Q}_p/\mathbb{Z}_p(i)) = H^2(G_\infty, \mathbb{Q}_p/\mathbb{Z}_p(i)) = 0$, since $i \neq 0$. Therefore, the restriction map

$$H^1(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(i)) \longrightarrow H^1(G_S(E_\infty), \mathbb{Q}_p/\mathbb{Z}_p(i))^{G_\infty} \tag{5}$$

is an isomorphism. Notice that the Galois group $G_S(E_\infty)$ acts trivially on $\mathbb{Q}_p/\mathbb{Z}_p(i)$, so we have

$$\begin{aligned} H^1(G_S(E_\infty), \mathbb{Q}_p/\mathbb{Z}_p(i))^{G_\infty} &= H^1(G_S(E_\infty), \mathbb{Q}_p/\mathbb{Z}_p)(i)^{G_\infty} \\ &\simeq \mathrm{Hom}(G_S(E_\infty), \mathbb{Q}_p/\mathbb{Z}_p)(i)^{G_\infty}. \end{aligned}$$

Then, by duality, the isomorphism (5) induces the following isomorphism:

$$\mathcal{X}_F^{(i)} \simeq \mathcal{X}_\infty(-i)_{G_\infty}, \tag{6}$$

where $\mathcal{X}_\infty = H_1(G_S(E_\infty), \mathbb{Z}_p)$ is the Galois group over E_∞ of E_∞^{ab}, the maximal abelian pro-p-extension which is unramified outside S.

Definition 2.5. *Let $i \neq 0$ be an integer. We define the field $\widehat{F}^{(i)}$ to be the subfield of E_∞^{ab} fixed by $I_{G_\infty}(\mathcal{X}_\infty(-i))$; hence*

$$\mathrm{Gal}(\widehat{F}^{(i)}/E_\infty) = \mathcal{X}_\infty(-i)_{G_\infty} \simeq \mathcal{X}_F^{(i)}.$$

When $i = 0$, we define $\widehat{F}^{(0)}$ as the composite of the fields E_∞ and $\widehat{F}$ i.e, $\widehat{F}^{(0)} = E_\infty\widehat{F}$.

For every integer i, we denote by $\widetilde{F}^{(i)}$ the subfield of $\widehat{F}^{(i)}$ fixed by the $\mathbb{Z}_p$-torsion sub-module $\mathcal{T}_F^{(i)}$ of $\mathcal{X}_F^{(i)}$; hence

$$\mathcal{T}_F^{(i)} \simeq \mathrm{Gal}(\widehat{F}^{(i)}/\widetilde{F}^{(i)}).$$

Remark 2.6. *In the case $i = 0$, we don't have the isomorphism (6) but we do have the following exact sequence:*

$$0 \longrightarrow (\mathcal{X}_\infty)_{G_\infty} \longrightarrow \mathcal{X}_F \longrightarrow \Gamma \longrightarrow 0.$$

It follows that the field $\widehat{F}^{(0)}$ is the maximal subfield of E_∞^{ab}, which is abelian over F.

Let X'_∞ be the Galois group $\mathrm{Gal}(L'_\infty/E_\infty)$, where L'_∞ is the maximal abelian unramified pro-p-extension of E_∞ which splits at p-adic primes of E_∞. We have a natural surjective map

$$\mathcal{X}_\infty(-i)_{G_\infty} \longrightarrow X'_\infty(-i)_{G_\infty}.$$

For $i \neq 0$, it is well known that $X'_\infty(-i)_{G_\infty}$ is isomorphic to the localization kernel

$$Ш^2(G_S(F), \mathbb{Z}_p(1-i)) := \ker(H^2(G_S(F), \mathbb{Z}_p(1-i)) \longrightarrow \bigoplus_{v|p} H^2(F_v, \mathbb{Z}_p(1-i))),$$

[28, Section 6, Lemma 1]. For $i \geq 2$, the group $Ш^2(G_S(F), \mathbb{Z}_p(i))$ is called the étale wild kernel and does not depend on S containing the p-adic places.

In the following proposition, we give an exact sequence which expresses the link between the $\mathbb{Z}_p$-torsion module $\mathcal{T}_F^{(i)}$ and the Pontryagin dual of $X'_\infty(i-1)_{G_\infty}$. Let $W^{(1-i)}$ be the co-kernel of the injective localization morphism

$$H^0(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \to \bigoplus_{v|p} H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i)),$$

so that $W^{(1-i)} \cong \left(\bigoplus_{v|p} H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))\right) / H^0(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(1-i))$.

Proposition 2.7. *Let F be a number field and let $i \neq 1$ be an integer such that F satisfies Conjecture $\mathrm{C}^{(i)}$. Then we have the following exact sequence:*

$$0 \longrightarrow W^{(1-i)} \longrightarrow \mathcal{T}_F^{(i)} \longrightarrow \mathrm{Hom}(X'_\infty(i-1)_{G_\infty}, \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow 0. \tag{7}$$

Proof. We start by recalling the first part of the Poitou-Tate exact sequence:

$$0 \to H^0(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \longrightarrow \bigoplus_{v|p} H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \longrightarrow H^2(G_S(F), \mathbb{Z}_p(i))^\vee \longrightarrow Ш^1(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \to 0.$$

Clearly, for $i \neq 1$, we have

$$Ш^1(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \cong \mathrm{Hom}(X'_\infty(i-1)_{G_\infty}, \mathbb{Q}_p/\mathbb{Z}_p).$$

Furthermore, if the field F satisfies Conjecture $\mathrm{C}^{(i)}$, Remark 2.3 gives an isomorphism

$$\mathcal{T}_F^{(i)} \simeq H^2(G_S(F), \mathbb{Z}_p(i))^\vee$$

Summarizing, we can rewrite the Poitou-Tate exact sequence as follows:

$$0 \to W^{(1-i)} \to \mathcal{T}_F^{(i)} \to \mathrm{Hom}(X'_\infty(i-1)_{G_\infty}, \mathbb{Q}_p/\mathbb{Z}_p) \to 0.$$

□

Note that the group $X'_\infty(-i)_{G_\infty}$ is a quotient of the Galois group X'_∞, thus it could be realized as a Galois group of an abelian and totally decomposed extension of E_∞ (this extension is denoted by $\mathcal{L}_\infty$ in [3, Section 2, page 653]).

Definition 2.8. *The field $L^{(i)}$ is the subfield of L'_∞ fixed by $I_{G_\infty}(X'_\infty(-i))$, hence*

$$\mathrm{Gal}(L^{(i)}/E_\infty) = X'_\infty(-i)_{G_\infty}.$$

In [3, Proposition 1], it is noticed that the extension $L^{(i)}$ is not in general abelian over F so we can not use the descent process to realize the group $X'_\infty(-i)_{G_\infty}$ as a Galois group over F. Using the same methods of Jaulent and Soriano [15, Section 3, page 3], one constructs a field $H^{(i)}$ (this field is denoted by $\widetilde{F}$ in [3, Section 2, page 653]) which is a Galois extension over F and the group $X'_\infty(-i)_{G_\infty}$ is isomorphic to the Galois group $\mathrm{Gal}(H^{(i)}/E_{n_0})$, where $E_{n_0} = H^{(i)} \cap E_\infty$ [3, Proposition 2]. Mention that in [3, page 653] the author assumes that $\mu_p \subseteq F$ but the generalization is easy. Let us recall the precise definition of the field $H^{(i)}$.

Definition 2.9. *The field $H^{(i)}$ is the composite of the fields F_γ, where F_γ is the subfield of $L^{(i)}$ fixed by a lifting of a topological generator γ of Γ.*

Remark 2.10. *Since the Galois groups $\mathrm{Gal}(L^{(i)}/E_\infty)$ and $\mathrm{Gal}(H^{(i)}/E_{n_0})$ are isomorphic, and $E_{n_0} = H^{(i)} \cap E_\infty$, we have $L^{(i)} = E_\infty H^{(i)}$.*

Let K/F be a cyclic p-extension of F. Following [9], we say that K is an infinitely embeddable extension of F if it is embeddable in a cyclic p-extension of F of arbitrary large degree. By class field theory, a p-extension K/F is infinitely embeddable if and only if for any place v of F, the local extension K_v/F_v is embeddable in a $\mathbb{Z}_p$-extension of F_v. We denote by F^{BP} the composite of all infinitely embeddable extensions of F. Obviously the field F^{BP} contains the composite $\widetilde{F}$ of all $\mathbb{Z}_p$-extensions of F. We set $\mathrm{T}_F := \mathrm{Gal}(\mathrm{F}^{\mathrm{BP}}/\widetilde{F})$ to be the Bertrandias-Payan module of F i.e., the $\mathbb{Z}_p$-torsion sub-module of $\mathrm{Gal}(\mathrm{F}^{\mathrm{BP}}/F)$. Let $\widehat{F}$ be the maximal abelian pro-p-extension of F which is unramified outside S. In view of [25, Theorem 4.2], we can see that F^{BP} is the subfield of $\widehat{F}$ fixed by the image of $W^{(1)} = \bigoplus_{v|p} \mu_p(F_v)/\mu_p(F)$ in $\mathcal{T}_F$, the $\mathbb{Z}_p$-torsion sub-module of $\mathcal{X}_F := \mathrm{Gal}(\widehat{F}/F)$. In a natural way, we define a twisted analogue of the Bertrandias-Payan field as follows:

Definition 2.11. *Let $i \neq 1$ be an integer such that F satisfies Conjecture $\mathrm{C}^{(i)}$. The twisted Bertrandias-Payan field $\mathrm{F}^{\mathrm{BP},(i)}$ is defined as the subfield of $\widehat{F}^{(i)}$ fixed by the image of $W^{(1-i)}$ in $\mathcal{T}_F^{(i)}$ in the exact sequence (7).*

Let $\mathrm{T}_F^{(i)}$ be the $\mathbb{Z}_p$-torsion of $\mathrm{Gal}(\mathrm{F}^{\mathrm{BP},(i)}/E_\infty)$. Assume that F satisfies Conjecture $\mathrm{C}^{(i)}$. By the definition of $\mathrm{F}^{\mathrm{BP},(i)}$ and the exact sequence (7), we have the following isomorphism:

$$\mathrm{T}_F^{(i)} \simeq \mathrm{Hom}(X'_\infty(i-1)_{G_\infty}, \mathbb{Q}_p/\mathbb{Z}_p).$$

In particular, if $i = 0$ we obtain the following isomorphism:

$$\mathrm{T}_F \simeq \mathrm{Hom}(X'_\infty(-1)_{G_\infty}, \mathbb{Q}_p/\mathbb{Z}_p),$$

(c.f. [25, Theorem 4.2]). Hence $\mathrm{T}_F^{(i)}$ is a twisted analogue of the Bertrandias-Payan module. In this context, we have the twist analogue of the exact sequence in [25, Theorem 4.2].

Corollary 2.12. *Let F be a number field and let $i \neq 1$ be an integer such that F satisfies Conjecture $\mathrm{C}^{(i)}$. Then, we have the following exact sequence:*

$$0 \to W^{(1-i)} \to \mathcal{T}_F^{(i)} \to \mathrm{T}_F^{(i)} \to 0. \tag{8}$$

Proposition 2.13. *For every integer $i \neq 0, 1$ such that F satisfies Conjecture $\mathrm{C}^{(i)}$, the twisted Bertrandias-Payan field $\mathrm{F}^{\mathrm{BP},(i)}$ contains the field $L^{(i)}$.*

Proof. Since $i \neq 0$, we have $Ш^2(G_S(F), \mathbb{Z}_p(1-i)) \simeq X'_\infty(-i)_{G_\infty}$. Thus, the Poitou-Tate exact sequence [19, page 682]

$$H^1(G_S(F), \mathbb{Z}_p(1-i)) \xrightarrow{\alpha^{(1-i)}} \bigoplus_{v|p} H^1(F_v, \mathbb{Z}_p(1-i)) \to \mathcal{X}_F^{(i)} \to Ш^2(G_S(F), \mathbb{Z}_p(1-i)) \to 0$$

induces a surjective homomorphism:

$$\mathcal{X}_F^{(i)} \twoheadrightarrow X'_\infty(-i)_{G_\infty}.$$

Its kernel $Y^{(i)} := \mathrm{Gal}(\widehat{F}^{(i)}/L^{(i)})$ is isomorphic to the co-kernel of the localization map

$$H^1(G_S(F), \mathbb{Z}_p(1-i)) \longrightarrow \bigoplus_{v|p} H^1(F_v, \mathbb{Z}_p(1-i)).$$

This map is injective since Conjecture $\mathrm{C}^{(i)}$ holds. Thus we have an exact sequence:

$$0 \to H^1(G_S(F), \mathbb{Z}_p(1-i)) \to \bigoplus_{v|p} H^1(F_v, \mathbb{Z}_p(1-i)) \to Y^{(i)} \to 0.$$

Taking the restriction to the $\mathbb{Z}_p$-torsion sub-modules, we obtain the following exact sequence:

$$\mathrm{tor}_{\mathbb{Z}_p} H^1(G_S(F), \mathbb{Z}_p(1-i)) \hookrightarrow \bigoplus_{v|p} \mathrm{tor}_{\mathbb{Z}_p} H^1(F_v, \mathbb{Z}_p(1-i)) \to \mathrm{tor}_{\mathbb{Z}_p} Y^{(i)}. \tag{9}$$

Moreover, we have the following well known isomorphisms

$$H^0(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \simeq \mathrm{tor}_{\mathbb{Z}_p} H^1(G_S(F), \mathbb{Z}_p(1-i))$$

and

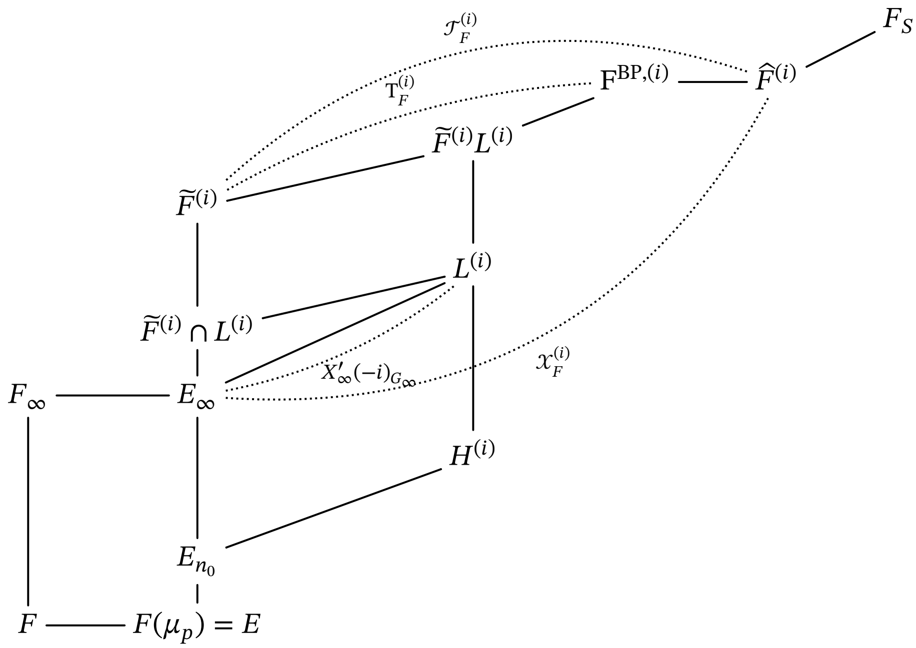
$$H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \simeq \mathrm{tor}_{\mathbb{Z}_p} H^1(F_v, \mathbb{Z}_p(1-i))$$

[31, Proposition 2.3]. The exact sequence (9) becomes

$$0 \to H^0(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \to \bigoplus_{v|p} H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \to \mathrm{tor}_{\mathbb{Z}_p} Y^{(i)}.$$

Then we obtain that the image of $W^{(1-i)}$ in $\mathcal{X}_F^{(i)}$ is contained in the $\mathbb{Z}_p$-torsion of the kernel $Y^{(i)} := \mathrm{Gal}(\widehat{F}^{(i)}/L^{(i)})$. This means that the field $L^{(i)}$ is contained in $\mathrm{F}^{\mathrm{BP},(i)}$. □

The following figure is an illustration of the situation in which we work:



Now, let K/F be a Galois p-extension of number fields, with Galois group G. If the extension K/F is unramified outside S, there exists a natural restriction map

$$f_i \,:\, H^2(G_S(F), \mathbb{Z}_p(i)) \longrightarrow H^2(G_S(K), \mathbb{Z}_p(i))^G.$$

We denote by $\hat{H}^{\cdot}(G, \cdot)$ the modified Tate cohomology groups (see [29]). If $i \neq 0, 1$, the kernel and co-kernel of the map f_i are given by

$$\ker(f_i) \cong H^1(G, H^1(G_S(K), \mathbb{Z}_p(i))) \cong \hat{H}^{-1}(G, H^2(G_S(K), \mathbb{Z}_p(i)))$$

and

$$\mathrm{coker}(f_i) \cong H^2(G, H^1(G_S(K), \mathbb{Z}_p(i))) \cong \hat{H}^0(G, H^2(G_S(K), \mathbb{Z}_p(i)))$$

[1, Proposition 3.1, page 41], [18, Theorem 1.2] and [16, Proposition 2.9] (the proof for $i \neq 0, 1$ is the same as for $i \geq 2$). If K satisfies Conjecture $\mathrm{C}^{(i)}$, the group $H^2(G_S(K), \mathbb{Z}_p(i))$ is finite and the above descriptions of the kernel and co-kernel of the map f_i show that, if G is cyclic, $\ker(f_i)$ and $\mathrm{coker}(f_i)$ have the same order.

Similarly for a prime v of F dividing p and a prime w of K above v, we have a restriction map [1, Chapter 3]

$$f_{i,v} \,:\, H^2(F_v, \mathbb{Z}_p(i)) \longrightarrow H^2(K_w, \mathbb{Z}_p(i))^{G_w}$$

where $G_w = \mathrm{Gal}(K_w/F_v)$ is the decomposition group of w in the extension K/F. Then exactly as in the global case, we have [1, Proposition 3.1, page 41]

$$\ker(f_{i,v}) \cong H^1(G_w, H^1(K_w, \mathbb{Z}_p(i))) \cong \hat{H}^{-1}(G_w, H^2(K_w, \mathbb{Z}_p(i)))$$

and

$$\mathrm{coker}(f_{i,v}) \cong H^2(G_w, H^1(K_w, \mathbb{Z}_p(i))) \cong \hat{H}^0(G_w, H^2(K_w, \mathbb{Z}_p(i))).$$

Consider the commutative diagram

$$\begin{array}{ccccc} H^2(G_S(K), \mathbb{Z}_p(i))^G & \longrightarrow & [\bigoplus_{v\in S, w\mid v} H^2(K_w, \mathbb{Z}_p(i))]^G & \cong & \bigoplus_{v\in S} H^2(K_w, \mathbb{Z}_p(i))^{G_w} \\ \uparrow{\scriptstyle f_i} & & & & \uparrow{\scriptstyle \bigoplus_{v\in S} f_{i,v}} \\ H^2(G_S(F), \mathbb{Z}_p(i)) & & \longrightarrow & & \bigoplus_{v\in S} H^2(F_v, \mathbb{Z}_p(i)) \end{array}$$

where for each $v \in S$, the isomorphism $[\bigoplus_{w\mid v} H^2(K_w, \mathbb{Z}_p(i))]^G \cong H^2(K_w, \mathbb{Z}_p(i))^{G_w}$ is a consequence of Shapiro's lemma, w being a prime of K above v. It follows that there exists a restriction map

$$j_i(K/F) : Ш^2(G_S(F), \mathbb{Z}_p(i)) \longrightarrow Ш^2(G_S(K), \mathbb{Z}_p(i))^G.$$

We are interested in the dual map

$$j_i^*(K/F) : (\mathrm{T}_K^{(i)})_G \longrightarrow \mathrm{T}_F^{(i)}$$

when K is contained in the cyclotomic $\mathbb{Z}_p$-extension F_∞ of F.

We need some additional notation. For all positive integer n, we denote by F_n the unique sub-extension of F_∞ such that $G_n := \mathrm{Gal}(F_n/F) \simeq \mathbb{Z}/p^n\mathbb{Z}$ and by $\mathrm{T}_n^{(i)} := \mathrm{T}_{F_n}^{(i)}$ the twisted Bertrandias-Payan module of F_n. We define the twisted Bertrandias-Payan module of F_∞ as the projective limit of $\mathrm{T}_n^{(i)}$ i.e., $\mathrm{T}_\infty^{(i)} := \varprojlim \mathrm{T}_n^{(i)}$, where the projective limit is taken via the natural maps $j_{i,n}^* := j_i^*(F_n/F) : (\mathrm{T}_n^{(i)})_{G_n} \to \mathrm{T}_m^{(i)}$ $(n \geq m)$. Let Γ be the Galois group $\mathrm{Gal}(F_\infty/F)$. Then we have a well-defined homomorphism

$$j_{i,\infty}^* : (\mathrm{T}_\infty^{(i)})_\Gamma \longrightarrow \mathrm{T}_F^{(i)}.$$

In the next lemma we show that $j_{i,\infty}^*$ is injective, or equivalently that the restriction map

$$j_{i,\infty} : Ш^2(G_S(F), \mathbb{Z}_p(i)) \longrightarrow (\varinjlim Ш^2(G_S(F_n), \mathbb{Z}_p(i)))^\Gamma$$

induced by the maps $j_i(F_n/F)$ is surjective provided that Conjecture $\mathrm{C}^{(i)}$ holds. More precisely,

Lemma 2.14. *Suppose that for every $n \geq 0$, the field F_n satisfies Conjecture* $\mathrm{C}^{(i)}$*,* $i \neq 0, 1$*. Then, we have a commutative diagram with exact lines*

$$\begin{array}{ccccc} \ker(j_{i,\infty}) & \hookrightarrow & Ш^2(G_S(F), \mathbb{Z}_p(i)) & \overset{j_{i,\infty}}{\twoheadrightarrow} & (\varinjlim Ш^2(G_S(F_n), \mathbb{Z}_p(i)))^{\Gamma} \\ \downarrow & & \downarrow g & & \downarrow \\ H^1(\Gamma, \varinjlim H^1(G_S(F_n), \mathbb{Z}_p(i))) & \hookrightarrow & H^2(G_S(F), \mathbb{Z}_p(i)) & \overset{f_{i,\infty}}{\twoheadrightarrow} & (\varinjlim H^2(G_S(F_n), \mathbb{Z}_p(i)))^{\Gamma} \end{array}$$

where $f_{i,\infty}$ is induced by the restriction maps

$$f_{i,n} : H^2(G_S(F), \mathbb{Z}_p(i)) \longrightarrow H^2(G_S(F_n), \mathbb{Z}_p(i))^{G_n}.$$

Proof. Let v be a p-adic prime of F and let $n \geq 0$. For commodity of notation, we denote also by v a prime of F_n above v and by $G_{n,v} = \mathrm{Gal}(F_{n,v}/F_v)$ its decomposition group in the extension F_n/F. Let us first show that the restriction homomorphism

$$f_i(F_{n,v}/F_v) : H^2(F_v, \mathbb{Z}_p(i)) \longrightarrow H^2(F_{n,v}, \mathbb{Z}_p(i))^{G_{n,v}}$$

is injective. The local duality theorem gives an isomorphim

$$H^2(F_{n,v}, \mathbb{Z}_p(i)) \cong H^0(F_{n,v}, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^{\vee} \cong H^0(F_{n,v}, \mathbb{Q}_p/\mathbb{Z}_p(i-1)).$$

Using Tate's lemma and the Hochschild-Serre spectral sequence associated to the extension groups

$$\mathrm{Gal}(E_{\infty,v}/F_{n,v}) \hookrightarrow G_{\infty,v} \twoheadrightarrow G_{n,v},$$

we see that the first cohomology group $H^1(G_{n,v}, H^0(F_{n,v}, \mathbb{Q}_p/\mathbb{Z}_p(i-1))) = 0$. Since $G_{n,v}$ is a cyclic group, it follows that

$$\hat{H}^{-1}(G_{n,v}, H^0(F_{n,v}, \mathbb{Q}_p/\mathbb{Z}_p(i-1))) = 0.$$

Summarizing, we obtain

$$\begin{aligned} \ker(\bigoplus_{v\in S} f_i(F_{n,v}/F_v)) &:= \bigoplus_{v\in S} \hat{H}^{-1}(G_{n,v}, H^2(F_{n,v}, \mathbb{Z}_p(i))) \\ &\simeq \bigoplus_{v\in S} \hat{H}^{-1}(G_{n,v}, H^0(F_{n,v}, \mathbb{Q}_p/\mathbb{Z}_p(i-1))) \\ &= 0. \end{aligned}$$

Now, the exact sequence

$$0 \to Ш^2(G_S(F_n), \mathbb{Z}_p(i)) \longrightarrow H^2(G_S(F_n), \mathbb{Z}_p(i)) \longrightarrow \bigoplus_{v\in S} H^2(F_{n,v}, \mathbb{Z}_p(i)) \longrightarrow H^0(G_S(F_n), \mathbb{Q}_p/\mathbb{Z}_p(1-i))^{\vee} \to 0$$

leads to the following commutative diagram:

$$\begin{array}{ccccc} Ш^2(G_S(F_n),\mathbb{Z}_p(i))^{G_n} & \hookrightarrow & H^2(G_S(F_n),\mathbb{Z}_p(i))^{G_n} & \to & \widetilde{\bigoplus_{v\in S}} H^2(F_{n,v},\mathbb{Z}_p(i))^{G_n} \\ \uparrow{\scriptstyle j_{i,n}} & & \uparrow{\scriptstyle f_{i,n}} & & \uparrow{\scriptstyle \widetilde{\bigoplus_{v\in S}} f_i(F_{n,v}/F_v)} \\ Ш^2(G_S(F),\mathbb{Z}_p(i)) & \hookrightarrow & H^2(G_S(F),\mathbb{Z}_p(i)) & \twoheadrightarrow & \widetilde{\bigoplus_{v\in S}} H^2(F_v,\mathbb{Z}_p(i)), \end{array} \tag{10}$$

where

$$\widetilde{\bigoplus_{v\in S}} H^2(F_v,\mathbb{Z}_p(i)) := \ker(\bigoplus_{v\in S} H^2(F_v,\mathbb{Z}_p(i)) \longrightarrow H^0(G_S(F),\mathbb{Q}_p/\mathbb{Z}_p(1-i))^\vee).$$

The map $\widetilde{\bigoplus_{v\in S}} f_i(F_{n,v}/F_v)$ is injective as the restriction of the map $\bigoplus_{v\in S} f_i(F_{n,v}/F_v)$ to $\widetilde{\bigoplus_{v\in S}} H^2(F_v,\mathbb{Z}_p(i))$. Since the fields F_n, $n\geq 0$, satisfy Conjecture $\mathrm{C}^{(i)}$, the group

$$\varinjlim \operatorname{coker}(f_{i,n}) = \varinjlim H^2(G_n, H^1(G_S(F_n),\mathbb{Z}_p(i)))$$

is trivial (the proof is exactly the same as [18, Proposition 3.2]). Taking the inductive limit in (10), we then obtain the following commutative diagram with exact lines and columns

$$\begin{array}{ccccc} \varinjlim Ш^2(G_S(F_n),\mathbb{Z}_p(i))^{G_n} & \hookrightarrow & \varinjlim H^2(G_S(F_n),\mathbb{Z}_p(i))^{G_n} & \to & \varinjlim \widetilde{\bigoplus_{v\in S}} H^2(F_{n,v},\mathbb{Z}_p(i))^{G_n} \\ \uparrow{\scriptstyle j_{i,\infty}} & & \uparrow{\scriptstyle f_{i,\infty}} & & \uparrow{\scriptstyle \widetilde{\bigoplus_{v\in S}} f_i(F_{\infty,v}/F_v)} \\ Ш^2(G_S(F),\mathbb{Z}_p(i)) & \hookrightarrow & H^2(G_S(F),\mathbb{Z}_p(i)) & \twoheadrightarrow & \widetilde{\bigoplus_{v\in S}} H^2(F_v,\mathbb{Z}_p(i)) \\ \uparrow & & \uparrow & & \\ \ker(j_{i,\infty}) & \hookrightarrow & H^1(\Gamma, \varinjlim H^1(G_S(F_n),\mathbb{Z}_p(i))) & & \end{array}$$

which shows that the map $Ш^2(G_S(F),\mathbb{Z}_p(i)) \longrightarrow \varinjlim Ш^2(G_S(F_n),\mathbb{Z}_p(i))^{G_n}$ is surjective. Therefore, we get the commutative diagram of the lemma. □

Theorem 2.15. *Let F be a number field and let $i\neq 0,1$ be an integer such that Conjecture $\mathrm{C}^{(i)}$ holds for all the fields F_n, $n\geq 0$. Then the homomorphism*

$$j^*_{i,\infty} : (\mathrm{T}^{(i)}_\infty)_\Gamma \longrightarrow \mathrm{T}^{(i)}_F$$

is injective. If we assume further that F is totally real and i is even, we get an isomorphism

$$(\mathrm{T}^{(i)}_\infty)_\Gamma \simeq \mathrm{T}^{(i)}_F.$$

Proof. The first claim follows from the Pontryagin dual of the top exact sequence in the commutative diagram of Lemma 2.14 and the isomorphisms

$$\mathrm{T}^{(i)}_F \simeq Ш^2(G_S(F),\mathbb{Z}_p(i))^\vee,$$

$$\mathrm{T}^{(i)}_\infty \simeq \varprojlim (Ш^2(G_S(F_n),\mathbb{Z}_p(i))^\vee) \simeq \operatorname{Hom}(\varinjlim Ш^2(G_S(F_n),\mathbb{Z}_p(i)),\mathbb{Q}_p/\mathbb{Z}_p).$$

Suppose now that F is totally real and i is even. Observe that, for every $n \geq 1$, F_n is also totally real. Using the exact sequence (1), we obtain

$$\begin{aligned}\mathrm{rank}_{\mathbb{Z}_p} H^1(G_S(F_n), \mathbb{Z}_p(i)) &= \text{co-rank} H^1(G_S(F_n), \mathbb{Q}_p/\mathbb{Z}_p(i)) \\ &= \mathrm{rank}_{\mathbb{Z}_p} H_1(G_S(F_n), \mathbb{Z}_p(-i)).\end{aligned}$$

Thus, the formula (3) shows that for $n \geq 1$, the group $H^1(G_S(F_n), \mathbb{Z}_p(i))$ is a $\mathbb{Z}_p$-torsion module. Note that for all $n \geq 1$, $H^0(G_S(F_n), \mathbb{Z}_p(i)) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ is trivial. From the exact sequence (1), it follows that the connecting homomorphism is an isomorphism

$$H^0(G_S(F_n), \mathbb{Q}_p/\mathbb{Z}_p(i)) \simeq H^1(G_S(F_n), \mathbb{Z}_p(i)).$$

Hence we have a commutative diagram

$$\begin{array}{ccc} H^0(G_S(F_n), \mathbb{Q}_p/\mathbb{Z}_p(i)) & \xrightarrow{\sim} & H^1(G_S(F_n), \mathbb{Z}_p(i)) \\ \uparrow & & \uparrow \\ H^0(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(i)) & \xrightarrow{\sim} & H^1(G_S(F), \mathbb{Z}_p(i)) \end{array}$$

where the vertical maps are the restriction maps. Taking the inductive limit, we get

$$\begin{aligned}\varinjlim H^1(G_S(F_n), \mathbb{Z}_p(i)) &\simeq \varinjlim H^0(G_S(F_n), \mathbb{Q}_p/\mathbb{Z}_p(i)) \\ &\simeq H^0(G_S(F_\infty), \mathbb{Q}_p/\mathbb{Z}_p(i)).\end{aligned}$$

Therefore,

$$H^1(\Gamma, \varinjlim H^1(G_S(F_n), \mathbb{Z}_p(i))) \simeq H^1(\Gamma, H^0(G_S(F_\infty), \mathbb{Q}_p/\mathbb{Z}_p(i))).$$

As explained after Lemma 2.4, $H^1(\Gamma, H^0(G_S(F_\infty), \mathbb{Q}_p/\mathbb{Z}_p(i)))$ is trivial. Thus, the cohomology group $H^1(\Gamma, \varinjlim H^1(G_S(F_n), \mathbb{Z}_p(i)))$ is trivial. Using this fact and Lemma 2.14, we obtain that

$$j_{i,\infty} : Ш^2(G_S(F), \mathbb{Z}_p(i)) \longrightarrow (\varinjlim Ш^2(G_S(F_n), \mathbb{Z}_p(i)))^\Gamma$$

is an isomorphism. Taking the Pontryagin dual we get the desired isomorphism. □

3. The twisted Kummer-Leopoldt's constant

Let F be a number field and let S be the set of p-adic and infinite places of F. We set by A_F the p-primary part of the (p)-class group of F. We denote by U the group of S-units of F and by $\widehat{U}$ the pro-p-completion of U.

A description of the Galois group $\mathcal{X}_F$ is given by the class field exact sequence relative to the decomposition

$$\widehat{U} \xrightarrow{\alpha} \bigoplus_{v|p} \widehat{F}_v \xrightarrow{\varphi} \mathcal{X}_F \longrightarrow A_F \longrightarrow 0, \tag{11}$$

where α is the natural pro-p-diagonal map and φ is the product of the local reciprocity homomorphisms which send each $\widehat{F}_v$ to the decomposition group in $\mathcal{X}_F$.

In Section 2, we noticed some equivalences formulations of Leopoldt's conjecture in terms of the $\mathbb{Z}_p$-rank of the p-ramified Iwasawa module and cohomology groups. Another formulation of this conjecture is the injectivity of the natural pro-p-diagonal map α or, equivalently, is the validity of the following property: For all integer $s \geq 1$, there exists an integer $t \geq 1$ such that:

$$\forall u \in U, \, (u \in {F_v}^{p^t}, \, \forall v \mid p) \Longrightarrow u \in U^{p^s},$$

[7, Section 4]. Using the isomorphism

$$\widehat{U} \cong H^1(G_S(F), \mathbb{Z}_p(1)),$$

the map α is nothing but the localization homomorphism:

$$\begin{array}{rrcl} \alpha^{(1)} = \bigoplus_{v|p} \alpha_v^{(1)} : & H^1(G_S(F), \mathbb{Z}_p(1)) & \longrightarrow & \bigoplus_{v|p} H^1(F_v, \mathbb{Z}_p(1)) \\ & x & \longmapsto & (\alpha_v^{(1)}(x))_v \end{array}$$

For an integer i, we consider the twisted analogue of the map α

$$\begin{array}{rrcl} \alpha^{(i)} = \bigoplus_{v|p} \alpha_v^{(i)} : & H^1(G_S(F), \mathbb{Z}_p(i)) & \longrightarrow & \bigoplus_{v|p} H^1(F_v, \mathbb{Z}_p(i)) \\ & x & \longmapsto & (\alpha_v^{(i)}(x))_v \end{array}$$

and if $x \in H^1(G_S(F), \mathbb{Z}_p(i))$, we keep (for simplicity) the notation $x := \alpha_v^{(i)}(x) \in H^1(F_v, \mathbb{Z}_p(i))$. Then, we consider the following property:

($\mathfrak{L}_i$) *For all integer $s \geq 1$, there exists an integer $t \geq 1$ such that:*

$$x \in H^1(G_S(F), \mathbb{Z}_p(i)) \, (x \in H^1(F_v, \mathbb{Z}_p(i))^{p^t}, \, \forall v \mid p) \Rightarrow x \in H^1(G_S(F), \mathbb{Z}_p(i))^{p^s}$$

Remark 3.1. *Notice that for all $t' \geq t$, we have*

$$H^1(F_v, \mathbb{Z}_p(i))^{p^{t'}} \subseteq H^1(F_v, \mathbb{Z}_p(i))^{p^t}.$$

Therefore, we can suppose that $t \geq s$ in the property ($\mathfrak{L}_i$).

For every integer i, the Poitou-Tate exact sequence with coefficients in the modules $\mathbb{Z}/p^n\mathbb{Z}(i)$ induces, by passing to the projective limit, the following exact sequence [19, page 682]

$$H^1(G_S(F), \mathbb{Z}_p(i)) \xrightarrow{\alpha^{(i)}} \bigoplus_{v|p} H^1(F_v, \mathbb{Z}_p(i)) \to \mathcal{X}_F^{(1-i)} \twoheadrightarrow Ш^2(G_S(F), \mathbb{Z}_p(i)) \tag{12}$$

When $i = 1$, $Ш^2(G_S(F), \mathbb{Z}_p(i)) \simeq A_F$ and the exact sequence (12) is nothing but the class field theory exact sequence (11). For $i \neq 1$,

$$Ш^2(G_S(F), \mathbb{Z}_p(i)) \simeq X'_\infty(i-1)_{G_\infty}$$

[28, Section 6, Lemma 1] and we have a twisted analogue of (11):

$$H^1(G_S(F), \mathbb{Z}_p(i)) \xrightarrow{\alpha^{(i)}} \bigoplus_{v|p} H^1(F_v, \mathbb{Z}_p(i)) \longrightarrow \mathcal{X}_F^{(1-i)} \longrightarrow X'_\infty(i-1)_{G_\infty} \longrightarrow 0.$$

In the next lemma, for $i \neq 0$, we show an equivalence between the validity of Conjecture $\mathrm{C}^{(1-i)}$ and the injectivity of the localization map:

$$\alpha^{(i)} : H^1(G_S(F), \mathbb{Z}_p(i)) \longrightarrow \bigoplus_{v|p} H^1(F_v, \mathbb{Z}_p(i)).$$

Lemma 3.2. *Let $i \neq 0$ be an integer. The following assertions are equivalent:*

i) The map $\alpha^{(i)}$ is injective.
ii) Conjecture $\mathrm{C}^{(1-i)}$ holds for (F, p).

Proof. Remark that for every p-adic prime v of F, the absolute Galois group of F_v acts non trivially on $\mathbb{Z}_p(i)$ when $i \neq 0$. Hence the cohomology group $H^0(F_v, \mathbb{Z}_p(i))$ is trivial for every p-adic prime v. Therefore, the Poitou-Tate exact sequence induces the following exact sequence

$$0 \longrightarrow H^2(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(1-i))^\vee \longrightarrow H^1(G_S(F), \mathbb{Z}_p(i)) \xrightarrow{\alpha^{(i)}} \bigoplus_{v|p} H^1(F_v, \mathbb{Z}_p(i)).$$

This shows that

$$\ker(\alpha^{(i)}) \cong H^2(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(1-i))^\vee.$$

□

Remark 3.3. *Although there is no Conjecture $\mathrm{C}^{(1)}$, the map $\alpha^{(0)}$ is always injective. Indeed, by the global Poitou-Tate duality, we have*

$$\ker(\alpha^{(0)}) := Ш^1(G_S(F), \mathbb{Z}_p) \simeq Ш^2(G_S(F), \mathbb{Q}_p/Z_p(1))^\vee.$$

Furthermore,

$$\begin{aligned} Ш^2(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(1)) &= \varinjlim Ш^2(G_S(F), \mu_{p^m}) \\ &= \varinjlim Ш^1(G_S(F), \mathbb{Z}/p^m\mathbb{Z})^\vee \\ &= \varinjlim Cl_S(F)/p^m \\ &= Cl_S(F) \otimes \mathbb{Q}_p/\mathbb{Z}_p \\ &= 0. \end{aligned}$$

In the next theorem we give other equivalences of the twisted Leopoldt's conjecture. The proof is an adaptation of that of [7, Proposition 1].

Theorem 3.4. *Let F be a number field. For all integer $i \neq 0$, the following properties are equivalent:*

(i) Conjecture $\mathrm{C}^{(1-i)}$ holds for (F, p).
(ii) The property $(\mathfrak{L}_i)$ is true.
(iii) There exists a positive integer c_i such that for all $n \geq 1$,

$$x \in H^1(G_S(F), \mathbb{Z}_p(i)) (x \in H^1(F_v, \mathbb{Z}_p(i))^{p^{c_i+n}}, \forall v \mid p) \Rightarrow x \in H^1(G_S(F), \mathbb{Z}_p(i))^{p^n}$$

(iv) There exists a positive integer c_i such that for all $n \gg 0$,

$$x \in H^1(G_S(F),\mathbb{Z}_p(i))(x \in H^1(F_v,\mathbb{Z}_p(i))^{p^{c_i+n}}, \forall v \mid p) \Rightarrow x \in H^1(G_S(F),\mathbb{Z}_p(i))^{p^n}$$

Proof. For a positive integer t, the homomorphism $\alpha^{(i)}$ induces the following one

$$\alpha_t^{(i)} : H^1(G_S(F),\mathbb{Z}_p(i))/p^t \longrightarrow \bigoplus_{v|p} H^1(F_v,\mathbb{Z}_p(i))/p^t.$$

For integers $t \geq s \geq 1$, we consider the following commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \ker(\alpha_t^{(i)}) & \longrightarrow & H^1(G_S(F),\mathbb{Z}_p(i))/p^t & \xrightarrow{\alpha_t^{(i)}} & \bigoplus_{v|p} H^1(F_v,\mathbb{Z}_p(i))/p^t \\
& \downarrow a_{s,t} & & \downarrow b_{s,t} & & \downarrow c_{s,t} \\
0 \longrightarrow & \ker(\alpha_s^{(i)}) & \longrightarrow & H^1(G_S(F),\mathbb{Z}_p(i))/p^s & \xrightarrow{\alpha_s^{(i)}} & \bigoplus_{v|p} H^1(F_v,\mathbb{Z}_p(i))/p^s
\end{array}$$

where the vertical maps are the natural ones. Since $\ker \alpha^{(i)} = \varprojlim \ker \alpha_t^{(i)}$, the homomorphism $\alpha^{(i)}$ is injective if and only if the homomorphism $a_{s,t}$ is trivial for $t \gg s$. According to Lemma 3.2, it follows that the validity of Conjecture $\mathrm{C}^{(1-i)}$ is equivalent to the triviality of the homomorphism $a_{s,t}$ for $t \gg s$. Hence we get the equivalence $(i) \Longleftrightarrow (ii)$.

Now we prove the implication $(ii) \Longrightarrow (iii)$. We suppose that the property $(\mathfrak{L}_i)$ holds and we proceed by induction over n. First let r be an integer such that $H^1(F_v,\mathbb{Z}_p(i))^{p^r}$ has no $\mathbb{Z}_p$-torsion for all prime v above p. By $(\mathfrak{L}_i)$ for $s = r+1$, there is an integer $c_i \geq r$ (see Remark 3.1) such that for all $x \in H^1(G_S(F),\mathbb{Z}_p(i))$:

$$(x \in H^1(F_v,\mathbb{Z}_p(i))^{p^{c_i+1}}, \forall v \mid p) \Longrightarrow x \in H^1(G_S(F),\mathbb{Z}_p(i))^{p^{r+1}}. \qquad (13)$$

The case $n = 1$ is deduced from (13). Let $n > 1$ and let $x \in H^1(G_S(F),\mathbb{Z}_p(i))$ such that x belongs to $H^1(F_v,\mathbb{Z}_p(i))^{p^{c_i+n}}$ for all v above p. According to (13), there is a $y \in H^1(G_S(F),\mathbb{Z}_p(i))$ such that $x = y^{p^{r+1}}$. Since $(y^{p^r})^p = x \in (H^1(F_v,\mathbb{Z}_p(i))^{p^{c_i+n-1}})^p$, we obtain that $y^{p^r} \in H^1(F_v,\mathbb{Z}_p(i))^{p^{c_i+n-1}}$, by the choice of r. Hence $y^{p^r} \in H^1(G_S(F),\mathbb{Z}_p(i))^{p^{n-1}}$, this implies that

$$x = (y^{p^r})^p \in H^1(G_S(F),\mathbb{Z}_p(i))^{p^n}.$$

The implications $(iii) \Longrightarrow (ii)$, $(iii) \Longrightarrow (iv)$ and $(iv) \Longrightarrow (i)$ are obvious. □

Remark 3.5. ***i)*** *From the proof of Theorem 3.4, we see that the truth of $(\mathfrak{L}_i)$ is equivalent to the injectivity of the map $\alpha^{(i)}$ also in the case where $i = 0$. As a consequence of Remark 3.3, the property $(\mathfrak{L}_0)$ is always true.*

ii) *The existence of the constant c_i is trivial in the case of totally real number field F and even integer i, since*

$$H^1(G_S(F),\mathbb{Z}_p(i)) = \mathrm{tor}_{\mathbb{Z}_p} H^1(G_S(F),\mathbb{Z}_p(i));$$

in particular, Conjecture $\mathrm{C}^{(1-i)}$ holds for (F, p).

Definition 3.6. *We define the twisted Kummer-Leopoldt constant $\varkappa_i = \varkappa_i(F)$ of the field F to be the minimal integer c_i satisfying the property (iv) of Theorem 3.4.*

The aim now is to determine the exact value of the twisted Kummer-Leopoldt constant. We shall express it as the exponent of a certain Galois module.

Lemma 3.7. *Let $i \neq 0, 1$ be an integer such that F satisfies Conjecture $\mathrm{C}^{(i)}$. The surjective homomorphism $\psi : \mathcal{X}_F^{(1-i)} \longrightarrow\!\!\!\!\rightarrow X'_\infty(i-1)_{G_\infty}$ factors through a homomorphism*

$$\Psi : \mathrm{T}_F^{(1-i)} \longrightarrow X'_\infty(i-1)_{G_\infty}$$

and $\ker(\Psi)$ *is isomorphic to the Galois group* $\mathrm{Gal}(\mathrm{F}^{\mathrm{BP},(1-i)}/\widetilde{F}^{(1-i)}L^{(1-i)})$.

Proof. First of all, we recall from the end of the proof of Proposition 2.13 that the image of $W^{(i)}$ in $\mathcal{T}_F^{(1-i)}$ is contained in the kernel

$$Y^{(1-i)} := \ker(\psi : \mathcal{X}_F^{(1-i)} \twoheadrightarrow X'_\infty(i-1)_{G_\infty}).$$

Therefore, taking the restriction of the surjective homomorphism

$$\psi : \mathcal{X}_F^{(1-i)} \longrightarrow\!\!\!\!\rightarrow X'_\infty(i-1)_{G_\infty}$$

to $\mathcal{T}_F^{(1-i)}$, we obtain the following commutative diagram with exact lines:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W^{(i)} & \longrightarrow & \mathcal{T}_F^{(1-i)} & \longrightarrow & \mathrm{T}_F^{(1-i)} & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\| & & \big\downarrow \Psi & & \\
0 & \longrightarrow & \mathrm{tor}_{\mathbb{Z}_p} Y^{(1-i)} & \longrightarrow & \mathcal{T}_F^{(1-i)} & \longrightarrow & X'_\infty(i-1)_{G_\infty} & &
\end{array}$$

Thus ψ induces the following homomorphism

$$\Psi : \mathrm{T}_F^{(1-i)} \longrightarrow X'_\infty(i-1)_{G_\infty}.$$

Furthermore, reading the figure in page 379, we see that the kernel $\ker(\Psi)$ is isomorphic to the Galois group $\mathrm{Gal}(\mathrm{F}^{\mathrm{BP},(1-i)}/\widetilde{F}^{(1-i)}L^{(1-i)})$. □

Theorem 3.8. *Let F be a number field and let $i \neq 0, 1$ be an integer such that F satisfies Conjecture $\mathrm{C}^{(1-i)}$. Let $\varkappa_i$ be the twisted Kummer-Leopoldt constant of F. Then $p^{\varkappa_i}$ is the exponent of the Galois group* $\mathrm{Gal}(\mathrm{F}^{\mathrm{BP},(1-i)}/\widetilde{F}^{(1-i)}L^{(1-i)})$.

Proof. Let us prove that $p^{\varkappa_i}$ is the exponent of

$$\ker(\Psi) \simeq \mathrm{Gal}(\mathrm{F}^{\mathrm{BP},(1-i)}/\widetilde{F}^{(1-i)}L^{(1-i)})$$

(Lemma 3.7). Let $j = 1 - i$ and recall that the kernel

$$Y^{(j)} = \ker(\mathcal{X}_F^{(j)} \twoheadrightarrow X'_\infty(-j)_{G_\infty})$$

is equal to the Galois group $\mathrm{Gal}(\widehat{F}^{(j)}/L^{(j)})$. For n sufficiently large such that p^n kills the $\mathbb{Z}_p$-torsion $\mathcal{T}_F^{(j)}$ of $\mathcal{X}_F^{(j)}$, the multiplication by p^n yields the following exact sequence

$$0 \longrightarrow Y^{(j)}[p^n] \longrightarrow \mathcal{T}_F^{(j)} \longrightarrow X'_\infty(-j)_{G_\infty}.$$

Comparing with the exact sequence of Corollary 2.12, we get a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W^{(1-j)} & \longrightarrow & \mathcal{T}_F^{(j)} & \longrightarrow & \mathrm{T}_F^{(j)} & \longrightarrow & 0 \\ & & \downarrow g_n & & \| & & \downarrow \Psi & & \\ 0 & \longrightarrow & Y^{(j)}[p^n] & \longrightarrow & \mathcal{T}_F^{(j)} & \longrightarrow & X'_\infty(-j)_{G_\infty} & & \end{array}$$

Using the snake lemma, we obtain that

$$\ker(\Psi) \simeq \mathrm{coker}(g_n). \tag{14}$$

Since Conjecture $\mathrm{C}^{(j)}$ holds for F, the map $\alpha^{(i)}$ is injective (recall that $j = 1-i$). Let us consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(G_S(F), \mathbb{Z}_p(i)) & \xrightarrow{\alpha^{(i)}} & \bigoplus_{v|p} H^1(F_v, \mathbb{Z}_p(i)) & \longrightarrow & Y^{(j)} & \longrightarrow & 0 \\ & & \downarrow p^n & & \downarrow p^n & & \downarrow p^n & & \\ 0 & \longrightarrow & H^1(G_S(F), \mathbb{Z}_p(i)) & \xrightarrow{\alpha^{(i)}} & \bigoplus_{v|p} H^1(F_v, \mathbb{Z}_p(i)) & \longrightarrow & Y^{(j)} & \longrightarrow & 0. \end{array}$$

By the snake lemma, we obtain the following exact sequence

$$0 \longrightarrow H^1(G_S(F), \mathbb{Z}_p(i))[p^n] \longrightarrow \bigoplus_{v|p} H^1(F_v, \mathbb{Z}_p(i))[p^n] \xrightarrow{\phi_n} Y^{(j)}[p^n] \longrightarrow$$
$$\longrightarrow H^1(G_S(F), \mathbb{Z}_p(i))/p^n \xrightarrow{\alpha_n^{(i)}} \bigoplus_{v|p} H^1(F_v, \mathbb{Z}_p(i))/p^n \longrightarrow \cdots$$

It follows that $\mathrm{coker}(\phi_n)$ is isomorphic to the kernel $\ker(\alpha_n^{(i)})$. Notice that for n large enough,

$$H^1(G_S(F), \mathbb{Z}_p(i))[p^n] \simeq H^0(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(i))$$

and

$$H^1(F_v, \mathbb{Z}_p(i))[p^n] \simeq H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(i))$$

for all v over p. Hence, we get that $\mathrm{coker}(\phi_n)$ is isomorphic to $\mathrm{coker}(g_n)$. Then, by (14)

$$\begin{aligned} \mathrm{coker}(g_n) &\simeq \ker(\alpha_n^{(i)}) \\ &\simeq \ker(\Psi). \end{aligned}$$

Since p^{κ_i} is the exponent of $\ker \alpha_n^{(i)}$, for n large enough, the result follows from Lemma 3.7. $\square$

We finish this section with the following proposition in which we consider the case of a CM-field.

Proposition 3.9. *Let F be a CM-field with totally real subfield F^+ and let i be an odd integer. Assume that the field F^+ satisfies Conjecture $C^{(1-i)}$. Then the twisted Kummer-Leopoldt constants $\kappa_i := \kappa_i(F)$ and $\kappa_i^+ := \kappa_i(F^+)$ are equal.*

Proof. Let n be an integer such that p^n kills both $\bigoplus_{v|p} H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(i))$ and $\mathcal{T}_F^{(1-i)}$. According to the end of the proof of Theorem 3.8, we know that p^{κ_i} is the exponent of

$$\ker(\alpha_n^{(i)} : H^1(G_S(F), \mathbb{Z}_p(i))/p^n \to \bigoplus_{v|p} H^1(F_v, \mathbb{Z}_p(i))/p^n).$$

Let $\tau \in \mathrm{Gal}(F/F^+)$ be the complex conjugation. Consider the decomposition

$$\ker(\alpha_n^{(i)}) = (\ker(\alpha_n^{(i)}))^+ \oplus (\ker(\alpha_n^{(i)}))^-,$$

where $(\ker(\alpha_n^{(i)}))^\pm = (1 \pm \tau)\ker(\alpha_n^{(i)})$. We have to show that $(\ker(\alpha_n^{(i)}))^-$ is trivial and that the exponent of $(\ker(\alpha_n^{(i)}))^+$ is $p^{\kappa_i^+}$. We start by observing that

$$H^1(G_S(F), \mathbb{Z}_p(i))^+ \simeq H^1(G_S(F^+), \mathbb{Z}_p(i)).$$

Since

$$\mathrm{rank}_{\mathbb{Z}_p} H^1(G_S(F), \mathbb{Z}_p(i)) = \mathrm{rank}_{\mathbb{Z}_p} H^1(G_S(F^+), \mathbb{Z}_p(i)),$$

it follows that $H^1(G_S(F), \mathbb{Z}_p(i))^-$ is a $\mathbb{Z}_p$-torsion module. Furthermore, notice that

$$H^0(G_S(F^+), \mathbb{Q}_p/\mathbb{Z}_p(i)) = 0 \iff i \not\equiv 0 \mod [F^+(\mu_p) : F^+].$$

Since $[F^+(\mu_p) : F^+]$ is even and i is odd, we get that $H^0(G_S(F^+), \mathbb{Q}_p/\mathbb{Z}_p(i))$ is trivial. This implies that

$$H^1(G_S(F), \mathbb{Z}_p(i))^- = \mathrm{tor}_{\mathbb{Z}_p} H^1(G_S(F), \mathbb{Z}_p(i)).$$

Using this fact and the choice of n, we see that the map

$$(\alpha_n^{(i)})^- : (H^1(G_S(F), \mathbb{Z}_p(i))/p^n)^- \to (\bigoplus_{v|p} H^1(F_v, \mathbb{Z}_p(i))/p^n)^-$$

is nothing but the injection

$$H^0(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(i)) \to (\bigoplus_{v|p} H^1(F_v, \mathbb{Z}_p(i))/p^n)^-.$$

Therefore, $(\ker(\alpha_n^{(i)}))^-$ is trivial for n large enough.

Also, using the isomorphism

$$(H^1(G_S(F), \mathbb{Z}_p(i))/p^n)^+ \simeq H^1(G_S(F^+), \mathbb{Z}_p(i))/p^n$$

we get that $(\ker(\alpha_n^{(i)}))^+$ is the kernel of the map

$$H^1(G_S(F^+), \mathbb{Z}_p(i))/p^n \to (\bigoplus_{v|p} H^1(F_v, \mathbb{Z}_p(i))/p^n)^+$$

which is of exponent $p^{\kappa_i^+}$. □

4. On the triviality of the twisted Kummer-Leopoldt constant

Let i be an integer and let p be an odd prime number. The (p,i)-regular number fields have been introduced in [2, Definition 1.1] as a generalization of p-rational fields [20, 21, 22]. Recall that a number field F is (p,i)-regular if the cohomology group $H^2(G_S(F),\mathbb{Z}/p\mathbb{Z}(i))$ is trivial, or equivalently if F satisfies Conjecture $\mathrm{C}^{(i)}$ and the $\mathbb{Z}_p$-module $\mathcal{T}_F^{(i)}$ is trivial. In particular, this triviality implies that of $\mathrm{Gal}(\mathrm{F}^{\mathrm{BP},(i)}/\widetilde{F}^{(i)}L^{(i)})$, where $\widetilde{F}^{(i)}$ is the subfield of $\widehat{F}^{(i)}$ fixed by $\mathcal{T}_F^{(i)}$ (Definition 2.5). Hence, by Theorem 3.8, we see that $\varkappa_{1-i}$ is trivial for (p,i)-regular number fields. In this section, we consider the other implication. Precisely, we give a characterization of the (p,i)-regularity in terms of the triviality of $\varkappa_{1-i}$.

Theorem 4.1. *Let $i\neq 0,1$ be an integer and let F be a number field satisfying Conjecture $\mathrm{C}^{(i)}$. Then F is (p,i)-regular if and only if the following three conditions hold:*

1) $\varkappa_{1-i}=0$;
2) *The injective map*
$$H^0(G_S(F),\mathbb{Q}_p/\mathbb{Z}_p(1-i))\longrightarrow\bigoplus_{v|p}H^0(F_v,\mathbb{Q}_p/\mathbb{Z}_p(1-i))$$
is an isomorphism;
3) $H^{(i)}\subset\widetilde{F}^{(i)}$.

Proof. Let us recall that $p^{\varkappa_{1-i}}$ is the exponent of $\mathrm{Gal}(\mathrm{F}^{\mathrm{BP},(i)}/\widetilde{F}^{(i)}L^{(i)})$ by Theorem 3.8 and that $\mathrm{Gal}(\mathrm{F}^{\mathrm{BP},(i)}/\widetilde{F}^{(i)}L^{(i)})\simeq\ker(\Psi:\mathrm{T}_F^{(i)}\longrightarrow X'_\infty(-i))$ by Lemma 3.7.

Suppose that F is (p,i)-regular. Then, the $\mathbb{Z}_p$-torsion module $\mathcal{T}_F^{(i)}$ is trivial. Using the exact sequence (8) of Corollary 2.12, we get that the groups $W^{(1-i)}$ and $\mathrm{T}_F^{(i)}$ are both trivial. Therefore, we obtain Condition 2) from the triviality of $W^{(1-i)}$, and Condition 1) from the triviality of $\mathrm{T}_F^{(i)}$. Furthermore, the vanishing of $\mathcal{T}_F^{(i)}$ shows that $\widehat{F}^{(i)}=\widetilde{F}^{(i)}$. Since $L^{(i)}$ is contained in $\hat{F}^{(i)}$, we have $L^{(i)}\subset\widetilde{F}^{(i)}$. This proves that $H^{(i)}\subset\widetilde{F}^{(i)}$.

Now assume that the three conditions are satisfied. Using again the exact sequence (8) of Corollary 2.12 we see that $\mathcal{T}_F^{(i)}$ and $\mathrm{T}_F^{(i)}$ are isomorphic, since $W^{(1-i)}$ is trivial by Condition 2). Further, using Remark 2.10 with Condition 3) we obtain that the field $L^{(i)}$ is contained in $\widetilde{F}^{(i)}$. Hence the morphism $\Psi:\mathrm{T}_F^{(i)}\longrightarrow X'_\infty(-i)$ is trivial. In particular, the kernel of Ψ equals to $\mathrm{T}_F^{(i)}$. Therefore, by Theorem 3.8, the Bertrandias-Payan module $\mathrm{T}_F^{(i)}$ is trivial because of the nullity of $\varkappa_{1-i}$. Hence the number field F is (p,i)-regular. □

Remark 4.2 (compare with [8, Proposition 2.3]). *For the case $i=0$, using the same arguments in the proof of Theorem 4.1, we can show that F is p-rational exactly when the three conditions hold:*

1) $\varkappa(F)=0$;

2) The map $\mu_p(F) \longrightarrow \bigoplus_{v|p} \mu_p(F_v)$ *is an isomorphism;*

3) $H_F \subset \widetilde{F}$.

Here $\kappa(F)$ *is the Kummer-Leopoldt constant for the units* [7, Definition 1], H_F *is the Hilbert class field of* F *and* $\widetilde{F}$ *is the composite of all* $\mathbb{Z}_p$*-extensions of* F.

It is well known that the field of rational numbers $\mathbb{Q}$ is p-rational for any prime number p. This is not the case for the (p, i)-regularity. For example, if the prime p is irregular, there is at least an integer i for which $\mathbb{Q}$ is not (p, i)-regular (a consequence of [2, (ii, β) Proposition 1.3]). It is also well known that all subfields of a (p, i)-regular number field are (p, i)-regular. Thus, to study the (p, i)-regularity of number fields we must suppose that $\mathbb{Q}$ is (p, i)-regular. From now on, we assume that $\mathbb{Q}$ is (p, i)-regular and we consider the case of quadratic number fields. The aim is to give a characterization of the (p, i)-regularity of a quadratic number field in the spirit of [12, §4.1].

We start with the following consequence of Theorem 4.1 and Proposition 3.9 that shows the triviality of some twisted Kummer-Leopoldt constants for imaginary quadratic fields.

Corollary 4.3. *Let* p *be an odd prime number and let* i *be an even integer such that* $\mathbb{Q}$ *is* (p, i)*-regular. Then, the Kummer-Leopoldt constant* $\kappa_{1-i}(F)$ *is zero for any imaginary quadratic field* F. □

Now, we prove the following helpful lemma in which we show that the morphism

$$H^0(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \longrightarrow \bigoplus_{v|p} H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))$$

is almost always an isomorphism.

Lemma 4.4. *Let* $F = \mathbb{Q}(\sqrt{d})$ *be a quadratic number field and let* i *be an integer. Then, the map* $H^0(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \longrightarrow \bigoplus_{v|p} H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))$ *is an isomorphism exactly in the following situations:*

i) the prime p *splits in* $F/\mathbb{Q}$ *and* $i \not\equiv 1 \mod (p-1)$;

ii) the prime p *is inert in* $F/\mathbb{Q}$;

iii) the prime p *ramifies in* $F/\mathbb{Q}$ *and* $\frac{2(i-1)}{p-1}$ *is even if* $i \equiv 1 \mod \frac{(p-1)}{2}$ *and* $(-1)^{\frac{p-1}{2}} \frac{d}{p}$ *is a square in* $\mathbb{Q}_p$.

Proof. We start with the following well known isomorphisms

$$H^0(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \simeq \mathbb{Z}/p^{w_i}\mathbb{Z} \text{ and } H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \simeq \mathbb{Z}/p^{w_{v,i}}\mathbb{Z},$$

where

$$w_i := \max\{n \mid i \equiv 1 \mod [F(\mu_{p^n}) : F]\}$$

and

$$w_{v,i} := \max\{n \mid i \equiv 1 \mod [F_v(\mu_{p^n}) : F_v]\}.$$

To prove the lemma, we discuss on the ramification of the prime p in $F/\mathbb{Q}$. We start with the case where p splits in $F/\mathbb{Q}$. Let v and v' be the primes of F above p. Observe that for all $n \geq 1$, we have

$$[F(\mu_{p^n}) : F] = [F_v(\mu_{p^n}) : F_v] = [F_{v'}(\mu_{p^n}) : F_{v'}] = p^{n-1}(p-1).$$

Hence, $H^0(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \to \bigoplus_{v|p} H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))$ is an isomorphism if and only if the groups $H^0(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(1-i))$, $H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))$ and $H^0(F_{v'}, \mathbb{Q}_p/\mathbb{Z}_p(1-i))$ are trivial. This is equivalent to $i \not\equiv 1 \mod (p-1)$.

Suppose now that p is inert in $F/\mathbb{Q}$ and let v be the unique prime of F above p.
Since $F \cap \mathbb{Q}(\mu_p) = \mathbb{Q}$ and $F_v \cap \mathbb{Q}_p(\mu_p) = \mathbb{Q}_p$, we have

$$[F(\mu_{p^n}) : F] = [F_v(\mu_{p^n}) : F_v] = p^{n-1}(p-1) \quad \text{for all } n \geq 1.$$

Thus the map $H^0(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \longrightarrow H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))$ is always an isomorphism.

The remainder case is when p ramifies in $F/\mathbb{Q}$. Let v be the unique prime of F above p. Suppose further that $d \neq (-1)^{\frac{p-1}{2}} p$ and $(-1)^{\frac{p-1}{2}} \frac{d}{p}$ is a square in $\mathbb{Q}_p$. On the one hand, since $\mathbb{Q}(\sqrt{(-1)^{\frac{p-1}{2}} p})$ is the unique quadratic subfield of $\mathbb{Q}(\mu_p)$, we can see that $F \cap \mathbb{Q}(\mu_p) = \mathbb{Q}$. On the other hand, the condition $(-1)^{\frac{p-1}{2}} \frac{d}{p}$ is a square in $\mathbb{Q}_p$ means that $F_v \cap \mathbb{Q}_p(\mu_p) = F_v$. Therefore for all $n \geq 1$, we have

$$[F(\mu_{p^n}) : F] = p^{n-1}(p-1) \text{ and } [F_v(\mu_{p^n}) : F_v] = p^{n-1}\frac{(p-1)}{2}.$$

Comparing the integers w_i and $w_{v,i}$, we get that $w_i = w_{v,i}$ if and only if either $i \not\equiv 1 \mod \frac{(p-1)}{2}$ or $i \equiv 1 \mod \frac{(p-1)}{2}$ and $\frac{2(i-1)}{p-1}$ is even.

To finish the proof we have to show that

$$H^0(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \to H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))$$

is an isomorphism when either $d = (-1)^{\frac{p-1}{2}} p$ or $d \neq (-1)^{\frac{p-1}{2}} p$ and $(-1)^{\frac{p-1}{2}} \frac{d}{p}$ is not a square in $\mathbb{Q}_p$. This is deduced from the fact that in both cases we have

$$[F(\mu_{p^n}) : F] = [F_v(\mu_{p^n}) : F_v] \quad \text{for all } n \geq 1.$$

□

Remark 4.5. ***a)*** *When the integer i satisfies* $i \not\equiv 1 \mod \frac{(p-1)}{2}$, *the localization map*

$$H^0(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \longrightarrow \bigoplus_{v|p} H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))$$

is always an isomorphism.

b) *If the integer i is even, then*

$$H^0(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \to \bigoplus_{v|p} H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))$$

is not an isomorphism exactly when p ramifies in $F/\mathbb{Q}$ and the following two conditions hold:

- $p \equiv 3 \mod (4)$, $d \neq -p$ *and* $\frac{-d}{p}$ *is a square in* $\mathbb{Q}_p$;
- $i \equiv 1 \mod \frac{(p-1)}{2}$ *and* $\frac{2(i-1)}{(p-1)}$ *is odd.*

According to Corollary 4.3 and *ii*) of Remark 3.5, we see that the twisted Kummer-Leopoldt constant $\varkappa_{1-i}(F)$ is always zero when F is an imaginary quadratic field and i is even or F is a real quadratic field and i is odd. Using Theorem 4.1 and Lemma 4.4, we get the following characterizations of the (p,i)-regularity for quadratic fields.

Proposition 4.6. *Let $i \geq 2$ be an integer such that $\mathbb{Q}$ is (p,i)-regular. For a square free integer $d > 0$, let $F = \mathbb{Q}(\sqrt{(-1)^{i+1}d})$. Suppose that F satisfies one of the three conditions in Lemma 4.4. Then, F is (p,i)-regular if and only if $H^{(i)}$ is contained in $\widetilde{F}^{(i)}$. In particular, F is (p,i)-regular when $X'_\infty(-i)_{G_\infty}$ is trivial.* □

Proposition 4.6 concerns only the cases when F is an imaginary quadratic field and i is even or F is a real quadratic field and i is odd. In the other cases, we have the following characterization.

Proposition 4.7. *Let $i \geq 2$ be an integer such that $\mathbb{Q}$ is (p,i)-regular. For a square free integer $d > 0$, let $F = \mathbb{Q}(\sqrt{(-1)^i d})$. Suppose that F satisfies one of the three conditions in Lemma 4.4. Then the quadratic field F is (p,i)-regular exactly when the following conditions hold:*

a) *The map $H^1(G_S(F), \mathbb{Z}_p(1-i))/p \longrightarrow \bigoplus_{v|p} H^1(F_v, \mathbb{Z}_p(1-i))/p$ is injective.*

b) *The field $H^{(i)}$ is contained in $\widetilde{F}^{(i)}$.*

Proof. Let $j = 1-i$. For simplicity we suppose that $H^0(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(j)) = 0$. Following Theorem 4.1 and Lemma 4.4, we have to prove the equivalence between the triviality of $\varkappa_j$ and the injectivity of the map

$$\alpha_1^{(j)} : H^1(G_S(F), \mathbb{Z}_p(j))/p \longrightarrow \bigoplus_{v|p} H^1(F_v, \mathbb{Z}_p(j))/p.$$

Recall that $\varkappa_j$ is trivial precisely when

$$\alpha_n^{(j)} : H^1(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(j))/p^n \longrightarrow \bigoplus_{v|p} H^1(F_v, \mathbb{Q}_p/\mathbb{Z}_p(j))/p^n$$

is injective for n large. Let's consider the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & (H^1(G_S(F),\mathbb{Z}_p(j)))^p & \to & H^1(G_S(F),\mathbb{Z}_p(j)) & \to & H^1(G_S(F),\mathbb{Z}_p(j))/p & \to & 0 \\ & & \downarrow p^{n-1} & & \downarrow p^{n-1} & & \downarrow & & \\ 0 & \to & (H^1(G_S(F),\mathbb{Z}_p(j)))^{p^n} & \to & H^1(G_S(F),\mathbb{Z}_p(j)) & \to & H^1(G_S(F),\mathbb{Z}_p(j))/p^n & \to & 0 \end{array}$$

where the right vertical map is defined by

$$x \mod H^1(G_S(F),\mathbb{Z}_p(j))^p \mapsto x^{p^{n-1}} \mod H^1(G_S(F),\mathbb{Z}_p(j))^{p^n}$$

and is clearly injective. Hence we have

$$H^1(G_S(F),\mathbb{Z}_p(j))/p^{n-1} \simeq \operatorname{coker}(H^1(G_S(F),\mathbb{Z}_p(j))/p \hookrightarrow H^1(G_S(F),\mathbb{Z}_p(j))/p^n).$$

Likewise, we see that for every p-adic place v

$$H^1(F_v,\mathbb{Z}_p(j))/p^{n-1} \simeq \operatorname{coker}(H^1(F_v,\mathbb{Z}_p(j))/p \hookrightarrow H^1(F_v,\mathbb{Z}_p(j))/p^n).$$

Therefore, the commutative diagram

$$\begin{array}{ccccccc} 0 & \to & \ker(\alpha_1^{(j)}) & \to & H^1(G_S(F),\mathbb{Z}_p(j))/p & \xrightarrow{\alpha_1^{(j)}} & \bigoplus_{v|p} H^1(F_v,\mathbb{Z}_p(j))/p \\ & & \downarrow & & \downarrow & & \\ 0 & \to & \ker(\alpha_n^{(j)}) & \to & H^1(G_S(F),\mathbb{Z}_p(j))/p^n & \xrightarrow{\alpha_n^{(j)}} & \bigoplus_{v|p} H^1(F_v,\mathbb{Z}_p(j))/p^n \end{array}$$

and the snake lemma induce the following exact sequence:

$$0 \longrightarrow \ker(\alpha_1^{(j)}) \longrightarrow \ker(\alpha_n^{(j)}) \longrightarrow \ker(\alpha_{n-1}^{(j)}) \longrightarrow 0. \tag{15}$$

An inductive process and the exact sequence (15) show that $\ker(\alpha_n^{(j)})$ is trivial for all $n \geq 2$ when $\ker(\alpha_1^{(j)})$ is. This means that $\varkappa_j = 0$ when $\ker(\alpha_1^{(j)})$ is trivial. Conversely, if $\varkappa_j = 0$, the exact sequence (15) shows that $\ker(\alpha_1^{(j)})$ is trivial. □

The main results of this section can be compared with [8, Proposition 2.3] and [12, Proposition 4.1]. In fact, Condition a) in the above proposition can be interpreted using Kummer theory. Indeed, it is well known that there is a subgroup $D_F^{(1-i)}$ of $E^\bullet := E \setminus \{0\}$, $E = F(\mu_p)$, such that

$$H^1(G_S(F),\mathbb{Z}_p(1-i))/p \cong D_F^{(1-i)}/E^{\bullet p}(-i)$$

and, for each prime v of F above p, a subgroup $D_v^{(1-i)}$ of $E_w^\bullet$, w being a prime of E above v, such that

$$H^1(F_v,\mathbb{Z}_p(1-i))/p \cong D_v^{(1-i)}/E_w^{\bullet\, p}(-i),$$

[10, 5, 6]. So, Condition a) asserts that the natural map

$$D_F^{(1-i)}/E^{\bullet p} \longrightarrow \bigoplus_{v|p} D_v^{(1-i)}/E_w^{\bullet\, p},$$

where for each v above p, w is a place of E dividing v, is injective.

Example. *The quadratic number field* $F = \mathbb{Q}(\sqrt{\pm p})$ *is* (p,i)*-regular for every integer* $i \equiv 1 \mod (p-1)$*. In fact, note that* F *has a unique* p*-adic prime and its class number is less than* p *e.g.,* [8, page 14]*. Hence according to* [2, (ii,α), Proposition 1.3]*,* F *is* (p,i)*-regular. Then the quadratic number field* F *satisfies Conjecture* $\mathrm{C}^{(i)}$ *and* $\kappa_{1-i} = 0$ *for all* $i \equiv 1 \mod (p-1)$*.*

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